On a weaker form of $\omega$-continuity

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Abstract. In [5], Hdeib introduced and investigated a new type of continuity called $\omega$-continuity. In [1], Al-Omari and Noorani have introduced the notion of almost weak $\omega$-continuity. It is the objective of this paper to study almost weak $\omega$-continuity and present some of its basic properties.

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1. Introduction

In this paper, a space will always mean a topological space on which no separation axioms assumed unless explicitly stated.

A subset $A$ of a space $(X, \tau)$ is called $\omega$-closed [4] if it contains all its condensation points. The complement of an $\omega$-closed set is called $\omega$-open, or equivalently, if for each $x \in A$ there exists an open set $U$ containing $x$ such that $|U \setminus A| \leq \aleph_0$ (see [8]). The family of all $\omega$-open subsets of a space $(X, \tau)$, denoted by $\omegaO(X)$, forms a topology on $X$ finer than $\tau$.

$\omega$-closure and $\omega$-interior of a subset $A$ of a space $X$, that were defined in an analogous manner to $cl(A)$ and $int(A)$, respectively, will be denoted by $\omega$-$cl(A)$ and $\omega$-$int(A)$, respectively.

Definition 1.1. A subset $A$ is said to be
(1) regular open [9] if $A = int(cl(A))$,
(2) regular closed [9] if $A = cl(int(A))$,
(3) preopen [7] if $A \subset int(cl(A))$.

A point $x \in X$ is said to be in the $\theta$-closure [10] of a subset $A$ of $X$, denoted by $\theta$-$cl(A)$, if $cl(G) \cap A \neq \emptyset$ for each open set $G$ of $X$ containing $x$. A subset $A$ of a space $X$ is called $\theta$-closed if $A = \theta$-$cl(A)$. The complement of a $\theta$-closed set is called $\theta$-open.

Lemma 1.1. ([4]) Let $A$ be a subset of a space $X$. Then
(1) $A$ is $\omega$-closed in $X$ if and only if $A = \omega$-$cl(A)$.
(2) $\omega$-$cl(X \setminus A) = X \setminus \omega$-$int(A)$.
(3) $\omega$-$cl(A)$ is $\omega$-closed in $X$.
(4) $x \in \omega$-$cl(A)$ if and only if $A \cap G \neq \emptyset$ for each $\omega$-open set $G$ containing $x$.

Definition 1.2. A function $f : X \to Y$ is said to be $\omega$-continuous [5] if $f^{-1}(A) \in \omegaO(X)$ for each open set $A$ of $Y$. 

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2. Almost weakly \( \omega \)-continuous functions

**Definition 2.1.** A function \( f : X \to Y \) is said to be

(1) almost weakly \( \omega \)-continuous at \( x \in X \) [1] if for each open set \( A \) of \( Y \) containing \( f(x) \), there exists an \( \omega \)-open set \( B \) containing \( x \) such that \( f(B) \subset \text{cl}(A) \).

(2) almost weakly \( \omega \)-continuous [1] if for each \( x \in X \), \( f \) is almost weakly \( \omega \)-continuous at \( x \in X \).

**Remark 2.1.** (1) Every weakly continuous function is almost weakly \( \omega \)-continuous [1].

(2) Every \( \omega \)-continuous function is almost weakly \( \omega \)-continuous.

(3) None of the above implications is reversible as shown in the following example and in [1].

**Example 2.1.** Let \( X = \{a, b, c, d\} \) and \( \sigma = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\} \). Consider the set of real numbers \( R \) with the standard topology \( \tau \). Then the function \( f : (R, \tau) \to (X, \sigma) \) defined by \( f(x) = \begin{cases} 1 & x \in R \setminus Q, \\ 0 & x \in Q \end{cases} \), where \( Q \) is the rational numbers is almost weakly \( \omega \)-continuous but it is not \( \omega \)-continuous.

**Theorem 2.1.** The following are equivalent for a function \( f : X \to Y \):

(1) \( f \) is almost weakly \( \omega \)-continuous,

(2) \( \omega\text{-cl}(f^{-1}(\text{int}(cl(A)))) \subset f^{-1}(\text{cl}(A)) \) for every subset \( A \) of \( Y \),

(3) \( \omega\text{-cl}(f^{-1}(\text{int}(K))) \subset f^{-1}(K) \) for every regular closed set \( K \) of \( Y \),

(4) \( \omega\text{-cl}(f^{-1}(B)) \subset f^{-1}(\text{cl}(B)) \) for every open set \( B \) of \( Y \),

(5) \( f^{-1}(B) \subset \omega\text{-int}(f^{-1}(\text{cl}(B))) \) for every open set \( B \) of \( Y \),

(6) \( \omega\text{-cl}(f^{-1}(B)) \subset f^{-1}(\text{cl}(B)) \) for each preopen set \( B \) of \( Y \),

(7) \( f^{-1}(B) \subset \omega\text{-int}(f^{-1}(\text{cl}(B))) \) for each preopen set \( B \) of \( Y \).

**Proof.**

(1) \( \Rightarrow \) (2) : Let \( A \subset Y \) and \( x \in X \setminus f^{-1}(\text{cl}(A)) \). We have \( f(x) \in Y \setminus \text{cl}(A) \). This implies that there exists an open set \( B \) containing \( f(x) \) such that \( B \cap A = \emptyset \). Also, \( \text{cl}(B) \cap \text{int}(\text{cl}(A)) = \emptyset \). Since \( f \) is almost weakly \( \omega \)-continuous, then there exists an \( \omega \)-open set \( S \) containing \( x \) such that \( f(S) \cap \text{cl}(B) = \emptyset \). We have \( S \cap f^{-1}(\text{int}(\text{cl}(A))) = \emptyset \) and hence \( x \in X \setminus \omega\text{-cl}(f^{-1}(\text{int}(\text{cl}(A)))) \). Thus, \( \omega\text{-cl}(f^{-1}(\text{int}(\text{cl}(A)))) \subset f^{-1}(\text{cl}(A)) \).

(2) \( \Rightarrow \) (3) : Let \( K \) be any regular closed set in \( Y \). We have

\[
\omega \text{-cl}(f^{-1}(\text{int}(K))) = \omega \text{-cl}(f^{-1}(\text{int}(\text{cl}(K)))) \subset f^{-1}(\text{cl}(\text{int}(K))) = f^{-1}(K).
\]

(3) \( \Rightarrow \) (4) : Let \( B \) be an open subset of \( Y \). Since \( \text{cl}(B) \) is regular closed in \( Y \), \( \omega\text{-cl}(f^{-1}(B)) \subset \omega\text{-cl}(f^{-1}(\text{cl}(B))) \subset f^{-1}(\text{cl}(B)) \).

(4) \( \Rightarrow \) (5) : Let \( B \) be any open set of \( Y \). Since \( Y \setminus \text{cl}(B) \) is open in \( Y \), by Lemma 1.1,

\[
X \setminus \omega \text{-int}(f^{-1}(\text{cl}(B))) = \omega \text{-cl}(f^{-1}(Y \setminus \text{cl}(B))) \subset f^{-1}(\text{cl}(Y \setminus \text{cl}(B))) \subset X \setminus f^{-1}(B).
\]

(5) \( \Rightarrow \) (1) : Let \( x \in X \) and \( B \) be any open subset of \( Y \) containing \( f(x) \). We have \( x \in f^{-1}(B) \subset \omega\text{-int}(f^{-1}(\text{cl}(B))) \). Take \( S = \omega\text{-int}(f^{-1}(\text{cl}(B))) \). This implies that \( f(S) \subset \text{cl}(B) \) and hence \( f \) is almost weakly \( \omega \)-continuous at \( x \in X \).

(1) \( \Rightarrow \) (6) : Let \( B \) be any preopen set of \( Y \) and \( x \in X \setminus f^{-1}(\text{cl}(B)) \). Then there exists an open set \( R \) containing \( f(x) \) such that \( R \cap B = \emptyset \). We have \( \text{cl}(R \cap B) = \emptyset \).
Since $B$ is preopen, then
\[ B \cap \text{cl}(R) \subset \text{int}(\text{cl}(B)) \cap \text{cl}(R) \subset \text{cl}(\text{int}(\text{cl}(B)) \cap R) \subset \text{cl}(\text{int}(\text{cl}(B)) \cap \text{cl}(B \cap R)) \subset \text{cl}(B \cap R) = \emptyset. \]

Since $f$ is almost weakly $\omega$-continuous and $R$ is an open set containing $f(x)$, there exists an $\omega$-open set $S$ in $X$ containing $x$ such that $f(S) \subset \text{cl}(R)$. We have $f(S) \cap B = \emptyset$ and hence $S \cap f^{-1}(B) = \emptyset$. This implies that $x \in X \setminus \omega\text{-cl}(f^{-1}(B))$ and hence $\omega\text{-cl}(f^{-1}(B)) \subset f^{-1}(\text{cl}(B))$.

(6) $\Rightarrow$ (7): Let $B$ be any preopen set of $Y$. Since $Y \setminus \text{cl}(B)$ is open in $Y$, by Lemma 1.1,
\[ X \setminus \omega\text{-int}(f^{-1}(\text{cl}(B))) = \omega\text{-cl}(f^{-1}(X \setminus \text{cl}(B))) \subset f^{-1}(\text{cl}(Y \setminus \text{cl}(B))) \subset X \setminus f^{-1}(B). \]

Thus, $f^{-1}(B) \subset \omega\text{-int}(f^{-1}(\text{cl}(B)))$.

(7) $\Rightarrow$ (1): Let $x \in X$ and $B$ any open set of $Y$ containing $f(x)$. Then $x \in f^{-1}(B) \subset \omega\text{-int}(f^{-1}(\text{cl}(B)))$. Take $S = \omega\text{-int}(f^{-1}(\text{cl}(B)))$. Then $f(S) \subset \text{cl}(B)$ and hence $f$ is almost weakly $\omega$-continuous at $x$ in $X$.

**Definition 2.2.** A function $f : X \to Y$ is said to be $(\omega, s)$-open if $f(B) \in SO(Y)$ for every $\omega$-open set $B$ of $X$.

**Definition 2.3.** A function $f : X \to Y$ is said to be neatly weak $\omega$-continuous if for each $x \in X$ and each open set $A$ of $X$ containing $f(x)$, there exists an $\omega$-open set $B$ containing $x$ such that $\text{Int}(f(B)) \subset \text{cl}(A)$.

**Theorem 2.2.** If a function $f : X \to Y$ is neatly weak $\omega$-continuous and $(\omega, s)$-open, then $f$ is almost weakly $\omega$-continuous.

**Proof.** Let $x \in X$ and $A \in SO(Y, f(x))$. Since $f$ is neatly weak $\omega$-continuous, there exists an $\omega$-open set $B$ of $X$ containing $x$ such that $\text{Int}(f(B)) \subset \text{cl}(A)$. Since $f$ is $(\omega, s)$-open, then $f(B) \in SO(Y)$. This implies that $f(B) \subset \text{cl}(\text{Int}(f(B))) \subset \text{cl}(A)$. Therefore $f$ is almost weakly $\omega$-continuous.

**Definition 2.4.** A function $f : X \to Y$ is relatively weak $\omega$-continuous if for each open set $A$ of $Y$, the set $f^{-1}(A)$ is $\omega$-open in the subspace $f^{-1}(\text{cl}(A))$.

**Theorem 2.3.** A function $f : X \to Y$ is $\omega$-continuous if and only if $f$ is almost weakly $\omega$-continuous and relatively weak $\omega$-continuous.

**Proof.** "Necessity". Obvious. "Sufficiency". Suppose that $A$ is an open set of $Y$. Since $f$ is relatively weak $\omega$-continuous, there exists an $\omega$-open set $B$ of $X$ such that $f^{-1}(A) = B \cap f^{-1}(\text{cl}(A))$. We have $f^{-1}(A) \subset \omega\text{-int}(f^{-1}(\text{cl}(A)))$. Therefore, $f^{-1}(A) = B \cap \omega\text{-int}(f^{-1}(\text{cl}(A)))$. This shows that $f^{-1}(A)$ is $\omega$-open in $X$ and hence $f$ is $\omega$-continuous.

**Definition 2.5.** A function $f : X \to Y$ is said to be $(\omega, p)$-continuous if $f^{-1}(A) \in \omega\text{O}(X)$ for each preopen set $A$ of $Y$.

**Definition 2.6.** A function $f : X \to Y$ is relatively weak $\omega$-continuous if for each preopen set $A$ of $Y$, the set $f^{-1}(A)$ is $\omega$-open in the subspace $f^{-1}(\text{cl}(A))$.

Observe that relatively weak $\omega$-continuous and relatively weak $\omega$-continuous are equivalent with each other.
Theorem 2.4. A function \( f : X \to Y \) is \((\omega, p)\)-continuous if and only if \( f \) is almost weakly \( \omega \)-continuous and relatively weak \( \omega p \)-continuous.

Proof. Similar to the proof of Theorem 2.3.

Theorem 2.5. The following are equivalent for a function \( f : X \to Y \):

(1) \( f \) is almost weakly \( \omega \)-continuous,

(2) \( f(\omega \text{-cl}(A)) \subseteq \theta \text{-cl}(f(A)) \) for each subset \( A \) of \( X \),

(3) \( \omega \text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\theta \text{-cl}(B)) \) for each subset \( B \) of \( Y \),

(4) \( \omega \text{-cl}(f^{-1}(\text{int}(\theta \text{-cl}(B)))) \subseteq f^{-1}(\theta \text{-cl}(B)) \) for every subset \( B \) of \( Y \).

Proof. (1) \( \Rightarrow \) (2) : Let \( A \subseteq X \) and \( x \in \omega \text{-cl}(A) \). Suppose that \( U \) is any open set of \( Y \) containing \( f(x) \). Then there exists an \( \omega \)-open set \( S \) containing \( x \) such that \( f(S) \subseteq \text{cl}(U) \). Since \( x \in \omega \text{-cl}(A) \), by Lemma 1.1, \( S \cap A \neq \emptyset \). Thus, \( \emptyset \neq f(S) \cap f(A) \subseteq \text{cl}(U) \cap f(A) \) and hence \( f(x) \in \theta \text{-cl}(f(A)) \). Thus, \( f(\omega \text{-cl}(A)) \subseteq \theta \text{-cl}(f(A)) \).

(2) \( \Rightarrow \) (3) : Let \( B \subseteq Y \). We have \( f(\omega \text{-cl}(f^{-1}(B))) \subseteq \theta \text{-cl}(B) \). Thus, \( \omega \text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\theta \text{-cl}(B)) \).

(3) \( \Rightarrow \) (4) : Let \( B \subseteq Y \). Since \( \theta \text{-cl}(B) \) is closed in \( Y \), then

\[
\omega \text{-cl}(f^{-1}(\text{int}(\theta \text{-cl}(B)))) \subseteq f^{-1}(\theta \text{-cl}(\text{int}(\theta \text{-cl}(B)))) \\
= f^{-1}(\text{cl}(\text{int}(\theta \text{-cl}(B)))) \\
\subseteq f^{-1}(\theta \text{-cl}(B)),
\]

(5)

(4) \( \Rightarrow \) (1) : Let \( U \) be any open set of \( Y \). Then \( U \subseteq \text{int}(\text{cl}(U)) = \text{int}(\theta \text{-cl}(U)). \)

Thus,

\[
\omega \text{-cl}(f^{-1}(U)) \subseteq \omega \text{-cl}(f^{-1}(\text{int}(\theta \text{-cl}(U)))) \subseteq f^{-1}(\theta \text{-cl}(U)) = f^{-1}(\text{cl}(U)).
\]

(6)

By Theorem 2.1, \( f \) is almost weakly \( \omega \)-continuous.

Theorem 2.6. The following hold for a function \( f : X \to Y \):

(1) If \( f \) is almost weakly \( \omega \)-continuous, then \( f^{-1}(A) \) is \( \omega \)-closed in \( X \) for every \( \theta \)-closed set \( A \) of \( Y \).

(2) If \( f \) is almost weakly \( \omega \)-continuous, then \( f^{-1}(A) \) is \( \theta \)-open in \( X \) for every \( \theta \)-open set \( A \) of \( Y \).

(3) If \( f^{-1}(\theta \text{-cl}(A)) \) is \( \omega \)-closed in \( X \) for every subset \( A \) of \( Y \), then \( f \) is almost weakly \( \omega \)-continuous.

Proof. (1) and (2) follows from Theorem 2.5.

(3) Let \( A \subseteq Y \). Since \( f^{-1}(\theta \text{-cl}(A)) \) is \( \omega \)-closed in \( X \), then \( \omega \text{-cl}(f^{-1}(A)) \subseteq \omega \text{-cl}(f^{-1}(\theta \text{-cl}(A))) = f^{-1}(\theta \text{-cl}(A)) \). By Theorem 2.5, \( f \) is almost weakly \( \omega \)-continuous.

A space \( X \) is called p-space if countable intersections of open subsets are open.

Theorem 2.7. The following are equivalent for a function \( f : X \to Y \) where \( X \) is a p-space:

(1) \( f \) is almost weakly \( \omega \)-continuous,

(2) \( f \) is weakly continuous.

Proof. It follows from the fact that \( \tau = \omega O(X) \) [2].

Lemma 2.1. If \( f : X \to Y \) is almost weakly \( \omega \)-continuous and \( g : Y \to Z \) is continuous, then the composition \( g \circ f : X \to Z \) is almost weakly \( \omega \)-continuous.
Let \( A \) be an open set of \( Z \) containing \( g(f(x)) \). Then \( g^{-1}(A) \) is an open set of \( Y \) containing \( f(x) \). This implies that there exists an \( \omega \)-open set \( B \) containing \( x \) such that \( f(B) \subseteq \text{cl}(g^{-1}(A)) \). Since \( g \) is continuous, then \( (gof)(B) \subseteq g(\text{cl}(g^{-1}(A))) \subseteq \text{cl}(A) \). Hence, \( gof \) is almost weakly \( \omega \)-continuous.

**Theorem 2.8.** Let \( \{A_i : i \in I\} \) be an \( \omega \)-open cover of a space \( X \). The following are equivalent for a function \( f : X \to Y \):

1. \( f \) is almost weakly \( \omega \)-continuous,
2. for each \( i \in I \), the restriction \( f_{A_i} : A_i \to Y \) is almost weakly \( \omega \)-continuous.

**Proof.** (1) \( \Rightarrow \) (2) : Let \( i \in I \) and \( A_i \) be an \( \omega \)-open set of \( X \). Suppose that \( x \in A_i \) and \( U \) is an open set of \( Y \) containing \( f_{A_i}(x) = f(x) \). Since \( f \) is almost weakly \( \omega \)-continuous, then there exists an \( \omega \)-open set \( B \) containing \( x \) such that \( f(B) \subseteq \text{cl}(U) \). On the other hand, \( B \cap A_i \) is \( \omega \)-open in \( A_i \) containing \( x \) and \( f_{A_i}(B \cap A_i) = f(B \cap A_i) \subseteq f(B) \subseteq \text{cl}(U) \). Thus, \( f_{A_i} \) is almost weakly \( \omega \)-continuous.

(2) \( \Rightarrow \) (1) : Let \( x \in X \) and \( U \) be an open set containing \( f(x) \). There exists \( i \in I \) such that \( x \in A_i \). Since \( f_{A_i} : A_i \to Y \) is almost weakly \( \omega \)-continuous, there exists an \( \omega \)-open set \( B \) in \( A_i \) containing \( x \) such that \( f_{A_i}(B) \subseteq \text{cl}(U) \). Since \( A_i \) is \( \omega \)-open in \( X \), then \( B \) is \( \omega \)-open in \( X \) containing \( x \) and \( f(B) \subseteq \text{cl}(U) \). Thus, \( f \) is almost weakly \( \omega \)-continuous.

**Definition 2.7.** A function \( f : X \to Y \) is said to be faintly \( \omega \)-continuous if for each \( x \in X \) and each \( \theta \)-open set \( V \) of \( Y \) containing \( f(x) \), there exists an \( \omega \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

**Theorem 2.9.** Let \( f : X \to Y \) be a function. The following are equivalent:

1. \( f \) is faintly \( \omega \)-continuous,
2. \( f^{-1}(A) \) is \( \omega \)-open in \( X \) for every \( \theta \)-open set \( A \) of \( Y \),
3. \( f^{-1}(B) \) is \( \omega \)-closed in \( X \) for every \( \theta \)-closed set \( B \) of \( Y \).

**Theorem 2.10.** Let \( f : X \to Y \) be a function. If \( Y \) is regular, then the following are equivalent. Otherwise, the implications \( (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \) hold:

1. \( f \) is \( \omega \)-continuous,
2. \( f^{-1}(\text{cl}(A)) \) is \( \omega \)-closed in \( X \) for every subset \( A \) of \( Y \),
3. \( f \) is almost weakly \( \omega \)-continuous,
4. \( f \) is faintly \( \omega \)-continuous.

**Proof.** (1) \( \Rightarrow \) (2) : Let \( A \subseteq Y \). Since \( \theta \)-cl \((A) \) is closed, then \( f^{-1}(\theta \text{-cl}(A)) \) is \( \omega \)-closed in \( X \).

(2) \( \Rightarrow \) (3) : It follows from Theorem 2.6.

(3) \( \Rightarrow \) (4) : Let \( A \) be a \( \theta \)-closed subset of \( Y \). By Theorem 2.5, \( \omega \text{-cl}(f^{-1}(A)) \subseteq f^{-1}(\theta \text{-cl}(A)) = f^{-1}(A) \). Hence, \( f(A) \) is \( \omega \)-closed. Thus, \( f \) is faintly \( \omega \)-continuous.

Let \( Y \) be regular and let \( A \) be any open set of \( Y \). Since \( Y \) is regular, \( A \) is \( \theta \)-open in \( Y \). Since \( f \) is faintly \( \omega \)-continuous, then \( f^{-1}(A) \) is \( \omega \)-open in \( X \). Hence, \( f \) is \( \omega \)-continuous. Thus, the implication \( (4) \Rightarrow (1) \) holds.

**Theorem 2.11.** The following properties equivalent for a function \( f : X \to Y \):

1. \( f : X \to Y \) is almost weakly \( \omega \)-continuous at \( x \in X \).
2. \( x \in \omega \text{-int}(f^{-1}(\text{cl}(A))) \) for each neighborhood \( A \) of \( f(x) \).

**Proof.** (1) \( \Rightarrow \) (2) : Let \( A \) be any neighborhood of \( f(x) \). There exists an \( \omega \)-open set \( B \) containing \( x \) such that \( f(B) \subseteq \text{cl}(A) \). We have \( B \subseteq f^{-1}(\text{cl}(A)) \). Since \( B \) is \( \omega \)-open, then \( x \in B \subseteq \omega \text{-int}(B) \subseteq \omega \text{-int}(f^{-1}(\text{cl}(A))) \).
Let $x \in \text{int}(f^{-1}(cl(A)))$ for each neighborhood $A$ of $f(x)$. Take $U = \text{int}(f^{-1}(cl(A)))$. Then $f(U) \subset cl(A)$. Moreover, $U$ is $\omega$-open. Thus, $f$ is almost weakly $\omega$-continuous at $x \in X$.

**Definition 2.8.** A subset $A$ is said to be $\omega$-semi-open if there exists $\omega$-open set $U$ such that $U \subset A \subset cl(U)$.

**Theorem 2.12.** Let $f : X \to Y$ be almost weakly $\omega$-continuous at $x \in X$. The following properties hold:

1. For each neighborhood $A$ of $f(x)$ and each $\omega$-neighborhood $B$ of $x$, there exists a nonempty $\omega$-open set $U \subset B$ such that $U \subset \omega-cl(f^{-1}(cl(A)))$.
2. For each neighborhood $A$ of $f(x)$, there exists a $\omega$-semi-open set $B$ containing $x$ such that $B \subset \omega-cl(f^{-1}(cl(A)))$.

**Proof.** (1) : Let $A$ be any neighborhood of $f(x)$ and $B$ be an open set of $X$ containing $x$. Since $x \in \text{int}(f^{-1}(cl(A)))$, then $B \cap \text{int}(f^{-1}(cl(A))) \neq \emptyset$. Take $U = B \cap \text{int}(f^{-1}(cl(A)))$. Thus, $U$ is a nonempty $\omega$-open set and hence $U \subset B$ and $U \subset \omega-cl(f^{-1}(cl(A))) \subset \omega-cl(f^{-1}(cl(A)))$.

(2) : Suppose that (1) holds. Let $B$ be $\omega$-open containing $x$ and $A$ be any neighborhood of $f(x)$. There exists a nonempty $\omega$-open set $U_B$ such that $U_B \subset \omega-cl(f^{-1}(cl(A)))$. Take $U = \cup \{U_B : B \text{ is open in } X \text{ containing } x\}$. Then $U$ is $\omega$-open, $x \in \omega-cl(U)$ and $U \subset \omega-cl(f^{-1}(cl(A)))$. Take $S = U \cup \{x\}$. Then $U \subset S \subset \omega-cl(U)$. Thus, $S$ is $\omega$-semi-open set containing $x$ and $S \subset \omega-cl(f^{-1}(cl(A)))$.

3. Further properties

Recall that a space is rim-compact [6] if it has a basis of open sets with compact boundaries. The graph of a function $f : X \to Y$, denoted by $G(f)$, is the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

**Theorem 3.1.** Let $f : X \to Y$ be a function with the closed graph. If $Y$ is a rim-compact space, then the following are equivalent:

1. $f$ is almost weakly $\omega$-continuous,
2. $f$ is $\omega$-continuous.

**Proof.** (2) $\Rightarrow$ (1) : Obvious.

(1) $\Rightarrow$ (2) : Let $x \in X$ and $A$ be any open set of $Y$ containing $f(x)$. Since $Y$ is rim-compact, there exists an open set $B$ of $Y$ such that $x \in B \subset A$ and $\partial B$ is compact. For each $y \in \partial B$, $(x, y) \in X \times Y \setminus G(f)$. This implies that there exist open sets $U_y \subset X$ and $V_y \subset Y$ such that $x \in U_y$, $y \in V_y$. Since $G(f)$ is closed, then $f(U_y) \cap V_y = \emptyset$. The family $\{V_y \subset \partial B : y \in \partial B\}$ is an open cover of $\partial B$. This implies that there exist a finite number of points of $\partial B$, say, $y_1$, $y_2$, ..., $y_n$ such that $\partial B \subset \cup_{i=1}^n V_{y_i}$. Take $S = \cap\{U_{y_i}\}_{i=1}^n$ and $R = \cup\{V_{y_i}\}_{i=1}^n$. (7)

Then $S$ and $R$ are open sets such that $x \in S$, $\partial B \subset R$ and $f(S) \cap \partial B \subset f(S) \cap R = \emptyset$. Since $f$ is almost weakly $\omega$-continuous, there exists an $\omega$-open set $N$ containing $x$ such that $f(N) \subset cl(B)$. Take $U = S \cap N$. Then, $U$ is $\omega$-open containing $x$, $f(U) \subset cl(B)$ and $f(U) \cap \partial B = \emptyset$. Thus, $f(U) \subset B \subset A$ and hence $f$ is $\omega$-continuous.

**Theorem 3.2.** Let $f : X \to Y$ be a function where $Y$ is a rim-compact Hausdorff space. Then the following are equivalent:

1. $f$ is $\omega$-continuous,
Proof. (1) ⇒ (2) : Obvious.
(2) ⇒ (1) : Since a rim-compact Hausdorff space is regular, by Theorem 2.10, \( f \) is \( \omega \)-continuous.

**Definition 3.1.** ([1]) If a space \( X \) can not be written as the union of two nonempty disjoint \( \omega \)-open sets, then \( X \) is said to be \( \omega \)-connected.

**Theorem 3.3.** If \( f : X \to Y \) is an almost weakly \( \omega \)-continuous surjection and \( X \) is \( \omega \)-connected, then \( Y \) is connected.

Proof. Suppose that \( Y \) is not connected. Then there exist nonempty open sets \( A \) and \( B \) of \( Y \) such that \( Y = A \cup B \) and \( A \cap B = \emptyset \). This implies that \( A \) and \( B \) are clopen in \( Y \). By Theorem 2.1, \( f^{-1}(A) \subset \omega\text{-int}(f^{-1}(\text{cl}(A))) = \omega\text{-int}(f^{-1}(\text{cl}(A))) \). Hence \( f^{-1}(A) \) is \( \omega \)-open in \( X \). Similarly, \( f^{-1}(B) \) is \( \omega \)-open in \( X \). We have \( f^{-1}(A) \cap f^{-1}(B) = \emptyset \), \( X = f^{-1}(A) \cup f^{-1}(B) \). Also, \( f^{-1}(A) \) and \( f^{-1}(B) \) are nonempty. Thus, \( X \) is not \( \omega \)-connected.

**Corollary 3.1.** If \( f : X \to Y \) is a \( \omega \)-continuous surjection and \( X \) is \( \omega \)-connected, then \( Y \) is connected.

For a function \( f : X \to Y \), the graph function \( g : X \to X \times Y \) of \( f \) is defined by \( g(x) = (x, f(x)) \) for each \( x \in X \).

**Theorem 3.4.** The following are equivalent for a function \( f : X \to Y \):
(1) \( f \) is almost weakly \( \omega \)-continuous,
(2) the graph function \( g \) is almost weakly \( \omega \)-continuous.

Proof. (1) ⇒ (2) : Let \( f \) be almost weakly \( \omega \)-continuous and \( x \in X \). Suppose that \( A \) is an open set containing \( g(x) \). There exist open sets \( B \subset X \) and \( C \subset Y \) such that \( g(x) = (x, f(x)) \in B \times C \subset A \). Since \( f \) is almost weakly \( \omega \)-continuous, there exists \( \omega \)-open set \( D \) containing \( x \) such that \( f(D) \subset \text{cl}(C) \). Take \( S = B \cap D \). Then \( S \) is an \( \omega \)-open set containing \( x \) and \( g(S) \subset \text{cl}(A) \). Thus, \( g \) is almost weakly \( \omega \)-continuous.

(2) ⇒ (1) : Let \( g \) be almost weakly \( \omega \)-continuous and \( x \in X \) and \( A \) be an open set of \( X \) containing \( f(x) \). Then \( X \times A \) is an open set containing \( g(x) \). There then exists an \( \omega \)-open set \( B \) containing \( x \) such that \( g(B) \subset \text{cl}(X \times A) = X \times \text{cl}(A) \). Thus, \( f(B) \subset \text{cl}(A) \) and hence \( f \) is almost weakly \( \omega \)-continuous.

**Theorem 3.5.** If \( f, g : X \to Y \) is almost weakly \( \omega \)-continuous and \( Y \) is Urysohn, then the set \( A = \{x \in X : f(x) = g(x)\} \) is \( \omega \)-closed in \( X \).

Proof. Let \( x \in X \setminus A \). Then \( f(x) \neq g(x) \). Since \( Y \) is Urysohn, then there exist open sets \( S \) and \( R \) of \( Y \) such that \( f(x) \in S \) and \( g(x) \in R \) and \( \text{cl}(S) \cap \text{cl}(R) = \emptyset \). Since \( f \) is almost weakly \( \omega \)-continuous, there exists \( \omega \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subset \text{cl}(S) \). Since \( g \) is almost weakly \( \omega \)-continuous, there exists \( \omega \)-open set \( B \) of \( X \) containing \( x \) such that \( g(B) \subset \text{cl}(R) \). Take \( H = U \cap B \). Then \( H \) is \( \omega \)-open containing \( x \) and \( f(H) \cap g(H) \subset \text{cl}(S) \cap \text{cl}(R) = \emptyset \). Thus, \( H \cap A = \emptyset \) and hence \( A \) is \( \omega \)-closed in \( X \).

**Theorem 3.6.** Let \( f : X \to Y \) be an almost weakly \( \omega \)-continuous function and \( A \) be a \( \theta \)-closed set of \( X \times Y \). Then \( p(A \cap G(f)) \) is \( \omega \)-closed in \( X \) where \( p \) is the projection of \( X \times Y \) onto \( X \) and \( G(f) = \{(x, f(x)) : x \in X\} \).
Proof. Let \( x \in \text{cl}(p(A \cap G(f))) \). Suppose that \( U \) is an open set of \( X \) containing \( x \) and \( V \) is an open set of \( Y \) containing \( f(x) \). Since \( f \) is almost weakly \( \omega \)-continuous, by Theorem 2.1, \( x \in f^{-1}(V) \subseteq \omega\text{-int}(f^{-1}(\text{cl}(V))) \). We have \( U \cap \omega\text{-int}(f^{-1}(\text{cl}(V))) \) and \( x \in U \cap \omega\text{-int}(f^{-1}(\text{cl}(V))) \). Since \( x \in \text{cl}(p(A \cap G(f))) \), then \( (U \cap \omega\text{-int}(f^{-1}(\text{cl}(V)))) \cap p(A \cap G(f)) \) contains a \( a \in X \). Then \( (a, f(a)) \in A \) and \( f(a) \in \text{cl}(V) \). Therefore

\[
\emptyset \neq (U \times \text{cl}(V)) \cap A \subseteq \text{cl}(U \times V) \cap A
\]

and \( (x, f(x)) \in \theta\text{-cl}(A) \). Since \( A \) is \( \theta \)-closed, \( (x, f(x)) \in A \cap G(f) \) and \( x \in p(A \cap G(f)) \). Thus, \( p(A \cap G(f)) \) is \( \omega \)-closed in \( X \).

**Corollary 3.2.** If \( f : X \to Y \) has the \( \theta \)-closed graph and \( g : X \to Y \) is almost weakly \( \omega \)-continuous, then the set \( A = \{ x \in X : f(x) = g(x) \} \) is \( \omega \)-closed in \( X \).

Proof. Let \( G(f) \) be \( \theta \)-closed. We have \( p(G(f) \cap G(g)) = \{ x \in X : f(x) = g(x) \} \). By Theorem 3.6, \( A = \{ x \in X : f(x) = g(x) \} \) is \( \omega \)-closed in \( X \).

**Theorem 3.7.** If \( f : X \to Y \) is almost weakly \( \omega \)-continuous and \( Y \) is Hausdorff, then for each \( (x, y) \notin G(f) \), there exist an \( \omega \)-open set \( A \subseteq X \) and an open set \( B \subseteq Y \) containing \( x \) and \( y \), respectively, such that \( f(A) \cap \text{int}(\text{cl}(B)) = \emptyset \).

Proof. Let \( (x, y) \notin G(f) \). Then \( y \neq f(x) \). Since \( Y \) is Hausdorff, there exist disjoint open sets \( B \) and \( C \) containing \( y \) and \( f(x) \), respectively. Thus, \( \text{int}(\text{cl}(B)) \cap \text{cl}(C) = \emptyset \). Since \( f \) is almost weakly \( \omega \)-continuous, there exists an \( \omega \)-open set \( A \) containing \( x \) such that \( f(A) \subseteq \text{cl}(C) \). Hence, \( f(A) \cap \text{int}(\text{cl}(B)) = \emptyset \).

**Definition 3.2.** A subset \( S \) of a space \( X \) is said to be \( N \)-closed relative to \( X \) \([3]\) if for each cover \( \{ A_i : i \in I \} \) of \( S \) by open sets of \( X \), there exists a finite subfamily \( I_0 \subseteq I \) such that \( S \subseteq \bigcup_{i \in I_0} \text{cl}(A_i) \).

**Theorem 3.8.** Let \( f : X \to Y \) be a function. Suppose that for each \( (x, y) \notin G(f) \), there exist an \( \omega \)-open set \( A \subseteq X \) and an open set \( B \subseteq Y \) containing \( x \) and \( y \), respectively, such that \( f(A) \cap \text{int}(\text{cl}(B)) = \emptyset \). Then inverse image of each \( N \)-closed set of \( Y \) is \( \omega \)-closed in \( X \).

Proof. Suppose that there exists a \( N \)-closed set \( S \subseteq Y \) such that \( f^{-1}(S) \) is not \( \omega \)-closed in \( X \). Then, there exists a point \( x \in \text{cl}(f^{-1}(S)) \setminus f^{-1}(S) \). Since \( f(x) \notin f^{-1}(S) \), then \( (x, y) \notin G(f) \) for each \( y \in S \). This implies that there exist \( \omega \)-open sets \( A_y(x) \subseteq X \) and an open set \( B(y) \subseteq Y \) containing \( x \) and \( y \), respectively, such that \( f(A_y(x)) \cap \text{int}(\text{cl}(B(y))) = \emptyset \). The family \( \{ B(y) : y \in S \} \) is a cover of \( S \) by open sets of \( Y \). Since \( S \) is \( N \)-closed, there exist a finite number of points \( y_1, y_2, ..., y_n \) in \( S \) such that \( S \subseteq \bigcup_{i=1}^{n} \text{int}(\text{cl}(B(y_i))) \). Take \( A = \bigcap_{i=1}^{n} A_{y_i}(x) \). Then \( f(A) \subseteq S \). Since \( x \in \text{cl}(f^{-1}(S)) \), then \( f(A) \cap S = \emptyset \). This is a contradiction.

**Corollary 3.3.** Let \( Y \) be Hausdorff such that every closed set is \( N \)-closed. Then the following are equivalent:

1. \( f \) is \( \omega \)-continuous,
2. \( f \) is almost weakly \( \omega \)-continuous.

Let \( \{ X_i \}_{i \in I} \) and \( \{ Y_i \}_{i \in I} \) be any two families of topological spaces. The product space of \( \{ X_i \}_{i \in I} \) and \( \{ Y_i \}_{i \in I} \) is denoted by \( \prod X_i \) and \( \prod Y_i \), respectively. Let \( f_i : X_i \to Y_i \) be a function for each \( i \in I \). Let \( f : \prod X_i \to \prod Y_i \) be the product function defined as follows: \( f_i(x_i) = (f_i(x_i)) \) for each \( (x_i) \in \prod X_i \). The projection of \( \prod X_i \) and \( \prod Y_i \) onto \( X_i \) and \( Y_i \), respectively is denoted by \( p_i \) and \( q_i \).
Theorem 3.9. If \( f_i : X_i \to Y_i \) is almost weakly \( \omega \)-continuous for each \( i \in I \), then a function \( f : \prod X_i \to \prod Y_i \) is almost weakly \( \omega \)-continuous.

Proof. Let \( x = (x_i) \in \prod X_i \). Suppose that \( A \) is an open set containing \( f(x) \). Then there exists an open set \( \prod B_i \) such that \( f(x) \in \prod_{i=1}^n B_i \times \prod_{i \neq j} Y_j \subset A \), where \( B_i \) is open in \( Y_i \). Since \( f_i \) is almost weakly \( \omega \)-continuous, there exists \( \omega \)-open sets \( S_i \) in \( X_i \) containing \( x_i \) such that \( f_i(S_i) \subset \text{cl}(B_i) \) for each \( i = 1, 2, \ldots, n \). Take \( S = \prod_{i=1}^n S_i \times \prod_{i \neq j} X_j \), then \( S \) is \( \omega \)-open in \( \prod X_i \) containing \( x \) and

\[
 f(S) \subset \prod_{i=1}^n f_i(S_i) \times \prod_{i \neq j} Y_j \subset \prod_{i=1}^n \text{cl}(B_i) \times \prod_{i \neq j} Y_j \subset \text{cl}(A). \tag{9}
\]

Thus, \( f \) is almost weakly \( \omega \)-continuous.

References


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