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# On the Adaptivity Analysis of the Wave Equation

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Abstract. The purpose of this work deals with the discretization of a second order linear wave equation by the implicit Euler scheme in time and by the spectral elements method in space. We prove that the adaptivity of the time steps can be combined with the adaptivity of the spectral mesh in an optimal way. Two families of error indicators, in time and in space, are proposed. Optimal estimates are obtained.

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## 1. Introduction

A posteriori error analysis of partial differential equation has gained much attention over the past twenty years. Elliptic, and parabolic problems have been widely developed by this theory in the context of the finite element approximation [27, 6, 12, 4, 3], and [7, 13, 14, 18, 19, 20, 28, 17]. However, the a posteriori analysis of hyperbolic problems either by the finite element method or by the spectral element method has not been well addressed in the literature [5, 10, 11, 16, 24, 25, 15, 1, 2].

The purpose of this work is to develop the a posteriori error analysis of initial-boundary-value problem for the second-order linear wave equation, discretized by the spectral elements method. The spectral element method consists of approximating the solution of partial differential equations with higher order polynomial functions on each element of the decomposition [8, 9, 23]. The discretization parameter is a $K$-tuple formed by the maximum polynomial degree $N_k$ on each element. However, as for the $h-p$ version of the finite element method, (see [6, 13]) this parameter is also a quantity $h_k$ representing the diameter of the element. To convert the second-order wave equation to a first-order system, we show that the time discretization is equivalent to the backward Euler-time discretization of the related first-order system.

This work is an extension to the spectral element method of the results obtained by Bernardi *and al.* [7] for the finite element method. More specifically, we present here two families of indicators, both of them are residual types. The first family of indicators is introduced in [20]. Those indicators are global with respect to spatial variable but local with respect to time discretization. Choosing the next time step is based on the time error indicator. The second family of indicators is an efficient tool for mesh adaptivity. These indicators are local for both temporal and spatial variables and can be computed explicitly as a function of the discrete solution and problem data. They are said to be optimal if their Hilbert sum is equivalent to the

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error and the equivalent constant is independent of the discretization parameter. This document is organized as follows:

Section 2 presents the second order linear wave equation and discusses the time-semi-discrete problem and its spatial discretization.

In section 3, we construct error indicators for the wave equation and prove upper and lower bounds based on time and space indicators.

## 2. Time and space discretization

**2.1. The continuous problem.** We consider $\Omega$ an open bounded connected domain of $\mathbb{R}^d$, for $d = 1$, $d = 2$ or $d = 3$. Let $\Gamma$ its Lipschitz continuous boundary and $T$ a positive real number.

For $f \in L^1(0,T;H_0^1(\Omega))$, we consider the following initial-boundary-value problem for the second-order linear wave equation,

$$\begin{cases} \partial_t^2 u - \Delta u = f & \text{in} \quad \Omega\times]0,\mathrm{T}[, \\ u = 0 & \text{on} \quad \Gamma\times]0,\mathrm{T}[, \\ u(.,0) = u_0 & \quad \text{in} \quad \Omega, \\ \partial_t u(.,0) = v_0 & \quad \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $u$ is the unknown defined on $\Omega\times]0,T[$, and $(u_0, v_0)$ are the data functions defined on $\Omega$.

**Proposition 2.1.** *For any data $f$ belongs to $L^1(0,T;H_0^1(\Omega))$ and $(u_0,v_0)$ belongs to $H_0^1(\Omega)\times L^2(\Omega)$, problem (1) has a unique solution $u$ in $\mathcal{C}^1(0,T;L^2(\Omega))\cap\mathcal{C}^0(0,T;H_0^1(\Omega))$ which satisfies the estimation for $0 \le t \le T$,*

$$\Big( \| \partial_t u \|^2 + \| \nabla u \|^2 \Big)^{\frac{1}{2}} \le \Big( \| v_0 \|^2 + \| \nabla u_0 \|^2 \Big)^{\frac{1}{2}} + \int_0^t \| f \| (s) ds. \tag{2}$$

The proof of the well posed-ness of the system (1) is based on the Cauchy-Lipschitz theorem and the estimate (2), see ([21], Chap. 1, Th. 12.3) for the detail of the proof. See also [22, 26, 29, 30, 31] for more general study about a non linear wave equation.

**2.2. The time semi discrete problem.** We make a partition of the interval $[0,T]$ into sub-intervals $[t_i, t_{i+1}]$, $1 \le i \le I$, where $0 = t_0 < t_1 < \ldots < t_I = T$. Let $\tau_i = t_{i+1} - t_i$, $\tau = (\tau_1, \ldots, \tau_i)$, $|\tau| = \max\limits_{1\le i\le I} |\tau_i|$, and

$$\sigma_\tau = \max_{2\le i\le I} \frac{\tau_i}{\tau_{i-1}}$$

the regularity parameter.

For any family $(u^i)_{1\le i\le I} = u(., t_i)$, we consider the function $u_\tau$, defined on the interval $[0,T]$, affine on each sub-interval $[t_{i-1}, t_i]$; $1 \le i \le I$, such that $u_\tau(t_i) = u(t_i)$, so

$$\forall t \in [t_{i-1}, t_i],\ u_\tau(t) = u^i - \frac{t_i - t}{\tau_i}(u^i - u^{i-1}).$$

Then, we use Euler implicit method for the discretization of the time derivative in problem (1), where the data $f = 0$ to simplify the analysis. The time discrete problem consists to find the sequence $u^i = u(x,t_i)_{0\le i\le I}$ in ${H_0^1(\Omega)}^{I+1}$ such that

$$\begin{cases} \dfrac{u^{i+1}-u^i}{\tau_i} - \dfrac{u^i-u^{i-1}}{\tau_{i-1}} - \tau_i \Delta u^{i+1} = 0 & \text{in} \quad \Omega, \quad 1\le \mathrm{i} \le \mathrm{I}, \\ u^{i+1} = 0 & \text{on} \quad \Gamma, \quad 1\le \mathrm{i} \le \mathrm{I}, \\ u^0 = u_0 & \text{in} \quad \Omega, \\ u^1 = u_0 + h_0 v_0 & \text{in} \quad \Omega, \end{cases} \tag{3}$$

where $(u_0, v_0)$ belongs to $H_0^1(\Omega)\times H_0^1(\Omega)$. If the value of $u^0$ and $v^0$ are known, we prove that $u^{i+1}$; $i\ge 1$ is a solution of the following variational formulation:

Find $u^{i+1}$ in $H_0^1(\Omega)$ such that for any $v\in H_0^1(\Omega)$ we have:

$$\int_\Omega u^{i+1}(\mathbf{x})v(\mathbf{x})d\mathbf{x} + \tau_i^2\int_\Omega \nabla u^{i+1}(\mathbf{x})\nabla v(\mathbf{x})d\mathbf{x} = \int_\Omega \Big(u^i + \frac{\tau_i}{\tau_{i-1}}(u^i - u^{i-1})\Big)(\mathbf{x})v(\mathbf{x})d\mathbf{x}. \tag{4}$$

**Proposition 2.2.** *If $(u_0, v_0)$ belongs to $\in H_0^1(\Omega)\times H_0^1(\Omega)$, problem (4) has a unique solution $u^{i+1}$; $i\ge 1$ belongs to $H_0^1(\Omega)$, which satisfies the following stability conditions*

$$\|\frac{u^{i+1}-u^i}{\tau_i}\|^2 + \|\nabla u^{i+1}\|^2 \le \|v_0\|^2 + 2\|\nabla u_0\|^2 + 2\tau_0^2\|\nabla v_0\|^2. \tag{5}$$

*and*

$$\|\ \frac{u^{i+1}-u^i}{\tau_i}\ \|^2 + \|\ \nabla u^{i+1}\ \|^2 \le 2\Big(\ \|\ v^1\ \|^2 + \|\ \nabla u^1\ \|^2\ \Big). \tag{6}$$

*Proof.* We use the Lax-Milgram theorem for easily proving that the variational formulation (4) has a unique solution. See [1] for the proof of the stability conditions (5) and (6), □

Now, we present in the following theorem the a priori time error estimate.

**Theorem 2.3.** *For the solution $u$ of the problem (1) and $(u^i)_{1\le i\le I}$ solution of the problem (3), the a priori error estimate holds for $0\le i\le I$:*

$$\|\frac{u(t_{i+1})-u(t_i)}{\tau_i} - \partial_t u(t_{i+1})\|^2 + \|\nabla(u(t_i)-u^i)\|^2 \le C\tau^2\Big(\int_0^{t_i}(\|\ \partial_t^3 u\ \| + \|\ \partial_t^2\nabla u\ \|)(s)ds\Big)^2, \tag{7}$$

*where $C$ is a positive constant independent of the step $\tau$.*

See [1] for the proof of theorem 2.3. The estimations (7) is of order 1 since the time discretization is based on the implicit Euler scheme.

**2.3. Spectral element discretization.** In the following, we will focus to the a posteriori analysis of the spectral element method in one dimension, since the polynomials inverse inequalities are not optimal for the spectral method in dimension $d\ge 2$. We now describe the discrete space. Let $\Lambda$ the interval $]-1,1[$. For each discrete time $t_i, 0\le i\le I$, we introduce a partition $P_i$ of the interval $\Lambda$ such that

$$-1 = a_0 \le a_1 \le \dots\dots\dots \le a_{K-1} \le a_K = 1,$$

and $\Lambda_k = ]a_{k-1}, a_k[, 1 \leq k \leq K$. Let $h_k$ the length of the sub-interval $\Lambda_k$, and $h = \max_{1\leq k\leq K} h_k$. The discrete parameter $\delta$ is a $K$-tuple of couples $(h_k, N_k), 1 \leq k \leq K$ where a integer $N_k \geq 2$.

First, we recall the following formulas which we will use after. Let $\xi_0 < ... < \xi_N$ be the zeros of the polynomial $(1-x^2)L_N^{'}$ and $\rho_j$ are its associated weights where $L_N$ is the Legendre polynomial defined on $\Lambda$. The Gauss-Lobatto quadrature formula on the interval $\Lambda = ]-1, 1[$ is written

$$\forall \phi \in \mathbb{P}_{2N-1}(\Lambda); \int_{-1}^{1} \phi(x)dx = \sum_{j=0}^{N} \phi(\xi_j^N)\rho_j^N, \tag{8}$$

where $\mathbb{P}_N(\Lambda)$ is the space of polynomials, defined on $\Lambda$, with degree $\leq N$.
We introduce a discrete scalar product for any $u$ and $v$ continuous on $\overline{\Lambda}$ by

$$(u, v)_\delta = \sum_{k=1}^{K}\sum_{j=0}^{N_k} u(\xi_j^{N_k})v(\xi_j^{N_k})\rho_j^{N_k}, \tag{9}$$

where $\xi_j^{N_k} = F_k^{-1}(\xi_j^N)$ and $\rho_j^{N_k} = (a_k - a_{k-1})\rho_j^N$, $0 \leq j \leq N$, such that $F_k$ is the bijection from $\Lambda_k$ into $\Lambda$.

We denote by $i_\delta$ the Lagrange interpolation operator on the set of nodes $\xi_j^{N_k}$ with values in

$$Y_\delta = \Big\{v_\delta \in H^1(\Lambda);\ v_{\delta|\Lambda_k} \in \mathbb{P}_{N_k}(\Lambda_k), 1 \leq k \leq K\Big\}.$$

Then, for each function $\varphi$ continuous on $\overline{\Lambda}_k$, $i_\delta(\varphi)_{|\Lambda_k}$ in $\mathbb{P}_{N_k}(\Lambda_k)$, and verify

$$i_\delta(\varphi)_{|\Lambda_k}(\xi_j^{N_k}) = \varphi_{|\Lambda_k}(\xi_j^{N_k}).$$

We consider the following property, which will be widely used in the following:

$$\forall\, u_\delta \in Y_\delta,\ \|u_\delta\|^2_{L^2(\Lambda)} \leq (u_\delta, u_\delta)_\delta \leq 3\|u_\delta\|^2_{L^2(\Lambda)}. \tag{10}$$

We define the discrete space as

$$X_\delta^i = \Big\{v_\delta \in H_0^1(\Lambda);\ \forall \Lambda_k \in P_i \quad v_{\delta|\Lambda_k} \in \mathbb{P}_{N_k}(\Lambda_k), 1 \leq k \leq K\Big\}. \tag{11}$$

We introduce the orthogonal projection operator $\Pi_\delta^i$ defined on $H_0^1(\Omega)$ into $X_\delta^i$. If $w \in H_0^1(\Omega)$, $\Pi_\delta^i w$ belongs to $X_\delta^i$ such that:

$$\forall t_\delta \in X_\delta^i, \qquad \Big(\frac{\partial(w - \Pi_\delta^i w)}{\partial x}, \frac{\partial t_\delta}{\partial x}\Big) = 0. \tag{12}$$

So using the Galerkin method combined with numerical integration, the discrete problem deduced from the problem (3) is written as: If $u_0$, and $v_0$ are continuous on $\overline{\Lambda}$, find $(u_\delta^i)_{0\leq i\leq I}$ in $\prod_{i=0}^{I} X_\delta^i$ such that:

$$u_\delta^0 = i_\delta u_0 \quad , \text{and} \quad u_\delta^1 = i_\delta u_0 + \tau_0 i_\delta v_0, \tag{13}$$

$$\forall v_\delta \in X_\delta^{i+1}, \Big(\frac{u_\delta^{i+1} - \Pi_\delta^{i+1}u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}, v_\delta\Big)_\delta + \tau_i(\frac{\partial u_\delta^{i+1}}{\partial x}\frac{\partial v_\delta}{\partial x})_\delta = 0. \tag{14}$$

As in the problem (4), we show that each $u_\delta^{i+1}$, $1 \le i \le I$ is the solution of the following discrete variational problem:

Find $u_\delta^{i+1}$ in $X_\delta^{i+1}$ such that:

$$\forall v_\delta \in X_\delta^{i+1}, \quad (u_\delta^{i+1}, v_\delta)_\delta + \tau_i^2 (\frac{\partial u_\delta^{i+1}}{\partial x}, \frac{\partial v_\delta}{\partial x})_\delta = \Big(\Pi_\delta^{i+1} u_\delta^i + \frac{\tau_i}{\tau_{i-1}} (u_\delta^i - \Pi_\delta^i u_\delta^{i-1}), v_\delta\Big)_\delta. \tag{15}$$

Thus, we easily prove using the Lax-Milgram theorem that the problem (13)-(14) has a unique solution.

**Remark 2.1.** The choice to work with different spectral meshes at each time step motivated us to use the $\Pi_\delta^i$ operators in contrary to the classical discretization of the spectral fixed-grid of the wave equation (see [1]),

## 3. A posteriori analysis of the discretization

In this section, we begin by defining two families of error indicators. The first relates to the discretization in time, and the second concerns the spectral discretization. We prove upper and lower bounds on the error, focusing first on the time discretization, and then on the spatial discretization.

### 3.1. A posteriori analysis of the time discretization.

We define the time indicators for each $1 \le i \le I$,

$$\kappa_i = \tau_i \parallel \frac{\partial (u_\delta^{i+1} - u_\delta^i)}{\partial x} \parallel + \tau_i \parallel \frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}} \parallel . \tag{16}$$

This type of time indicators was first used in [20]. We refer also to [7] for their use in a posteriori analysis of the finite element discretization of some parabolic problem (Heat equation). We remark that if the values of the discrete solutions $u_\delta^{i+1}$, $u_\delta^i$ and $u_\delta^{i-1}$ are known, we can easily compute the time indicator $\kappa_i$.

Let $v^i = \frac{u^i - u^{i-1}}{\tau_{i-1}}$ for $1 \le i \le I$. Thus the residual problem if $U = \begin{pmatrix} u \\ v \end{pmatrix}$, and $U_\tau = \begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix}$ is

$$\begin{cases} \partial_t (U - U_\tau) - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} (U - U_\tau) = \begin{pmatrix} D_u \\ D_v \end{pmatrix} & \text{in} \quad \Omega \times ]0, \mathrm{T}[, \\ u - u_\tau = 0 & \text{on} \quad \Gamma \times ]0, \mathrm{T}[, \\ (U - U_\tau)(.,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{17}$$

where $D_u(x,t) = v - v_\tau$, for $t_i \le t \le t_{i+1}$, $1 \le i \le I-1$, and $D_u(x,t) = 0$, for $0 \le t \le t_1$ likewise $D_v(x,t) = \frac{\partial^2 (u - u_\tau)}{\partial x^2}$, for $t_i \le t \le t_{i+1}$, $1 \le i \le I-1$ and $D_v(x,t) = \frac{\partial^2 u_\tau}{\partial x^2}$, for $0 \le t \le t_1$.

**Proposition 3.1.** *The a posteriori error estimate between the solution $u$ of problem (1), when the data $f = 0$, and the solution $(u^i)_{0 \le i \le I}$ of problem (3) holds for*

$0 \leq i \leq I$:

$$\begin{aligned}
&\| (\partial_t u)(t_{i+1}) - \frac{u^{i+1} - u^i}{\tau_i} \|_{H^{-1}(\Lambda)} + \| u(t_{i+1}) - u^{i+1} \| \\
&\leq C \Bigg( \sum_{j=1}^{i} \tau_j ( \| \frac{\partial(u^{j+1} - u_\delta^{j+1})}{\partial x} \| + \| \frac{\partial(u^j - u_\delta^j)}{\partial x} \| ) \\
&\quad + \| (u^{j+1} - u_\delta^{j+1}) - (u^j - \Pi_\delta^{j+1} u_\delta^j) \| \\
&\quad + (\frac{\tau_j}{\tau_{j-1}}) \| (u^j - u_\delta^j) - (u^{j-1} - \Pi_\delta^j u_\delta^{j-1}) \| + \kappa_j + \tau_0 \|\nabla u_0\| + \tau_0^2 \|\nabla v_0\| \Bigg).
\end{aligned} \tag{18}$$

*Proof.* We make the inner product of (17) with $\begin{pmatrix} u - u_\delta \\ \Delta^{-1}(v - v_\delta) \end{pmatrix}$. Let

$$\Sigma(t) = \Big( \| u - u_\delta \|^2 + \| v - v_\delta \|_{H^{-1}(\Omega)}^2 \Big)^{\frac{1}{2}}.$$

Then, we conclude that

$$\frac{1}{2} \frac{d^2 \Sigma}{dt} = (D_u, u - u_\delta) + (D_v, \Delta^{-1}(v - v_\delta)) \leq \Big( \| D_u \|^2 + \| D_v \|_{H^{-1}(\Omega)}^2 \Big)^{\frac{1}{2}} \Sigma.$$

So,

$$\frac{d\Sigma}{dt} \leq \Big( \| D_u \|^2 + \| D_v \|_{H^{-1}(\Omega)}^2 \Big)^{\frac{1}{2}} \leq \| D_u \| + \| D_v \|_{H^{-1}(\Omega)} . \tag{19}$$

Since $\Sigma(0) = 0$, then by integration of (19) between 0 and $t_{i+1}$, we obtain

$$\Sigma(t_{i+1}) \leq \int_0^{t_{i+1}} (\| D_u \| + \| D_v \|_{H^{-1}(\Omega)}) dt.$$

We know that

$$\forall t \in [t_j, t_{j+1}],\ u_\tau(t) = u^{j+1} - \frac{t_{j+1} - t}{\tau_j} (u^{j+1} - u^j),$$

then

$$\begin{aligned}
\int_{t_j}^{t_{j+1}} \| D_v \|_{H^{-1}(\Omega)} \, dt = \frac{\partial^2 (u^{j+1} - u^j)}{\partial x^2} \int_{t_j}^{t_{j+1}} (\frac{t_{j+1} - t}{\tau_j}) dt \\
= (\frac{\tau_j}{2}) (\frac{\partial^2 (u^{j+1} - u^j)}{\partial x^2}).
\end{aligned}$$

Thus, we conclude using the triangular inequality

$$\| \frac{\partial^2 (u^{j+1} - u^j)}{\partial x^2} \| \leq \| \frac{\partial^2 (u_\delta^{j+1} - u_\delta^j)}{\partial x^2} \| + \| \frac{\partial^2 (u^{j+1} - u_\delta^{j+1})}{\partial x^2} \| + \| \frac{\partial^2 (u^j - u_\delta^j)}{\partial x^2} \| .$$

We use the same arguments to evaluate $\int_{t_j}^{t_{j+1}} \| D_u \| \, dt$. Combining all these inequalities leads to the desired result (18). □

In the following proposition, we prove an upper bound of the error indicators $\kappa_i$ for each $0 \leq i \leq I$.

**Proposition 3.2.** *For each indicators $\kappa_i$, $0 \le i \le I$, we have the following estimate*

$$\begin{aligned}
\kappa_i \le & \| \int_{t_i}^{t_{i+1}} \frac{\partial^2 (u - u_\delta)}{\partial x^2} \, dt \| + \| \int_{t_i}^{t_{i+1}} (v - v_\delta) \, dt \| \\
& + \sum_{k=0}^{1} \| (\partial_t u)(t_{i+1-k}) - \frac{u^{i+1-k} - u^{i-k}}{\tau_{i-k}} \|_{H^{-1}(\Omega)} + \| u(t_{i+1-k}) - u^{i+1-k} \| \\
& + \tau_i \sum_{k=0}^{1} \| \frac{\partial^2 (u^{i+1-k} - u_\delta^{i+1-k})}{\partial x^2} \| + \\
& + \| \frac{(u^{i+1-k} - u_\delta^{i+1-k}) - (u^{i-k} - \Pi_\delta^{i+1-k} u_\delta^{i-k})}{\tau_{i-k}} \| .
\end{aligned} \tag{20}$$

*Proof.* Using triangular inequality, it is enough to bound the following terms

$$\tau_i \| \frac{\partial^2 (u^{i+1} - u^i)}{\partial x^2} \|, \quad \tau_i \| v^{i+1} - v^i \| . \tag{21}$$

i) To bound the first term of (21), we make the inner product of the second line of (17) by $(u^{i+1} - u^i)$ and we integrate between the time $t_i$ and $t_{i+1}$. So, we have

$$\begin{aligned}
\frac{\tau_i}{2} \| \frac{\partial^2 (u^{i+1} - u^i)}{\partial x^2} \|^2 \le & \int_{t_i}^{t_{i+1}} (\partial_t (v - v_\tau), u^{i+1} - u^i) dt \\
& + ( \int_{t_i}^{t_{i+1}} \frac{\partial^2 (u - u_\tau)}{\partial x^2} dt, \frac{\partial^2 (u^{i+1} - u^i)}{\partial x^2} ).
\end{aligned}$$

Then by integrating by parts, we conclude that

$$\begin{aligned}
\frac{\tau_i}{2} \| \frac{\partial^2 (u^{i+1} - u^i)}{\partial x^2} \|^2 \le & \Big( (\partial_t u)(t_{i+1}) - \frac{u^{i+1} - u^i}{\tau_i}, u^{i+1} - u^i \Big) \\
& - \Big( (\partial_t u)(t_i) - \frac{u^i - u^{i-1}}{\tau_{i-1}}, u^{i+1} - u^i \Big) \\
& + \Big( \int_{t_i}^{t_{i+1}} \frac{\partial^2 (u - u_\tau)}{\partial x^2} dt, \frac{\partial^2 (u^{i+1} - u^i)}{\partial x^2} \Big).
\end{aligned}$$

Applying Cauchy-Schwarz inequality we conclude

$$\begin{aligned}
\frac{\tau_i}{2} \| \frac{\partial^2 (u^{i+1} - u^i)}{\partial x^2} \| \le & \| (\partial_t u)(t_{i+1}) - \frac{u^{i+1} - u^i}{\tau_i} \|_{H^{-1}(\Omega)} \\
& + \| (\partial_t u)(t_i) - \frac{u^i - u^{i-1}}{\tau_{i-1}} \|_{H^{-1}(\Omega)} + \| \int_{t_i}^{t_{i+1}} \frac{\partial^2 (u - u_\tau)}{\partial x^2} dt \| .
\end{aligned}$$

ii) As in the estimation of the first term of (21), to bound the second term of (21), we make the inner product of the first equation of (17) with $v^{i+1} - v^i$, and integrating between the time $t_i$ and $t_{i+1}$. This leads to

$$\begin{aligned}
\frac{\tau_i}{2} \| v^{i+1} - v^i \|^2 \le & \Big( u(t_{i+1}) - u^{i+1}, v^{i+1} - v^i \Big) - \Big( u(t_i) - u^i, v^{i+1} - v^i \Big) \\
& - \Big( \int_{t_i}^{t_{i+1}} (v - v_\tau) dt, v^{i+1} - v^i \Big).
\end{aligned}$$

This permits us to conclude the estimate (20). □

**3.2. A posteriori analysis of the spectral discretization.** For each $1 \le i \le I$ and each $\Lambda_k, 1 \le k \le K$, we define the refinement spectral indicators

$$\beta_i^k = \| u_\delta^i - \Pi_\delta^{i+1} u_\delta^i \| + N_k^{-1} \| \frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}} \| . \tag{22}$$

These indicators are local and respect both the time and spaces variables and depends of the local discrete solution. So, they can be computed explicitly for each iteration of time. We remark that the first term in (22) appears since we use different spatial meshes on the various time levels, while the other term is the same as those in standard residual-based error bounds for the elliptic equation (see [27]). Then, the residual problem deduced from the system (13)-(14). We define for each $1 \le i \le I$

$$v^i = \frac{u^i - u^{i-1}}{\tau_{i-1}}, \quad v_\delta^i = \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}, \quad eu_\delta^i = u^i - u_\delta^i, \quad ev_\delta^i = v^i - v_\delta^i. \tag{23}$$

Then, we conclude from the two problems 3 and 13-14 that the error vector $E_\delta^i = \begin{pmatrix} eu_\delta^i \\ ev_\delta^i \end{pmatrix}$ is the solution of the following residual problem

$$\begin{cases} \dfrac{E_\delta^{i+1} - E_\delta^i}{\tau_i} - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} E_\delta^{i+1} = \begin{pmatrix} \xi u_\delta^i \\ \xi v_\delta^i \end{pmatrix} & \text{in} \quad \Omega, \quad 0 \le \mathrm{i} \le \mathrm{I}, \\ eu_\delta^{i+1} = 0 & \text{on} \quad \Gamma, \quad 0 \le \mathrm{i} \le \mathrm{I}, \\ E_\delta^1 = \begin{pmatrix} u^0 - u_\delta^0 + \tau_0 (v^0 - v_\delta^0) \\ v^0 - v_\delta^0 \end{pmatrix} & \text{in} \quad \Omega. \end{cases} \tag{24}$$

The two terms $\xi u_\delta^i$ and $\xi v_\delta^i$ belongs to $H^{-1}(\Omega)$ and are defined as

$$\begin{aligned} < \xi u_\delta^i, v > &= \Big( \frac{u_\delta^i - \Pi_\delta^{i+1} u_\delta^i}{\tau_i}, v \Big) \\ < \xi v_\delta^i, v > &= -\frac{1}{\tau_i} \Big( \frac{u_\delta^i - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}, v \Big) - \Big( \frac{\partial u_\delta^{i+1}}{\partial x}, \frac{\partial v}{\partial x} \Big), \end{aligned} \tag{25}$$

where $< \cdot, \cdot >$ is the duality product between $H^{-1}(\Omega)$, and $H_0^1(\Omega)$. The bound of the error estimate by the refinement spectral indicators is the subject of the following proposition.

**Proposition 3.3.** *The a posteriori error estimate between the solution* $(u^i)$ *of the problem* (3)*, and the solution* $(u_\delta^i)$ *of the problem* (13)-(14) *holds for all* $1 \le i \le I - 1$

$$\begin{aligned} & \| \frac{(u^{i+1} - u_\delta^{i+1}) - (u^i - \Pi_\delta^{i+1} u_\delta^i)}{\tau_i} \|_{H^{-1}(\Omega)} + \| u^{i+1} - u_\delta^{i+1} \| \\ & \quad \le C \Big( \sum_{j=1}^{i} \big( \sum_{k=1}^{K} (\beta_j^k)^2 \big)^{\frac{1}{2}} + \| u^0 - u_\delta^0 \| + \tau_0 \| v^0 - v_\delta^0 \| \Big). \end{aligned} \tag{26}$$

*Proof.* By using the inequality (6) applied to the residual problem (24), and knowing that for any $a \geq 0$, $b \geq 0$, $\frac{1}{\sqrt{2}}(a+b) \leq \sqrt{a^2+b^2} \leq a+b$, we obtain that

$$\| ev_\delta^{i+1} \|_{H^{-1}(\Omega)} + \| eu_\delta^{i+1} \| \leq C\Big( \| ev_\delta^1 \|_{H^{-1}(\Omega)} + \| eu_\delta^1 \| + \sum_{j=1}^{i} \tau_j \big( \| \xi v_\delta^{j+1} \|_{H^{-1}(\Omega)} + \| \xi u_\delta^{j+1} \| \big)\Big). \quad (27)$$

Then, we have to bound the terms in the right hand side of inequality (27). The bounds of $\| eu_\delta^1 \|_{H^{-1}(\Omega)}$, and $\| ev_\delta^1 \|$ are done using the last equation of the system (24). From the definition of $\xi u_\delta^i$, we show that

$$\| \xi u_\delta^{j+1} \| = \| \frac{u_\delta^j - \Pi_\delta^{j+1} u_\delta^j}{\tau_j} \| = \frac{1}{\tau_j} \big( \sum_{k=1}^{K} \| u_\delta^j - \Pi_\delta^{j+1} u_\delta^j \|^2_{L^2(\Lambda_k)} \big)^{\frac{1}{2}}.$$

Since,

$$\| \xi v_\delta^{j+1} \|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{< \xi v_\delta^i, v >}{\| \nabla v \|},$$

and using the equality (14), we have for any $v \in H_0^1(\Omega)$ and $v_\delta \in X_\delta^i$

$$\| \xi v_\delta^{j+1} \|_{H^{-1}(\Omega)} = -\frac{1}{\tau_i} \Big( \frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}, v - v_\delta \Big) - \Big( \frac{\partial u_\delta^i}{\partial x}, \frac{\partial (v - v_\delta)}{\partial x} \Big).$$

We consider for any function $v \in H_0^1(\Omega)$ the function

$$v_\delta = \sum_{k=1}^{K} \pi^{1,0}_{N_k-1} \big( v - v(a_{k-1}) \psi_{k-1} - v(a_k) \psi_k \big) + \sum_{k=1}^{K} v(a_k) \psi_k,$$

where $\psi_k$ are an affine functions on $\Lambda_k$ equal to 1 on the node $a_k$, and equal to 0 on the other nodes $a_l, l \neq k$. $\pi^{1,0}_{N_k-1}$ is the orthogonal projection operator from $H_0^1(\Lambda_k)$ into $\mathbb{P}_{N_k}(\Lambda_k) \cap H_0^1(\Lambda_k)$, we refer the reader to [8] for the properties of this operator. The function $v_\delta$ is in the space $X_\delta$ since $v \in H_0^1(\Omega)$. Then, we make an integration by part we obtain that

$$\| \xi v_\delta^{j+1} \|_{H^{-1}(\Omega)} = -\frac{1}{\tau_i} \Big( \frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}, v - v_\delta \Big).$$

So, we conclude the result (26) using the Cauchy Schwarz inequality. □

The upper bound estimate of the spectral refinement indicators is subject of the following proposition.

**Proposition 3.4.** *The following estimate holds for the indicators $\beta_i^k$, $1 \leq i \leq I$,*

$$\beta_i^k \leq C\Big( \sum_{j=0}^{1} \big( \| \frac{(u^{i+1-j} - u_\delta^{i+1-j}) - (u^{i-j} - \Pi_\delta^{i+1-j} u_\delta^{i-j})}{\tau_{i-j}} \|_{H^{-1}(\Lambda_k)} + \| u^{i+1-j} - u_\delta^{i+1-j} \|_{L^2(\Lambda_k)} \big) + \tau_i \big( \| \frac{(u^{i+1} - u_\delta^{i+1}) - (u^i - \Pi_\delta^{i+1} u_\delta^i)}{\tau_i} \|_{L^2(\Lambda_k)} + \| \frac{\partial (u^{i+1} - u_\delta^{i+1})}{\partial x} \|_{L^2(\Lambda_k)} \big) \Big), \quad (28)$$

*where $C$ is a constant independent of $\tau$ and $\delta$.*

*Proof.* We bound successively the two terms in $\beta_i^k$ denoted by ${\beta_1}_i^k$ and ${\beta_2}_i^k$. We have from the first equation of the system (24)

$$\frac{eu_\delta^{i+1} - eu_\delta^i}{\tau_i} - ev_\delta^{i+1} = \xi u_\delta^{i+1} = \frac{u_\delta^i - \Pi_\delta^{i+1} u_\delta^i}{\tau_i}.$$

Now, we make the $L^2$ norm of this equation and we multiply by $\tau_i$ leads to

$${\beta_1}_i^k \leq \sum_{j=0}^{1} \| u^{i+1-j} - u_\delta^{i+1-j} \|_{L^2(\Lambda_k)} + \tau_i \| \frac{(u^{i+1} - u_\delta^{i+1}) - (u^i - \Pi_\delta^{i+1} u_\delta^i)}{\tau_i} \|_{L^2(\Lambda_k)} . \tag{29}$$

Let $v_\delta$ be the function equal to $\Big(\frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}\Big)\psi_k$ on the interval $\Lambda_k$, and equal to 0 on $\Lambda \backslash \Lambda_k$, where $\psi_k$ is an affine functions on $\Lambda_k$ equal to 1 on the node $a_k$, and equal to 0 on the other nodes $a_l, l \neq k$. Then, we show by integration by part that

$$< \xi v_\delta^{i+1}, v_\delta >= -\frac{1}{\tau_i} \| \Big(\frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}\Big)\psi_k^{\frac{1}{2}} \|^2_{L^2(\Lambda_k)} .$$

Thus, making the inner product of the second equation of the system (24) with $-\tau_i v_\delta$ gives

$$\begin{aligned} -\frac{1}{\tau_i} \| \Big(\frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}\Big)\psi_k^{\frac{1}{2}} \|^2_{L^2(\Lambda_k)} \\ \leq \Big( \sum_{j=0}^{1} \| \frac{(u^{i+1-j} - u_\delta^{i+1-j}) - (u^{i-j} - \Pi_\delta^{i+1-j} u_\delta^{i-j})}{\tau_{i-j}} \|_{H^{-1}(\Lambda_k)} \\ + \| \frac{\partial (u^{i+1} - u_\delta^{i+1})}{\partial x} \|_{L^2(\Lambda_k)} \Big) \| \frac{\partial v_\delta}{\partial x} \|_{L^2(\Lambda_k)} . \end{aligned}$$

Now, we use the following two inverse inequality (see [8] and [9] for the proof). For all $\varphi_N \in \mathbb{P}_N(\Lambda)$, we have

$$\int_{-1}^{1} (\varphi_N')^2(\zeta)(1-\zeta^2)^2 d\zeta \leq \; c \, N^2 \int_{-1}^{1} \varphi_N^2(\zeta)(1-\zeta^2) d\zeta,$$

and

$$\int_{-1}^{1} \varphi_N^2(\zeta) d\zeta \leq \; c \, N^2 \int_{-1}^{1} \varphi_N^2(\zeta)(1-\zeta^2) d\zeta.$$

Then, we combine all this inequality we conclude that there exists a constant $C$ such that

$$\begin{aligned} {\beta_2}_i^k \leq C\Big( \sum_{j=0}^{1} \| \frac{(u^{i+1-j} - u_\delta^{i+1-j}) - (u^{i-j} - \Pi_\delta^{i+1-j} u_\delta^{i-j})}{\tau_{i-j}} \|_{H^{-1}(\Lambda_k)} \\ + \| \frac{\partial (u^{i+1} - u_\delta^{i+1})}{\partial x} \|_{L^2(\Lambda_k)} \Big). \end{aligned} \tag{30}$$

Finally, from (29) and (30) we have the desired result (28). □

## Conclusion

For the discretization of the a partial differential equations the a posteriori analysis is a very efficient tool for mesh adaptivity. We interested in this work to the a posteriori analysis of the discretization of the second order wave equation by the spectral element method. We constructed two residual kind of indicators and we proved their optimal upper and lower error bounds. The algorithm of resolution and the implementation of these results will be the subject of our future paper.


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