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Fixed point of φ-contraction in metric spaces endowed with a graph

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Abstract. The purpose of this paper is to present some fixed point results for self-generalized contractions in metric spaces. We obtain sufficient conditions for the existence of a fixed point of the mapping $T : X \to X$ in the metric space X endowed with a graph G such that the set $V(G)$ of vertices of G coincides with X.

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1. Introduction

Let T be a selfmap of a metric space (X, d). Following Petruşel and Rus [5], we say that T is a Picard operator (abbr., PO) if T has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$ and T is weakly Picard operator (abbr. WPO) if the sequence $(T^n x)_{n\in\mathbb{N}}$ converges, for all $x \in X$ and the limit (which depends on x) is a fixed point of T.

Let (X, d) be a metric space. Let Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [[4], p. 309]) by assigning to each edge the distance between its vertices. By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$E\left(G^{-1}\right) = \{(x, y) \,|\, (y, x) \in G\}.$$

The letter $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E\left(\tilde{G}\right) = E(G) \cup E\left(G^{-1}\right) \tag{1}$$

We call (V', E') a subgraph of G if $V' \subseteq V(G)$, $E' \subseteq E(G)$ and for any edge $(x, y) \in E'$, $x, y \in V'$. Now we recall a few basic notions concerning connectivity of graphs. All of them can be found, e.g., in [4]. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^N$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, ..., N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path

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beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Clearly, G_x is connected.

Recently, two results have appeared, giving sufficient conditions for f to be a PO if (X,d) is endowed with a graph. The first result in this direction was given by J. Jakhymski [3] who also presented its applications to the Kelisky-Rivlin theorem on iterates of the Bernstein operators on the space $C[0,1]$.

Definition 1.1 ([3], Def. 2.1)**.** *We say that a mapping $f : X \to X$ is a Banach G-contraction or simply G-contraction if f preserves edges of G, i.e.,*

$$\forall x, y \in X\ ((x,y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G)) \tag{2}$$

and f decreases weights of edges of G in the following way:

$$\exists \alpha \in (0,1), \forall x, y \in X\ ((x,y) \in E(G) \Rightarrow d(f(x), f(y)) \leqslant \alpha d(x,y)) \tag{3}$$

Theorem 1.1 ([3], Th 3.2)**.** *Let (X,d) be complete, and let the triple (X,d,G) have the following property:*

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Let $f : X \to X$ be a G-contraction, and $X_f = \{x \in X \,|(x, fx) \in E(G)\}$. Then the following statements hold.

1. $\mathrm{card} Fix\, f = \mathrm{card}\,\{[x]_{\tilde{G}}\,|x \in X_f\}$.
2. $\mathrm{Fix}\, f \neq \emptyset$ iff $X_f \neq \emptyset$.
3. *f has a unique fixed point iff there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\tilde{G}}$.*
4. *For any $x \in X_f$, $f\,\big|_{[x]_{\tilde{G}}}$ is a PO.*
5. *If $X_f \neq \emptyset$ and G is weakly connected, then f is a PO.*
6. *If $X' := \cup\{[x]_{\tilde{G}}\,|x \in G\}$ then $f\,|_{X'}$ is a WPO.*
7. *If $f \subseteq E(G)$, then f is a WPO.*

Subsequently, Bega, Butt and Radojević extended Theorem 1.1 for set valued mappings.

Definition 1.2 ([1], Def. 2.6)**.** *Let $F : X \rightsquigarrow X$ be a set valued mapping with nonempty closed and bounded values. The mapping F is said to be a G-contraction if there exists a $k \in (0,1)$ such that*

$$D(Fx, Fy) \leqslant kd(x,y) \text{ for all } x, y \in E(G)$$

and if $u \in Fx$ and $v \in Fy$ are such that

$$d(u,v) \leqslant kd(x,y) + \alpha, \text{ for each } \alpha > 0$$

then $(u,v) \in E(G)$.

Theorem 1.2. *Let (X,d) be a complete metric space and suppose that the triple (X,d,G) has the property:*

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Let $F : X \rightsquigarrow X$ be a G-contraction and $X_f = \{x \in X : (x,u) \in E(G) \text{ for some } u \in F(x)\}$. Then the following statements hold:

1. *For any $x \in X_F$, $F\,\big|_{[x]_{\tilde{G}}}$ has a fixed point.*

2. *If $X_F \neq \emptyset$ and G is weakly connected, then F has a fixed point in X.*
3. *If $X' := \cup \{[x]_{\tilde{G}} : x \in X_F\}$, then $F|_{X'}$ has a fixed point.*
4. *If $F \subseteq E(G)$ then F has a fixed point.*
5. *$Fix\, F \neq \emptyset$ if and only if $X_F \neq \emptyset$.*

We recall that:

Definition 1.3. *A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*
i. *φ is monotone increasing, i.e., $t_1 \leqslant t_2$ implies $\varphi(t_1) \leqslant \varphi(t_2)$;*
ii. *$(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0 for all $t > 0$;*
is said to be a comparison function.

Definition 1.4. *A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*
i. *φ is monotone increasing, i.e., $t_1 \leqslant t_2$ implies $\varphi(t_1) \leqslant \varphi(t_2)$;*
ii. *$\sum\limits_{n=0}^{\infty} \varphi^n(t)$ converges for all $t > 0$;*
is said to be a $(c) - comparison\ function$.

Remark 1.1. *Any (c)-comparison function is a comparison function.*

Remark 1.2. *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function then $\varphi(t) < t$,* for all $t > 0$, *$\varphi(0) = 0$ and φ is right continuous at 0.*

Example 1.1. $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \begin{cases} \frac{1}{2}t; & t \in [0,1] \\ t - \frac{1}{2}; & t > 1 \end{cases}$ *is a (c)-comparison function.*

Example 1.2. $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \frac{t}{1+t}$ *is a comparison function but not a (c)-comparison function.*

We refer to Rus [7] and Berinde [2] for a detailed study of φ-contractions.

Definition 1.5. *Let (X,d) a metric space. A mapping $T : X \to X$ is a φ-contraction if there exists a comparison function $\varphi : R_+ \to \mathbb{R}_+$ such that:*

$$d(Tx, Ty) \leqslant \varphi(d(x,y)), \text{ for all } x, y \in X.$$

Now we discuss some types of continuity of mappings. The first of them is well known and often used in the metric fixed point theory.

Definition 1.6. *A mapping $T : X \to X$ is called orbitally continuous if for all $x \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers, $T^{k_n}x \to y \in X$ implies $T(T^{k_n}x) \to Ty$ as $n \to \infty$.*

Definition 1.7. *A mapping $T : X \to X$ is called orbitally G-continuous if given $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$,*

$$x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for } n \in \mathbb{N} \text{ imply } Tx_n \to Tx$$

The aim of this paper is to study the existence of fixed points for $(G,\varphi)-$*contraction* in metric spaces endowed with a graph G by defining the $(G,\varphi)-$ *contraction*.

2. Main Results

Throughout this section we assume that (X,d) is a metric space, and G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. The set of all fixed points of a mapping T is denoted by $FixT$.

By using the idea of Jakhymski [3], we will say that:

Definition 2.1. *Let (X,d) be a metric space and G a graph. The mapping $T : X \to X$ is said to be a $(G,\varphi)-$contraction if:*
1. $\forall\, x, y \in X\ ((x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G))$.
2. *there exists a comparison function $\varphi : R_+ \to \mathbb{R}_+$ such that:*

$$d(Tx,Ty) \leqslant \varphi(d(x,y))$$

for all $(x,y) \in E(G)$.

Remark 2.1. *If T is a $(G,\varphi)-$contraction, then T is both a $\left(G^{-1},\varphi\right)-$contraction and a $\left(\tilde{G},\varphi\right)-$contraction. This is consequence of symmetry of d and 1.*

Example 2.1. *Any $\varphi-$contraction is a $(G_0,\varphi)-$contraction, where the graph G_0 is defined by $E(G_0) = X \times X$.*

Example 2.2. *Any $G-$contraction is a $(G,\varphi)-$contraction, where the comparison function is $\varphi : \mathbb{R}_+ \to \mathbb{R}_+,\ \varphi(t) = at$.*

Definition 2.2. *We say that sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, elements of X, are Cauchy equivalent if each of them is a Cauchy sequence and $d(x_n,y_n) \to 0$.*

The first main result of this section is a fixed point theorem for $(G,\varphi)-$*contraction* on an complete metric space endowed with a graph.

Theorem 2.1. *Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be an operator. We suppose that:*
- (i.) *G is weakly connected;*
- (ii.) *for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $d(x_n,x_{n+1}) \to 0$ there exists k, $n_0 \in \mathbb{N}$ such that $(x_{kn},x_{km}) \in E(G)$ for all $m,n \in \mathbb{N}$ $m,n \geqslant n_0$;*
- $(\text{iii.})_a$ *T is orbitally continuous*
  or
- $(\text{iii.})_b$ *T is orbitally G-continuous and there exists a subsequence $(T^{n_k}x_0)_{k\in\mathbb{N}}$ of $(T^n x_0)_{n\in\mathbb{N}}$ such that $(T^{n_k}x_0,x^*) \in E(G)$ for each $k \in N$;*
- (iv.) *there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is a $(G,\varphi)-$contraction;*
- (v.) *the metric d is complete.*

Then T is a PO.

Proof. Let $x_0 \in X$ be such that $(x_0,Tx_0) \in E(G)$. Then, from the definition and an easy induction we obtain

$$\left(T^n x_0, T^{n+1}x_0\right) \in E(G) \text{ and } d\left(T^n x_0, T^{n+1}x_0\right) \leqslant \varphi^n\left(d(x_0,Tx_0)\right) \text{ for all } n \in \mathbb{N}.$$

So $\lim\limits_{n\to\infty} d\left(T^n x_0, T^{n+1}x_0\right) = 0$ and by (ii.) there exists $k, n_0 \in \mathbb{N}$ such that

$$\left(T^{kn}x_0, T^{km}x_0\right) \in E(G) \text{ for all } m,n \in \mathbb{N}\ m,n \geqslant n_0.$$

Since $d\left(T^{kn}x_0, T^{k(n+1)}x_0\right) \to 0$, for an arbitrary $\varepsilon > 0$, we can choose $N \in \mathbb{N}$, $N \geqslant n_0$ such that

$$d\left(T^{kn}x_0, T^{k(n+1)}x_0\right) < \varepsilon - \varphi(\varepsilon) \text{ for each } n \geqslant N.$$

Since $\left(T^{kn}x_0, T^{k(n+1)}x_0\right) \in E(G)$ we have for any $n \geqslant N$ that

$$\begin{aligned} d\left(T^{kn}x_0, T^{k(n+2)}x_0\right) &\leqslant d\left(T^{kn}x_0, T^{k(n+1)}x_0\right) + d\left(T^{k(n+1)}x_0, T^{k(n+2)}x_0\right) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi^k\left(d\left(T^{kn}x_0, T^{k(n+1)}x_0\right)\right) < \varepsilon. \end{aligned}$$

Now since $\left(T^{kn}x_0, T^{k(n+2)}x_0\right) \in E(G)$ we have for any $n \geqslant N$ that

$$d\left(T^{kn}x_0, T^{k(n+3)}x_0\right) \leqslant d\left(T^{kn}x_0, T^{k(n+1)}x_0\right) + d\left(T^{k(n+1)}x_0, T^{k(n+3)}x_0\right)$$
$$< \varepsilon - \varphi(\varepsilon) + \varphi^k\left(d\left(T^{kn}x_0, T^{k(n+2)}x_0\right)\right) < \varepsilon.$$

By induction we have

$$d\left(T^{kn}x_0, T^{k(n+m)}x_0\right) < \varepsilon, \text{ for any } m \in \mathbb{N} \text{ and } \ n \geqslant N.$$

Hence $\left(T^{kn}x_0\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). From (v.) we have $T^{kn}x_0 \to x^*$, as $n \to \infty$. Because $d\left(T^n x_0, T^{n+1}x_0\right) \to 0$, we get $T^n x_0 \to x^*$, as $n \to \infty$.

Let $x \in X$ be arbitrarily chosen. Then:

(1) If $(x, x_0) \in E(G)$, then $(T^n x, T^n x_0) \in E(G)$, $\forall n \in \mathbb{N}$ and thus $d(T^n x, T^n x_0) \leqslant \varphi(d(x, x_0))$, $\forall n \in \mathbb{N}$. Letting $n \to \infty$ we obtain that $T^n x \to x^*$.

(2) If $(x, x_0) \notin E(G)$, then, from (i.), there exists a path $(x_i)_{i=0}^M$ in $\tilde{G}$ from x_0 to x, i.e., $x_M = x$ and $(x_{i-1}, x_i) \in E\left(\tilde{G}\right)$ for $i = 1, ..., M$. An easy induction shows $(T^n x_{i-1}, T^n x_i) \in E\left(\tilde{G}\right)$ for $i = 1, ..., M$ and

$$d(T^n x_0, T^n x) \leqslant \sum_{i=1}^{M} \varphi^n\left(d(x_{i-1}, x_i)\right)$$

so $d(T^n x, T^n y) \to 0$ and we obtain $T^n x \to x^*$.

Now we will prove that $x^* \in F_T$. If $(iii.)_a$ holds, then clearly $x^* \in F_T$. If we suppose that $(iii.)_b$ takes place, then since $(T^{n_k}x_0)_{k\in\mathbb{N}} \to x^*$ and $(T^{n_k}x_0, x^*) \in E(G)$ for all $k \in \mathbb{N}$ we obtain, from the orbitally G-continuity of T , that $T^{n_k+1}x_0 \to Tx^*$ as $k \to \infty$. Thus $x^* = Tx^*$. If we have $Ty = y$ for some $y \in X$, then from above, we must have $T^n y \to x^*$, so $y = x^*$.

□

Remark 2.2. *The Theorem 2.1 is a generalization of Theorem 3.3 from* [6].

Now if we improve the properties of the operator T then we can drop some of the conditions of the graph G. From now on we will consider that the function φ is a $(c) - comparison$ function.

In the following we will show that the convergence of successive approximations for $(G, \varphi) - contraction$ is closely related to the connectivity of a graph. We say that sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, elements of X, are Cauchy equivalent if each of them is a Cauchy sequence and $d(x_n, y_n) \to 0$.

Theorem 2.2. *The following statements are equivalent:*

(i) *G is weakly connected;*

(ii) *for any $(G, \varphi) - contraction$ $T : X \to X$, given $x, y \in X$, the sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent;*

(iii) *for any $(G, \varphi) - contraction$ $T : X \to X$,* $\operatorname{card}(\operatorname{Fix} T) \leqslant 1$.

Proof. $(i) \Rightarrow (ii)$: Let T be a $(G, \varphi) - contraction$ and $x, y \in X$. By hypothesis, $[x]_{\tilde{G}} = X$, so $y \in [x]_{\tilde{G}}$. Then there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x to y, i.e., $x_0 = x, x_N = y$ and $(x_{i-1}, x_i) \in E\left(\tilde{G}\right)$ for $i = 1, ..., N$. An easy induction shows $(T^n x_{i-1}, T^n x_i) \in E\left(\tilde{G}\right)$ for $i = 1, ..., N$ and

$$d(T^n x, T^n y) \leqslant \sum_{i=1}^{N} \varphi^n\left(d(x_{i-1}, x_i)\right)$$

so $d(T^n x, T^n y) \to 0$.

In the same way, there is a path $(z_i)_{i=0}^M$ in $\tilde{G}$ from x to Tx, i.e., $z_0 = x, z_M = Tx$ and $(z_{i-1}, z_i) \in E\left(\tilde{G}\right)$ for $i = 1, ..., M$. Then we have

$$d\left(T^n x, T^{n+1} x\right) \leqslant \sum_{i=1}^{M} \varphi^n \left(d\left(z_{i-1}, z_i\right)\right)$$

Hence

$$\sum_{n=0}^{\infty} d\left(T^n x, T^{n+1} x\right) = \sum_{i=1}^{M} \sum_{n=0}^{\infty} \varphi^n \left(d\left(z_{i-1}, z_i\right)\right) < \infty$$

and a standard argument shows $(T^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence, so is $(T^n y)_{n\in\mathbb{N}}$.

$(ii) \Rightarrow (iii)$: Let T be a $(G, \varphi) - contraction$ and $x, y \in \operatorname{Fix} T$. By (ii), $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent which yields $x = y$.

$(iii) \Rightarrow (i)$: Suppose, on the contrary, G is not weakly connected, i.e., $\tilde{G}$ is disconnected. So, there exists an $x_0 \in X$ such that the both sets $[x_0]_{\tilde{G}}$ and $X \backslash [x_0]_{\tilde{G}}$ are nonempty. Let $y_0 \in X \backslash [x_0]_{\tilde{G}}$ and define

$$Tx = x_0 \text{ if } x \in [x_0]_{\tilde{G}} \quad \text{and} \quad Tx = y_0 \text{ if } x \in X \backslash [x_0]_{\tilde{G}}$$

Clearly, $Fix\, T = \{x_0, y_0\}$. We show T is a $(G, \varphi) - contraction$. Let $(x, y) \in E(G)$. Then $[x]_{\tilde{G}} = [y]_{\tilde{G}}$, so either $x, y \in [x]_{\tilde{G}}$, or $x, y \in X \backslash [x]_{\tilde{G}}$. Hence in both cases $Tx = Ty$, so $(Tx, Ty) \in E(G)$ since $E(G) \supseteq \Delta$, and $d(Tx, Ty) = 0 \leqslant \varphi(d(x, y))$. Thus T is a $(G, \varphi) - contraction$ having two fixed points which violates (iii). □

As an immediate consequence of Theorem 2.2, we obtain the following

Corollary 2.1. *Let (X, d) be a complete metric space and G a graph weakly connected. For any $(G, \varphi) - contraction$ $T : X \to X$, there is $x^* \in X$ such that $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*

The next example shows that one cannot improve Corollary 2.1 by adding that x^* is a fixed point of T.

Example 2.3. *Let $X := [0, 1]$ be endowed with the Euclidean metric d_E. Define the graph G by*

$$E(G) = \{(x, y) \in (0, 1] \times (0, 1] \,|x \geqslant y\} \cup \{(0, 0), (0, 1)\}$$

Set

$$Tx = \frac{x}{4} \text{ for } x \in (0, 1], \text{ and } T0 = \frac{1}{4}$$

It is easy to verify G is weakly connected and T is a $(G, \varphi) - contraction$ with $\varphi(t) = \frac{t}{4}$. Clearly, $T^n x \to 0$ for all $x \in X$, but T has no fixed points.

The proofs of our fixed point theorems depend on the following

Proposition 2.1. *Assume that $T : X \to X$ is a $(G, \varphi) - contraction$ such that for some $x_0 \in X$, $Tx_0 \in [x_0]_{\tilde{G}}$. Let $\tilde{G}_{x_0}$ be the component of $\tilde{G}$ containing x_0. Then $[x_0]_{\tilde{G}}$ is T-invariant and $T|_{[x_0]_{\tilde{G}}}$ is a $\left(\tilde{G}_{x_0}, \varphi\right) - contraction$. Moreover, if $x, y \in [x_0]_{\tilde{G}}$, then $(T^n x)_{n\in\mathbb{N}}$ and $(T^n x)_{n\in\mathbb{N}}$ are Cauchy equivalent.*

Proof. Let $x \in [x_0]_{\tilde{G}}$. Then there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x_0 to x, i.e., $x_N = x$ and $(x_{i-1}, x_i) \in E\left(\tilde{G}\right)$ for $i = 1, ..., N$. But T is a $(G, \varphi) - contraction$ which yields

$(Tx_{i-1},Tx_i)\in E\left(\tilde{G}\right)$ for $i=1,...,N$, i.e., $(Tx_i)_{i=0}^N$ is a path in $\tilde{G}$ from Tx_0 to Tx. Thus $Tx\in[Tx_0]_{\tilde{G}}$. Since, by hypothesis, $Tx_0\in[x_0]_{\tilde{G}}$, i.e., $[Tx_0]_{\tilde{G}}=[x_0]_{\tilde{G}}$, we infer $Tx\in[x_0]_{\tilde{G}}$. Thus $[x_0]_{\tilde{G}}$ is T-invariant.

Now let $(x,y)\in E\left(\tilde{G}_{x_0}\right)$. This means there is a path $((x_i)_{i=0}^N$ in $\tilde{G}$ from x_0 to y such that $x_{N-1}=x$. Let $(y_i)_{i=0}^M$ be a path in $\tilde{G}$ from x_0 to Tx_0. Repeating the argument from the first part of the proof, we infer $(y_0,y_1,...y_M,Tx_1,Tx_2,...Tx_N)$ is a path in $\tilde{G}$ from x_0 to Ty; in particular, $(Tx_{N-1},Tx_N)\in E\left(\tilde{G}_{x_0}\right)$, i.e., $(Tx,Ty)\in E\left(\tilde{G}_{x_0}\right)$. Moreover, since $E\left(\tilde{G}x_0\right)\subseteq E\left(\tilde{G}\right)$ and T is a $\left(\tilde{G},\varphi\right)-contraction$, we infer $T|_{[x_0]_{\tilde{G}}}$ is a $\left(\tilde{G}_{x_0},\varphi\right)-contraction$. Finally, in view of Theorem 2.2, the second statement follows immediately from the first one since $\tilde{G}x_0$ is connected. □

Theorem 2.3. *Let (X,d) be complete, and let the triple (X,d,G) have the following property:*

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n\to x$ and $(x_n,x_{n+1})\in E(G)$ for $n\in\mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n},x)\in E(G)$ for $n\in\mathbb{N}$.

Let $T:X\to X$ be a $(G,\varphi)-contraction$, and $X_T=\{x\in X\,|(x,Tx)\in E(G)\}$. Then the following statements hold.

(1) $\mathrm{card}Fix\,T=\mathrm{card}\,\{[x]_{\tilde{G}}\,|x\in X_T\}$.
(2) $\mathrm{Fix}\,T\neq\emptyset$ iff $X_T\neq\emptyset$.
(3) *T has a unique fixed point iff there exists $x_0\in X_f$ such that $X_T\subseteq[x_0]_{\tilde{G}}$.*
(4) *For any $x\in X_T$, $T\,\big|_{[x]_{\tilde{G}}}$ is a PO.*
(5) *If $X_T\neq\emptyset$ and G is weakly connected, then T is a PO.*
(6) *If $X':=\cup\{[x]_{\tilde{G}}\,|x\in G\}$ then $T\,|_{X'}$ is a WPO.*
(7) *If $T\subseteq E(G)$, then T is a WPO.*

Proof. We begin with points (4) and (5). Let $x\in X_f$. Then $Tx\in[x]_{\tilde{G}}$, so by Proposition 2.1, if $y\in[x]_{\tilde{G}}$, then $(T^nx)_{n\in\mathbb{N}}$ and $(T^ny)_{n\in\mathbb{N}}$ are Cauchy equivalent. By completeness, $(T^nx)_{n\in\mathbb{N}}$ converges to some $x^*\in X$. Clearly, also $\lim\limits_{n\to\infty}T^ny=x^*$. Since $(x,Tx)\in E(G)$, then by induction we have that

$$\left(T^nx,T^{n+1}x\right)\in E(G)\,,\,\text{for all}\,n\in\mathbb{N}.\tag{4}$$

By hypothesis, there is a subsequence $\left(T^{k_n}x\right)_{n\in\mathbb{N}}$ such that $\left(T^{k_n}x,x^*\right)\in E(G)$ for all $n\in\mathbb{N}$. Hence and by (4), we infer $\left(x,Tx,T^2x,...,T^{k_1}x,x^*\right)$ is a path in G (hence also in $\tilde{G}$) from x to x^*, i.e., $x^*\in[x]_{\tilde{G}}$. Moreover, because T is a $(G,\varphi)-contraction$ we have

$$d\left(T^{k_n+1}x,Tx^*\right)\leqslant\varphi\left(d\left(T^{k_n}x,x^*\right)\right)<d\left(T^{k_n}x,x^*\right)$$

for all $n\in\mathbb{N}$. Hence, letting n tend to ∞ we conclude $x^*=Tx^*$. Thus $T\,\big|_{[x]_{\tilde{G}}}$ is a PO. Moreover, if G is weakly connected, then $[x]_{\tilde{G}}=X$, so T is a PO.

Now (6) is an easy consequence of (4). To show (7) observe that $T\subseteq E(G)$ means $X_T=X$. This yields $X'=X$, so T is a WPO in view of (6).

To prove (1), consider a mapping π defined by

$$\pi(x)=[x]_{\tilde{G}}\ \text{for all}\ x\in\mathrm{Fix}\,T.$$

It suffices to show π is a bijection of $\mathrm{Fix}\,T$ onto $\Omega=\{\,[x]_{\tilde{G}}|\ x\in X_T\}$. Since $E(G)\supseteq\Delta$, we infer $\mathrm{Fix}\,T\subseteq X_T$ which yields $\pi(FixT)\subseteq\Omega$. On the other hand, if $x\in X_T$, then by (4), $\lim\limits_{n\to\infty}T^nx\in[x]_{\tilde{G}}\cap Fix\,T$ which implies $\pi\left(\lim\limits_{n\to\infty}T^nx\right)\in[x]_{\tilde{G}}$. Thus π is a surjection of Fix T onto Ω. Now, if $x_1, x_2 \in \mathrm{Fix}\, T$ are such that $\pi(x_1) = \pi(x_2)$, i.e., $[x_1]_{\tilde{G}} = [x_2]_{\tilde{G}}$, then $x_2 \in [x_1]_{\tilde{G}}$, so by (4),

$$\lim_{n\to\infty} T^n x_2 \in [x_1]_{\tilde{G}} \cap Fix\, T = \{x_1\},$$

i.e., $x_2 = x_1$ since $T^n x_2 = x_2$. Consequently, T is injective. Thus (1) is proved. Finally, observe that (2) and (3) are simple consequences of (1).

□

Corollary 2.2. *Let (X, d) be complete and ε-chainable for some $\varepsilon > 0$, i.e., given $x, y \in X$, there is $N \in \mathbb{N}$ and a sequence $(x_i)_{i=0}^N$ such that*

$$x_0 = x, x_N = y \text{ and } d(x_{i-1}, x_i) < \varepsilon \text{ for } i = 1, ..., N.$$

Let $T : X \to X$ be a function and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a $(c) - comparison\ function$ such that

$$\forall x, y \in X\, (d(x, y) < \varepsilon \Rightarrow d(Tx, Ty) \le \varphi(d(x, y))) \tag{5}$$

Then T is a PO.

Proof. Consider the graph G with $V(G) = X$, and $E(G) = \{(x, y) \in X \times X |\ d(x, y) < \varepsilon\}$. Then ε-chainability of (X, d) means G is connected. If $(x, y) \in E(G)$, then

$$d(Tx, Ty) \le \varphi(d(x, y)) < d(x, y) < \varepsilon$$

so $(Tx, Ty) \in E(G)$, hence T is a $(G, \varphi) - contraction.$

Let $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to x$, then $d(x_n, x) < \varepsilon$ for sufficiently large n, so there is $(x_{k_n})_{n\in\mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$. Thus by Theorem 2.3, T is PO. □

References

[1] I. Bega, A.R. Butt and S. Radojević, The contraction principle for set valued mappings on a metric space with a graph, *Comput. Math. Appl.* **60** (2010), 1214–1219.

[2] V. Berinde, *Iterative Approximation of Fixed Points*, Springer, 2007.

[3] J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Amer. Math. Soc.* **1** (2008), no. 136, 1359–1373.

[4] R. Johnsonbaugh, *Discrete Mathematics*, Prentice-Hall, Inc., New Jersey, 1997.

[5] A. Petruşel and I.A. Rus, Fixed point theorems in ordered L-spaces, *Proc. Amer. Math. Soc.* **134** (2006), 411–418.

[6] D. O'Regan and A. Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, *J. Math. Anal. Appl.* **341** (2008), 1241-1252.

[7] I.A. Rus, *Generalized Contractions and Applications*, Cluj Univ. Press, 2001.

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