Localization of $MTL$ - algebras

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Abstract. The aim of the present paper is to define the localization $MTL$ - algebra of a $MTL$ - algebra $A$ with respect to a topology $F$ on $A$. In the last part of the paper is proved that the maximal $MTL$ - algebra of quotients (defined in [15]) and the $MTL$ - algebra of fractions relative to an $\land$ - closed system (defined in [3]) are $MTL$ - algebras of localization.

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Basic Fuzzy logic ($BL$ from now on) is the many-valued residuated logic introduced by Hájek in [10] to cope with the logic of continuous t-norms and their residua. Monoidal logic ($ML$ from now on), is a logic whose algebraic counterpart is the class of residuated; $MTL$-algebras (see [9]) are algebraic structures for the Esteva-Godo monoidal t-norm based logic ($MTL$), a many-valued propositional calculus that formalizes the structure of the real unit interval $[0,1]$, induced by a left–continuous t-norm. $MTL$ algebras were independently introduced in [6] under the name weak-$BL$ algebras.

A remarkable construction in ring theory is the localization ring $A_F$ associated with a Gabriel topology $F$ on a ring $A$.

Using the model of localization ring, in [9], G. Georgescu defined for a bounded distributive lattice $L$ the localization lattice $L_F$ of $L$ with respect to a topology $F$ on $L$ and prove that the maximal lattice of quotients for a distributive lattice is a lattice of localization (relative to the topology of regular ideals); analogous results we have for lattices of fractions of bounded distributive lattices relative to $\land$ - closed systems.

The main aim of this paper is to develop a theory of localization for $MTL$ - algebras. Since $BL$ - algebras are particular classes of $MTL$ - algebras, the results of this paper generalize a part of the results from [2] for $BL$ - algebras. The main difference is that the axiom $x \odot (x \rightarrow y) = x \land y$ is not valid for $MTL$-algebras.

1. Definitions and preliminaries

Definition 1.1. A residuated lattice ([1], [18]) is an algebra $(A, \land, \lor, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:

$(a_1)$ $(A, \land, \lor, 0, 1)$ is a bounded lattice relative to the order $\leq$;

$(a_2)$ $(A, \odot, 1)$ is a commutative ordered monoid;

$(a_3)$ $(\odot, \rightarrow)$ is an adjoint pair, i.e. $z \leq x \rightarrow y$ if $x \odot z \leq y$ for every $x, y, z \in A$.

The class $RL$ of residuated lattices is equational (see [11]).

For examples of residuated lattices see [3] and [18].
In what follows by $A$ we denote the universe of a residuated lattice. For $x \in A$, we denote $x^* = x \to 0$ and $(x^*)^* = x^{**}$.

We review some rules of calculus for residuated lattices $A$ used in this paper:

**Theorem 1.1.** ([1], [18]) Let $x, y, z \in A$. Then we have the following:

(c1) $1 \to x = x, x \to x = 1, y \leq x \to y, x \odot (x \to y) \leq y, x \to 1 = 1, 0 \to x = 1, x \odot 0 = 0$;

c2) $x \leq y$ iff $x \to y = 1$;

c3) $x \leq y$ implies $x \odot z \leq y \odot z, z \to x \leq z \to y$ and $y \to z \leq x \to z$;

c4) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$, so $(x \odot y)^* = x \to y^* = y \to x^*$;

c5) $x \odot x^* = 0$ and $x \odot y = 0$ iff $x \leq y^*$;

If $A$ is a complete residuated lattice and $(y_i)_{i \in I}$ is a family of elements of $A$, then:

(c6) $x \odot \left( \bigvee_{i \in I} y_i \right) = \bigvee_{i \in I} (x \odot y_i)$;

(c7) $x \to \left( \bigwedge_{i \in I} y_i \right) = \bigwedge_{i \in I} (x \to y_i)$.

By $B(A)$ we denote the set of all complemented elements in the lattice $L(A) = (A, \land, \lor, 0, 1)$. Complements are generally not unique, unless the lattice is distributive; in the case of residuated lattices, however, although the underlying lattices need not be distributive, the complements are unique ([8]). Also, if $b$ is the complement of $a$, then $a$ is the complement of $b, b = a^*, a^2 = a$ and $a^{**} = a$ ([1], [3]). So, $B(A)$ is a Boolean subalgebra of $A$, called the Boolean center of $A$.

**Theorem 1.2.** ([3]) For $e \in A$ the following assertions are equivalent:

(i) $e \in B(A)$;

(ii) $e \lor e^* = 1$.

**Theorem 1.3.** ([3]) If $e, f \in B(A)$ and $x, y \in A$, then:

(c8) $e \odot x = e \land x$;

c9) $x \odot (x \to e) = e \land x, e \odot (e \to x) = e \land x$;

(c10) $e \odot (x \to y) = e \odot [(e \odot x) \to (e \odot y)]$;

(c11) $x \odot (e \to f) = x \odot [(x \odot e) \to (x \odot f)]$.

**Definition 1.2.** ([5], [6], [7]) A MTL-algebra is a residuated lattice satisfying the prelinearity equation:

(c12) $(x \to y) \lor (y \to x) = 1$.

The variety of MTL-algebras will be denoted by $\mathbb{MTL}$.

**Proposition 1.1.** ([5]) For a residuated lattice, the following conditions are equivalent:

(i) $A \in \mathbb{MTL}$;

(ii) $A$ is a subdirect product of linearly ordered residuated lattices;

(iii) For every $x, y, z \in A$ we have:

(c13) $x \to (y \lor z) = (x \to y) \lor (x \to z)$;

(iv) For every $x, y, z \in A$ we have:

(c14) $(x \land y) \to z = (x \to z) \lor (y \to z)$.

**Corollary 1.1.** ([5]) Let $A \in \mathbb{MTL}$. Then for every $x, y, z \in A$ we have:

(c15) $(x \land y)^* = x^* \lor y^*$;

(c16) $x \odot (y \land z) = (x \odot y) \land (x \odot z)$;

(c17) $x \land (y \lor z) = (x \land y) \lor (x \land z)$;

(c18) $x \lor y = ((x \to y) \to y) \land ((y \to x) \to x)$.
Remark 1.1. From (c18) we deduce that a MTL-algebra is a semi-Boolean lattice (see [13]).

Remark 1.2. Every linearly ordered residuated lattice is a MTL-algebra. A MTL-algebra \( A \) is a BL-algebra iff in \( A \) is verified the divisibility condition: \( x \circ (x \rightarrow y) = x \land y \). So, BL-algebras are examples of MTL-algebras; for an example of MTL-algebra which is not BL-algebra consider the residuated lattice defined on the unit interval \( A = [0, 1] \), for all \( x, y \in A \), such that

\[
x \circ y = 0 \text{ if } x + y \leq \frac{1}{2} \text{ and } x \land y \text{ elsewhere,}
\]

\[
x \rightarrow y = 1 \text{ if } x \leq y \text{ and } \max \left\{ \frac{1}{2} - x, y \right\} \text{ elsewhere (see [18], p.16).}
\]

Let \( 0 < y < x, x + y < \frac{1}{2} \). Then \( y < \frac{1}{2} - x \) and \( 0 \neq y = x \land y \), but \( x \circ (x \rightarrow y) = x \circ (\frac{1}{2} - x) = 0 \). This residuated lattice is a chain, so is a MTL-algebra, but the divisibility condition not hold.

Definition 1.3. Let \( (P, \leq) \) an ordered set. A nonempty subset \( I \) of \( P \) is called order ideal if, whenever \( x \in I, y \in P \) and \( y \leq x \), we have \( y \in I \); we denote by \( I(P) \) the set of all order ideals of \( P \).

For a MTL-algebra \( A \) we denote by \( \text{Id}(A) \) the set of all ideals of the lattice \( L(A) \).

Remark 1.3. Clearly, \( \text{Id}(A) \subseteq I(A) \) and if \( I_1, I_2 \subseteq I(A) \), then \( I_1 \cap I_2 \in I(A) \). Also, if \( I \in I(A) \), then \( 0 \in I \).

2. Topologies on a MTL-algebra

Definition 2.1. A non-empty set \( F \) of elements \( I \subseteq I(A) \) will be called a topology on \( A \) if the following axioms hold:

\( (a_1) \) If \( I_1 \in F, I_2 \subseteq I(A) \) and \( I_1 \subseteq I_2 \), then \( I_2 \in F \) (hence \( A \in F \));

\( (a_5) \) If \( I_1, I_2 \in F \), then \( I_1 \cap I_2 \in F \).

Remark 2.1. 1. \( F \) is a topology on \( A \) iff \( F \) is a filter of the lattice of power set of \( A \); for this reason a topology on \( I(A) \) is usually called a Gabriel filter on \( I(A) \).

2. Clearly, if \( F \) is a topology on \( A \), then \( (A, F \cup \{\emptyset\}) \) is a topological space.

Any intersection of topologies on \( A \) is a topology; so, the set \( T(A) \) of all topologies of \( A \) is a complete lattice with respect to inclusion.

Example 2.1. If \( I \in I(A) \), then the set \( F(I) = \{I' \in I(A) : I \subseteq I'\} \) is a topology on \( A \).

Remark 2.2. If in particular \( A = [0, 1] \) is the MTL-algebra from Remark 1.2, then \( I(A) = \{[0, x] : x \in A\} \). For \( x = 0 \), \( F(\{0\}) = I(A) \); for \( x \in (0, 1) \), \( F([0, x]) = \{[0, y] : x \leq y, y \in A\} \).

Definition 2.2. ([15]) A non-empty set \( I \subseteq A \) will be called regular if for every \( x, y \in A \) such that \( e \land x = e \land y \) for every \( e \in I \cap B(A) \), then \( x = y \).

Example 2.2. If we denote \( R(A) = \{I \subseteq A : I \text{ is a regular subset of } A\} \), then \( I(A) \cap R(A) \) is a topology on \( A \).

Remark 2.3. Clearly, if \( A = [0, 1] \) is the MTL-algebra from Remark 1.2, since \( B(A) = \{0, 1\} = L_2 \) then only \( I = A \) is a regular subset of \( A \) \( (I = [0, x] \text{ with } x \neq 1 \) are non regular because contain 0 and for example we have \( 0 \land a = 0 \land b \) for every \( a, b \in A \) and \( a \neq b \). So, in this case \( F = I(A) \cap R(A) = \{A\} \).
Example 2.3. A nonempty set $I \subseteq A$ will be called dense (see [9]) if for $x \in A$ such that $e \wedge x = 0$ for every $e \in I \cap B(A)$, then $x = 0$. If we denote by $D(A)$ the set of all dense subsets of $A$, then $R(A) \subseteq D(A)$ and $F = I(A) \cap D(A)$ is a topology on $A$.

Remark 2.4. As above, for MTL-algebra $A = [0, 1]$ from Remark 1.2, $D(A) = \{A\}$ (because $I \in D(A)$ if $1 \in I$).

Definition 2.3. ([3]) A subset $S \subseteq A$ is called $\wedge$-closed if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.

Example 2.4. For any $\wedge$-closed subset $S$ of $A$, the set $F_S = \{I \in I(A) : I \cap S \cap B(A) \neq \emptyset\}$ is a topology on $A$.

Remark 2.5. In the case of MTL-algebra $A = [0, 1]$ from Remark 1.2, $S \subseteq [0, 1]$ is a $\wedge$-closed subset if $1 \in S$. Since $B(A) = \{0, 1\} = L_2$ then for $S \subseteq A$ a $\wedge$-closed system, $F_S = \{I \in I(A) : I \cap S \cap \{0, 1\} \neq \emptyset\}$.

1. If $S$ is a $\wedge$-closed system of $A$ such that $0 \in S$ we have $I \cap S \cap B(A) \neq \emptyset$ for every $I \in I(A)$, so $F_S = I(A)$.

2. If $0 \notin S$ then $F_S = \{A\}$ (because, if $I \in I(A)$ and $1 \in I$ implies $I = A$).

3. $F$-multipliers and localization MTL-algebras

Let $F$ be a topology on a MTL-algebra $A$ and we consider the relation $\theta_F$ of $A$ defined in the following way: $(x, y) \in \theta_F$ there exists $I \in F$ such that $e \wedge x = e \wedge y$ for any $e \in I \cap B(A)$.

Lemma 3.1. $\theta_F$ is a congruence on $A$.


We shall denote by $a/\theta_F$ the congruence class of an element $a \in A$ and by $p_F : A \rightarrow A/\theta_F$ the canonical morphism of MTL-algebras.

Proposition 3.1. For $a \in A$, $a/\theta_F \in B(A/\theta_F)$ iff there exists $I \in F$ such that $a \vee a^* \geq e$ for every $e \in I \cap B(A)$. So, if $a \in B(A)$, then $a/\theta_F \in B(A/\theta_F)$.

Proof. Using Theorem 1.2, for $a \in A$, we have $a/\theta_F \in B(A/\theta_F) \Leftrightarrow a/\theta_F \vee (a/\theta_F)^* = 1/\theta_F \Leftrightarrow (a \vee a^*)/\theta_F = 1/\theta_F \Leftrightarrow$ there exist $I \in F$ such that $(a \vee a^*) \wedge e = 1 \wedge e = e$, for every $e \in I \cap B(A) \Leftrightarrow a \vee a^* \geq e$, for every $e \in I \cap B(A)$. If $a \in B(A)$, then for every $I \in F_0 = 1 = a \vee a^* \geq e$, for every $e \in I \cap B(A)$, hence $a/\theta_F \in B(A/\theta_F)$.

Corollary 3.1. If $F = I(A) \cap R(A)$, then for $a \in A$, $a \in B(A)$ iff $a/\theta_F \in B(A/\theta_F)$.

Definition 1.1. Let $F$ be a topology on $A$. A $F$-multiplier is a mapping $f : I \rightarrow A/\theta_F$ where $I \in F$ and for every $x \in I$ and $e \in B(A)$ the following axioms are fulfilled:

1. $f(e \circ x) = e/\theta_F \wedge f(x) = e/\theta_F \circ f(x)$;
2. $f(e) \leq x/\theta_F$;
3. $x/\theta_F \circ (x/\theta_F \rightarrow f(x)) = f(x)$.

Remark 3.1. If $A$ is a BL-algebra, then the axiom (3.3) is a consequence of (a7) (because in this case $x/\theta_F \circ (x/\theta_F \rightarrow f(x)) = x/\theta_F \wedge f(x)$, for every $x \in I$).

By $dom(f) \in F$ we denote the domain of $f$; if $dom(f) = A$, we called $f$ total.

To simplify language, we will use $F$-multiplier instead partial $F$-multiplier, using total to indicate that the domain of a certain $F$-multiplier is $A$. 

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If $F = \{ \mathcal{A} \}$, then $\theta_F$ is the identity congruence of $A$ so a $F$– multiplier is a total multiplier in sense of [15], Definition 3, which verify the conditions $M_1, M_2$ and $M_3$.

The maps $0, 1 : A \to A/\theta_F$ defined by $0(x) = 0/\theta_F$ and $1(x) = x/\theta_F$ for every $x \in A$ are $F$– multipliers in the sense of Definition 3.1.

Also, for $a \in B(A)$, $f_0 : A \to A/\theta_F$ defined by $f_0(a)(x) = a/\theta_F \land x/\theta_F$ for every $x \in A$, is a $F$– multiplier. If $\text{dom}(f_0) = A$, we denote $f_0$ by $f_0 \circ \theta_F$; clearly, $f_0 \circ \theta_F = 0$.

We shall denote by $M(I, A/\theta_F)$ the set of all the $F$– multipliers having the domain $I \in F$ and $M(A/\theta_F) = \bigcup_{I \in F} M(I, A/\theta_F)$. If $I_1, I_2 \in F$, $I_1 \subseteq I_2$ we have a canonical mapping $\varphi_{I_1, I_2} : M(I_2, A/\theta_F) \to M(I_1, A/\theta_F)$ defined by $\varphi_{I_1, I_2}(f) = f|_{I_1}$ for $f \in M(I_2, A/\theta_F)$. Let us consider the directed system of sets

$\{\{M(I, A/\theta_F)\}_{I \in F}, \{\varphi_{I_1, I_2}\}_{I_1, I_2 \in F, I_1 \subseteq I_2}\}$

and denote by $A_F$ the inductive limit (in the category of sets) $A_F = \lim_{I \in F} M(I, A/\theta_F)$. For any $F$– multiplier $f : I \to A/\theta_F$ we shall denote by $(\widehat{f})$ the equivalence class of $f$ in $A_F$.

**Remark 3.2.** If $f_i : I_i \to A/\theta_F$, $i = 1, 2$, are $F$– multipliers, then $(I_1, f_1) = (I_2, f_2)$ (in $A_F$) if there exists $I \in F$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.

**Proposition 3.2.** If $I_1, I_2 \in F$ and $f_i \in M(I, A/\theta_F)$, $i = 1, 2$, then $(c_{19})$ $f_1(x) \circ [x/\theta_F \to f_2(x)] = f_2(x) \circ [x/\theta_F \to f_1(x)]$, for every $x \in I_1 \cap I_2$.

**Proof.** For $x \in I_1 \cap I_2$ we have $f_1(x) \circ [x/\theta_F \to f_2(x)] = x/\theta_F \circ (x/\theta_F \to f_1(x)) \circ (x/\theta_F \to f_2(x)) = [x/\theta_F \circ (x/\theta_F \to f_2(x))] \circ (x/\theta_F \to f_1(x)) = f_2(x) \circ [x/\theta_F \to f_1(x)]$.

Let $f_i : I_i \to A/\theta_F$, (with $I_i \in F$, $i = 1, 2$), $F$– multpliers. Let us consider the mappings $f_1 \land f_2, f_1 \lor f_2, f_1 \circ f_2, f_1 \to f_2 : I_1 \cap I_2 \to A/\theta_F$ defined by

$(f_1 \land f_2)(x) = f_1(x) \oplus f_2(x), (f_1 \lor f_2)(x) = f_1(x) \circ f_2(x), (f_1 \circ f_2)(x) = f_1(x) \circ f_2(x), (f_1 \to f_2)(x) = x/\theta_F \circ [f_1(x) \to f_2(x)],$

for any $x \in I_1 \cap I_2$, and let

$(I_1, f_1) \land (I_2, f_2) = (I_1 \cap I_2, f_1 \oplus f_2), (I_1, f_1) \lor (I_2, f_2) = (I_1 \cap I_2, f_1 \circ f_2),

(I_1, f_1) \circ (I_2, f_2) = (I_1 \cap I_2, f_1 \circ f_2), (I_1, f_1) \to (I_2, f_2) = (I_1 \cap I_2, f_1 \to f_2).$

Clearly, the definitions of the operations $\land, \lor, \circ$ and $\to$ on $A_F$ are correct.

**Lemma 3.2.** $f_1 \land f_2 \in M(I_1 \cap I_2, A/\theta_F)$.

**Proof.** It is suffic to verify only $a_8$ (for $a_6$ and $a_7$, see [2]).

For every $x \in I_1 \cap I_2$ we have $x/\theta_F \circ [x/\theta_F \to (f_1 \land f_2)(x)] = x/\theta_F \circ (f_1(x) \land f_2(x)) = x/\theta_F \circ [(x/\theta_F \to f_1(x)) \land (x/\theta_F \to f_2(x))] = x/\theta_F \circ (x/\theta_F \to f_1(x)) \land (x/\theta_F \to f_2(x)) = (f_1(x) \land f_2(x)) = (f_1 \land f_2)(x)$, that is, $f_1 \land f_2 \in M(I_1 \cap I_2, A/\theta_F)$.

**Lemma 3.3.** $f_1 \lor f_2 \in M(I_1 \cap I_2, A/\theta_F)$.

**Proof.** The axioms $a_6$ and $a_7$ are verified as in the case of $BL$– algebras (see [2]). To verify $a_8$, let $x \in I_1 \cap I_2$. Then $x/\theta_F \circ [x/\theta_F \to (f_1 \lor f_2)(x)] = x/\theta_F \circ (f_1(x) \lor f_2(x)) = x/\theta_F \circ [(x/\theta_F \to f_1(x)) \lor (x/\theta_F \to f_2(x))] = x/\theta_F \circ [(x/\theta_F \to f_1(x)) \lor (x/\theta_F \to f_2(x))] = f_1(x) \lor f_2(x) = (f_1 \lor f_2)(x)$, that is, $f_1 \lor f_2 \in M(I_1 \cap I_2, A/\theta_F)$. 

\[ \square \]
Lemma 3.4. \( f_1 \odot f_2 \in M(I_1 \cap I_2, A/\theta_F) \).

Proof. By using \( c_{10} \), \( a_6 \) and \( a_7 \) are verified as in the case of \( BL \)-algebras (see [2]). For \( a_8 \) let \( x \in I_1 \cap I_2 \) and denote \( f = f_1 \odot f_2 \).

To prove the equality \( x/\theta_F \odot (x/\theta_F \multimap f(x)) = f(x) \) it is sufficient (using \( c_1 \)) to prove that \( f(x) \leq x/\theta_F \odot (x/\theta_F \multimap f(x)) \). We have \( f(x) = f_1(x) \odot (x/\theta_F \multimap f_2(x)) = x/\theta_F \odot (x/\theta_F \multimap f_1(x)) \odot (x/\theta_F \multimap f_2(x)) \) and \( x/\theta_F \odot (x/\theta_F \multimap f(x)) = x/\theta_F \odot (x/\theta_F \multimap f_1(x)) \odot (x/\theta_F \multimap f_2(x)) \).

So, to prove that \( f(x) \leq x/\theta_F \odot (x/\theta_F \multimap f(x)) \) it is sufficient to prove that \( x/\theta_F \odot (x/\theta_F \multimap f_1(x)) \odot (x/\theta_F \multimap f_2(x)) \leq x/\theta_F \odot (x/\theta_F \multimap f_1(x)) \odot (x/\theta_F \multimap f_2(x)) \). That is, \( \alpha \leq x/\theta_F \multimap (x/\theta_F \odot \alpha) \) (with \( \alpha \odot x/\theta_F \multimap f_1(x) \odot (x/\theta_F \multimap f_2(x)) \)), which is clearly, since \( \alpha \leq x/\theta_F \multimap (x/\theta_F \odot \alpha) \) (\( \alpha \odot x/\theta_F \multimap (x/\theta_F \odot \alpha) = 1 \)), that \( f_1 \odot f_2 \in M(I_1 \cap I_2, A/\theta_F) \). \( \square \)

Lemma 3.5. \( f_1 \rightarrow f_2 \in M(I_1 \cap I_2, A/\theta_F) \).

Proof. By using \( c_{10} \), \( a_6 \) and \( a_7 \) are verified as in the case of \( BL \)-algebras (see [2]). For \( a_8 \), let \( x \in I_1 \cap I_2 \) and denote \( f = f_1 \rightarrow f_2 \); then \( f(x) = x/\theta_F \multimap f_1(x) \odot f_2(x) \).

We have \( f_1(x) \odot f_2(x) \leq x/\theta_F \multimap f_1(x) \odot f_2(x) \), hence \( x/\theta_F \multimap f_1(x) \odot f_2(x) \leq f_1(x) \odot f_2(x) \). That is, \( f_1 \odot f_2 \leq f_3 \). Conversely, if \( f_1 \odot f_2 \leq f_3 \), we have \( f_1(x) \odot f_2(x) \leq f_3(x) \), for every \( x \in I_1 \cap I_2 \). Obviously \( f_1 \odot f_2 \leq f_3 \) if \( f_1 \odot f_2 \leq f_3 \) for all \( x \in I_1 \cap I_2 \). Since \( f \in M(I_1 \cap I_2, A/\theta_F) \) and \( f_1 \odot f_2 \leq f_3 \), we have \( f_1 \odot f_2 \leq f_3 \) if \( f_1 \odot f_2 \leq f_3 \) for all \( x \in I_1 \cap I_2 \). Obviously, \( x/\theta_F \multimap f_1(x) \leq f_3(x) \). That is, \( f_1 \odot f_2 \leq f_3 \) if \( f_1 \odot f_2 \leq f_3 \) for all \( x \in I_1 \cap I_2 \). Since \( f_1 \odot f_2 \leq f_3 \) if \( f_1 \odot f_2 \leq f_3 \), we deduce that \( f_1 \odot f_2 \leq f_3 \). Since the prelinarity equation \( c_{10} \) is proved as in the case of \( BL \)-algebras (see [2]) we deduce that \( (A, \multimap, 0, 1) = (\widehat{A}, \widehat{0}, \widehat{1}) \) is a \( MTL \)-algebra. \( \square \)

Proposition 3.3. \( (A_F, \land, \lor, \neg, 0, 1) = (\widehat{A}, \widehat{0}, \widehat{1}) \) is an \( MTL \)-algebra.

Proof. We verify the axioms of \( MTL \)-algebras.

(a1). Obviously \( (A_F, \land, \lor, \neg, 0, 1) = (\widehat{A}, \widehat{0}, \widehat{1}) \) is a bounded lattice.

(a2). As in the case of \( BL \)-algebras (see [2]), by using \( c_{19} \) and \( a_8 \).

(a3). \( f_i \in M(I_1, A/\theta_F) \) where \( I_1 \in \mathcal{F}, i = 1, 2, 3 \).

Since \( f_1 \leq f_2 \rightarrow f_3 \) for \( x \in I_1 \cap I_2 \cap I_3 \), we have \( f_1(x) \leq (f_2 \rightarrow f_3)(x) \iff f_1(x) \leq x/\theta_F \multimap f_2(x) \). So, by \( c_3 \), \( f_1(x) \multimap x/\theta_F \multimap f_2(x) \leq f_1(x) \multimap f_2(x) \). That is, \( f_1 \odot f_2 \leq f_3 \). Conversely, if \( f_1 \odot f_2 \leq f_3 \), we have \( f_1(x) \multimap f_2(x) \leq f_3(x) \), for every \( x \in I_1 \cap I_2 \cap I_3 \). Obviously, \( x/\theta_F \multimap f_1(x) \leq f_3(x) \iff x/\theta_F \multimap (f_2 \rightarrow f_3)(x) \iff f_1(x) \leq (f_2 \rightarrow f_3)(x) \). That is, \( f_1 \odot f_2 \leq f_3 \) if \( f_1 \odot f_2 \leq f_3 \) for all \( x \in I_1 \cap I_2 \cap I_3 \). Since \( f_1 \odot f_2 \leq f_3 \), we deduce that \( f_1 \odot f_2 \leq f_3 \). Since the prelinarity equation \( c_{10} \) is proved as in the case of \( BL \)-algebras (see [2]) we deduce that \( (A, \land, \lor, \neg, 0, 1) = (\widehat{A}, \widehat{0}, \widehat{1}) \) is a \( MTL \)-algebra. \( \square \)

Remark 3.3. \( (M(A/\theta_F), \land, \lor, \neg, 0, 1) \) is a \( MTL \)-algebra.

Definition 3.2. The \( MTL \)-algebra \( A_F \) will be called the localization \( MTL \)-algebra of \( A \) with respect to the topology \( \mathcal{F} \).

Definition 3.3. ([5], [7]) A \( MTL \)-algebra \( A \) is called

(i) An \( IMTL \)-algebra (involutive \( MTL \)-algebra) if it satisfies the equation

\( x^{**} = x \);

(ii) a \( SMTL \)-algebra if it satisfies the equation

\( (S) \ (x \land x^*) = 0; \)

(iii) a \( WNM \)-algebra (weak nilpotent minimum) if it satisfies the equation

\( (W) \ (x \land y)^* \lor [(x \land y) \rightarrow (x \land y)] = 1; \)
(iv) a $\Pi$SMTL-algebra if it is a SMTL-algebra satisfying the equation

$$(\Pi) \quad [z^* \circ ((x \circ z) \rightarrow (y \circ z))] \rightarrow (x \rightarrow y) = 1.$$
If $f$ is a Boolean algebra. For example, if $f$ is an algebra. Also, hence $f$ is not a principal multiplier (because $f(1) = 1$). But $f$ is not a Boolean algebra. Also, hence $f$ is not a principal multiplier (because $f(1) = 1$).

Remark 3.4. If MTL-algebra $(A, \land, \lor, \to, 0, 1)$ is a BL-algebra (resp. an IMTL-algebra, a SMTL-algebra, a WNM-algebra, a ISMTL-algebra), then MTL-algebra $(M(A/\theta_f), \land, \lor, \to, 0, 1)$ is a BL-algebra (resp. an IMTL-algebra, a SMTL-algebra, a WNM-algebra, a ISMTL-algebra).

Remark 3.5. If MTL-algebra $(A, \land, \lor, \to, 0, 1)$ is a BL-algebra in [2] will be called $(A, \land, \lor, \to, 0, 1)$ the localization BL-algebra of $A$ with respect to the topology $F$.

Lemma 3.6. Let the map $v_F : B(A) \to A_F$ defined by $v_F(a) = (A, \overline{a})$ for every $a \in B(A)$. Then:

(i) $v_F$ is a morphism of MTL-algebras;
(ii) For $a \in B(A)$, $(A, \overline{a}) \in B(A_F)$;
(iii) $v_F(B(A)) \in B(A_F)$.

Proof. (i), (iii). As in the case of BL-algebras (see [2]), (ii). For $a \in B(A)$ we have $a \lor a^* = 1$, hence $(a \land x) \lor (x \lor (a \land x))^* \cong (a \land x) \lor (x \lor (a \land x))^* \cong (a \land x) \lor (x \lor (a \land x))^* \cong (a \land x) \lor (x \lor (a \land x))^*$.

4. Applications

In the following we describe the localization MTL-algebra $A_F$ in some special instances.

1. If $I \in I(A)$, and $F$ is the topology $F(I) = \{I' \in I(A) : I \subseteq I'\}$ (see Example 2.1), then $A_F$ is isomorphic with $M(I, A/\theta_f)$ and $v_F : B(A) \to A_F$ is defined by $v_F(a) = (A, \overline{a})$ for every $a \in B(A)$.

If $I$ is a regular subset of $A$, then $\theta_f$ is the identity, hence $A_F$ is isomorphic with $M(I, A)$ (see [15], Definition 3, conditions $M_1$, $M_2$ and $M_3$), which in generally is not a Boolean algebra. For example, if $I = A = [0, 1]$ is the Lukasiewicz structure (see [18]) then $A_F$ is not a Boolean algebra (see [2]).

Remark 4.1. If consider MTL-algebra $A = [0, 1]$ from Remark 1.2, then

1. If $I = \{0\}$, then $F(\{0\}) = I(A)$ (see Remark 2.2), so $A_F \cong M(I, A/\theta_f) = M(\{0\}, A/\theta_f) = 0$.
2. If $I = A$, then $F(A) = \{A\}$ and $\theta_f$ is the identity, hence $A_F$ is isomorphic with $M(I, A/\theta_f)$. Since $B(A) = L_2 = \{0, 1\}$, then $f \in M(A, A)$ iff $f(x) \leq x$ for every $x \in A$. So, $f(0) = 0$. For $x \geq 1$ if we denote $f(x) = y$, then $y \leq x$ and we deduce that $x \lor (x \rightarrow f(x)) = x \lor (x \rightarrow y) = x \lor \max(\frac{1}{2} - x, y) = x \lor y = x \lor y = f(x)$, so for $x \geq 1$ if $f \in M(A, A)$ and $f(x) \leq y$. If consider $f \in A_F = M(A, A)$ such that $f(\frac{1}{2}) = \frac{1}{2}$, then $(f \lor f')(\frac{1}{2}) = f(\frac{1}{2}) \lor f'(\frac{1}{2}) = \frac{1}{2} \lor (\frac{1}{2} \lor \frac{1}{2}) = \frac{1}{2} \lor \frac{1}{2} = \frac{1}{2}$. Hence $f$ is not a Boolean element in $A_F$ (hence in this case $A_F$ is not a Boolean algebra).
3. If $I = [0, x]$ with $x \neq 0, 1$, $\mathcal{F}(I) = \{[0, a] : x \leq a, a \in (0, 1]\}$. Since $0 \in [0, a], a \neq 1$ and $0 \wedge x = 0 \wedge y$, then $(x, y) \in \theta_\mathcal{F}$ for every $x, y \in A$, hence in this case $A_\mathcal{F} \approx M(I, 0) = 0$.

2. **Main remark.** To obtain the maximal MTL-algebra of quotients $Q(A)$ as a localization relative to a topology $\mathcal{F}$ we have to develop another theory of multipliers (meaning we add new axioms for $\mathcal{F}$-multipliers).

**Definition 4.1.** Let $\mathcal{F}$ be a topology on $A$. A strong - $\mathcal{F}$- multiplier is a mapping $f : I \to A/\theta_\mathcal{F}$ (where $I \in \mathcal{F}$) which verifies the axioms $a_6, a_7$ and $a_8$ (see Definition 3.1) and

\[(a_9) \quad f(e) = B(A/\theta_\mathcal{F}); \]

\[(a_{10}) \quad (x/\theta_\mathcal{F}) \wedge f(e) = (e/\theta_\mathcal{F}) \wedge f(x), \text{ for every } e \in I \cap B(A) \text{ and } x \in I.\]

**Remark 4.2.** If $(\wedge, \vee, \circ, \to, 0, 1)$ is a MTL-algebra, the maps $0, 1 : A \to A/\theta_\mathcal{F}$ defined by $0(x) = 0/\theta_\mathcal{F}$ and $1(x) = x/\theta_\mathcal{F}$ for every $x \in A$ are strong - $\mathcal{F}$- multipliers. We recall that if $f_1 : I_1 \to A/\theta_\mathcal{F}$, $f_2 : I_2 \to A/\theta_\mathcal{F}$ defined by $f_1(x) = f_1(x), f_2(x) = f_2(x)$, then $f_1 \wedge f_2$ is a strong - $\mathcal{F}$- multiplier if $f_1 \wedge f_2 : I_1 \cap I_2 \to A/\theta_\mathcal{F}$ defined by $f_1 \wedge f_2(x) = f_1(x) \wedge f_2(x) \in A/\theta_\mathcal{F}$ and $(f_1 \wedge f_2)(x) = (f_1(x) \wedge f_2(x))$. For any $x \in I_1 \cap I_2$ are $\mathcal{F}$-multipliers.

If $f_1, f_2$ are strong - $\mathcal{F}$- multipliers then the multipliers $f_1 \wedge f_2, f_1 \vee f_2, f_1 \circ f_2, f_1 \to f_2$ are also strong - $\mathcal{F}$- multipliers (the proof is as in the case of BL-algebras, see [2]).

**Remark 4.3.** Analogous as in the case of $\mathcal{F}$- multipliers if we work with strong-$\mathcal{F}$- multipliers we obtain a MTL-subalgebra of $A_\mathcal{F}$ denoted by $s - A_\mathcal{F}$ which will be called the strong-localization MTL-algebra of $A$ with respect to the topology $\mathcal{F}$.

So, if $\mathcal{F} = I(A) \cap R(A)$ is the topology of regular ideals, then $\theta_\mathcal{F}$ is the identity congruence of $A$ and we obtain the definition for multipliers on $A$, so

$$s - A_\mathcal{F} = \lim_{I \in \mathcal{F}} (s - M(I, A)),$$

where $s - M(I, A)$ is the set of strong multipliers of $A$ having the domain $I$ (see [15], Definition 3, conditions $M_1 - M_5$).

In this situation we obtain:

**Proposition 4.1.** In the case $\mathcal{F} = I(A) \cap R(A)$, $A_\mathcal{F}$ is exactly the maximal MTL-algebra $Q(A)$ of quotients of $A$ (introduced in [15]) which is a Boolean algebra (for the proof, see [14] Proposition 0.12, p.194, for the case of BL-algebras). If MTL-algebra $A$ is a BL-algebra, $A_\mathcal{F}$ is exactly the maximal BL-algebra $Q(A)$ of quotients of $A$.

**Remark 4.4.** If consider in particular MTL-algebra $A = [0, 1]$ from Remark 1.2, then $\mathcal{F} = \{\}$, hence $A_\mathcal{F} \approx s - M(A, A)$. Consider $f \in s - M(A, A)$. Clearly, $f(0) = 0$ and $f(1) \in \{0, 1\}$. If $f(1) = 0$, then for every $x \in A$, $x \wedge f(1) = 0 \wedge f(1) = f(x) \Leftrightarrow f(x) = 0 \Rightarrow f = 0$. If $f(1) = 1$ then from $a_{10}$, $f(x) = x \in 1(x)$, hence $f = 1$. So, in this case $s - A_\mathcal{F} \approx s - M(A, A) = L_2$.

3. Denoting by $\mathcal{D}$ the topology of dense ordered ideals of $A$, then (since $R(A) \subseteq D(A)$) there exists a morphism of MTL-algebras $\alpha : Q(A) \to s - A_\mathcal{D}$ such that the diagram

$$
\begin{array}{ccc}
B(A) & \xrightarrow{\pi_\mathcal{D}} & Q(A) \\
\downarrow v_\mathcal{D} \quad & & \uparrow \alpha \\
s - A_\mathcal{D}
\end{array}
$$

is commutative (i.e. $\alpha \circ \tau^{-1} = \nu_{D}$). Indeed, if $[f, I] \in Q(A)$ (with $I \in I(A) \cap R(A)$ and $f : I \rightarrow A$ a strong multiplier in the sense of [15]) we denote by $f_{D}$ the strong - $D$–multiplier $f_{D} : I \rightarrow A/\theta_{D}$ defined by $f_{D}(x) = f(x)/\theta_{D}$ for every $x \in I$. Thus, $\alpha$ is defined by $\alpha([f, I]) = [f_{D}, I]$.

4. Let $S \subseteq A$ a $\wedge$–closed system of $MTL$–algebra $A$. Consider the following congruence on $A : (x, y) \in \theta_{S} \Leftrightarrow$ there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ (see [3]). $A[S] = A/\theta_{S}$ is called in [3] the $MTL$-algebra of fractions of $A$ relative to the $\wedge$–closed system $S$.

As in the case of $BL$–algebras we obtain the following result:

**Proposition 4.2.** If $F_{S}$ is the topology associated with a $\wedge$–closed system $S \subseteq A$, then the $MTL$-algebra $s - A_{F_{S}}$ is isomorphic with $B(A[S])$.

**Remark 4.5.** In the proof of Proposition 4.2 the axiom $\alpha_{10}$ is not necessarily.

**Remark 4.6.** If $A$ is $MTL$–algebra $A = [0, 1]$, from Remark 1.2, since $B(A) = \{0, 1\} = L_{2}$ then for $S \subseteq A$ a $\wedge$–closed system, $F_{S} = \{I \in I(A) : I \cap S \cap \{0, 1\} \neq \emptyset\}$ and $s - A_{F_{S}}$ is isomorphic with $B(A[S])$:

1. If $S$ is a $\wedge$–closed systems of $A$ such that $0 \in S$ , then $F_{S} = I(A)$ (see Remark 2.5 ) and $s - A_{F_{S}} = s - A_{I(A)} \approx B(A[S]) = B(0) = \emptyset$.
2. If $0 \notin S$, $F_{S} = A$ (see Remark 2.5) and $s - A_{F_{S}} = s - A_{A} \approx B(A[S]) = B(A) = \{0, 1\} = L_{2}$.

**Concluding remarks**

Since in particular a $MTL$–algebra is a $BL$–algebra we obtain a part of the results about localization of $BL$–algebras (see [2]), so we deduce that the main results of this paper are generalization of the analogous result relative to $BL$–algebras from [2].

We use in the construction of localization $MTL$–algebra $A_{F}$ the Boolean center $B(A)$ of $MTL$–algebra $A$; as a consequence of this fact, $s - A_{F}$ is a Boolean algebra in some particular cases.

A very interesting subject for future research would be a treatment of the localization for $MTL$ algabras or residuated lattices without use the Boolean center.

**References**


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