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Schur convexity of a class of symmetric functions

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Abstract. In this paper we derive some general conditions in order to prove the Schur-convexity of a class of symmetric functions. The log-convexity conditions which appear in this paper will contradicts one of the results of K. Guan from [2]. Also, we prove that a special class of rational maps are Schur-convex functions in R^n_+. As an application, Ky-Fan's inequality is generalized.

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1. Introduction

The Schur-convex functions were introduced by I. Schur in 1923 and have important applications in analytic inequalities, elementary quantum mechanics and quantum information theory. See [4].

The aim of our paper is to derive some general conditions under which the symmetric functions play the property of Schur-convexity. In order to state our results we need some preparation.

Consider two vectors $x = (x_1, ..., x_n),\ y = (y_1, ..., y_n) \in \mathbb{R}^n$.

Definition 1.1. *We say that x is majorized by y, denote it by $x \prec y$, if the rearrangement of the components of x and y such that $x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}$, $y_{[1]} \geq y_{[2]} \geq ... \geq y_{[n]}$ satisfy $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$, $(1 \leq k \leq n-1)$ and $\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}$.*

Definition 1.2. *The function $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}^n$, is called Schur-convex if $x \prec y$ implies $f(x) \leq f(y)$.*

Theorem 1.1. *(see [12]) Let $f(x) = f(x_1, ..., x_n)$ be a symmetric function with continuous partial derivative on $I^n = I \times I \times ... \times I$, where I is an open interval. Then $f : I^n \to \mathbb{R}$ is Schur convex if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0, \qquad (1)$$

on I^n. It is strictly convex if inequality 1 is strict for $x_i \neq x_j$, $1 \leq i, j \leq n$.

An important source of Schur-convex functions is given in [6] by Merkle in the following way:

Theorem 1.2. *Let f be a differentiable function defined on an interval I. Define the function F of two variables by*

$$F(x, y) = \frac{f(x) - f(y)}{x - y}\ (x \neq y), \quad F(x, x) = f'(x),$$

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where $(x, y) \in I^2$. If $x \hookrightarrow f'''(x)$ is continuous then following statements are equivalent:

1) f' is convex on I,

2) $F(x, y) \leq \frac{f'(x)+f'(y)}{2}$, for all $x, y \in I$,

3) $f'(\frac{x+y}{2}) \leq F(x, y)$, for all $x, y \in I$,

4) F is Schur-convex on I^2.

Remark 1.1. *If we consider $f(x) = x^{n+1}$ in Theorem 1.2 we obtain that the elementary symmetric function $\sum_{i=0}^{n} x^i y^{n-i}$ is Schur-convex. See also* [7].

2. Proof of the main results

In this paper we investigate Schur-convexity of the following symmetric functions:

$$\mathcal{F}_n^k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^{k} f(x_{i_j}),\ k = 1, 2, ..., n,$$

where f is a positive function which satisfies certain conditions.

For the case $k = 1$, if f is a convex function, Schur convexity is obvious. (See Hardy-Littlewood-Polya inequality [11]).

In the following we say that a function $f : \Omega \to \mathbb{R}_+$ is log-convex if the function $\log f$ is convex.

Remark 2.1. *If f is a log-convex function then f is also a convex function. See* [11]*. Ostrowski in* [7]*, seems to be the first who noticed the importance of log-convexity in deriving the property of Schur convexity.*

Theorem 2.1. *Let $\Omega \subset \mathbb{R}$ a convex set with nonempty interior. If $f : \Omega \to \mathbb{R}_+$ is a differentiable function in the interior of Ω, continuous on Ω, positive and log-convex, then $\mathcal{F}_n^2(x) = \sum_{1 \leq i < j \leq n} f(x_i) f(x_j)$, $x_i, x_j \in \Omega$, is Schur strictly convex in Ω^n.*

Proof.

$$\mathcal{F}_n^2(x) = f(x_1)f(x_2) + (f(x_1) + f(x_2)) \sum_{i=3}^{n} f(x_i) + G(x_3, ..., x_n).$$

Thus, we have

$$(x_1 - x_2)\Big(\frac{\partial \mathcal{F}_n^2(x)}{\partial x_1} - \frac{\partial \mathcal{F}_n^2(x)}{\partial x_2}\Big)$$
$$= (x_1 - x_2)\Big(f'(x_1)f(x_2) - f'(x_2)f(x_1) + (f'(x_1) - f'(x_2)) \sum_{i=3}^{n} f(x_i)\Big).$$

Since f is a log-convex function ($\frac{f'}{f}$ is monotone), and also convex we have

$$(x_1 - x_2)(f'(x_1)f(x_2) - f'(x_2)f(x_1)) \geq 0.$$

respectively,

$$(x_1 - x_2)(f')(x_1) - f'(x_2) \geq 0.$$

In conclusion,

$$(x_1 - x_2)\Big(\frac{\partial \mathcal{F}_n^2(x)}{\partial x_1} - \frac{\partial \mathcal{F}_n^2(x)}{\partial x_2}\Big) \geq 0,$$

condition which assure the Schur-convexity. □

Theorem 2.2. *Let $\Omega \subset \mathbb{R}$ a convex set with nonempty interior. If $f : \Omega \to \mathbb{R}_+$ is a differentiable function in the interior of Ω, continuous on Ω, positive and log-convex then $\mathcal{F}_n^{n-1}(x) = \sum_{1\le i_1,\ldots<i_{n-1}\le n} \prod_{j=1}^{n-1} f(x_{i_j})$ is strictly Schur-convex in Ω^n .*

Proof.

$$\mathcal{F}_n^{n-1}(x) = \prod_{i=3}^{n} f(x_i)\Big(f(x_1) + f(x_2) + f(x_1)f(x_2)\sum_{j=3}^{n}\frac{1}{f(x_j)}\Big).$$

Hence,

$$(x_1 - x_2)\left(\frac{\partial\mathcal{F}_n^{n-1}(x)}{\partial x_1} - \frac{\partial\mathcal{F}_n^{n-1}(x)}{\partial x_2}\right)$$

$$= (x_1-x_2)\prod_{i=3}^{n} f(x_i)\left(f'(x_1) - f'(x_2) + ... + \sum_{j=3}^{n}\frac{1}{f(x_j)}\left(f'(x_1)f(x_2) - f'(x_2)f(x_1)\right)\right) \ge 0,$$

by the same arguments as in the proof from above. □

In the same hypotheses it follows Schur-geometric-convexity of this family of functions. See [13].

3. Applications

Proposition 3.1. *Let f be a log-convex convex function defined on an interval I. Then the Jensen inequality embedds into a string of inequalities*

$$\begin{aligned} f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) &\le \left(\prod_{k=1}^{n} f(x_k)\right)^{1/n} \\ &\le \left(\frac{1}{\binom{n}{k}}\sum_{1\le i_1<\cdots<i_j\le n} f(x_{i_1})\cdots f(x_{i_j})\right)^{1/k} \\ &\le \frac{1}{n}\sum_{k=1}^{n} f(x_k) \end{aligned}$$

Proof. The first one is motivated by the log-convexity of f and the fact that we have the following majorization $(\frac{1}{n}\sum_{i=1} x_i, ..., \frac{1}{n}\sum_{i=1} x_i) \prec (x_1, ..., x_n)$. The others is motivated by Newton's inequalities. See [11], Appendix B, for a survey on Newton's inequalities. Also using the log-concavity if the function $g(x) = x, x_i \to f(x_i)$ should be obtained the inequality between every middle term and right hand term. □

Remark 3.1. *Among the many example of log-convex functions we recall here: x^{-2}, $\frac{1}{e^x-1}$ and Γ (on $(0,\infty)$), $\frac{x+1}{1-x}$ (on $(0,1)$) and $\frac{x}{\sin x}$ (on $(0,\pi)$). As well known, every log-convex function is convex too. See* [11]*, p. 66.*

If the function f take any positive small values then the Schur-convexity of $\mathcal{F}_n^k$ is equivalent with the log-convexity of the function f.

In [2], K. Guan consider the particular case $f(x) = \frac{x}{1-x}$ on $(0,1)$, which is not log-convex on $(0,1/2)$ and $f(0) = 0$. This contradicts our theory. Moreover, the error in [2] is in the proof of Theorem 2.4 , see the case $x_1 = 1/2$, $x_2 = 1/4$, $x_3 = 1/10$. If we consider in [2] the function $f(x) = \frac{1+x}{1-x}$ all the results became true (f is log-convex on $(0,1)$).

4. Further results and applications

In this section we prove that the function $x \hookrightarrow \frac{ac_{r+1}(x)+bc_r(x)}{\alpha c_r(x)+\beta c_{r-1}(x)}$ is Schur-convex, where $c_r(x) = \sum_{i_1+...+i_n=r} x_1^{i_1} \cdots x_n^{i_n}$, $i_1, ..., i_n$ are nonnegative integers, $r \in \mathbb{N}$ and $a, b, \alpha, \beta \in \mathbb{R}_+$. We extend the inequalities from [3].

In order to prove some further results we present three lemmas.

Lemma 4.1. *Suppose that* $x_i > 0$, $i = 1, ..., n$, $\sum_{i=1}^n x_i = s$, $c \geq s$, *then*

$$\frac{c-x}{nc/s-1} = \Big(\frac{c-x_1}{nc/s-1}, ..., \frac{c-x_n}{nc/s-1}\Big) \prec (x_1, ..., x_n) = x. \tag{2}$$

Lemma 4.2. *Suppose that* $x_i > 0$, $i = 1, ..., n$, $\sum_{i=1}^n x_i = s$, $c \geq s$, *then*

$$\frac{c+x}{s+nc} = \Big(\frac{c+x_1}{s+nc}, ..., \frac{c+x_n}{s+nc}\Big) \prec (\frac{x_1}{s}, ..., \frac{x_n}{s}) = \frac{x}{s}. \tag{3}$$

Lemma 4.3. *Suppose that* $x_i > 0$, $i = 1, ..., n$, $\sum_{i=1}^n x_i = s$, *then*

$$\frac{s}{n} = \Big(\frac{s}{n}, ..., \frac{s}{n}\Big) \prec (x_1, ..., x_n) = x. \tag{4}$$

K. Guan [3] were proved also two lemmas:

Lemma 4.4. *Suppose that* $x_i > 0$, $i = 1, ..., n$. *Let*

$$\overline{x_i} = (x_1, ...x_{i-1}, x_{i+1}, ..., x_n). \tag{5}$$

Then we have

$$c_r(x) = x_i c_{r-1}(x) + c_r(\overline{x_i}). \tag{6}$$

Lemma 4.5. *(See* [3]*) Suppose that* $a = (a_1, ..., a_n)$, $a_i \geq 0$, $i = 1, ..., n$ *and* $r \in \mathbb{N}^*$. *Then we have*

$$D_r^2(a) \leq D_{r-1}(a) D_{r+1}(a), \tag{7}$$

where $D_r(x) = \binom{r+n-1}{n-1}^{-1} C_{[r]}^n(x)$, $\binom{r+n-1}{n-1} = \frac{(n+r-1)!}{(n-1)!r!}$.

Theorem 4.1. *The function* $f(x) = \frac{ac_{r+1}(x)+bc_r(x)}{\alpha c_r(x)+\beta c_{r-1}(x)}$ *is a Schur-convex function in* $\mathbb{R}_+^n$, *where* $r \geq 1$ *is a positive integer and* $a, b, \alpha, \beta \in \mathbb{R}_+$. *Moreover, the function* $f(x)$ *is also increasing in* x_i, $i = 1, ..., n$.

Proof. It is obvious that the function $f(x)$ is symmetric and have continuous partial derivatives in $\mathbb{R}_+^n$. Differentiating f with respect x_i we have

$$\begin{aligned} \frac{\partial f(x)}{\partial x_i} &= \frac{a\alpha\big(\frac{\partial c_{r+1}(x)}{\partial x_i}c_r(x) - \frac{\partial c_r(x)}{\partial x_i}c_{r+1}(x)\big)}{(c_r(x)+c_{r+1}(x))^2} + \frac{b\beta\big(\frac{\partial c_r(x)}{\partial x_i}c_{r-1}(x) - \frac{\partial c_{r-1}(x)}{\partial x_i}c_r(x)\big)}{(c_r(x)+c_{r+1}(x))^2} \\ &+ \frac{a\beta(\frac{\partial c_{r+1}(x)}{\partial x_i}c_{r-1}(x) - \frac{\partial c_{r-1}(x)}{\partial x_i}c_{r+1}(x))}{(c_r(x)+c_{r+1}(x))^2} \end{aligned} \tag{8}$$

We denote the first term from right hand side of 8 by $A(x_i)$, the second by $B(x_i)$ and the third by $C(x_i)$.

From (6) it follows that

$$A(x_i) - A(x_j) = \frac{a\alpha\big(\frac{\partial c_r(x)}{\partial x_j}c_{r+1}(\overline{x_j}) - \frac{\partial c_r(x)}{\partial x_i}c_{r+1}(\overline{x_i})\big)}{(c_r(x)+c_{r+1}(x))^2}.$$

Clearly,

$$\frac{\partial c_{r+1}(x)}{\partial x_i} = c_r(x) + x_i\frac{\partial c_r(x)}{x_i} = c_r(x) + x_i\big(c_{r-1}(x) + x_i\frac{\partial c_{r-1}(x)}{\partial x_i}\big)$$
$$= c_r(x) + x_i c_{r-1}(x) + x_i^2\frac{\partial c_{r-1}(x)}{\partial x_i} = ...$$
$$= c_r(x) + x_i c_{r-1}(x) + x_i^2 c_{r-2}(x) + \cdots + x_i^{r-1}c_1(x) + x_i^r.$$

Using (6), we obtain

$$A(x_i) = \Big((c_r(x)c_r(x) - c_{r+1}(x)c_{r-1}(x)) + x_i(c_r(x)c_{r-1}(x) - c_{r+1}(x)c_{r-2}(x))$$
$$+ \cdots + x_i^{r-2}(c_r(x)c_1(x) - c_{r+1}(x)c_0(x)) + c_r(x)x_i^r\Big)\frac{1}{(c_r(x) + c_{r+1}(x))^2}.$$

Hence,

$$A(x_i) - A(x_j) = \frac{1}{(c_r(x) + c_{r+1}(x))^2}\Big[(c_{r+1}(x) - x_j c_r(x))(c_{r-1}(x) + x_j c_{r-2}(x) + x_j^2 c_{r-3}(x) +$$
$$\cdots + x_j^{r-2}c_1(x) + x_j^{r-1}) - (c_{r+1}(x) - x_i c_r(x))(c_{r-1}(x) + x_i c_{r-2}(x) + x_i^2 c_{r-3}(x) + \cdots$$
$$+ x_i^{r-2}c_1(x) + x_i^{r-1})\Big]$$
$$= \frac{1}{(c_r(x) + c_{r+1}(x))^2}\Big[(c_r(x)c_{r-1}(x) - c_{r+1}(x)c_{r-2}(x))(x_i - x_j) + (c_r(x)c_{r-2}(x)$$
$$- c_{r+1}(x)c_{r-3}(x))(x_i^2 - x_j^2) + \cdots + (c_r(x)c_1(x) - c_{r+1}(x)c_0(x))(x_i^{r-1} - x_j^{r-1})$$
$$+ c_r(x)(x_i^r - x_j^r)\Big].$$

Menon in [5] has proved the following result:

$$\frac{c_r(x)}{c_{r+1}(x)} > \frac{c_{r-2}(x)}{c_{r-1}(x)}, \frac{c_r(x)}{c_{r+1}(x)} > \frac{c_{r-3}(x)}{c_{r-2}(x)}, ..., \frac{c_r(x)}{c_{r+1}(x)} > \frac{c_0(x)}{c_1(x)}. \tag{9}$$

Therefore

$$A(x_i) \geq 0.$$

Notice that

$$(x_i - x_j)(x_i^k - x_j^k) \geq 0, \ (1 \leq k \leq r). \tag{10}$$

From (9) and (10) we get

$$(x_i - x_j)(A(x_i) - A(x_j)) \geq 0.$$

In a similar way we can prove that $B(x_i) \geq 0$ and $(x_i - x_j)(B(x_i) - B(x_j)) \geq 0$. For $C(x_i)$ the proof is different. We rewrite $C(x_i)$ in the form

$$C(x_i) = \frac{1}{(c_r(x) + c_{r+1}(x))^2}\Big(\frac{\partial c_{r+1}}{\partial x_i}c_r(x) - \frac{\partial c_r}{\partial x_i}c_{r+1}(x)$$
$$+ \frac{\partial c_r}{\partial x_i}c_{r+1}(x) - \frac{\partial c_{r-1}}{\partial x_i}c_{r+1}(x)\Big).$$

We study the sign of

$$\frac{\partial c_r(x)}{\partial x_i} - \frac{\partial c_{r-1}(x)}{\partial x_i} = c_{r-1}(x) + (x_i - 1)\frac{\partial c_{r-1}(x)}{\partial x_i}$$
$$= \big((x_i - 1)c_{r-1}(x)\big)'(x_i) > 0.$$

The positivity of last term is fulfilled because the function $x_i \hookrightarrow (x_i - 1)c_{r-1}(x)$ is increasing.

Clearly we have

$$(x_i - x_j)(C(x_i) - C(x_j)) \geq 0.$$

By Theorem 1.1 $f(x)$ is Schur-convex. □

Theorem 4.2. *Suppose that $x_i > 0$, $i = 1, ..., n$, $\sum_{i=1}^{n} x_i = s$, $c \geq s$. Then we have*

$$\frac{ac_{r+1}(c-x) + (nc/s - 1)bc_r(c-x)}{ac_{r+1}(x) + bc_r(x)} \leq \Big(\frac{nc}{s} - 1\Big)\frac{\alpha c_r(c-x) + (nc/s-1)\beta c_{r-1}(x)}{\alpha c_r(x) + \beta c_{r-1}(x)}.$$

Proof. Apply Theorem 4.1 and Lemma 4.1. □

Theorem 4.3. *Suppose that $x_i > 0$, $i = 1, ..., n$, $\sum_{i=1}^{n} x_i = s$, $c \geq 0$. Then we have*

$$\frac{ac_{r+1}(c+x) + (nc/s + 1)bc_r(c+x)}{ac_{r+1}(x) + bc_r(x)} \leq \Big(\frac{nc}{s} + 1\Big)\frac{\alpha c_r(c+x) + (nc/s+1)\beta c_{r-1}(x)}{\alpha c_r(x) + \beta c_{r-1}(x)}.$$

Proof. Apply Theorem 4.1 and Lemma 4.2. □

Corollary 4.1. *Suppose that $x_i > 0$, $i = 1, ..., n$, $\sum_{i=1}^{n} x_i = s$, $c \geq s$. Then we have*

$$\frac{ac_{r+1}(c-x) + (nc/s - 1)bc_r(c-x)}{ac_{r+1}(x) + bc_r(x)} \leq \Big(\frac{nc}{s} - 1\Big)^r,$$

where $a, b \in \mathbb{R}_+$.

Remark 4.1. *If we take $c = 1$ we obtain a new Ky-Fan type inequality of the form*

$$\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}(1 - x_i)} \leq \Big(\frac{ac_{r+1}(1-x) + (nc/s - 1)bc_r(1-x)}{ac_{r+1}(x) + bc_r(x)}\Big)^{\frac{1}{r}}.$$

More interesting results about other forms of Fan's inequality and valuable applications in spaces with nonpositive curvature (NPC spaces) can be found in [9] and [8].

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