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Several characterizations of the 4–valued modal algebras

Aldo Victorio Figallo and Paolo Landini

Abstract. A. Monteiro, in 1978, defined the algebras he named tetravalent modal algebras, that will be called 4−*valued modal algebras* in this work. These algebras constitute a generalization of the 3−valued Lukasiewicz algebras defined by Moisil.

The theory of the 4−valued modal algebras has been widely developed by I. Loureiro in [6, 7, 8, 9, 10, 11, 12] and by A. V. Figallo in [2, 3, 4, 5].

J. Font and M. Rius indicated, in the introduction to the important work [1], a brief but detailed review on the 4−valued modal algebras.

In this work varied characterizations are presented that show the "closeness" this variety of algebras has with other well–known algebras related to the algebraic counterparts of certain logics.

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1. Introduction

In 1940 G. C. Moisil [13] introduced the notion of three–valued Łukasiewicz algebra. In 1963, A.Monteiro [14] characterized these algebras as algebras $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ of type $(2, 2, 1, 1, 0)$ which verify the following identities:

(A1) $x \vee 1 = 1$,
(A2) $x \wedge (x \vee y) = x$,
(A3) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,
(A4) $\sim\sim x = x$,
(A5) $\sim (x \vee y) = \sim x \wedge \sim y$,
(A6) $\sim x \vee \nabla x = 1$,
(A7) $x \wedge \sim x = \sim x \wedge \nabla x$,
(A8) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

L.Monteiro [15] proved that A1 follows from A2, $\cdots$, A8, and that A2, $\cdots$, A8, are independent.

From A2, $\cdots$, A5 it follows that $\langle A, \wedge, \vee, \sim, 1\rangle$ is a De Morgan algebra with last element 1 and first element $0 = \sim 1$.

In Lemma 1.1 we will indicate other properties valid in the variety of 4–valued modal algebras necessary for the development that follows.

Lemma 1.1. *In every 4-valued modal algebra* $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ *we have :* A9-A17.

(A9) $x \leq \nabla x$,
(A10) $\nabla 1 = 1$,
(A11) $\nabla x \leq \nabla\nabla x$,
(A12) $\nabla x \vee \sim \nabla x = 1$,

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(A13) $\nabla x \wedge \sim \nabla x = 0$,
(A14) $\nabla\nabla x = \nabla x$,
(A15) *If* $x \leq y$, *then* $\nabla x \leq \nabla y$,
(A16) $\nabla(\nabla x \vee \nabla y) = \nabla(x \vee y)$,
(A17) $\nabla x \vee \nabla y = \nabla(x \vee y)$,

the proof of which we will indicate in the section that follows.

In 1969 J. Varlet [16] characterized three–valued Łukasiewicz algebras by means of other operations. Let $\langle A, \wedge, \vee, *, +, 0, 1\rangle$ be an algebra of type $(2, 2, 1, 1, 0, 0)$ where $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice with least element 0, greatest element 1 and the following properties are satisfied:

(V1) $x \wedge x^* = 0$,
(V2) $(x \wedge y)^* = x^* \wedge y^*$,
(V3) $0^* = 1$,
(V4) $x \vee x^+ = 1$,
(V5) $(x \vee y)^+ = x^+ \wedge y^+$,
(V6) $1^+ = 0$,
(V7) If $x^* = y^*$ and $x^+ = y^+$, then $x = y$.

About these algebras he proved that it is posible to define, in the sense of [14, 15] a structure of three-valued Łukasiewicz algebra by taking $\sim x = (x \vee x^*) \wedge x^+$ and $\triangledown x = x^{**}$.

Furthermore it holds $x^* = \sim \triangledown x$ and $x^+ = \triangledown \sim x$. Therefore three–valued Łukasiewicz are double Stone lattices which satisfy the determination principle V7. Moreover V7 may be replaced by the identity

$$(x \wedge x^+) \wedge (y \vee y^*) = x \wedge x^+.$$

Later, in 1963, A. Monteiro [14] considered the 4–valued modal algebras $\langle A, \wedge, \vee, \sim, \triangledown, 1\rangle$ of type $(2, 2, 1, 1, 0)$ which satisfy A2, $\cdots$, A7 as an abstraction of three–valued Łukasiewicz algebras.

In this paper we give several characterizations of the 4–valued modal algebras. In the first one we consider the operations $\wedge, \vee, \neg, \Gamma, 0, 1$ where $\neg x = \sim \triangledown x$, $\Gamma x = \triangledown \sim x$ are called strong and weak negation respectively.

2. A characterization of the 4–valued modal algebras

Before working on Theorem 2.1 we will indicate proofs from A9 through A16. Then,

(A9) $x \leq \nabla x$:
(1) $(x \wedge \sim x) \vee \nabla x = (\nabla x \wedge \sim x) \vee \nabla x = \nabla x$, [A7]
(2) $(x \vee \nabla x) \wedge (\sim x \vee \nabla x) = \nabla x$,
(3) $(x \vee \nabla x) \wedge 1 = \nabla x$, [A6]
(4) $x \vee \nabla x = \nabla x$,
(5) $x \leq \nabla x$.

(A10) $\nabla 1 = 1$: [A9]

(A11) $\nabla x \leq \nabla\nabla x$: [A9]

(A12) $\nabla x \vee \sim \nabla x = 1$:
(1) $\nabla x \wedge \sim \nabla x = \sim \nabla x \wedge \nabla\nabla x$, [A7]
(2) $\sim \nabla x \vee \nabla x = \nabla x \vee \sim \nabla\nabla x$ [(1), A5]
$= (\nabla x \vee \sim \nabla\nabla x) \wedge 1$
$= (\nabla x \vee \sim \nabla\nabla x) \wedge (\sim x \vee \nabla x)$ [A6]

$= ((\nabla x \vee \sim \nabla\nabla x) \wedge \sim x) \vee ((\nabla x \vee \sim \nabla\nabla x) \wedge \nabla x)$
$= ((\nabla x \vee \sim x) \wedge (\sim \nabla\nabla x \vee \sim x)) \vee \nabla x$
$= (1 \wedge (\sim \nabla\nabla x \vee \sim x)) \vee \nabla x$ [A6]
$= \sim \nabla\nabla x \vee \sim x \vee \nabla x$
$= 1.$ [A6]

(A13) $\nabla x \wedge \sim \nabla x = 0$: [A12]

(A14) $\nabla\nabla x = \nabla x$:

$\nabla\nabla x = \nabla\nabla x \wedge 1$
$= \nabla\nabla x \wedge (\nabla x \vee \sim \nabla x)$ [A12]
$= (\nabla\nabla x \wedge \nabla x) \vee (\nabla\nabla x \wedge \sim \nabla x)$
$= \nabla x \vee (\nabla\nabla x \wedge \sim \nabla x)$ [A11]
$= \nabla x \vee (\sim \nabla x \wedge \nabla x)$ [A7]
$= \nabla x.$

(A15) If $x \leq y$, then $\nabla x \leq \nabla y$:

(1) $x \leq y$, [Hip.]
(2) $\sim y \leq \sim x$, [(1)]
(3) $\sim y \vee \nabla y \leq \sim x \vee \nabla y$, [(2)]
(4) $1 = \sim x \vee \nabla y$, [A6]
(5) $0 = x \wedge \sim \nabla y$, [A6]
(6) $\sim \nabla x = \sim \nabla x \vee (x \wedge \sim \nabla y)$ [(5)]
$= (\sim \nabla x \vee x) \wedge (\sim \nabla x \vee \sim \nabla y)$
$= 1 \wedge (\sim \nabla x \vee \sim \nabla y)$ [A6]
$= \sim \nabla x \vee \sim \nabla y,$
(7) $\nabla x = \nabla x \wedge \nabla y$, [(6)]
(8) $\nabla x \leq \nabla y.$ [(7)]

(A16) $\nabla(\nabla x \vee \nabla y) = \nabla(x \vee y)$:

(1) $x \leq x \vee y$,
(2) $y \leq x \vee y$,
(3) $\nabla x \leq \nabla(x \vee y)$, [(1), A15]
(4) $\nabla y \leq \nabla(x \vee y)$, [(2), A15]
(5) $\nabla x \vee \nabla y \leq \nabla(x \vee y)$, [(3), (4)]
(6) $x \leq \nabla x$, [A9]
(7) $y \leq \nabla y$, [A9]
(8) $x \vee y \leq \nabla x \vee \nabla y$, [(6), (7)]
(9) $\nabla(x \vee y) \leq \nabla(\nabla x \vee \nabla y)$, [(8), A15]
(10) $\nabla(\nabla x \vee \nabla y) \leq \nabla\nabla(x \vee y)$, [(5), A15]
(11) $\nabla(\nabla x \vee \nabla y) \leq \nabla(x \vee y)$, [(10), A14]
(12) $\nabla(\nabla x \vee \nabla y) = \nabla(x \vee y).$ [(9), (11)]

(A17) $\nabla x \vee \nabla y = \nabla(x \vee y)$:

(1) $\sim (\nabla x \vee \nabla y) \wedge \nabla(\nabla x \vee \nabla y) = (\nabla x \vee \nabla y) \wedge \sim (\nabla x \vee \nabla y)$ [A7]
$= (\nabla x \vee \nabla y) \wedge \sim \nabla x \wedge \sim \nabla y$ [A5]
$= ((\nabla x \wedge \sim \nabla x) \vee (\nabla y \wedge \sim \nabla x)) \wedge \sim \nabla y$
$= \nabla y \wedge \sim \nabla x \wedge \sim \nabla y$ [A13]
$= 0,$ [A13]
(2) $\sim (\nabla x \vee \nabla y) \wedge \nabla(x \vee y) = 0,$ [(1), A16]
(3) $(\nabla x \vee \nabla y) \vee \sim \nabla(x \vee y) = 1,$
(4) $((\nabla x \vee \nabla y) \vee \sim \nabla(x \vee y)) \wedge \nabla(x \vee y) = \nabla(x \vee y),$ [(3)]
(5) $((\nabla x \vee \nabla y) \wedge \nabla(x \vee y)) \vee (\sim \nabla(x \vee y) \wedge \nabla(x \vee y)) = \nabla(x \vee y),$
(6) $(\nabla x \vee \nabla y) \wedge \nabla(x \vee y) = \nabla(x \vee y),$ [(5), A13]

(7) $\nabla(x \vee y) \leq \nabla x \vee \nabla y$, [(6)]
(8) $\nabla x \vee \nabla y = \nabla(x \vee y)$. [(7), (5) of A16]

Theorem 2.1. *Let $\langle A, \wedge, \vee, \neg, \Gamma, 0, 1\rangle$ be an algebra of type $(2, 2, 1, 1, 0, 0)$ where $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice with least element 0, greatest element 1 and the operators $\nabla, \sim$ are defined on A by means of the formulas:*
(D1) $\nabla x = \neg\neg x$,
(D2) $\sim x = (x \vee \neg x) \wedge \Gamma x$.
Then (i) *and* (ii) *are equivalent:*
(i) $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ *is a 4–valued modal algebra.*
(ii) $\langle A, \wedge, \vee, \neg, \Gamma, 0, 1\rangle$ *verifies these properties:*
(B1) $\neg\neg 1 = 1$,
(B2) $x \wedge \neg x = 0$,
(B3) $x \vee \Gamma x = 1$,
(B4) $\neg x \wedge \Gamma\neg x = 0$,
(B5) $\Gamma x \vee \neg\Gamma x = 1$,
(B6) $\Gamma(x \wedge y) = \Gamma x \vee \Gamma y$,
(B7) $\neg(x \vee y) = \neg x \wedge \neg y$,
(B8) $\neg(x \wedge \neg y) = \neg x \vee \neg\neg y$,
(B9) $\Gamma(x \vee \Gamma y) = \Gamma x \wedge \Gamma\Gamma y$,
(B10) $(x \vee y) \wedge \Gamma(x \vee y) \leq x \vee \neg x$,
(B11) $x \wedge \Gamma x \wedge y \wedge \Gamma y \leq \Gamma(x \vee y)$.
Where $a \leq b$ if and only if $a \wedge b = a$ or $a \vee b = b$. Moreover, the operators $\neg, \Gamma$ are defined on A by means of the formulas:
(D3) $\neg x =\sim \nabla x$,
(D4) $\Gamma x = \nabla \sim x$.

Proof. (i) $\Longrightarrow$ (ii)
(B1) $\neg\neg 1 = 1$: [D1, A10]
(B2) $x \wedge \neg x = 0$:
$x \wedge \neg x = x \wedge \sim \nabla x$ [D3]
$=\sim (\sim x \vee \nabla x)$
$=\sim 1$ [A6]
$= 0$.
(B3) $x \vee \Gamma x = 1$:
$x \vee \Gamma x = x \vee \nabla \sim x$ [D4]
$=\sim\sim x \vee \nabla \sim x$
$= 1$. [A6]
(B4) $\neg x \wedge \Gamma\neg x = 0$:
$\neg x \wedge \Gamma\neg x =\sim \nabla x \wedge \nabla \sim\sim \nabla x$ [D3, D4]
$=\sim \nabla x \wedge \nabla\nabla x$ [A5]
$=\sim \nabla x \wedge \nabla x$ [A14]
$= 0$. [A13]
(B5) $\Gamma x \vee \neg\Gamma x = 1$:
$\Gamma x \vee \neg\Gamma x = \nabla \sim x \vee \sim \nabla\nabla \sim x$ [D3, D4]
$= \nabla \sim x \vee \sim \nabla \sim x$ [A14]
$= 1$. [A12]
(B6) $\Gamma(x \wedge y) = \Gamma x \vee \Gamma y$:
$\Gamma(x \wedge y) = \nabla \sim (x \wedge y)$ [D4]
$= \nabla(\sim x \vee \sim y)$
$= \nabla \sim x \vee \nabla \sim y$ [A17]

$= \Gamma x \vee \Gamma y.$ [D4]

(B7) $\neg(x \vee y) = \neg x \wedge \neg y$:

$\neg(x \vee y) = \sim \nabla(x \vee y)$ [D3]
$= \sim (\nabla x \vee \nabla y)$ [A17]
$= \sim \nabla x \wedge \sim \nabla y$
$= \neg x \wedge \neg y.$ [D3]

(B8) $\neg(x \wedge \neg y) = \neg x \vee \neg\neg y$:

$\neg(x \wedge \neg y) = \sim \nabla(x \wedge \sim \nabla y)$ [D3]
$= \sim (\nabla x \wedge \nabla \sim \nabla y)$ [A8]
$= \sim \nabla x \vee \sim \nabla \sim \nabla y$
$= \neg x \vee \neg\neg y.$ [D3]

(B9) $\Gamma(x \vee \Gamma y) = \Gamma x \wedge \Gamma\Gamma y$:

$\Gamma(x \vee \Gamma y) = \nabla \sim (x \vee \nabla \sim y)$ [D4]
$= \nabla(\sim x \wedge \sim \nabla \sim y)$
$= \nabla \sim x \wedge \nabla \sim \nabla \sim y$ [A8]
$= \Gamma x \wedge \Gamma\Gamma y.$ [D4]

(B10) $(x \vee y) \wedge \Gamma(x \vee y) \leq x \vee \neg x$:

(1) $x \wedge \Gamma x = x \wedge \sim x$:

$x \wedge \Gamma x = x \wedge \nabla \sim x$ [D4]
$= \sim (\sim x) \wedge \nabla(\sim x)$ [A4]
$= \sim\sim x \wedge \sim x$ [A7]
$= x \wedge \sim x,$ [A4]

(2) $(x \vee y) \wedge \Gamma(x \vee y) \leq \sim x$:

$(x \vee y) \wedge \Gamma(x \vee y) = (x \vee y) \wedge \sim (x \vee y)$ [(1)]
$= (x \vee y) \wedge \sim x \wedge \sim y)$ [A5]
$\leq \sim x,$

(3) $\sim x \leq x \vee \neg x$:

$\sim x \wedge (x \vee \neg x) = \sim x \wedge (x \vee \sim \nabla x)$ [D3]
$= (\sim x \wedge x) \vee (\sim x \wedge \sim \nabla x)$
$= (\sim x \wedge \nabla x) \vee \sim (x \vee \nabla x)$ [A7]
$= (\sim x \wedge \nabla x) \vee \sim \nabla x$ [A9]
$= (\sim x \vee \sim \nabla x) \wedge (\nabla x \vee \sim \nabla x)$
$= \sim (x \wedge \nabla x)$ [A5, A12]
$= \sim x,$ [A9]

(4) $(x \vee y) \wedge \Gamma(x \vee y) \leq x \vee \neg x.$ [(2), (3)]

(B11) $x \wedge \Gamma x \wedge y \wedge \Gamma y \leq \Gamma(x \vee y)$:

$(x \wedge \Gamma x) \wedge (y \wedge \Gamma y) = (x \wedge \sim x) \wedge (y \wedge \sim y)$ [(1) of B10]
$= (x \wedge y) \wedge (\sim x \wedge \sim y)$
$\leq (x \vee y) \wedge \sim (x \vee y)$
$= (x \vee y) \wedge \Gamma(x \vee y)$ [(1) of B10]
$\leq \Gamma(x \vee y).$

(ii) $\Longrightarrow$ (i)

(B12) $\Gamma 0 = 1$:

(1) $x \vee \Gamma x = 1,$ [B3]
(2) $\Gamma 0 = 1.$ [(1)]

(B13) $\neg 1 = 0$:

(1) $1 \wedge \neg 1 = 0,$ [B2]
(2) $\neg 1 = 0.$ [(1)]

(B14) $\neg x \leq \Gamma x$:

$$\begin{aligned}\neg x \wedge \Gamma x &= (\neg x \wedge \Gamma x) \vee 0 \\ &= (\neg x \wedge \Gamma x) \vee (x \wedge \neg x) && \text{[B2]} \\ &= \neg x \wedge (\Gamma x \vee x) \\ &= \neg x \wedge 1 && \text{[B3]} \\ &= \neg x.\end{aligned}$$

(B15) $\neg 0 = 1$:

(1) $0 = 1 \wedge \neg 1$, [B2]

(2) $\begin{aligned}\neg 0 &= \neg(1 \wedge \neg 1) && \text{[(1)]} \\ &= \neg 1 \vee \neg\neg 1 && \text{[B8]} \\ &= 1. && \text{[B1]}\end{aligned}$

(B16) $\Gamma 1 = 0$:

$\begin{aligned}\neg 0 \wedge \Gamma\neg 0 &= 0, && \text{[B4]} \\ 1 \wedge \Gamma 1 &= 0, && \text{[B15]} \\ \Gamma 1 &= 0.\end{aligned}$

(B17) $\Gamma x \wedge \Gamma\Gamma x = 0$:

$\begin{aligned}\Gamma x \wedge \Gamma\Gamma x &= \Gamma(x \vee \Gamma x) && \text{[B9]} \\ &= \Gamma 1 && \text{[B3]} \\ &= 0. && \text{[B16]}\end{aligned}$

(B18) $\neg x \vee \neg\neg x = 1$:

$\begin{aligned}\neg x \vee \neg\neg x &= \neg(x \wedge \neg x) && \text{[B8]} \\ &= \neg 0 && \text{[B2]} \\ &= 1. && \text{[B15]}\end{aligned}$

(B19) $\neg\neg x = \Gamma\neg x$:

(1) $\begin{aligned}\Gamma\neg x &= \Gamma\neg x \wedge 1 \\ &= \Gamma\neg x \wedge (\neg x \vee \neg\neg x) && \text{[B18]} \\ &= (\Gamma\neg x \wedge \neg x) \vee (\Gamma\neg x \wedge \neg\neg x) \\ &= 0 \vee (\Gamma\neg x \wedge \neg\neg x) && \text{[B4]} \\ &= \Gamma\neg x \wedge \neg\neg x,\end{aligned}$

(2) $\Gamma\neg x \leq \neg\neg x$, [(1)]

(3) $\neg\neg x \leq \Gamma\neg x$, [B14]

(4) $\neg\neg x = \Gamma\neg x$. [(2), (3)]

(B20) $\Gamma\Gamma x = \neg\Gamma x$:

(1) $\neg\Gamma x \leq \Gamma\Gamma x$, [B14]

(2) $\Gamma x \wedge \Gamma\Gamma x = 0$, [B17]

(3) $\Gamma x \vee \neg\Gamma x = 1$, [B5]

(4) $\Gamma\Gamma x = \Gamma\Gamma x \wedge 1 = \Gamma\Gamma x \wedge (\Gamma x \vee \neg\Gamma x) = (\Gamma\Gamma x \wedge \Gamma x) \vee (\Gamma\Gamma x \wedge \neg\Gamma x) = \Gamma\Gamma x \wedge \neg\Gamma x$, [(2)]

(5) $\Gamma\Gamma x \leq \neg\Gamma x$, [(4)]

(6) $\Gamma\Gamma x = \neg\Gamma x$. [(1), (5)]

(B21) $\neg x \wedge \Gamma\Gamma x = 0$:

(1) $\neg x \leq \Gamma x$, [B14]

(2) $\neg x \wedge \Gamma\Gamma x \leq \Gamma x \wedge \Gamma\Gamma x$, [(1)]

(3) $\neg x \wedge \Gamma\Gamma x = 0$. [(2), B17]

(B22) $x \leq \neg\neg x$:

(1) $\begin{aligned}x \wedge 1 &= x \wedge (\neg x \vee \neg\neg x) && \text{[B18]} \\ &= (x \wedge \neg x) \vee (x \wedge \neg\neg x) \\ &= x \wedge \neg\neg x, && \text{[B2]}\end{aligned}$

(2) $x \leq \neg\neg x$. [(1)]

(B23) $\Gamma\Gamma x \leq x$:

(1) $x = x \vee \Gamma 1$ [B16]
$= x \vee \Gamma(x \vee \Gamma x)$ [B3]
$= x \vee (\Gamma x \wedge \Gamma\Gamma x)$ [B9]
$= (x \vee \Gamma x) \wedge (x \vee \Gamma\Gamma x)$
$= x \vee \Gamma\Gamma x,$ [B3]
(2) $\Gamma\Gamma x \leq x.$ [(1)]

(B24) $\neg\neg\neg x = \neg x$:
(1) $\neg\neg\neg x = \neg(\neg\neg x)$
$= \neg(\neg\neg x \vee x)$ [B22]
$= \neg\neg\neg x \wedge \neg x,$ [B7]
(2) $\neg\neg\neg x \leq \neg x,$ [(1)]
(3) $\neg x \leq \neg\neg\neg x,$ [B22]
(4) $\neg\neg\neg x = \neg x.$ [(2), (3)]

(B25) $\Gamma\Gamma\Gamma x = \Gamma x$:
(1) $\Gamma\Gamma\Gamma x \leq \Gamma x,$ [B23]
(2) $\Gamma\Gamma\Gamma x = \Gamma(\Gamma\Gamma x)$
$= \Gamma(\Gamma\Gamma x \wedge x)$ [B23]
$= \Gamma\Gamma\Gamma x \vee \Gamma x,$ [B6]
(3) $\Gamma x \leq \Gamma\Gamma\Gamma x,$ [(2)]
(4) $\Gamma x = \Gamma\Gamma\Gamma x.$ [(1), (3)]

(B26) $\neg\Gamma x \leq x$:
(1) $\Gamma\Gamma x = \neg\Gamma x,$ [B20]
(2) $\Gamma\Gamma x \leq x,$ [B23]
(3) $\neg\Gamma x \leq x.$ [(1), (2)]

(B27) $\Gamma\Gamma\neg x = \neg x$:
$\Gamma\Gamma\neg x = \neg\Gamma\neg x$ [B20]
$= \neg\neg\neg x$ [B19]
$= \neg x.$ [B24]

(B28) $\Gamma\Gamma\Gamma\neg x = \neg\neg x$:
$\Gamma\Gamma\Gamma\neg x = \Gamma\neg x$ [B27]
$= \neg\neg x.$ [B19]

(B29) $\Gamma((x \vee \neg x) \wedge \Gamma x) = \neg\neg x$:
$\Gamma((x \vee \neg x) \wedge \Gamma x) = \Gamma(x \vee \neg x) \vee \Gamma\Gamma x$ [B6]
$= \Gamma(x \vee \Gamma\Gamma\neg x) \vee \Gamma\Gamma x$ [B27]
$= (\Gamma x \wedge \neg\neg x) \vee \Gamma\Gamma x$ [B9, B28]
$= (\Gamma x \vee \Gamma\Gamma x) \wedge (\neg\neg x \vee \Gamma\Gamma x)$
$= \neg\neg x.$ [B3, B22, B23]

(B30) $\neg\neg\Gamma x = \Gamma x$:
$\neg\neg\Gamma x = \Gamma\neg\Gamma x$ [B19]
$= \Gamma\Gamma\Gamma x$ [B20]
$= \Gamma x.$ [B25]

(B31) $\neg\neg\neg\Gamma x = \Gamma\Gamma x$:
$\neg\neg\neg\Gamma x = \Gamma\Gamma\Gamma\neg\neg\Gamma x$ [B28]
$= \Gamma\neg\neg\Gamma x$ [B25]
$= \neg\neg\neg\Gamma x$ [B19]
$= \neg\Gamma x$ [B24]
$= \Gamma\Gamma x.$ [B20]

(B32) $\neg((x \wedge \Gamma x) \vee \neg x) = \Gamma\Gamma x$:
$\neg((x \wedge \Gamma x) \vee \neg x) = \neg(x \wedge \Gamma x) \wedge \neg\neg x$ [B7]

$= \neg(x \wedge \neg\neg\Gamma x) \wedge \neg\neg x$ [B30]
$= (\neg x \vee \Gamma\Gamma x) \wedge \neg\neg x$ [B8, B31]
$= \Gamma\Gamma x.$ [B2, B22, B23]

(B33) $\Gamma\neg\neg x = \neg x$:

$\Gamma\neg\neg x = \neg\neg\neg x$ [B19]
$= \neg x.$ [B24]

Now we are able to prove the axioms A4, A6 and A7.

Axiom A4 $\sim\sim x = x$:

First, we observe that from B14 and D2 we obtain: (D3) $\sim x = (x \wedge \Gamma x) \vee \neg x$.
Then

(1) $\sim\sim x = (((x \wedge \Gamma x) \vee \neg x) \wedge \Gamma((x \wedge \Gamma x) \vee \neg x)) \vee \neg((x \wedge \Gamma x) \vee \neg x),$ [D3]
(2) $\Gamma((x \wedge \Gamma x) \vee \neg x)) = \Gamma((x \vee \neg x) \wedge (\Gamma x \vee \neg x))$
$= \Gamma((x \vee \neg x) \wedge \Gamma x)$ [B14]
$= \neg\neg x,$ [B29]
(3) $\sim\sim x = (((x \wedge \Gamma x) \vee \neg x) \wedge \neg\neg x) \vee \Gamma\Gamma x$ [(1), (2), B32]
$= (((x \wedge \Gamma x) \wedge \neg\neg x) \vee (\neg x \wedge \neg\neg x)) \vee \Gamma\Gamma x$
$= ((x \wedge \Gamma x) \wedge \neg\neg x) \vee \Gamma\Gamma x$ [B2]
$= ((x \wedge \Gamma x) \vee \Gamma\Gamma x) \wedge (\neg\neg x \vee \Gamma\Gamma x)$
$= ((x \wedge \Gamma x) \vee \Gamma\Gamma x) \wedge \neg\neg x$ [B22, B23]
$= ((x \vee \Gamma\Gamma x) \wedge (\Gamma x \vee \Gamma\Gamma x)) \wedge \neg\neg x$
$= (x \vee \Gamma\Gamma x) \wedge \neg\neg x$ [B3]
$= x \wedge \neg\neg x$ [B23]
$= x.$ [B22]

Axiom A6 $\sim x \vee \nabla x = 1$:

$\sim x \vee \nabla x = (x \wedge \Gamma x) \vee \neg x \vee \neg\neg x$ [D3, D1]
$= (x \wedge \Gamma x) \vee 1 = 1.$ [B18]

Axiom A7 $x\wedge \sim x = \sim x \wedge \nabla x$:

$\sim x \wedge \nabla x = ((x \wedge \Gamma x) \vee \neg x) \wedge \neg\neg x$ [D3, D1]
$= (x \wedge \Gamma x \wedge \neg\neg x) \vee (\neg x \wedge \neg\neg x)$
$= (x \wedge \Gamma x) \vee 0$ [B22, B2]
$= (x \wedge \Gamma x) \vee (x \wedge \neg x)$ [B2]
$= ((x \wedge \Gamma x) \vee x) \wedge ((x \wedge \Gamma x) \vee \neg x)$
$= x\wedge \sim x.$ [D3]

(B34) If $x \leq y$ then $\neg y \leq \neg x$ and $\Gamma y \leq \Gamma x$:

(1) $x \leq y,$ [Hip.]
(2) $x \vee y = y,$ [(1)]
(3) $\neg(x \vee y) = \neg y,$ [(2)]
(4) $\neg x \wedge \neg y = \neg y,$ [(3), B7]
(5) $\neg y \leq \neg x,$ [(4)]
(6) $x \wedge y = x,$ [(1)]
(7) $\Gamma(x \wedge y) = \Gamma x,$ [(6)]
(8) $\Gamma x \vee \Gamma y = \Gamma x,$ [(7), B6]
(9) $\Gamma y \leq \Gamma x.$ [(8)]

(B35) $\sim \Gamma x = \Gamma\Gamma x$:

$\sim \Gamma x = \Gamma\Gamma x \wedge (\Gamma x \vee \neg\Gamma x)$ [D2]
$= \Gamma\Gamma x \wedge (\Gamma x \vee \Gamma\Gamma x)$ [B20]
$= \Gamma\Gamma x.$ [A2]

(B36) $\sim (\neg x \wedge \Gamma y) = \neg\neg x \vee \Gamma\Gamma y$:

(1) $\sim (\neg x \wedge \Gamma y) = \Gamma (\neg x \wedge \Gamma y) \wedge ((\neg x \wedge \Gamma y) \vee \neg (\neg x \wedge \Gamma y)),$ [D2]

On the other hand

(2) $\Gamma(\neg x \wedge \Gamma y) = \Gamma\neg x \vee \Gamma\Gamma y$ [B6]
$= \neg\neg x \vee \Gamma\Gamma y,$ [B19]

and

(3) $\neg(\neg x \wedge \Gamma y) = \neg\Gamma y \vee \neg\neg x$ [B8]
$= \neg\neg x \vee \Gamma\Gamma y,$ [B20]

Then B36 follows from (1), (2) and (3).

(B37) $\Gamma\Gamma(y \vee \neg\neg x) = \Gamma\Gamma y \vee \neg\neg x$:

$\Gamma\Gamma y \vee \neg\neg x = \Gamma\Gamma y \vee \Gamma\Gamma\neg\neg x$ [B27]
$= \Gamma(\Gamma y \wedge \Gamma\neg\neg x)$ [B6]
$= \Gamma(\Gamma y \wedge \neg x)$ [B33]
$= \Gamma(\Gamma y \wedge \Gamma\Gamma\neg x)$ [B27]
$= \Gamma\Gamma(y \vee \Gamma\neg x)$ [B9]
$= \Gamma\Gamma(y \vee \neg\neg x).$ [B19]

(B38) If $x \leq y$ then $\sim y \leq \sim x$:

Let x, y be such that

(1) $x \leq y$,

Then

(2) $\sim y \vee \sim x = (y \wedge \Gamma y) \vee \neg y \vee \neg x \vee (x \wedge \Gamma x)$ [D3]
$= (y \wedge \Gamma y) \vee \neg x \vee (x \wedge \Gamma x)$ [(1), B34]
$= \neg x \vee ((y \vee x) \wedge (y \vee \Gamma x) \wedge (\Gamma y \vee x) \wedge (\Gamma y \vee \Gamma x))$
$= \neg x \vee (y \wedge (y \vee \Gamma x) \wedge (\Gamma y \vee x) \wedge \Gamma x),$ ((1), B34)

(3) $\sim y \vee \sim x = \neg x \vee (y \wedge (\Gamma y \vee x) \wedge \Gamma x)$ [(2)]
$= \neg x \vee (\Gamma x \wedge ((y \wedge \Gamma y) \vee (y \wedge x)))$
$= \Gamma x \wedge (\neg x \vee (y \wedge \Gamma y) \vee x),$ [(1), B14]

(4) $y \wedge \Gamma y \leq x \vee \neg x,$ [(1), B10]

Then

(5) $\sim y \vee \sim x = \Gamma x \wedge (\neg x \vee x) = \sim x,$ [(3), (4), D2]

(6) $\sim y \leq \sim x.$ [(5)]

(B39) $\neg x \wedge \Gamma y \leq \Gamma(x \vee y)$:

(1) $\neg x \wedge \Gamma y = \Gamma\Gamma\neg x \wedge \Gamma y$ [B27]
$= \Gamma(y \vee \Gamma\neg x),$ [B9]

(2) $x \leq \neg\neg x$ [B22]
$= \Gamma\neg x,$ [B19]

(3) $\Gamma(y \vee \Gamma\neg x) \vee \Gamma(x \vee y) = \Gamma((y \vee \Gamma\neg x) \wedge (x \vee y))$ [B6]
$= \Gamma(((y \vee \Gamma\neg x) \wedge x) \vee ((y \vee \Gamma\neg x) \wedge y))$
$= \Gamma(((y \vee \Gamma\neg x) \wedge x) \vee y)$
$= \Gamma(((y \vee \Gamma\neg x) \vee y) \wedge (x \vee y))$
$= \Gamma((y \vee \Gamma\neg x) \wedge (x \vee y))$
$= \Gamma(y \vee (\Gamma\neg x \wedge x))$
$= \Gamma(y \vee x),$ [(2)]

(4) $\Gamma(y \vee \Gamma\neg x) \leq \Gamma(x \vee y),$ [(3)]

(5) $\neg x \wedge \Gamma y \leq \Gamma(x \vee y).$ [(1), (4)]

(B40) $\neg x \wedge \sim y \leq (x \vee y) \vee \neg(x \vee y)$:

(1) $\neg x \wedge \sim y \wedge ((x \vee y) \vee \neg(x \vee y)) = \sim y \wedge ((\neg x \wedge (x \vee y)) \vee (\neg x \wedge \neg(x \vee y)))$
$= \sim y \wedge ((\neg x \wedge y) \vee (\neg x \wedge \neg y))$ [B2, B7]
$= \sim y \wedge \neg x \wedge (y \vee \neg y)$
$= \neg x \wedge \sim y,$ [D2]

(2) $\neg x \wedge \sim y \leq (x \vee y) \vee \neg(x \vee y).$ [(1)]

(B41) $x \wedge \Gamma x \wedge \sim y \leq \Gamma(x \vee y)$:

$$\begin{aligned} x \wedge \Gamma x \wedge \sim y &= x \wedge \Gamma x \wedge \Gamma y \wedge (y \vee \neg y) && \text{[D2]} \\ &= (x \wedge \Gamma x \wedge \Gamma y \wedge y) \vee (x \wedge \Gamma x \wedge \Gamma y \wedge \neg y) \\ &= (x \wedge \Gamma x \wedge \Gamma y \wedge y) \vee (x \wedge \Gamma x \wedge \neg y) && \text{[B14]} \\ &\leq \Gamma(x \vee y) \vee (x \wedge \Gamma x \wedge \neg y) && \text{[B11]} \\ &\leq \Gamma(x \vee y) \vee (x \wedge \Gamma(x \vee y)) && \text{[B39]} \\ &= \Gamma(x \vee y). && \text{[A2]} \end{aligned}$$

(B42) $\neg x \wedge \sim y \leq \Gamma(x \vee y)$:

$$\begin{aligned} \text{(1)}\ \neg x \wedge \sim y \wedge \Gamma(x \vee y) &= \neg x \wedge (y \vee \neg y) \wedge \Gamma y \wedge \Gamma(x \vee y) && \text{[D2]} \\ &= \neg x \wedge \Gamma y \wedge (y \vee \neg y) && \text{[B39]} \\ &= \neg x \wedge \sim y, && \text{[D2]} \end{aligned}$$

(2) $\neg x \wedge \sim y \leq \Gamma(x \vee y)$. [(1)]

Now we can prove the axiom A5.

Axiom A5 $\sim(x \vee y) = \sim x \wedge \sim y$:

$$\begin{aligned} \text{(1)}\ \sim x \wedge \sim y &= ((x \vee \neg x) \wedge \Gamma x) \wedge \sim y && \text{[D2]} \\ &= (x \vee \neg x) \wedge (\Gamma x \wedge \sim y) \\ &= (x \wedge \Gamma x \wedge \sim y) \vee (\sim y \wedge \neg x \wedge \Gamma x) \\ &= (x \wedge \Gamma x \wedge \sim y) \vee (\sim y \wedge \neg x), && \text{[B14]} \end{aligned}$$

(2) $x \wedge \Gamma x \wedge \sim y \leq \Gamma(x \vee y)$, [B41]

(3) $\sim y \wedge \Gamma x \leq (x \vee y) \vee \sim (x \vee y)$, [B40]

(4) $\sim x \wedge \sim y \leq \Gamma(x \vee y) \wedge ((x \vee y) \vee \neg(x \vee y))$, [(1), (2), (3)]

(5) $\sim x \wedge \sim y \leq \sim (x \vee y)$, [(4), D2]

(6) $x \leq x \vee y$,

(7) $y \leq x \vee y$,

(8) $\sim (x \vee y) \leq \sim x$, [(6), B38]

(9) $\sim (x \vee y) \leq \sim y$, [(7), B38]

(10) $\sim (x \vee y) \leq \sim x \wedge \sim y$, [(8), (9)]

(11) $\sim x \wedge \sim y = \sim (x \vee y)$. [(5), (10)]

Therefore, $\langle A, \wedge, \vee, \sim, \nabla, 1 \rangle$ is a 4-valued modal algebra. □

3. Other characterizations

The following characterization of 4–valued modal algebras is easier than that given in Theorem 2.1.

Theorem 3.1. *Let* $(A, \wedge, \vee, \sim, \neg, 1)$ *be an algebra of type* (2,2,1,1,0) *where* $(A, \wedge, \vee, \sim, 1)$ *is a De Morgan algebra with last element* 1 *and first element* $0 = \sim 1$. *If* ∇ *is an unary operation defined on* A *by means of the formula* $\nabla x = \sim \neg x$, *then* A *is a* 4*–valued modal algebra if and only if it verifies:*

(T1) $x \wedge \neg x = 0$,

(T2) $x \vee \neg x = x \vee \sim x$.

Furthermore $\neg x = \sim \nabla x$.

Proof. We check only sufficient condition

(A6) $\sim x \vee \nabla x = \sim x \vee \sim \neg x = \sim (x \wedge \neg x) = 1$ [T1]

(A7)
$$\begin{aligned} \sim x \wedge \nabla x &= \sim x \wedge \sim \neg x && \text{[T2]} \\ &= \sim (x \vee \neg x) \\ &= \sim (x \vee \sim x) \\ &= x \wedge \sim x \end{aligned}$$

□

Remark 3.1. In a 4–valued modal algebra the operation considered in Theorem 2.1, generally does not coincide with the pseudo-complement * as we can verify in the following example:

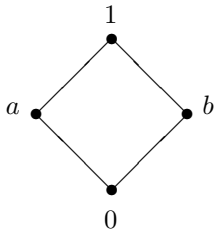

Figure 1.

x	$\sim x$	∇x
0	1	0
a	a	1
b	b	1
1	0	1

Table 1.

we have

x	$\neg x$	x^*
0	1	1
a	0	b
b	0	a
1	0	0

Table 2.

However, every finite 4–valued modal algebra is a distributive lattice pseudo complemented. We do not know whether this situation holds in the non-finite case. This suggests that we consider a particular class of De Morgan algebras.

Definition 3.1. An algebra $(A, \wedge, \vee, \sim, ^*, 1)$ of type (2,2,1,1,0) is a *modal De Morgan p–algebra* if the reduct $(A, \wedge, \vee, \sim, 1)$ is a De Morgan algebra with last element 1 and first element $0 = \sim 1$, the reduct is a pseudo–complemented meet–lattice and the following condition is verified

$$\text{H1)}\ x \vee \sim x \leq x \vee x^*$$

Example 3.1. The De Morgan algebra whose Hasse diagram is given in Figure 2 and the operations $\sim$ and * are defined in Table 3.

Figure 2.

x	$\sim x$	x^*
0	1	1
a	b	0
b	a	0
1	0	0

Table 3.

is not a modal De Morgan p–algebra because $b = (a \vee \sim a) \not\leq a \vee a^* = a$.

Theorem 3.2. *If we define on a modal De Morgan p–algebra* $\langle A, \wedge, \vee, \sim, ^*, 1\rangle$ *the operation* $\neg$ *by means of the formula* $\neg x = x^* \wedge \sim x$ *then the algebra* $\langle A, \wedge, \vee, \neg, 1\rangle$ *verifies the identities* T1 *and* T2.

Proof.
(T1) $x \wedge \neg x = x \wedge x^* \wedge \sim x = 0 \wedge \sim x = 0$
(T2) $x \vee \neg x = x \vee (x^* \wedge \sim x) = (x \vee x^*) \wedge (x \vee \sim x)$
$= (x \vee \sim x)$ [H1]
□

Remark 3.2. By [4] we know that every finite modal 4–valued algebra A is direct product of copies of T2, T3 and T4, where T2=$\{0,1\}$, T3=$\{0,a,1\}$ and T4=$\{0,a,b,1\}$ are modal De Morgan p–algebra we conclude that A is also a modal De Morgan p–algebra.

We do not know whether this situation holds in the non-finite case.

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Congruence relations on pseudo BE–algebras

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Abstract. In this paper, we consider the notion of congruence relation on pseudo BE–algebras and construct quotient pseudo BE–algebra via this congruence relation. Also, we use the notion of normal pseudo filters and get a congruence relation.

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1. Introduction

Some recent researchers led to generalizations of some types of algebraic structures by pseudo structures. G Georgescu and A. Iorgulescu [3], and independently J. Rachunek [11], introduced pseudo MV–algebras which are a non-commutative generalization of MV–algebras. The notions of pseudo BL–algebras and pseudo BCK–algebras were introduced and studied by G. Georgescu and A. Iorgulescu [9, 10, 4]. A. Walendziak gave a system of axioms defining pseudo BCK– algebras [12]. Y, B. Jun and et al. introduced the concepts of pseudo-atoms, pseudo BCI–ideals and pseudo BCI–homomorphisms in pseudo BCI– algebras and characterizations of a pseudo BCI–ideal, and provide conditions for a subset to be a pseudo BCI–ideal [5]. Y. H. Kim and K. S. So [7], discuss on minimal elements in pseudo BCI–algebras.

The notion of BE–algebras was introduced by H. S. Kim and Y. H. Kim [6]. We generalized the notion of BE–algebras and introduced the notion of pseudo BE–algebras, pseudo subalgebras, pseudo filters and investigated some related properties [1]. We introduced the notion of distributive pseudo BE–algebra and normal pseudo filters and prove some basic properties. Furthermore, the notion of pseudo upper sets in pseudo BE– algebras introduced and prove that the every pseudo filter F of X is union of pseudo upper sets. We show that in distributive pseudo BE–algebras normal pseudo filters and pseudo filters are equivalent [2].

In the present paper, we apply the notion of congruence relations to pseudo BE–algebras and discuss on the quotient algebras via this congruence relations. It is a natural question which is the relationships between congruence relations on pseudo BE–algebras and (normal)pseudo filters. From here comes the main motivation for this. We show that quotient of a pseudo BE–algebra via a congruence relation is a pseudo BE–algebra and prove that, if X is a distributive pseudo BE–algebra, then it becomes to a BE–algebra.

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2. Preliminaries

In this section we review the basic definitions and some elementary aspects that are necessary for this paper.

Definition 2.1. [1] An algebra $(X; *, \diamond, 1)$ of type $(2, 2, 0)$ is called a *pseudo BE–algebra* if it satisfies in the following axioms:

$(pBE1)$ $x * x = 1$ and $x \diamond x = 1$,
$(pBE2)$ $x * 1 = 1$ and $x \diamond 1 = 1$,
$(pBE3)$ $1 * x = x$ and $1 \diamond x = x$,
$(pBE4)$ $x * (y \diamond z) = y \diamond (x * z)$,
$(pBE5)$ $x * y = 1 \Leftrightarrow x \diamond y = 1$, for all $x, y, z \in X$.

In a pseudo BE–algebra, one can introduce a binary relation " $\leq$ " by $x \leq y \Leftrightarrow x * y = 1 \Leftrightarrow x \diamond y = 1$, for all $x, y \in X$. From now on X is a pseudo BE–algebra, unless otherwise is stated and we note that if $(X; *, \diamond, 1)$ is a pseudo BE–algebra, then $(X; \diamond, *, 1)$ is a pseudo BE–algebra, too.

Remark 2.1. If X is a pseudo BE–algebra satisfying $x * y = x \diamond y$, for all $x, y \in X$, then X is a BE–algebra.

Proposition 2.1. [1, 2] *The following statements hold:*
(1) $x * (y \diamond x) = 1$, $x \diamond (y * x) = 1$,
(2) $x \diamond (y \diamond x) = 1$, $x * (y * x) = 1$,
(3) $x \diamond ((x \diamond y) * y) = 1$, $x * ((x * y) \diamond y) = 1$,
(4) $x * ((x \diamond y) * y) = 1$, $x \diamond ((x * y) \diamond y) = 1$,
(5) *If* $x \leq y * z$, *then* $y \leq x \diamond z$,
(6) *If* $x \leq y \diamond z$, *then* $y \leq x * z$,
(7) $1 \leq x$, *implies* $x = 1$.
(8) *If* $x \leq y$, *then* $x \leq z * y$ *and* $x \leq z \diamond y$,
for all $x, y, z \in X$.

Definition 2.2. [1] A non-empty subset F of X is called a *pseudo filter* of X if it satisfies in the following axioms:

$(pF1)$ $1 \in F$,
$(pF2)$ $x \in F$ and $x * y \in F$ imply $y \in F$.

Proposition 2.2. [1] *Let $F \subseteq X$ and $1 \in F$. F is a pseudo filter if and only if $x \in F$ and $x \diamond y \in F$ imply $y \in F$, for all $x, y \in X$.*

Theorem 2.3. [1] *Let X be a pseudo BE-algebra. Then every pseudo filter of X is a pseudo sub-algebra.*

Definition 2.3. [2] X is said to be *distributive* if it satisfies in the following condition:

$$x * (y \diamond z) = (x * y) \diamond (x * z), \ \textit{for all } x, y, z \in X.$$

Theorem 2.4. [2] *Let X be a distributive and $x \leq y$. Then*
(i) $z * x \leq z * y$, *and* $z * x \leq z \diamond y$,
(ii) $z \diamond x \leq z * y$, *and* $z \diamond x \leq z \diamond y$,
for all $x, y, z \in X$.

Proposition 2.5. [2] *Let X be a distributive. Then*
(i) $y * z \leq (x * y) * (x * z)$, *and* $y * z \leq (x * y) \diamond (x * z)$,
(ii) $y \diamond z \leq (x * y) * (x * z)$, *and* $y \diamond z \leq (x * y) \diamond (x * z)$,
(iii) $A(x * y) = A(x \diamond y)$,

for all $x, y, z \in X$.

Definition 2.4. [2] A pseudo filter F is said to be *normal*, if for all $x, y \in X$

$$x * y \in F \text{ if and only if } x \diamond y \in F.$$

Theorem 2.6. [2] *Let X be distributive. Then every pseudo filter is normal.*

Theorem 2.7. [2] *Let $(X; *, \diamond, 1)$ be a distributive pseudo BE–algebra. $(X, \diamond, *, 1)$ is a distributive pseudo BE–algebra if and only if $(X; *, 1)$ is a BE–algebra (i. e. $x * y = x \diamond y$, for all $x, y \in X$).*

3. Congruences relations on pseudo BE–algebras

Quotient algebras are a basic tool for exploring the structures of pseudo BE–algebras. There are some relations between pseudo filters, pseudo congruence and quotient pseudo BE–algebras. We define the notion of congruence relations on pseudo BE–algebras and prove that the quotient algebra $(X/\theta; *, \diamond, C_1)$ is a pseudo BE–algebra.

Definition 3.1. Let "θ" be an equivalence relation on X. "θ" is called:

(i) *Left congruence* relation on X if $(x, y) \in \theta$ implies $(u * x, u * y) \in \theta$ and $(u \diamond x, u \diamond y) \in \theta$, for all $u \in X$.

(ii) *Right congruence* relation on X if $(x, y) \in \theta$ implies $(x * v, y * v) \in \theta$ and $(x \diamond v, y \diamond v) \in \theta$, for all $v \in X$.

(iii) *Congruence* relation on X if has the substitution property with respect to " $*$ " and "$\diamond$", that is, for any $(x, y), (u, v) \in \theta$ we have $(x*u, y*v) \in \theta$ and $(x \diamond u, y \diamond v) \in \theta$.

Example 3.1. (i). It is obvious that $\nabla = X \times X$ and $\triangle = \{(x, x) \mid x \in X\}$ is a congruence relation on X.
(ii). Let $X = \{1, a, b, c, d\}$ and operations " $*$ " and " $\diamond$ " defined as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	1	1
b	1	1	1	1	1
c	1	a	a	1	1
d	1	a	b	c	1

$\diamond$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	1	1
b	1	1	1	1	1
c	1	a	b	1	1
d	1	a	b	c	1

Set $\theta_1 := \triangle \cup \{(d, 1), (1, d)\}$ and $\theta_2 := \triangle \cup \{(1, a), (a, 1)\}$. We can see that θ_1 is a congruence relation on X and θ_2 is a left congruence relation on X. Since $(1, a) \in \theta_2$, and $(b, a) = (1 * b, a * b) \notin \theta_2$, it follows that θ_2 is not a right congruence relation.
(iii). Let $X = \{1, a, b, c\}$, operations " $*$ " and " $\diamond$ " defined as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	1	1	1
c	1	1	1	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	1	b
b	1	1	1	1
c	1	1	1	1

Then, $(X; *, \diamond, 1)$ is a pseudo BE–algebra. If set $\theta_3 = \Delta \cup \{(b, c), (c, b)\}$, then θ_3 is a right congruence relation. Since $(b, c) \in \theta_3$ and $a \in X$, but $(1, a) = (a * b, a * c) \notin \theta_3$, it follows that θ_3 is not a left congruence neither a congruence relation.

For any $x \in X$, we define

$$\phi_x = \{(a,b) \in X \times X : x * a = x * b \text{ and } x \diamond a = x \diamond b\}.$$

Proposition 3.1. *ϕ_x is a left congruence relation on X, for all $x \in X$.*

Proof. It is obvious that ϕ_x is an equivalence relation on X. Let $(a,b) \in \phi_x$ and $u \in X$. Hence $x * a = x * b$. Now, we have $x * (u * a) = u * (x * a) = u * (x * b) = x * (u * b)$. Therefore, $(u * a, u * b) \in \phi_x$. By a similar way $(u \diamond a, u \diamond b) \in \phi_x$. □

The following example shows that ϕ_x is not a right congruence relation on X, in general.

Example 3.2. Let $X = \{1, a, b, c\}$ and operations " $*$ " and " $\diamond$ " defined as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	b	1	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	c
c	1	c	1	1

Then, $(X; *, \diamond, 1)$ is a pseudo BE–algebra. It can be seen that

$$\phi_c = \{(1,1), (a,a), (b,b), (c,c), (1,b), (b,1), (b,c), (c,b)\}$$

is a left congruence relation on X, but it is not right congruence relation because $(c,b) \in \phi_c$ but $(c * a, b * a) = (b,a) \notin \phi_c$.

Proposition 3.2. *Let X be distributive. Then ϕ_x is a right congruence relation on X, for all $x \in X$.*

Proof. It is sufficient to show that if $(a,b) \in \phi_x$ and $v \in X$, then $(a * v, b * v) \in \phi_x$. Let $(a,b) \in \phi_x$ and $v \in X$. Hence $x * a = x * b$ and $x \diamond a = x \diamond b$. Now, by using distributivity of X we have $x * (a \diamond v) = (x * a) \diamond (x * v) = (x * b) \diamond (x * v) = x * (b \diamond v)$. Therefore, $(a \diamond v, b \diamond v) \in \phi_x$. By a similar way $(a * v, b * v) \in \phi_x$. □

Example 3.3. In Example 3.2, consider

$$c * (c \diamond a) = c * c = 1 \neq b = 1 \diamond b = (c * c) \diamond (c * a),$$

then X is not distributive. Also we showed that ϕ_c is not a right congruence relation.

Let $pCon(X)$ be the set of all congruence relations on X and respectively $pCon_L(X)$ ($pCon_R(X)$) be the set of all the left (right) congruence relations on X. It is clear that $pCon(X) = pCon_L(X) \cap pCon_R(X)$. For $\theta \in pCon(X)$ we will denote $C_x(\theta) = \{y \in X : y \sim_\theta x\}$, abbreviated by C_x. We will call C_x the equivalence class containing x and so $X/\theta = \{C_x : x \in X\}$.

Theorem 3.3. *Let $\theta \in pCon(X)$. Then $C_1 = \{x \in X : x \sim_\theta 1\}$ is a pseudo filter of X.*

Proof. Since θ is a reflexive relation, we see that $(1,1) \in \theta$ and so $1 \sim_\theta 1$. Thus $1 \in C_1$. Now, let $x, y \in X$. Assume that $a \in C_1$, $a * x \in C_1$. Then $a * x \sim_\theta 1$. Now, we have $x \diamond (a * x) \sim_\theta x \diamond 1$. Thus $1 \sim_\theta a \sim_\theta x$ and so $x \in C_1$. This shows that C_1 is a pseudo filter of X. □

Note. Let $\theta \in pCon(X)$. Define operations " $*$ " and " $\diamond$ " on X/θ by $C_x * C_y = C_{x*y}$ and $C_x \diamond C_y = C_{x \diamond y}$. Let $\nu : X \to X/\theta$ be such that $\nu(x) = C_x$ for all $x \in X$. Then, ν is an epimorphism. In fact $\nu(x * y) = C_{x*y} = C_x * C_y = \nu(x) * \nu(y)$ and $\nu(x \diamond y) = C_{x \diamond y} = C_x \diamond C_y = \nu(x) \diamond \nu(y)$. ν is called the *natural homomorphism* from X to X/θ.

Proposition 3.4. *The following statements hold:*
(*i*) *if* $\theta = X \times X$, *then* $X/\theta = \{C_1\}$,
(*ii*) *if* $\theta = \triangle_X$, *then* $X/\theta = \{X\}$,
(*iii*) *if* $x \leq y$, *then* $C_x \leq C_y$.

Proof. (*i*). Let $C_x \in X/\theta$, for some $x \in X$. Since $\theta = X \times X$, we have $(x, y) \in \theta$ for all $y \in X$. Hence $C_x = C_y$. Putting $y := 1$, then $C_x = C_1$. Therefore, $X/\theta = \{C_1\}$.

(*ii*). Let $C_x \in X/\theta$, for some $x \in X$. Since $\theta = \triangle_X$, we have $C_x = \{x\}$. Therefore, $X/\theta = \{X\}$.

(*iii*). Since $x \leq y$, we get that $x * y = 1$ and $x \diamond y = 1$. Hence $C_{x*y} = C_1 = C_x * C_y$ and $C_{x\diamond y} = C_1 = C_x \diamond C_y$. Therefore, $C_x \leq C_y$. □

Proposition 3.5. *Let* $\theta \in pCon(X)$. *Then* $(X/\theta; *, \diamond, C_1)$ *is a pseudo* BE*–algebra.*

Proof. If $C_x, C_y, C_z \in X/\theta$, then we have
($pBE1$) $C_x * C_x = C_1$ and $C_x \diamond C_x = C_1$,
($pBE2$) $C_x * C_1 = C_1$ and $C_x \diamond C_1 = C_1$,
($pBE3$) $C_1 * C_x = C_x$ and $C_1 \diamond C_x = C_x$,
($pBE4$) $C_x * (C_y \diamond C_z) = C_y \diamond (C_x * C_z)$,
($pBE5$) $C_x \leq C_y \Leftrightarrow C_x * C_y = C_1 \Leftrightarrow C_x \diamond C_y = C_1$.
Then, $(X/\theta; *, \diamond, C_1)$ is a pseudo BE–algebra. □

Example 3.4. Consider congruence relation θ_1 in Example 3.1(ii), then

$$X/\theta_1 = \{C_1 = C_d = \{1, d\}, C_a = \{a\}, C_b = \{b\}, C_c = \{c\}\},$$

with the operations " $*$ " and " $\diamond$ " defined by following table is a pseudo BE–algebra.

$*$	C_1	C_a	C_b	C_c
C_1	C_1	C_a	C_b	C_c
C_a	C_1	C_1	C_a	C_1
C_b	C_1	C_1	C_1	C_1
C_c	C_1	C_a	C_a	C_1

$\diamond$	C_1	C_a	C_b	C_c
C_1	C_1	C_a	C_b	C_c
C_a	C_1	C_1	C_c	C_1
C_b	C_1	C_1	C_1	C_1
C_c	C_1	C_a	C_b	C_1

Theorem 3.6. *Let* X *be distributive and* $\theta \in pCon(X)$. *Then* $(X/\theta; *, \diamond, C_1)$ *is too.*

Proof. Let $C_x, C_y, C_z \in X/\theta$, for any $x, y, z \in X$. Then

$$\begin{aligned} C_x * (C_y \diamond C_z) = C_x * C_{y\diamond z} &= C_{x*(y\diamond z)} \\ &= C_{(x*y)\diamond(x*z)} \\ &= C_{x*y} \diamond C_{x*z} \\ &= (C_x * C_y) \diamond (C_x * C_z). \end{aligned}$$

Therefore, X/θ is distributive. □

Proposition 3.7. *Let* $f : X \to Y$ *be a homomorphism. Then*
(*i*) $f(1) = 1$,
(*ii*) f *has the isotonic property, i. e., if* $x \leq y$, *then* $f(x) \leq f(y)$, *for all* $x, y \in X$.

Proof. (*i*). Let $x \in X$. Since $x * x = x \diamond x = 1$ and f is a homomorphism, we see that $f(1) = f(x * x) = f(x) * f(x) = 1$ and $f(1) = f(x \diamond x) = f(x) \diamond f(x) = 1$. Hence $f(1) = 1$.

(*ii*). If $x \leq y$, Then $x * y = x \diamond y = 1$. So, (*i*) implies

$$f(x) * f(y) = f(x * y) = f(1) = 1, \text{ and } f(x) \diamond f(y) = f(x \diamond y) = f(1) = 1.$$

Hence $f(x) \leq f(y)$. Therefore, f has the isotonic property. □

Proposition 3.8. *Let $f : X \to Y$ be a homomorphism and $\theta = \{(x, y) : f(x) = f(y)\}$. Then*
(i) θ is a congruence relation on X,
(ii) $X/\theta \cong f(X)$.

Proof. *(i)*. It is obvious θ is an equivalence relation on X. We only show that θ satisfies the substitution property. Assume that (x, y) and $(u, v) \in \theta$. Then we have $f(x) = f(y)$ and $f(u) = f(v)$. Since f is a homomorphism and above argument yields,

$$f(x * u) = f(x) * f(u) = f(y) * f(v) = f(y * v).$$

and

$$f(x \diamond u) = f(x) \diamond f(u) = f(y) \diamond f(v) = f(y \diamond v).$$

Then $(x*u, y*v), (x\diamond u, y\diamond v) \in \theta$. In the same way we have $(u*x, v*y), (u\diamond x, v\diamond y) \in \theta$. Hence θ is a congruence relation on X.

(ii). By using the Proposition 3.5, we have $(X/\theta; *, \diamond, C_1)$ is a pseudo BE–algebra. Let $\nu : X/\theta \to f(X)$ be such that $\nu(C_x) = f(x)$, for all $C_x \in X/\theta$. Then

(i). v is well defined, because if $C_x = C_y$, for any $x, y \in X$, then $(x, y) \in \theta$. Therefore, $f(x) = f(y)$. Hence $\nu(C_x) = \nu(C_y)$.

(ii). $\ker \nu = \{C_x : \nu(C_x) = f(x) = 1\} = \{C_x : f(x) = f(1)\} = \{C_x : (x, 1) \in \theta\} = C_1$. Then v is one to one.

(iii). $\nu(C_x * C_y) = \nu(C_{x*y}) = f(x * y) = f(x) * f(y) = \nu(C_x) * \nu(C_y)$ and $\nu(C_x \diamond C_y) = \nu(C_{x\diamond y}) = f(x \diamond y) = f(x) \diamond f(y) = \nu(C_x) \diamond \nu(C_y)$. Thus ν is a homomorphism. Therefore, $X/\theta \cong f(X)$. □

4. Congruence relations induced by pseudo filters

In this section we assume that X is a distributive pseudo BE–algebra, unless otherwise is stated.

Proposition 4.1. *Let F be a pseudo filter of X. Define*

$$x \sim_F y \quad \textit{if and only if} \quad x * y,\ y * x \in F.$$

Then $\sim_F \in pCon(X)$.

Proof. *(i)*. Since $1 \in F$, we have $x * x = 1 \in F$, i.e., $x \sim_F x$. This means that " $\sim_F$ " is reflexive. Now, if $x \sim_F y$ and $y \sim_F z$, then $x * y, y * x \in F$ and $y * z, z * y \in F$. By Proposition 2.5(i), $y * z \leq (x * y) * (x * z)$. Now, since $y * z \in F$ and F is a pseudo filter, it follows that $(x * y) * (x * z) \in F$. So $x * z \in F$. By a similar way we see that $z * x \in F$. This shows that " $\sim_F$ " is transitive. The symmetry of " $\sim_F$ " is immediate from the definition. Therefore, " $\sim_F$ " is an equivalence relation on X.

(ii). Let $x \in X$ and $u \sim_F v$. Then by Proposition 2.5(i), $v * u \leq (x * v) * (x * u)$. Now, since $v * u \in F$ and F is a pseudo filter, $(x * v) * (x * u) \in F$. By a similar way, $(x * u) * (x * v) \in F$. Therefore, $x * v \sim_F x * u$. Also, by Proposition 2.5(ii), $u * v \leq (x \diamond u) * (x \diamond v)$. Now, since $u * v \in F$ and F is a pseudo filter, we see that $(x \diamond u) * (x \diamond v) \in F$. By a similar way, $(x \diamond v) * (x \diamond u) \in F$. Therefore, $x \diamond v \sim_F x \diamond u$.

By using Proposition 2.5(ii), we have $x * u \leq (y \diamond x) * (y \diamond u)$, then $(x * u) \diamond ((y \diamond x) * (y \diamond u)) = 1$ and so by $(pBE4)$ we have $(y \diamond x) * ((x * u) \diamond (y \diamond u)) = 1$, which implies that $(x * u) \diamond (y \diamond u) \in F$, because F is pseudo filter $y * x \in F$ and by Theorem 2.6, F is normal, then $y \diamond x \in F$. Hence $(x * u) * (y \diamond u) \in F$. On the other hand, we have

$x * y \leq (y \diamond u) * (x * u)$, because

$$\begin{aligned}(x * y) \diamond ((y \diamond u) * (x * u)) &= (y \diamond u) * ((x * y) \diamond (x * u)) \\ &= (y \diamond u) * (x * (y \diamond u)) = 1.\end{aligned}$$

Hence $(y \diamond u) * (x * u) \in F$, because F is pseudo filter and $x * y \in F$. Thus $x * u \sim_F y \diamond u$. Finally, since $y \diamond u \sim_F y \diamond v$ and by a similar way, $y \diamond v \sim_F y * v$. By the transitivity " $\sim_F$ " we get $x * u \sim_F y * v$. By the same manner $x \diamond u \sim_F y \diamond v$. Therefore, $\sim_F \in pCon(X)$. □

Note. Now, let F be a pseudo filter of X. Denote the equivalence class of x by C_x. Then $F = C_1$. In fact, if $x \in F$, then $x * 1 = x \diamond 1 = 1 \in F$ and $1 * x = 1 \diamond x = x \in F$, i.e., $x \sim_F 1$. Hence $x \in C_1$.

Conversely, let $x \in C_1$. Then $x = 1 * x = 1 \diamond x \in F$, and so $x \in F$. Hence $F = C_1$. Denote $X/F = \{C_x : x \in X\}$ and define that $C_x * C_y = C_{x*y}$ and $C_x \diamond C_y = C_{x \diamond y}$. Since " $\sim_F$ " is a congruence relation on X, the operations " $*$ " and " $\diamond$ " are well defined.

Example 4.1. Let $X = \{1, a, b, c, d\}$ and operations " $*$" and " $\diamond$" defined as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	c	c	1

$\diamond$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	b	c	1

Then, $(X; *, \diamond, 1)$ is a distributive pseudo BE–algebra. It can be easily seen that $F = \{1, a, d\}$ is a pseudo filter. We have

$$\sim_F = \{(1,1), (a,a), (b,b), (c,c), (d,d), (1,a), (a,1), (d,1), (1,d)(a,d), (d,a), (b,c), (c,b)\}$$

and so $\sim_F \in pCon(X)$.

Theorem 4.2. *Let $F \in pF(X)$. Then*

(*i*) $(X/F; *, C_1) = (X/F; \diamond, C_1)$ *is a BE–algebra (which is called quotient pseudo BE–algebra via F, and $C_1 = F$.)*

(*ii*) $(X/F; \diamond, *, 1)$ *is a distributive pseudo BE–algebra if and only if $(X/F; *, 1)$ is a BE–algebra (i. e. $C_{x*y} = C_{x \diamond y}$, for all $x, y \in X$).*

Proof. (*i*). By similar way of the proof of Proposition 3.5, $(X/F; *, \diamond, C_1)$ is a distributive pseudeo BE-algebra. To prove X/F is a BE-algebra it is sufficient to prove, $C_x * C_y = C_x \diamond C_y$, for all $C_x, C_y \in X/F$. By Proposition 2.5 (*iii*), $A(x * y) = A(x \diamond y)$. By definition of $A(x)$, it is obvious that $x * y \in A(x * y)$ and $x \diamond y \in A(x \diamond y)$. Thus $x * y \in A(x * y) = A(x \diamond y)$ and so $(x * y) * (x \diamond y) = 1 \in F$. By similar way, $x \diamond y \in A(x \diamond y) = A(x * y)$ and so $(x \diamond y) * (x * y) = 1 \in F$. Hence $x * y \sim_F x \diamond y$ and so $C_{x*y} = C_{x \diamond y}$, which means $C_x * C_y = C_x \diamond C_y$.

(*ii*). By (*i*) and Theorem 2.7, the proof is obvious. □

Example 4.2. Let $X = \{1, a, b, c, d, e\}$. Define the operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	c	c	d	1
b	1	a	1	1	d	e
c	1	a	1	1	d	e
d	1	a	1	1	1	e
e	1	a	c	c	d	1

$\diamond$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	d	1
b	1	a	1	1	d	e
c	1	a	1	1	d	e
d	1	a	1	1	1	e
e	1	a	c	c	d	1

Then $(X; *, \diamond, 1)$ is a distributive pseudo BE–algebra. By consider pseudo filter $F = \{1, e\}$, we have $X/F = \{C_1 = C_e = F, C_a = \{a\}, C_b = C_c = \{b, c\}, C_d = \{d\}\}$ with the operations " $*$ " and " $\diamond$ " defined by following table is a pseudo BE–algebra.

$* = \diamond$	C_1	C_a	C_b	C_d
C_1	C_1	C_a	C_b	C_d
C_a	C_1	C_1	C_b	C_d
C_b	C_1	C_a	C_1	C_d
C_d	C_1	C_a	C_1	C_1

Proposition 4.3. *Let $\theta \in pCon(X)$. Then*
(*i*) $F_\theta \in pF(X)$,
(*ii*) $F_\theta = \{x | (x, 1) \in \theta\}$.

Proof. (*i*). Since $(x, x) \in \theta$, we have $x * x = 1 \in F_\theta$. Suppose that $x * y, x \in F_\theta$. There are $(u, v), (p, q) \in \theta$ such that $x * y = u * v$ and $x = p * q$. Since $(u, v) \in \theta \in pCon(X)$, we have $(u * v, v * v) = (x * y, 1) \in \theta$ and by a similar way $(x, 1) \in \theta$. Now, $(x*y, 1*y) = (x*y, y) \in \theta$. Hence $(y, 1) \in \theta$. This yields that $y*1 = 1, 1*y = y \in F_\theta$. That is F_θ is a pseudo filter of X. Furthermore, we can see that, F_θ is normal pseudo filter from Theorem 2.6.

(*ii*). Put $F := \{x | (x, 1) \in \theta\}$. Let $x \in F_\theta$. There is $(u, v) \in \theta$ such that $x = u * v$. Since θ is a congruence, we have $(x, 1) = (u * v, 1) = (u * v, v * v) \in \theta$. Hence $F_\theta \subseteq F$.

Now, let $x \in F$. Hence $(x, 1) \in \theta$ and so $x * 1 = 1, 1 * x = x \in F_\theta$. Hence $F \subseteq F_\theta$. Therefore, $F = F_\theta$. □

In [8], M. Kondo proved that θ is a regular cogruence relation on BCI–algebra if and only if $\theta = \theta_{I_\theta}$. Now, it is natural to ask whether $\theta = \theta_{F_\theta}$ in pseudo BE–algebras, for all $\theta \in pCon(X)$. We shall investigate the relation between the congruences θ and θ_{F_θ}.

Theorem 4.4. *Let $\theta \in pCon(X)$. Then $\theta_{F_\theta} = \theta$.*

Proof. Let $(x, y) \in \theta$. Then $x * y, y * x \in F_\theta$. Since F_θ is a normal pseudo filter by Proposition 4.3(*i*), we have $(x, y) \in \theta_{F_\theta}$. Therefore $\theta \subseteq \theta_{F_\theta}$. Now, it is sufficient to show that $\theta_{F_\theta} \subseteq \theta$. Let $(x, y) \in \theta_{F_\theta}$. By definition, we have $x * y, y * x \in F_\theta$. Hence there are $(u, v), (p, q) \in \theta$ such that $x * y = u * v, y * x = p * q$. Since $\theta \in pCon(X)$, we have

$$(x * y, 1) = (u * v, 1) = (u * v, v * v) \in \theta.$$

By a similar way $(y * x, 1) \in \theta$. Using Proposition 4.3(*ii*), $x * y, y * x \in F_\theta$. Hence $(x, y) \in \theta$ and so $\theta_{F_\theta} \subseteq \theta$. Therefore, $\theta_{F_\theta} = \theta$. □

Proposition 4.5. *Let $f : X \to Y$ be a homomorphism. Then*
(*i*) *f is epimorphic if and only if $Im(f) = Y$,*
(*ii*) *f is monomorphic if and only if* $\ker(f) = \{0\}$,

(*iii*) *f is isomorphic if and only if the inverse mapping f^{-1} is isomorphic.*
(*iv*) $\ker(f)$ *is a closed pseudo filter of X,*
(*v*) *$Im(f)$ is a pseudo subalgebra of Y.*

Proof. (*iv*). By Proposition 3.7(i), $1 \in \ker(f)$. Let $x, x * y \in \ker(f)$, then $f(x) = f(x * y) = 1$, and so

$$1 = f(x * y) = f(x) * f(y) = 1 * f(y) = f(y).$$

Thus $y \in \ker(f)$. Now, let $x * y \in \ker(f)$. Then $f(x * y) = f(x) * f(y) = 1$, and so by $(pBE5)$ we have $f(x) \diamond f(y) = f(x \diamond y) = 1$. Therefore, $x \diamond y \in F$. By a similar way we can prove if $x \diamond y \in F$, then $x * y \in F$. Hence $\ker(f)$ is a closed pseudo filter of X.

(*v*). Obviously, $Im(f)$ is a non-vacuous set. If y_1, $y_2 \in Im(f)$, then there exist x_1, $x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$, thus

$$y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2) \in Im(f),$$

and

$$y_1 \diamond y_2 = f(x_1) \diamond f(x_2) = f(x_1 \diamond x_2) \in Im(f).$$

Consequently, $Im(f)$ is a pseudo subalgebra of Y. □

Note. In general, $Im(f)$ may not be a pseudo filter.

Example 4.3. Let $X = \{1, a, b, c\}$ and $Y = \{1, a, b, c, d\}$. Define operations " $*$ " and " $\diamond$ " on X and Y as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	c	1
b	1	1	1	1
c	1	a	b	1

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	1	a
b	1	1	1	1	a
c	1	a	a	1	a
d	1	1	1	1	1

$\diamond$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	1	c
b	1	1	1	1	c
c	1	a	b	1	d
d	1	1	1	1	1

Then $(X; *, \diamond, 1)$ and $(Y; *, \diamond, 1)$ are pseudo BE–algebras and $\{1, a, b, c\}$ is a pseudo filter of X. Now, if we consider $f : X \to Y$ as the identity map, then f is a homomorphism and $f(X) = X$. We can see that $X = \{1, a, b, c\}$ is a trivial pseudo filter of X, but $f(X)$ is not a pseudo filter of Y, because

$$a * d = a \in f(X),\ a \in f(X) \text{ but } d \notin f(X).$$

Proposition 4.6. *Let $f : X \to Y$ be an epimorphism. If F is a pseudo filter of X, then $f(F)$ is a pseudo filter of Y.*

Proof. $f(F)$ is nonempty subset of Y because $1 \in f(F)$. Let $y \in Y$ and $a \in f(F)$ such that $a * y, \in f(F)$. Then there exist $x \in X$ and $a_1 \in F$ such that $f(x) = y$ and $f(a_1) = a$. Now, we have $a * y = f(a_1) * f(x) = f(a_1 * x) \in f(F)$. Hence $a_1 * x \in F$. Since F is a pseudo filter and $a_1 \in F$, we have $x \in F$. Therefore, $y = f(x) \in f(F)$. □

Theorem 4.7. *Let F be a closed pseudo filter of X. Then there is a canonical surjective homomorphism $\varphi : X \to X/F$ by $\varphi(x) = C_x$, and $\ker \varphi = F$, where $\ker \varphi = \varphi^{-1}(C_1)$.*

Proof. It is clear that φ is well-defined. Let $x, y \in X$. Then

$$\varphi(x * y) = C_{x*y} = C_x * C_y = \varphi(x) * \varphi(y)$$

and

$$\varphi(x \diamond y) = C_{x\diamond y} = C_x \diamond C_y = \varphi(x) \diamond \varphi(y).$$

Hence φ is homomorphism.

Clearly φ is onto. Also, we have

$$\begin{aligned} \ker\varphi = \{x \in X : \varphi(x) = C_1\} &= \{x \in X : C_x = C_1\} \\ &= \{x \in X : x * 1, 1 * x, x \diamond 1, 1 \diamond x \in F\} \\ &= \{x \in X : x \in F\} = F. \end{aligned}$$

□

5. Conclusion

In this paper, we consider the relation between congruence relations on pseudo BE–algebras and (normal) pseudo filters. Also, we show that the quotient of a pseudo BE–algebra via a congruence relation is a pseudo BE–algebra and prove that, if X is a distributive pseudo BE–algebra and F is a normal pseudo filter, then the quotient algebra via this filter is a BE–algebra.

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Strongly nonlinear variational parabolic initial-boundary value problems

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Abstract. We prove an existence result for a strongly nonlinear parabolic equations with dual data in Sobolev spaces.

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1. Introduction

We deal with the following boundary value problems

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\Big(a(x,t,u,\nabla u)\Big) + g(x,t,u,\nabla u) + H(x,t,\nabla u) = f & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \end{cases} \tag{1}$$

where the cylinder $Q = \Omega \times [0,T]$ with a given real number $T > 0$ and Ω is a bounded domain of $\mathbb{R}^N$, $a : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, measurable with respect to x in Ω for every (t,s,ξ) in $[0,T] \times \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) such that for all ξ, η in $\mathbb{R}^N$, $\xi \neq \eta$,

$$|a(x,t,s,\xi)| \leq \beta\Big[k(x,t) + |s|^{p-1} + |\xi|^{p-1}\Big], \tag{2}$$

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) > 0, \tag{3}$$

$$a(x,t,s,\xi)\xi \geq \alpha|\xi|^p, \tag{4}$$

where the function $k(x,t) \in L^{p'}(Q)$ and β, α are positives constants.

Furthermore, let $g(x,t,s,\xi) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, and $H(x,t,\xi) : \Omega \times [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be two Carathéodory functions which satisfy, for all $(x,t) \in Q$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, the following conditions

$$|g(x,t,s,\xi)| \leq L_1(|s|)\Big(L_2(x,t) + |\xi|^p\Big), \tag{5}$$

$$g(x,t,s,\xi)s \geq 0, \tag{6}$$

$$|H(x,t,\xi)| \leq h(x,t)|\xi|^{p-1}, \tag{7}$$

where $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function in $L^1(\mathbb{R})$, while $L_2(x,t)$ belongs to $L^1(Q)$, and $h(x,t) \in L^r(Q)$ with $r > \max(N,p)$.

$$u(x,0) = u_0 \in L^1(\Omega). \tag{8}$$

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Our purpose in this paper is to prove the existence of solutions for the initial-boundary value problems (1) in the setting Sobolev space, in the case where $H \neq 0$ and f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$, where the principal part $-\mathrm{div}\Big(a(x,t,u,\nabla u)\Big)$, the nonlinearity g and H satisfying some general growth conditions. Note that, a little information is known for the parabolic case.

The study of the nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field.

For some classical and recent results on parabolic problems in Orlicz and Sobolev spaces see Dall'Aglio-Orsina [11] and Porretta [18] who proved the existence of solutions for the following Cauchy-Dirichlet problem (1) where $H = 0$ and the right hand side f is assumed to belong to $L^1(Q)$. This result generalizes analogous one of Boccardo-Gallouët [7], J. L. Lions [17], Landes [14] with $g = 0$, and of Landes-Mustonen [15, 16] with $g = g(x,t,u)$. See also [8] and [2, 4, 9, 10, 12, 19, 20, 21, 22] for related topics. In all of these results, the function a is supposed to satisfy a polynomial growth condition with respect to u and ∇u.

2. Main result

Firstly, we give the following lemma which will be used in our main result.

Lemma 2.1. *Given the functions $\lambda, \gamma, \varphi, \rho$ defined on $[a,+\infty[$, suppose that $a \geq 0$, $\lambda \geq 0$, $\gamma \geq 0$ and that $\lambda\gamma, \lambda\varphi$ and $\lambda\rho$ belong to $L^1(a,+\infty)$. If for a.e., $t \geq 0$ we have $\varphi(t) \leq \rho(t) + \gamma(t)\displaystyle\int_t^{+\infty} \lambda(\tau)\varphi(\tau)d\tau$, then for a.e., $t \geq 0$*

$$\varphi(t) \leq \rho(t) + \gamma(t)\int_t^{+\infty} \rho(\tau)\lambda(\tau)\left(\int_t^{\tau} \lambda(r)\gamma(r)dr\right)d\tau.$$

For the proof of this Lemma see [3]. Now we shall prove the following existence theorem:

Theorem 2.2. *Let $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and assume that (2)-(8) hold. Then there exists at least one solution of the problem (1), in the following sense:*

$$\begin{cases} \displaystyle\int_\Omega S_k(u-v)(T)\,dx + \Big\langle \frac{\partial v}{\partial t}, T_k(u-v)\Big\rangle + \int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)\,dx\,dt \\ \qquad + \displaystyle\int_Q g(x,t,u,\nabla u)T_k(u-v)\,dx\,dt + \int_Q H(x,t,\nabla u)T_k(u-v)\,dx\,dt \\ \qquad = \displaystyle\int_Q fT_k(u-v)\,dx\,dt + \int_\Omega S_k(u_0 - v(0))\,dx, \end{cases} \tag{9}$$

for all $v \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, where $S_k(s) = \displaystyle\int_0^s T_k(r)\,dr$.

Proof. We divide the proof of this Theorem in four steps.

Step 1: Approximate problem and Energy estimate. For $n > 0$, let us define the following approximation of u_0, g and H. Set

$$g_n(x,t,s,\xi) = \frac{g(x,t,s,\xi)}{1 + \frac{1}{n}|g(x,t,s,\xi)|} \quad \text{and} \quad H_n(x,t,\xi) = \frac{H(x,t,\xi)}{1 + \frac{1}{n}|H(x,t,\xi)|},$$

and $\{u_{0n}\}$ be a sequence in $L^2(\Omega)$ such that $u_{0n} \to u_0$ in $L^1(\Omega)$.

Let us now consider the following regularized problems:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)\Big) + g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n) = f \text{ in } \mathcal{D}'(Q), \\ u_n(x,t=0) = 0 \qquad\qquad \text{in } \Omega, \\ u_n(x,t) = 0 \qquad\qquad \text{on } \partial\Omega \times (0,T). \end{cases} \tag{10}$$

Note that $g_n(x,t,s,\xi)$ and $H_n(x,t,\xi)$ are satisfying the following conditions

$$|g_n(x,t,s,\xi)| \le \max\Big\{|g(x,t,s,\xi)| \ ; \ n\Big\} \text{ and } |H_n(x,t,\xi)| \le \max\Big\{|H(x,t,\xi)| \ ; \ n\Big\}.$$

Moreover, since $f \in L^{p'}(0,T;W^{-1,\,p'}(\Omega))$, proving existence of a weak solution $u_n \in L^p(0,T;W_0^{1,\,p}(\Omega))$ of (10) is an easy task (see e.g. [17]).
For $\varepsilon > 0$ and $s \ge 0$, we define

$$\varphi_\varepsilon(r) = \begin{cases} \operatorname{sign}(r) & \text{if} \quad |r| > s+\varepsilon, \\ \dfrac{\operatorname{sign}(r)(|r|-s)}{\varepsilon} & \text{if} \quad s < |r| \le s+\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

We choose $v = \varphi_\varepsilon(u_n)$ as test function in (10), we have

$$\begin{aligned} &\Big[\int_\Omega B^n_{\varphi_\varepsilon}(u_n)\,dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla(\varphi_\varepsilon(u_n))\,dx\,dt \\ &\quad + \int_Q g_n(x,t,u_n,\nabla u_n)\varphi_\varepsilon(u_n)\,dx\,dt + \int_Q H_n(x,t,\nabla u_n)\varphi_\varepsilon(u_n)\,dx\,dt \\ &\quad = \int_0^T \langle f;\varphi_\varepsilon(u_n)\rangle\,dt, \end{aligned}$$

where $B^n_{\varphi_\varepsilon}(r) = \displaystyle\int_0^r \varphi_\varepsilon(s)\,ds$. Using $B^n_{\varphi_\varepsilon}(r) \ge 0$, $g_n(x,t,u_n,\nabla u_n)\varphi_\varepsilon(u_n) \ge 0$, (7) and Hölder's inequality, we obtain

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{\{s<|u_n|\le s+\varepsilon\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \\ &\quad \le \left(\int_{\{s<|u_n|\le s+\varepsilon\}} |f|^{p'}\,dx\,dt\right)^{\frac{1}{p'}} \left(\int_{\{s<|u_n|\le s+\varepsilon\}} \left(\frac{|\nabla u_n|}{\varepsilon}\right)^p\,dx\,dt\right)^{\frac{1}{p}} \\ &\qquad + \int_{\{s<|u_n|\}} h(x,t)|\nabla u_n|^{p-1}dx\,dt. \end{aligned}$$

Observe that,

$$\begin{aligned} &\int_{\{s<|u_n|\}} h(x,t)|\nabla u_n|^{p-1}dx\,dt \\ &\quad \le \int_s^{+\infty} \left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} h^p\,dx\,dt\right)^{\frac{1}{p}} \left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} d\sigma. \end{aligned} \tag{11}$$

Because,

$$\begin{aligned}
\int_{\{s<|u_n|\}} h(x,t)|\nabla u_n|^{p-1}dx\,dt &= \int_s^{+\infty} \frac{-d}{d\sigma}\left(\int_{\{\sigma<|u_n|\}} h(x,t)|\nabla u_n|^{p-1}\,dx\,dt\right) d\sigma \\
&= \int_s^{+\infty} \lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{\{\sigma<|u_n|\le\sigma+\delta\}} h(x,t)|\nabla u_n|^{p-1}\,dx\,dt\right) d\sigma \\
&\le \int_s^{+\infty} \lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{\{\sigma<|u_n|\le\sigma+\delta\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\int_{\{\sigma<|u_n|\le\sigma+\delta\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} d\sigma \\
&= \int_s^{+\infty} \left(\lim_{\delta\to 0}\frac{1}{\delta}\int_{\{\sigma<|u_n|\le\sigma+\delta\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\lim_{\delta\to 0}\frac{1}{\delta}\int_{\{\sigma<|u_n|\le\sigma+\delta\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} d\sigma \\
&= \int_s^{+\infty} \left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} d\sigma.
\end{aligned}$$

By (4) and (11), we deduce that

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\{s<|u_n|\le s+\varepsilon\}} \alpha|\nabla u_n|^p\,dx\,dt \\
&\quad\le \left(\frac{1}{\varepsilon}\int_{\{s<|u_n|\le s+\varepsilon\}} |f|^{p'}\,dx\,dt\right)^{\frac{1}{p'}}\left(\frac{1}{\varepsilon}\int_{\{s<|u_n|\le s+\varepsilon\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}} \\
&\qquad + \int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} d\sigma.
\end{aligned}$$

Letting ε go to zero, we obtain

$$\begin{aligned}
&\frac{-d}{ds}\int_{\{s<|u_n|\}} \alpha|\nabla u_n|^p\,dx\,dt \\
&\quad\le \left(\frac{-d}{ds}\int_{\{s<|u_n|\}} |f|^{p'}\,dx\,dt\right)^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}} \\
&\qquad + \int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} d\sigma,
\end{aligned} \tag{12}$$

where $\{s<|u_n|\}$ denotes the set $\{(x,t)\in Q,\ s<|u_n(x,t)|\}$ and $\mu(s)$ stands for the distribution function of u_n, that is $\mu(s)=|\{(x,t)\in Q,\ |u_n(x,t)|<s\}|$ for all $s\ge 0$.

Now, we recall the following inequality (see for example [13]), we have for almost every $s>0$

$$1\le \left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(-\frac{d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p dx\,dt\right)^{\frac{1}{p}}. \tag{13}$$

Using (13), we have

$$\begin{aligned}
&\frac{-d}{ds}\int_{\{s<|u_n|\}} \alpha|\nabla u_n|^p\,dx\,dt \\
&\quad= \alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leq \left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|f|^{p'}\,dx\,dt\right)^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}}$$
$$+\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}}$$
$$\times\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}d\sigma.$$

Which implies that,

$$\alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}\leq\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|f|^{p'}\,dx\,dt\right)^{\frac{1}{p'}}+\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}$$
$$\times(-\mu'(s))^{\frac{1}{p'}}\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}d\sigma. \tag{14}$$

Now, we consider two functions $B(s)$ and $F(s)$ (see Lemma 2.2 of [1]) defined by

$$\int_{\{s<|u_n|\}}h^p(x,t)dxdt=\int_0^{\mu(s)}B^p(\sigma)d\sigma, \tag{15}$$
$$\int_{\{s<|u_n|\}}|f|^{p'}dxdt=\int_0^{\mu(s)}F^{p'}(\sigma)d\sigma. \tag{16}$$

$$||B||_{L^p(0,T;W_0^{1,p}(\Omega))}\leq||h||_{L^p(0,T;W_0^{1,p}(\Omega))}\ \text{ and }\ ||F||_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\leq||f||_{L^{p'}(0,T;W^{-1,p'}(\Omega))}. \tag{17}$$

From (14), (15) and (16) we get

$$\alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p dxdt\right)^{\frac{1}{p'}}\leq F(\mu(s))(-\mu'(s))^{\frac{1}{p'}}$$
$$+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\int_s^{+\infty}B(\mu(\nu))(-\mu'(\nu))^{\frac{1}{p}}\left(-\frac{d}{d\nu}\int_{\{\nu<|u_n|\}}|\nabla u_n|^p dxdt\right)^{\frac{1}{p'}}d\nu.$$

From Lemma 2.1, we obtain

$$\alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}\leq F(\mu(s))(-\mu'(s))^{\frac{1}{p'}}+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}$$
$$\times\int_s^{+\infty}F(\mu(\sigma))B(\mu(\sigma))(-\mu'(\sigma))\exp\left(\int_s^{\sigma}(NC_N^{\frac{1}{N}})^{-1})B(\mu(r))(\mu(r))^{\frac{1}{N}-1}(-\mu'(r))dr\right)d\sigma.$$

Raising to the power p', integrating between 0 and $+\infty$ and by a variable change we have

$$\alpha^{p'}\int_Q|\nabla u_n|^p\,dx\,dt\leq c_0\int_0^{|Q|}F^{p'}(\lambda)\,d\lambda$$
$$+c_0\int_0^{|Q|}\lambda^{(\frac{1}{N}-1)p'}\left[\int_0^{\lambda}F(z)B(z)\exp\left(\int_z^{\lambda}(NC_N^{\frac{1}{N}})^{-1}B(v)v^{\frac{1}{N}-1}dv\right)dz\right]^{p'}d\lambda.$$

Using Hölder's inequality and (17), then we get

$$||u_n||_{L^p(0,T;W_0^{1,p}(\Omega))}\quad\leq\quad c_1, \tag{18}$$

where c_i is some positive constant not depending of n. Then there exists $u \in L^p(0,T;W_0^{1,p}(\Omega))$ such that, for some subsequence

$$u_n \rightharpoonup u \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{19}$$

we conclude that

$$||T_k(u_n)||^p_{L^p(0,T;W_0^{1,p}(\Omega))} \leq c_2 k. \tag{20}$$

Then for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that u_n converges almost everywhere to some measurable function u in Q. Thus by using the same argument as in [5, 6], we can show that

$$u_n \to u \text{ a.e. in } Q, \tag{21}$$

and we can deduce from (20) that

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)). \tag{22}$$

Which implies, by using (2), for all $k>0$ that there exists a function $\overline{a} \in (L^{p'}(Q))^N$, such that

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \overline{a} \text{ weakly in } (L^{p'}(Q))^N. \tag{23}$$

Finally, denoting $u'_n = f + \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)\Big) - g_n(x,t,u_n,\nabla u_n) - H_n(x,t,\nabla u_n)$ we observe that, $f + \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)\Big)$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $-g_n(x,t,u_n,\nabla u_n) - H_n(x,t,\nabla u_n)$ is bounded in $L^1(Q)$. Then we can conclude that $(u_n)_n$ is relatively compact in $L^p_{loc}(Q)$, thus we can deduce $u_n \to u$ in $L^p_{loc}(Q)$, and $u_n \to u$ strongly in $L^1(Q)$.

Step 2: Almost everywhere convergence of the gradients. This step is devoted to introduce for $k \geq 0$ fixed a time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes (see Lemma 6 and Proposition 3, p. 230, and Proposition 4, p. 231, in [14]). For $k>0$ fixed, and let $\varphi(t) = te^{\gamma t^2}$, $\gamma > 0$. It is well known that when $\gamma > \left(\frac{L_1(k)}{2\alpha}\right)^2$, one has

$$\varphi'(s) - (\tfrac{L_1(k)}{\alpha})|\varphi(s)| \geq \tfrac{1}{2}, \text{ for all } s \in \mathbb{R}. \tag{24}$$

Let $\psi_i \in \mathcal{D}(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$. Note that w^i_μ is a smooth function having the following properties:

$$\frac{\partial w^i_\mu}{\partial t} = \mu(T_k(u) - w^i_\mu), \quad w^i_\mu(0) = T_k(\psi_i), \quad \left|w^i_\mu\right| \leq k, \tag{25}$$

$$w^i_\mu \to T_k(u) \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)), \text{ as } \mu \to \infty. \tag{26}$$

We introduce the following function:

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \leq m, \\ 0 & \text{if } |s| \geq m+1, \\ m+1-|s| & \text{if } m \leq |s| \leq m+1, \end{cases}$$

where $m > k$. Let $\theta^{\mu,i}_n = T_k(u_n) - w^i_\mu$ and $z^{\mu,i}_{n,m} = \varphi(\theta^{\mu,i}_n)h_m(u_n)$.

Using in (10) the test function $z_{n,m}^{\mu,i}$, we obtain

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t} \ ; \ \varphi(T_k(u_n) - w_\mu^i)h_m(u_n)\rangle \, dt \\
&\quad + \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla w_\mu^i]\varphi'(\theta_n^{\mu,i})h_m(u_n)\, dx\, dt \\
&\quad + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \varphi(\theta_n^{\mu,i})h_m'(u_n)\, dx\, dt \\
&\quad + \int_Q \Big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\Big) z_{n,m}^{\mu,i}\, dx\, dt \\
&= \int_0^T \langle f \ ; \ z_{n,m}^{\mu,i}\rangle \, dt,
\end{aligned}$$

which implies since $g_n(x,t,u_n,\nabla u_n)\varphi(T_k(u_n) - w_\mu^i)h_m(u_n) \geq 0$ on $\{|u_n| > k\}$

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t} \ ; \ \varphi(T_k(u_n) - w_\mu^i)h_m(u_n)\rangle \, dt \\
&\quad + \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla w_\mu^i]\varphi'(\theta_n^{\mu,i})h_m(u_n)\, dx\, dt \\
&\quad + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \varphi(\theta_n^{\mu,i})h_m'(u_n)\, dx\, dt \\
&\quad + \int_{\{|u_n|\leq k\}} g_n(x,t,u_n,\nabla u_n)\varphi(T_k(u_n) - w_\mu^i)h_m(u_n)\, dx\, dt \\
&\leq \int_0^T \langle f \ ; \ z_{n,m}^{\mu,i}\rangle \, dt + \int_Q |H_n(x,t,\nabla u_n) z_{n,m}^{\mu,i}|\, dx\, dt.
\end{aligned} \tag{27}$$

In the sequel and throughout the paper, we will omit for simplicity the denote $\varepsilon(n,\mu,i,m)$ all quantities (possibly different) such that

$$\lim_{m\to\infty}\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{n\to\infty} \varepsilon(n,\mu,i,m) = 0,$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then μ, i and finally m. Similarly we will write only $\varepsilon(n)$, or $\varepsilon(n,\mu)$,... to mean that the limits are made only on the specified parameters.

We will deal with each term of (27). First of all, observe that

$$\int_0^T \langle f \ ; \ z_{n,m}^{\mu,i}\rangle \, dt + \int_Q |H_n(x,t,\nabla u_n) z_{n,m}^{\mu,i}|\, dx\, dt = \varepsilon(n,\mu), \tag{28}$$

since $\varphi(T_k(u_n) - w_\mu^i)h_m(u_n)$ converges to $\varphi\big(T_k(u) - (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)\big)h_m(u)$ strongly in $L^p(Q)$ and weakly$-*$ in $L^\infty(Q)$ as $n \to \infty$ and finally $\varphi\big(T_k(u) - (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)\big)h_m(u)$ converges to 0 strongly in $L^p(Q)$ and weakly$-*$ in $L^\infty(Q)$ as $\mu \to \infty$.

On the one hand, the definition of the sequence w_μ^i makes it possible to establish the following Lemma 2.3.

Lemma 2.3. *For $k \geq 0$ we have*

$$\int_0^T \langle \frac{\partial u_n}{\partial t} \ ; \ \varphi(T_k(u_n) - w_\mu^i)h_m(u_n)\rangle \, dt \geq \varepsilon(n,m,\mu,i). \tag{29}$$

Proof. (see Blanchard, Murat and Redwane [6]). □

Then, the second term of the left hand side of (27) can be written

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\quad=\int_{\{|u_n|\le k\}} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\qquad+\int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\quad=\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)\,dx\,dt\\
&\qquad+\int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt,
\end{aligned}$$

since $m>k$ and $h_m(u_n)=1$ on $\{|u_n|\le k\}$, we deduce that

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\quad=\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\
&\qquad\qquad(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-w_\mu^i)\,dx\,dt\\
&\qquad+\int_Q a(x,t,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\qquad+\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\qquad-\int_Q a(x,t,u_n,\nabla u_n)\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\quad=K_1+K_2+K_3+K_4.
\end{aligned}$$

Using (2), (23) and Lebesgue theorem we have $a(x,t,T_k(u_n),\nabla T_k(u))$ converges to $a(x,t,T_k(u),\nabla T_k(u))$ strongly in $(L^{p'}(Q))^N$ and $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ weakly in $(L^p(Q))^N$, then $K_2=\varepsilon(n)$. Using (23) and (26) we have $K_3=\displaystyle\int_Q \overline{a}\ \nabla T_k(u)\,dx\,dt+\varepsilon(n,\mu)$. For what concerns K_4 can be written, since $h_m(u_n)=0$ on $\{|u_n|>m+1\}$

$$\begin{aligned}
K_4&=-\int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&=-\int_{\{|u_n|\le k\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\quad-\int_{\{k<|u_n|\le m+1\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt,
\end{aligned}$$

and, as above, by letting n to $+\infty$ we get

$$\begin{aligned}
K_4=&-\int_{\{|u|\le k\}}\overline{a}\nabla w_\mu^i\varphi'(T_k(u)-w_\mu^i)\,dx\,dt\\
&-\int_{\{k<|u|\le m+1\}}\overline{a}\nabla w_\mu^i\varphi'(T_k(u)-w_\mu^i)h_m(u)\,dx\,dt+\varepsilon(n),
\end{aligned}$$

so that, by letting μ to $+\infty$ we get

$$K_4 \;=\; -\int_Q \overline{a}\nabla T_k(u)\,dx\,dt+\varepsilon(n,\mu).$$

We conclude then that

$$\begin{aligned}\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\ =\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\ (\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-w_\mu^i)\,dx\,dt+\varepsilon(n,\mu).\end{aligned}\tag{30}$$

To deal with the third term of the left hand side of (27), observe that

$$\begin{aligned}\left|\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\varphi(\theta_n^{\mu,i})h_m'(u_n)\,dx\,dt\right|\\ \le\varphi(2k)\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt.\end{aligned}$$

By (2) and (18), we obtain

$$\left|\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\varphi(\theta_n^{\mu,i})h_m'(u_n)\,dx\,dt\right|\ \le\ \varepsilon(n,m).\tag{31}$$

We now turn to fourth term of the left hand side of (27), can be written

$$\begin{aligned}&\left|\int_{\{|u_n|\le k\}}g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt\right|\\ &\le\int_{\{|u_n|\le k\}}L_1(k)(L_2(x,t)+|\nabla T_k(u_n)|^p|\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\ &\le L_1(k)\int_Q L_2(x,t)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\ &\quad+\frac{L_1(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt,\end{aligned}\tag{32}$$

since $L_2(x,t)$ belong to $L^1(Q)$ it is easy to see that

$$L_1(k)\int_Q L_2(x,t)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\ =\ \varepsilon(n,\mu).$$

On the other hand, the second term of the right hand side of (32), can be written

$$\begin{aligned}&\frac{L_1(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\ &=\frac{L_1(k)}{\alpha}\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\ &\qquad\qquad(\nabla T_k(u_n)-\nabla T_k(u))|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\ &\quad+\frac{L_1(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\ &\quad+\frac{L_1(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u))\nabla T_k(u)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt,\end{aligned}$$

and, as above, by letting first n then finally μ to infinity, we can easily see, that each one of last two integrals is of the form $\varepsilon(n,\mu)$. This implies that

$$\left|\int_{\{|u_n|\le k\}} g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt\right|$$
$$\le \frac{L_1(k)}{\alpha}\int_Q \Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)$$
$$(\nabla T_k(u_n)-\nabla T_k(u))|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt+\varepsilon(n,\mu). \tag{33}$$

Combining (27), (29), (30), (31) and (33), we get

$$\int_Q \Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)$$
$$(\nabla T_k(u_n)-\nabla T_k(u))\left(\varphi'(T_k(u)-w_\mu^i)-\tfrac{L_1(k)}{\alpha}|\varphi(T_k(u_n)-w_\mu^i)|\right)dx\,dt$$
$$\le \varepsilon(n,\mu,i,m),$$

and so, thanks to (24), we have

$$\int_Q \Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)(\nabla T_k(u_n)-\nabla T_k(u))\,dx\,dt\le\varepsilon(n). \tag{34}$$

Hence by passing to the limit sup over n, we get

$$\limsup_{n\to\infty}\int_Q \Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)(\nabla T_k(u_n)-\nabla T_k(u))\,dx\,dt=0$$

This implies that

$$T_k(u_n)\to T_k(u)\text{ strongly in }L^p(0,T;W_0^{1,p}(\Omega))\text{ for all }k. \tag{35}$$

Now, observe that for every $\sigma>0$,

$$\begin{aligned}&\operatorname{meas}\Big\{(x,t)\in Q:\ |\nabla u_n-\nabla u|>\sigma\Big\}\\ &\quad\le \operatorname{meas}\Big\{(x,t)\in Q:\ |\nabla u_n|>k\Big\}+\operatorname{meas}\Big\{(x,t)\in Q:\ |u|>k\Big\}\\ &\qquad+\operatorname{meas}\Big\{(x,t)\in Q:\ \big|\nabla T_k(u_n)-\nabla T_k(u)\big|>\sigma\Big\}\end{aligned}$$

then as a consequence of (35) we have that ∇u_n converges to ∇u in measure and therefore, always reasoning for a subsequence,

$$\nabla u_n\to\nabla u\text{ a. e. in }Q. \tag{36}$$

Which implies

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\rightharpoonup a(x,t,T_k(u),\nabla T_k(u))\text{ weakly in }(L^{p'}(Q))^N. \tag{37}$$

Step 3: Equi-integrability of H_n and g_n. We shall now prove that $H_n(x,t,\nabla u_n)$ converges to $H(x,t,\nabla u)$ and $g_n(x,t,u_n,\nabla u_n)$ converges to $g(x,t,u,\nabla u)$ strongly in $L^1(Q)$ by using Vitali's theorem. Since $H_n(x,t,\nabla u_n)\to H(x,t,\nabla u)$ a.e. Q and $g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u)$ a.e. Q, thanks to (5) and (7), it suffices to prove that $H_n(x,t,\nabla u_n)$ and $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q. We will now prove that $H_n(x,\nabla u_n)$ is uniformly equi-integrable, we use Hölder's inequality

and (18), we have

$$\begin{aligned}\int_E |H_n(x,\nabla u_n)| &\leq \left(\int_E h^p(x,t)\,dx\,dt\right)^{\frac{1}{p}}\left(\int_Q |\nabla u_n|^p\right)^{\frac{1}{p'}}\\ &\leq c_1\left(\int_E h^p(x,t)\,dx\,dt\right)^{\frac{1}{p}}.\end{aligned} \tag{38}$$

which is small uniformly in n when the measure of E is small.

To prove the uniform equi-integrability of $g_n(x,t,u_n,\nabla u_n)$. For any measurable subset $E\subset Q$ and $m\geq 0$,

$$\int_E |g(x,t,u_n,\nabla u_n)|\,dx\,dt = \int_{E\cap\{|u_n|\leq m\}} |g(x,t,u_n,\nabla u_n)|\,dx\,dt + \int_{E\cap\{|u_n|>m\}} |g(x,t,u_n,\nabla u_n)|\,dx\,dt$$

$$\begin{aligned}&\leq L_1(m)\int_{E\cap\{|u_n|\leq m\}} [L_2(x,t)+|\nabla u_n|^p]\,dx\,dt + \int_{E\cap\{|u_n|>m\}} |g(x,t,u_n,\nabla u_n)|\,dx\,dt \quad (39)\\ &\leq L_1(m)\int_{E\cap\{|u_n|\leq m\}} [L_2(x,t)+|\nabla T_m(u_n)|^p]\,dx\,dt + \int_{E\cap\{|u_n|>m\}} |g(x,t,u_n,\nabla u_n)|\,dx\,dt\\ &= K_1+K_2.\end{aligned}$$

For fixed m, we get

$$K_1 \leq L_1(m)\int_E [L_2(x,t)+|\nabla T_m(u_n)|^p]\,dx\,dt,$$

which is thus small uniformly in n for m fixed when the measure of E is small (recall that $T_m(u_n)$ tends to $T_m(u)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$). We now discuss the behavior of the second integral of the right hand side of (39), let ψ_m be a function such that

$$\begin{cases}\psi_m(s)=0 & \text{if} \quad |s|\leq m-1,\\ \psi_m(s)=\operatorname{sign}(s) & \text{if} \quad |s|\geq m,\\ \psi_m'(s)=1 & \text{if} \quad m-1<|s|<m.\end{cases} \tag{40}$$

We choose $\psi_m(u_n)$ as a test function for $m>1$ in (10), we obtain

$$\begin{aligned}&\Big[\int_\Omega B_m^n(u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\psi_m'(u_n)\,dx\,dt\\ &\qquad + \int_Q g_n(x,t,u_n,\nabla u_n)\psi_m(u_n)\,dx\,dt + \int_Q H_n(x,t,\nabla u_n)\psi_m(u_n)\,dx\,dt\\ &\qquad = \int_0^T \langle f\ ;\ \psi_m(u_n)\rangle\,dt,\end{aligned}$$

where $B_m^n(r)=\displaystyle\int_0^r \psi_m(s)ds$, which implies, since $B_m^n(r)\geq 0$ and using (4), Hölder's inequality

$$\begin{aligned}\int_{\{m-1\leq|u_n|\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt &\leq \int_E |H_n(x,t,\nabla u_n)|\,dx\,dt\\ &\quad + \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\Big(\int_{\{m-1\leq|u_n|\leq m\}} |\nabla u_n|^p\,dx\,dt\Big)^{\frac{1}{p}}.\end{aligned}$$

By (18), we have

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>m-1\}}|g_n(x,t,u_n,\nabla u_n)|\,dx\,dt=0.$$

Thus we proved that the second term of the right hand side of (39) is also small, uniformly in n and in E when m is sufficiently large. Which shows that $g_n(x,t,u_n,\nabla u_n)$ and $H_n(x,t,\nabla u_n)$ are uniformly equi-integrable in Q as required, we conclude that

$$\begin{cases} H_n(x,t,\nabla u_n)\to H(x,t,\nabla u) & \text{strongly in} \quad L^1(Q),\\ g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u) & \text{strongly in} \quad L^1(Q). \end{cases} \tag{41}$$

Step 4: Passing to the limit. Going back to approximate equations (10) and using $v\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ as the test function, one has

$$\begin{aligned}&\int_\Omega S_k(u_n-v)(T)\,dx+\Big\langle\frac{\partial v}{\partial t},T_k(u_n-v)\Big\rangle+\int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx\,dt\\ &\quad+\int_Q g(x,t,u_n,\nabla u_n)T_k(u_n-v)\,dx\,dt+\int_Q H(x,t,\nabla u_n)T_k(u_n-v)\,dx\,dt\\ &=\int_Q fT_k(u_n-v)\,dx\,dt+\int_\Omega S_k(u_{n0}-v(0))\,dx,\end{aligned}$$

in which we can pass to the limit thanks to the previous results, we prove the existence of a solution u of the nonlinear parabolic problems (9). This completes the proof of Theorem 2.2. □

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Solving nonlinear fractional differential equations using multi-step homotopy analysis method

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Abstract. This paper presents a numerical technique for solving fractional differential equation by employing the multi-step homotopy analysis method (MHAM). It is known that the corresponding numerical solution obtained using the HAM is valid only for a short time. On the contrary, the results obtained using the MHAM are more valid and accurate during a long time, and are highly agreement with the exact solutions in the case of integer-order systems. The objective of the present paper is to modify the HAM to provide symbolic approximate solution for linear and nonlinear of fractional differential equations. The efficient and accuracy of the method used in this paper will be demonstrated by comparison with the known methods and with the known exact solutions in the non fractional case. The fractional derivatives are described in the Caputo sense.

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1. Introduction

The solutions of fractional differential equations are much involved. In general, there exists no method that yields an exact solution for fractional differential equations. Only approximate solution can be derived using linearization or perturbation methods. In recent years, many researchers have focused on the numerical solution of ordinary differential equations of fractional order and some numerical methods have been developed such as Fourier transform method [1], Adomian decomposition method [2, 3] and Homotopy perturbation method [4, 5]. Recently, the Homotopy analysis method (HAM) has been proposed by Liao [6, 7, 8, 9, 10, 11]. Based on homotopy of topology, the validity of the HAM is independent of whether there exist small parameters or not in the considered equation. The HAM has been used to investigate a variety of mathematical and physical problems [7]. The homotopy analysis method contains a certain auxiliary parameter h and auxiliary linear operator L which provides us with a simple way to control and adjust the rate of convergence of the series solution [12, 13]. The objective of the present paper is to modify the HAM to provide approximate solution for linear and nonlinear fractional differential equations. In this paper we investigate the applicability and effectiveness of the HAM when treated as an algorithm in a sequence of intervals (i.e. time step) for finding accurate approximate solutions to the fractional differential equation of the form

$$D_*^{\alpha}\ u(t) + a_n\ u^{(n)}(t) + a_{n-1}\ u^{(n-1)}(t) + ... + a_0\ u(t) + N(u(t),\ u^{'}(t)) = f(t),$$
$$t \geq 0, \quad n-1 < \alpha \leq n, \tag{1.1}$$

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subject to the initial conditions

$$u^{(i)}(0) = b_i, \qquad i = 0, 1, ..., n-1. \tag{1.2}$$

Here D_*^α is the fractional derivative in the Caputo sense. This modified method is the multi-step homotopy analysis method. It can be found that the corresponding numerical solutions obtained by using the HAM are valid only for a short time [13]. While the ones obtained by using the MHAM are more valid and accurate during a long time and are highly agreement with the exact solutions in the case of integer-order systems. Some examples are given to illustrate this method. The paper is organized as follows. A brief review of the fractional calculus is given in Section 2. In Section 3, the proposed method is described. In Section 4, we investigate the applicability and effectiveness of the multi-step homotopy analysis method for finding accurate approximate solutions to the fractional differential equation and we will present a comparison between our results with the exact solution by plotting the exact solution and the approximate solution. Conclusions are presented in Section 5.

2. Fractional calculus

In this section, we introduce the linear operators of fractional integration and fractional differentiation in the framework of the Riemann-Liouville and Caputo fractional calculus [14, 15, 16, 17].

Definition 2.1. A real function $f(x),\ x > 0$, is said to be in the space $C_\alpha,\ \alpha \in \mathbb{R}$ if it can be written as $f(x) = x^p f_1(x)$, for some $p > \alpha$ where $f_1(x)$ is continous in $[0, \infty)$, and it is said to be in the space C_α^m if $f^{(m)} \in C_\alpha,\ m \in \mathbb{N}$.

Definition 2.2. A function $f(x), x > 0$, is said to be in the space $C_\mu^m,\ m \in \mathbb{N} \cup \{0\}$, if $f^m \in C_\mu$.

Definition 2.3. The Riemann-Liouville fractional integral of order $\alpha > 0$ is

$$\begin{aligned} J^\alpha u(t) &= u_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, \quad t > 0,\ \alpha \in \mathbb{R}^+, \\ J^0 u(t) &= u(t). \end{aligned}$$

Definition 2.4. The Caputo fractional derivative of $u(t)$ is defined as:

$$D_*^\alpha\ u(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{u^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\ d\tau, & m-1 < \alpha < m, \\ \dfrac{d^m u(t)}{dt^m}, & \alpha = m, \end{cases}$$

Hence, we have the following properties

1. $J^\alpha J^\beta u(t) = J^{\alpha+\beta} u(t) = J^\alpha J^\beta u(t),\ \alpha,\ \beta \geq 0,\ u \in C\mu,$
2. $J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)},\ \alpha > 0,\ \gamma > -1,\ t > 0,$
3. $D_*^\alpha J^\alpha u(t) = u(t),$
4. $J^\alpha D_*^\alpha u(t) = u(t) - \displaystyle\sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{t^k}{k!},\ t > 0,\ m-1 < \alpha \leq m.$

3. Multi-step homotopy analysis method algorithm

The HAM has been extended by many authors to solve linear and nonlinear problems in terms of convergent series with easily computable components, it has some drawbacks: the series solution always converges in a small region and it has slow convergent rate or completely divergent in the wider region. In this section, we present the basic ideas of the MHAM that have been developed for the numerical solution of our problem (1.1), (1.2). It is only a simple modification of the standard HAM and can ensure the validity of the approximate solution for large time t. To extend this solution over the interval $[0,\ t]$, we divide the interval I into $r-$subintervals of equal length $\triangle t$, $[t_0,\ t_1)$, $[t_1,\ t_2), ...,\ [t_{r-1},\ t_r]$ with $t_0 = 0,\ t_r = t$. Let t^* be the initial value and $u_j(t)$ be approximate solutions in each subinterval $[t_{j-1},\ t_j)$, $j = 1, 2, \ldots, r$, then the equations (1.1), (1.2) can be transformed into the following system

$$D_*^{\alpha}\ u_j(t) + a_n\ u_j^{(n)}(t) + a_{n-1}\ u_j^{(n-1)}(t) + ... + a_0\ u_j(t) + N(u_j(t),\ u_j^{'}(t)) = f(t),$$
$$t \geq 0, \quad n-1 < \alpha \leq n,\ , j = 1, 2, \ldots, r, \tag{3.1}$$

subject to the initial conditions

$$u_1^{(i)}(t^*) = b_i,\ \ u_j^{(i)}(t^*) = u_{j-1}^{(i)}(t_{j-1}) = c_{j,i} \quad i = 0, 1, ..., n-1,\ , j = 1, 2, \ldots, r. \tag{3.2}$$

The zero-order deformation equation of system (3.1) has the form

$$(1-q)L[\phi_j(t,q) - u_j(t^*)] = qh[D_*^{\alpha}\phi_j(t,q) + a_n\ \frac{d^n}{dt^n}\phi_j(t,q) \tag{3.3}$$
$$+a_{n-1}\frac{d^{n-1}}{dt^{n-1}}\phi_j(t,q) + ... + a_0\phi_j(t,q) + N(\phi_j(t,q),\ \frac{d}{dt}\phi_j(t,q)) - f(t)].$$

Where $q \in [0,1]$ is an embedding parameter, L is an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter, $\phi_j(t;q)$ is unknown function, $u_j(t^*)$ be the initial guess of $u_j(t)$ which satisfy the initial condition and $f(t)$ is known function. Obviously, when $q = 0$ we have

$$\phi_1(t,0) = u_1(t^*),\ \phi_j(t,0) = u_j(t^*), \quad j = 2, 3, ..., r. \tag{3.4}$$

When $q = 1$, we have

$$\phi_j(t,1) = u_j(t), \quad j = 1, 2, ..., r. \tag{3.5}$$

Expanding $\phi_j(t,q),\ \ j = 1, 2, ..., r$, in Taylor series with respect to q, we get

$$\phi_j(t,q) = u_j(t^*) + \sum_{m=1}^{\infty} u_{j,m}(t)q^m,\ \ j = 1, 2, ..., r, \tag{3.6}$$

where

$$u_{j,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_j(t,q)}{\partial q^m}|_{q=0}. \tag{3.7}$$

If the initial guess $u_j(t^*)$, the auxiliary linear operator L and the nonzero auxiliary parameter h are properly chosen so that the power series (3.6) converges at $q = 1$, one has

$$u_j(t) = \phi_j(t;1) = u_j(t^*) + \sum_{m=1}^{\infty} u_{j,m}(t).$$

For brevity, define the vector

$$\overrightarrow{u}_{j,m}(t) = \{u_{j,0}(t), u_{j,1}(t), \ldots, u_{j,m}(t)\},$$

Differentiating the zero-order deformation equation (3.3) m times with respective to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called $mth-$order deformation equations

$$L[u_{j,m}(t) - \chi_m u_{j,m-1}(t)] = h\ \Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)), \tag{3.8}$$

where

$$\Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)) = D_*^{\alpha}\ u_{j,m-1}(t) + a_n\ u_{j,m-1}^{(n)}(t) + a_{n-1}\ u_{j,m-1}^{(n-1)}(t) + ... + a_0\ u_{j,m-1}(t)$$
$$+\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}N(\phi_j(t,q), \frac{d}{dt}\phi_j(t,q))|_{q=0} - f(t)(1-\chi_m), \tag{3.9}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}. \tag{3.10}$$

Select the auxiliary linear operator $L = D_*^{\alpha}$, then the mth-order deformation equations (3.8) can be written in the form

$$u_{j,m}(t) = \chi_m u_{j,m-1}(t) + h\ J^{\alpha}[\ \Re_{j,m}(\overrightarrow{u}_{j,m-1}(t))], \tag{3.11}$$

and a power series solution has the form

$$u_j(t) = \sum_{m=0}^{\infty} u_{j,m}(t), \quad j = 1, 2, ..., r.$$

Finally, the solutions of system (1.1) has the form

$$u(t) = \begin{cases} u_1(t), & t \in [t_0, t_1] \\ u_2(t), & t \in [t_1, t_2] \\ & \vdots \\ u_r(t), & t \in [t_{r-1}, t_r] \end{cases}.$$

4. Numerical results

To demonstrate the effectiveness of the method for solving nonlinear fractional differential equations, we consider here the following four examples.

4.1. Example 1. Consider the following simple harmonic fractional oscillator

$$D_*^{\alpha}u(t) + (0.5)^{\alpha}\ u(t) = 0, \quad t \geq 0, \quad 1 < \alpha \leq 2, \tag{4.1}$$

subject to the initial condition

$$u(0) = 1,\ u^{'}(0) = 0. \tag{4.2}$$

The exact solutions of this equation when $\alpha = 2$ is $u(t) = \cos 0.5t$. Let $u_j(t)$ be the approximate solutions in the subinterval $[t_{j-1}, t_j)$, then equation (4.1), is transformed into the following system

$$D_*^{\alpha}u_j(t) + (0.5)^{\alpha}\ u_j(t) = 0, \quad t \geq 0, \quad 1 < \alpha \leq 2,\ j = 1, 2, ..., r. \tag{4.3}$$

Let $u_j(t^*) = c_j$, with $c_1 = 1$ are the initial guesses of $u_j(t)$, then we can construct the MHAM (3.11) where

$$\Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)) = D_*^{\alpha}u_{j,m-1}(t) + (0.5)^{\alpha}u_{j,m-1}(t). \tag{4.4}$$

The series solution for equation (4.3) is given by,

$$u_j(t) = c_j + \frac{h(0.5)^{\alpha}c_j(1+(1+h)+(1+h)^2)}{\Gamma(\alpha+1)}(t-t^*)^{\alpha} + \frac{h^2(0.5)^{2\alpha}c_j(1+2(1+h))}{\Gamma(2\alpha+1)}(t-t^*)^{2\alpha} + \frac{h^3(0.5)^{3\alpha}c_j}{\Gamma(3\alpha+1)}(t-t^*)^{3\alpha} + ...$$

To demonstrate the effectiveness of the proposed algorithm as an approximate tool for solving the fractional differential equations (4.1), (4.2) for larger t, we apply the proposed algorithm on the interval $[0,\ 100]$. We choose to divide the interval $[0,\ 100]$ to subintervals with time step $\triangle t = 0.1$. Figure 1 shows the series solution exhibit the periodic behavior which is the characteristic of the simple harmonic Equations (4.1), (4.2) obtained for $\alpha = 2,\ 1.6,\ 1.5$ and when $h = -1$. It can be seen that the results obtained using the MHAM (when $\alpha = 2$) match the results of the exact solution ($u(t) = \cos 0.5t$) very well, and are highly in agreement during a long time. It is clear that the numerical results obtained using the MHAM has the same trajectories for various values of α and all its solutions are expected to oscillate with decreasing to zero when the value of α is decreasing.

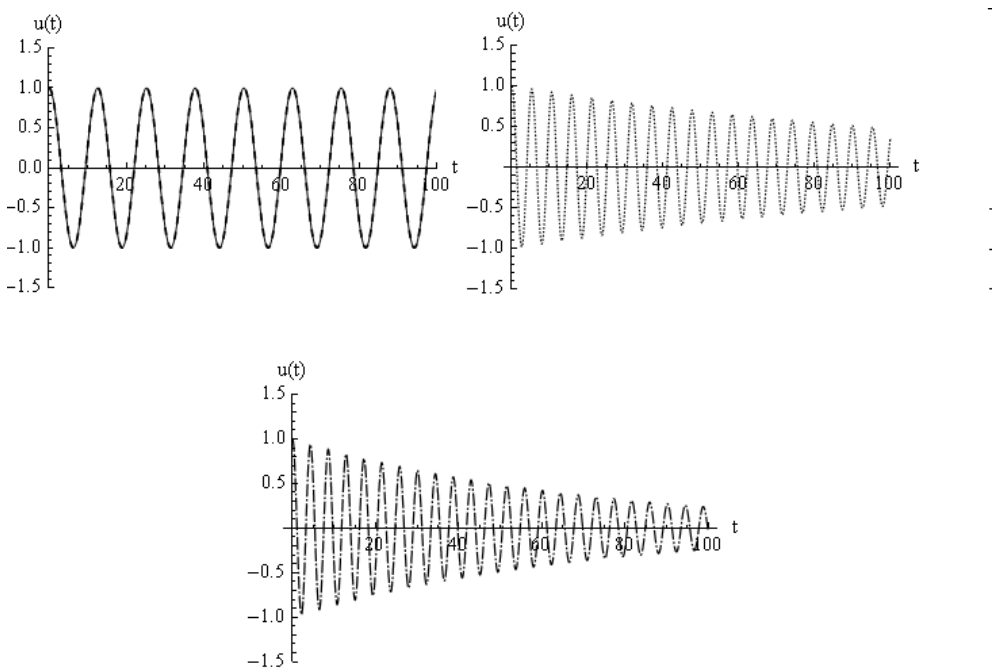

FIGURE 1. The displacement for Example 1: Solid line: exact solution, Dashed line: MHAM solution when $\alpha = 2$, Dotted line: MHAM solution when $\alpha = 1.6$, Dashed dotted line: MHAM solution when $\alpha = 1.5$.

4.2. Example 2. Consider the following nonlinear fractional Riccati equation

$$D_*^{\alpha}u(t) + u^2(t) - 1 = 0, \quad t \geq 0, \quad 0 < \alpha \leq 1, \tag{4.5}$$

subject to the initial condition

$$u(0) = 0. \tag{4.6}$$

The exact solutions of this equation when $\alpha = 1$ is $u(t) = \frac{e^{2t}-1}{e^{2t}+1}$. Let $u_j(t)$ be the approximate solutions in the subinterval $[t_{j-1}, t_j)$, then equation (4.5), is transformed into the following system

$$D_*^{\alpha} u_j(t) + u_j^2(t) - 1 = 0, \quad t \geq 0, \quad 0 < \alpha \leq 1, \ j = 1, 2, ..., r. \tag{4.7}$$

Let $u_j(t^*) = c_j$, with $c_1 = 0$ are the initial guesses of $u_j(t)$, then we can construct the MHAM (3.11) where

$$\Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)) = D_*^{\alpha} u_{j,m-1}(t) + \sum_{i=0}^{m-1} u_{j,i}(t) u_{j,m-i-1}(t) - (1 - \chi_m). \tag{4.8}$$

The series solution for equation (4.7) is given by

$$u_j(t) = c_j + \frac{h(c_j^2 - 1)(1 + (1+h) + (1+h)^2)}{\Gamma(\alpha+1)}(t - t^*)^{\alpha}$$
$$+ \frac{2h^2 c_j (c_j^2 - 1)(1 + 2(1+h))}{\Gamma(2\alpha+1)}(t - t^*)^{2\alpha} + \frac{h^3 (c_j^2 - 1)^2 \Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}(t - t^*)^{3\alpha} + ...$$

In this example we apply the proposed algorithm on the interval $[0,\ 20]$. We choose to divide the interval $[0,\ 20]$ to subintervals with time step $\triangle t = 0.1$. Figure 2 shows the series solution of the MHAM of the nonlinear fractional Riccati equations (4.5), (4.6) (when $\alpha = 1,\ 0.9,\ 0.7$ and $h = -1$) and the displacement of the exact solution ($u(t) = \frac{e^{2t}-1}{e^{2t}+1}$). From the graphical results it can be seen that the results obtained using the MHAM (when $\alpha = 1$) match the results of the exact solution very well. Therefore, the proposed method is very effcient and accurate method that can be used to provide analytical solutions for linear and nonlinear fractional differential equations. Also as the previous example, the numerical results obtained using the MHAM has the same trajectories for various values of α.

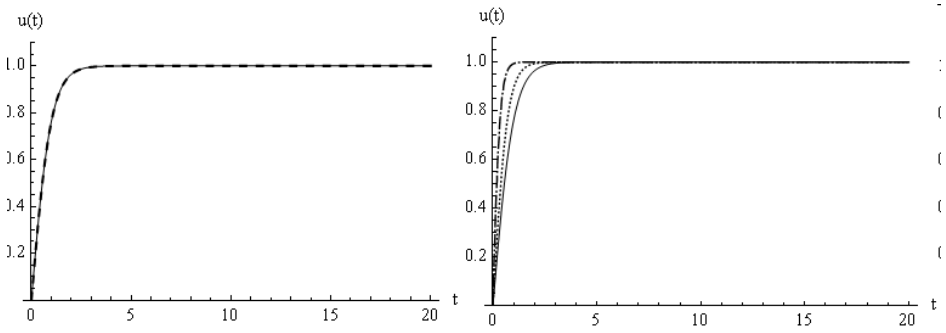

FIGURE 2. The displacement for Example 2: Solid line: exact solution, Dashed line: MHAM solution when $\alpha = 1$, Dotted line: MHAM solution when $\alpha = 0.9$, Dashed dotted line: MHAM solution when $\alpha = 0.7$.

4.3. Example 3.

$$D_*^{\alpha} u(t) - 2u(t) + u^2(t) - 1 = 0, \quad t \geq 0, \quad 0 < \alpha \leq 1, \tag{4.9}$$

subject to the initial condition

$$u(0) = 0. \tag{4.10}$$

The exact solutions of this equation when $\alpha = 1$ is $u(t) = 1 + \sqrt{2}\ \tanh(\sqrt{2}\ t + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1}))$. Let $u_j(t)$ be the approximate solutions in the subinterval $[t_{j-1}, t_j)$, then equation (4.9), is transformed into the following system

$$D^{\alpha}_{*}u_j(t) - 2u_j(t) + u_j^2(t) - 1 = 0, \quad t \geq 0, \quad 0 < \alpha \leq 1,\ j = 1, 2, ..., r. \tag{4.11}$$

Let $u_j(t^*) = c_j$, with $c_1 = 0$ are the initial guesses of $u_j(t)$, then we can construct the MHAM (3.11) where

$$\Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)) = D^{\alpha}_{*}u_{j,m-1}(t) - 2\ u_{j,m-1}(t) + \sum_{i=0}^{m-1} u_{j,i}(t)u_{j,m-i-1}(t) - (1 - \chi_m). \tag{4.12}$$

Then the analytic solution for system (4.11) is derived as followsis

$$\begin{aligned} u_j(t) = c_j &+ \frac{h(c_j^2 - 2c_j - 1)(1 + (1+h) + (1+h)^2)}{\Gamma(\alpha+1)}(t-t^*)^{\alpha} \\ &+ \frac{2h^2(c_j^2 - 2c_j - 1)(c_j - 1)(1 + 2(1+h))}{\Gamma(2\alpha+1)}(t-t^*)^{2\alpha} \\ &+ \frac{h^3(c_j^2 - 2c_j - 1)}{\Gamma(3\alpha+1)}(\frac{(c_j^2 - 2c_j - 1)\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + (2c_j^2 - 6c_j + 4))(t-t^*)^{3\alpha} + ... \end{aligned}$$

In this example we apply the proposed algorithm on the interval $[0,\ 20]$. We choose to divide the interval $[0,\ 20]$ to subintervals with time step $\triangle t = 0.1$. Figure 3 shows the series solution of the MHAM of the nonlinear fractional Riccati equations (4.9), (4.10) (when $\alpha = 1,\ 0.9,\ 0.7$ and $h = -1$) and the displacement of the exact solution $(u(t) = 1+\sqrt{2}\ \tanh(\sqrt{2}\ t+\frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})))$. Also the results of our computations (when $\alpha = 1$) are in excellent agreement with the results obtained by the exact solution and are highly in agreement during a long time. Therefore, the proposed method is very efficient and accurate method that can be used to provide analytic solutions for linear and nonlinear fractional differential equations. Also as example (2), the numerical results obtained using the MHAM has the same trajectories for various values of α.

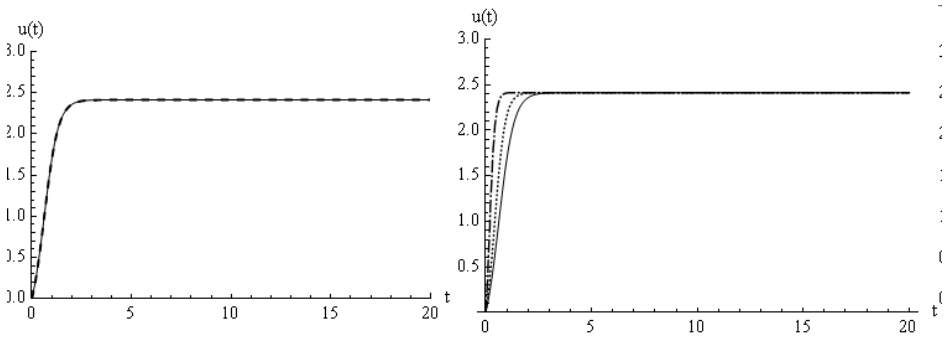

FIGURE 3. The displacement for Example 3: Solid line: exact solution, Dashed line: MHAM solution when $\alpha = 1$, Dotted line: MHAM solution when $\alpha = 0.9$, Dashed dotted line: MHAM solution when $\alpha = 0.7$.

4.4. Example 4. Consider the following nonlinear fractional equation

$$D_*^{\alpha}u(t) + 2u(t) + u^2(t) = 0, \quad t \geq 0, \quad 1 < \alpha \leq 2, \tag{4.13}$$

subject to the initial condition

$$u(0) = 0.1,\ u^{'}(0) = 0. \tag{4.14}$$

Let $u_j(t)$ be the approximate solutions in the subinterval $[t_{j-1}, t_j)$, then equation (4.13), is transformed into the following system

$$D_*^{\alpha}u_j(t) + 2\ u_j(t) + u_j^2(t) = 0, \quad t \geq 0, \quad 1 < \alpha \leq 2,\ j = 1, 2, ..., r. \tag{4.15}$$

Let $u_j(t^*) = c_j$, with $c_1 = 0$ are the initial guesses of $u_j(t)$, then we can construct the MHAM (3.11) where

$$\Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)) = D_*^{\alpha}u_{j,m-1}(t) + 2\ u_{j,m-1}(t) + \sum_{i=0}^{m-1} u_{j,i}(t)u_{j,m-i-1}(t). \tag{4.16}$$

The analytic solutions for system (4.15) derived by

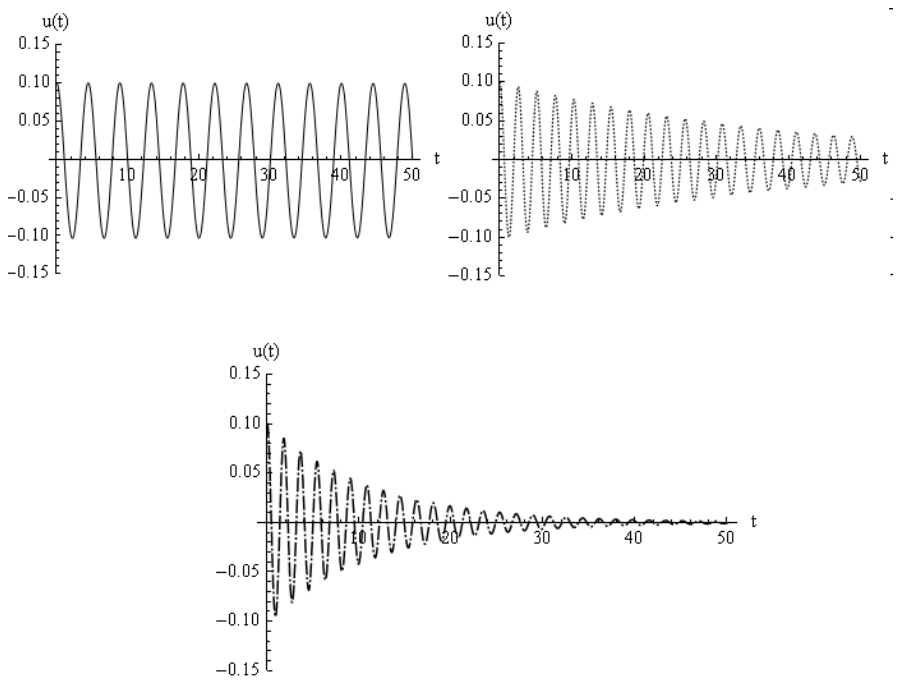

FIGURE 4. The displacement for Example 4: Solid line: MHAM solution when $\alpha = 2$, Dotted line: MHAM solution when $\alpha = 1.7$, Dashed dotted line: MHAM solution when $\alpha = 1.5$.

$$\begin{aligned} u_j(t) &= c_j + \frac{h(c_j^2 + 2c_j)(1 + (1+h) + (1+h)^2)}{\Gamma(\alpha+1)}(t-t^*)^{\alpha} \\ &+ \frac{2h^2(c_j^2 + 2c_j)(c_j+1)(1+2(1+h))}{\Gamma(2\alpha+1)}(t-t^*)^{2\alpha} \\ &+ \frac{h^3(c_j^2 + 2c_j)}{\Gamma(3\alpha+1)}\left(\frac{(c_j^2 + 2c_j)\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + (2c_j^2 + 6c_j + 4)\right)(t-t^*)^{3\alpha} + ... \end{aligned}$$

Let $[0,\ 50]$ be the interval over which we want to find the solution of the initial value problem (4.13), (4.14). Assume that the interval $[0,\ 50]$ is divided into subintervals of equal length $\triangle t = 0.1$. Figure 4 shows the series solution exhibit the periodic behavior which is the characteristic of the nonlinear fractional differential equations (4.13), (4.14) obtained for $\alpha = 2,\ 1.7,\ 1.5$ and when $h = -1$. It is clear that the numerical results obtained using the MHAM have the same trajectories for various values of α and all its solutions are expected to oscillate with decreasing to zero when the value of α is decreasing.

5. Conclusions

The fundamental goal of this work has been to propose an efficient algorithm for the solution of linear and nonlinear fractional differential equation. Based on some numerical and analytical techniques, we discussed in this paper the MHAM. The MHAM is an efficient modification of the HAM which introduces an efficient tool for calculating approximate solution for linear and nonlinear fractional differential equation. Our method is a direct method, further it is simple and accurate. It is a practical method and can easily be implemented on computer to solve such problems. We have used the method with four examples. The main advantage of the method is a fast convergence to the solution. Moreover, it avoids amount of calculations required by the other existing analytical methods. The new method leads to higher accuracy and simplicity, and in all cases the solutions obtained are easily programmable approximates to the analytic solution of the original problems with the accuracy required. The proposed scheme can be applied for other nonlinear equations. It can be found that the corresponding numerical solutions obtained by using HAM are valid only for a short time. While the ones obtained by using MHAM are more valid and accurate during a long time. A comparison between the graph of the numerical result with the graph of the exact solution indicates that the MHAM method is powerful analytic method for handling differential equations of fractional order.

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Growth of certain combinations of entire solutions of higher order linear differential equations

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Abstract. The main purpose of this paper is to study the growth of certain combinations of entire solutions of higher order complex linear differential equations.

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1. Introduction and main results

In this paper, we assume that the reader is familiar with the fundamental results and standard notations of the Nevanlinna theory [4, 13] . In addition, we will use $\rho(f)$ to denote the order and $\rho_2(f)$ to denote the hyper-order of f. See, [4, 6, 13] for notations and definitions.
We consider the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0, \tag{1}$$

where $A_j(z)$ $(j = 0, \cdots, k-1)$ are entire functions. Suppose that $\{f_1, f_2, \cdots, f_k\}$ is the set of fundamental solutions of (1). It is clear that $f = c_1 f_1 + c_2 f_2 + \cdots + c_k f_k$ where c_i $(i = 1, \cdots, k)$ are complex numbers is a solution of (1), but what about the properties of $f = c_1 f_1 + c_2 f_2 + \cdots + c_k f_k$ if c_i $(i = 1, \cdots, k)$ are non-constant entire functions? In [7] , the authors gave answer to this question for the case $k = 2$, and obtained the following results.

Theorem 1.1. [7] *Let $A(z)$ be transcendental entire function of finite order. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically such that $\max\{\rho(d_1), \rho(d_2)\} < \rho(A)$. If f_1 and f_2 are two linearly independent solutions of*

$$f'' + A(z) f = 0, \tag{2}$$

then the polynomial of solutions $g_f = d_1 f_1 + d_2 f_2$ satisfies

$$\rho(g_f) = \rho(f_j) = \infty \quad (j = 1, 2)$$

and

$$\rho_2(g_f) = \rho_2(f_j) = \rho(A) \quad (j = 1, 2).$$

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Theorem 1.2. [7] *Let $A(z)$ be a polynomial of $\deg A = n$. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically such that $\max\{\rho(d_1), \rho(d_2)\} < \frac{n+2}{2}$ and $h \not\equiv 0$, where*

$$h = \begin{vmatrix} d_1 & 0 & d_2 & 0 \\ d_1' & d_1 & d_2' & d_2 \\ d_1'' - d_1 A & 2d_1' & d_2'' - d_2 A & 2d_2' \\ d_1''' - 3d_1' A - d_1 A' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2' A - d_2 A' & d_2'' - d_2 A + 2d_2'' \end{vmatrix}.$$

If f_1 and f_2 are two linearly independent solutions of (2), then the polynomial of solutions $g_f = d_1 f_1 + d_2 f_2$ satisfies

$$\rho(g_f) = \rho(f_j) = \frac{n+2}{2} \quad (j = 1, 2).$$

The aim of this paper is to study the growth of

$$g_k = d_1 f_1 + d_2 f_2 + \cdots + d_k f_k,$$

where $\{f_1, f_2, \cdots, f_k\}$ is any set of fundamental solutions of (1) and $d_j(z)$ $(j = 1, 2, \cdots, k)$ are finite order entire functions that are not all vanishing identically. In fact, we give sufficient conditions on $A_j(z)$ $(j = 0, \cdots, k-1)$ and $d_j(z)$ $(j = 1, 2)$ to prove that for any two solutions f_1 and f_2 of (1), the growth of $g_2 = d_1 f_1 + d_2 f_2$ is the same as the growth of f_j $(j = 1, 2)$, and we obtain the following results.

Theorem 1.3. *Let $A_j(z)$ $(j = 0, \cdots, k-1)$ be entire functions of finite order such that $\max\{\rho(A_j) : j = 1, \cdots, k-1\} < \rho(A_0)$. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically such that $\max\{\rho(d_1), \rho(d_2)\} < \rho(A_0)$. If f_1 and f_2 are any two linearly independent solutions of (1), then the combination of solutions $g_2 = d_1 f_1 + d_2 f_2$ satisfies*

$$\rho(g_2) = \rho(f_j) = \infty \quad (j = 1, 2)$$

and

$$\rho_2(g_f) = \rho_2(f_j) = \rho(A) \quad (j = 1, 2).$$

Theorem 1.4. *Let $A_0(z)$ be transcendental entire function with $\rho(A_0) = 0$, and let $A_1, \cdots, A_{k-1}$ be polynomials. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically. If f_1 and f_2 are any two linearly independent solutions of (1), then the combination of solutions $g_2 = d_1 f_1 + d_2 f_2$ satisfies*

$$\rho(g_2) = \rho(f_j) = \infty \quad (j = 1, 2).$$

Return now to the differential equation

$$f^{(k)} + p_{k-1}(z) f^{(k-1)} + \cdots + p_0(z) f = 0, \tag{3}$$

where $p_j(z)$ $(j = 0, \cdots, k-1)$ are polynomials with $p_0(z) \not\equiv 0$. It is well-known that every solution f of (3) is an entire function of finite rational order; see, [10], [11], [5, pp. 199 − 209], [9, pp. 106 − 108], [12, pp. 65 − 67]. For equation (3), set

$$\lambda = 1 + \max_{0 \le j \le k-1} \frac{\deg p_j}{k-j}. \tag{4}$$

It is known [6, p. 127] that for any solution f of (3), we have

$$\rho(f) \le \lambda.$$

As we have seen in Theorem 1.3 and [7], it is clear that the study of the growth of g_k where $k > 2$, is more difficult than the case where $k = 2$. For that, we give in the following result some sufficient conditions to prove that g_k keeps the same order of growth of solutions of (3) for $k \ge 2$, and we obtain the following result.

Theorem 1.5. *Let $p_j(z)$ $(j=0,\cdots,k-1)$ be polynomials, and let $d_i(z)$ $(1\leq i\leq k)$ be entire functions that are not all vanishing identically such that $\max\{\rho(d_i): 1\leq i\leq k\}<\lambda$. If $\{f_1,f_2,\cdots,f_k\}$ is any set of fundamental solutions of (3), then the combination of solutions g_k satisfies*

$$\rho(g_k)=1+\max_{0\leq j\leq k-1}\frac{\deg p_j}{k-j}.$$

Remark 1.1. The proof of Theorems 1.3-1.5 is quite different from that in the proof of Theorems 1.1-1.2 (see, [7]) . The main ingredient in the proof is Lemma 2.1. By the proof of Theorem 1.5, we can deduce that Theorem 1.2 holds without the additional condition $h\not\equiv 0$.

Corollary 1.6. *Let $A(z)$ be a nonconstant polynomial and let $d_i(z)$ $(1\leq i\leq k)$ be entire functions that are not all vanishing identically such that*

$$\max\{\rho(d_i):\ 1\leq i\leq k\}<\frac{\deg(A)+k}{k}.$$

If $\{f_1,f_2,\cdots,f_k\}$ is any set of fundamental solutions of

$$f^{(k)}+A(z)f=0, \tag{5}$$

then the combination of solutions g_k satisfies

$$\rho(g_k)=\frac{\deg(A)+k}{k}.$$

2. Preliminary lemmas

Lemma 2.1. [8] *(i) Let $f(z)$ be an entire function with $\rho_2(f)=\alpha>0$, and let $L(f)=a_kf^{(k)}+a_{k-1}f^{(k-1)}+\cdots+a_0f$, where a_0, $a_1,\cdots,$ a_k are entire functions which are not all equal zero and satisfy $b=\max\{\rho(a_j):\ j=0,\cdots,k\}<\alpha$. Then $\rho_2(L(f))=\alpha$.*
(ii) Let $f(z)$ be an entire function with $\rho(f)=\alpha\leq\infty$, and let $L(f)=a_kf^{(k)}+a_{k-1}f^{(k-1)}+\cdots+a_0f$, where a_0, $a_1,\cdots,$ a_k are entire functions which are not all equal zero and satisfy $b=\max\{\rho(a_j):\ j=0,\cdots,k\}<\alpha$. Then $\rho(L(f))=\alpha$.

Lemma 2.2. [3] *For any given equation of the form (3), there must exists a solution of (3) that satisfies $\rho(f)=\lambda$, where λ is the constant in (4).*

Lemma 2.3. [1] *Let $A_j(z)$ $(j=0,\cdots,k-1)$ be entire functions of finite order such that*

$$\max\{\rho(A_j): j=1,\cdots,k-1\}<\rho(A_0).$$

Then every solution $f\not\equiv 0$ of (1) satisfies $\rho(f)=\infty$ and $\rho_2(f)=\rho(A_0)$.

Lemma 2.4. [2] *Let $A_0(z)$ be transcendental entire function with $\rho(A_0)=0$, and let $A_1,\cdots,A_{k-1}$ be polynomials. Then every solution $f\not\equiv 0$ of (1) satisfies $\rho(f)=\infty$.*

By using similar proofs as in the proofs of Proposition 1.5 and Proposition 5.5 in [6] , we easily obtain the following lemma.

Lemma 2.5. *For all non-trivial solutions f of (5). If A is a polynomial with $\deg A=n\geq 1$, then we have*

$$\rho(f)=\frac{n+k}{k}.$$

Lemma 2.6. *Let f be any nontrivial solution of* (1). *Then the following identity holds*

$$\sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)=\sum_{j=1}^{k}\left(A_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right),$$

where $A_k(z)\equiv 1$ *and* $C_j^i=\dfrac{j!}{i!\,(j-i)!}$.

Proof. We have

$$\begin{aligned}\sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)&=A_0\frac{d_1}{d_2}f+\sum_{j=1}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)\\&=A_0\frac{d_1}{d_2}f+\sum_{j=1}^{k}\left(A_jC_j^0\left(\frac{d_1}{d_2}\right)f^{(j)}+A_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)\\&=A_0\frac{d_1}{d_2}f+\sum_{j=1}^{k}A_j\left(\frac{d_1}{d_2}\right)f^{(j)}+\sum_{j=1}^{k}\left(A_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)\\&=\frac{d_1}{d_2}\left(f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_0f\right)+\sum_{j=1}^{k}\left(A_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)\\&=\sum_{j=1}^{k}\left(A_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right).\end{aligned}$$

□

Lemma 2.7. *Let f be any nontrivial solution of* (1). *Then the following identity holds*

$$\sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)=\frac{\sum_{i=0}^{k-1}D_if^{(i)}}{d_2^{2k}},$$

where D_i $(i=0,\cdots,k-1)$ *are entire functions depending on* d_1,d_2 *and* A_j $(j=1,\cdots,k-1)$, $A_k(z)\equiv 1$.

Proof. It is clear that we can express the double sum

$$\sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)=\sum_{j=1}^{k}A_j\left(\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)$$

in the form of differential polynomial in f of order $k-1$. By mathematical induction we can prove that

$$\sum_{j=1}^{k}A_j\left(\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)=\sum_{i=0}^{k-1}\alpha_if^{(i)},\tag{6}$$

where

$$\alpha_i=\sum_{p=i+1}^{k}A_pC_p^{p-i}\left(\frac{d_1}{d_2}\right)^{(p-i)}.\tag{7}$$

Also, we have

$$\left(\frac{d_1}{d_2}\right)^{(j)} = \frac{\beta_j}{d_2^{2^j}}, \tag{8}$$

where β_j is entire function. Hence, we deduce from (6)-(8) that

$$\sum_{i=0}^{k-1}\alpha_i f^{(i)} = \frac{\sum\limits_{i=0}^{k-1} D_i f^{(i)}}{d_2^{2^k}},$$

where D_i $(i = 0, \cdots, k-1)$ are entire functions depending on d_1, d_2 and A_j $(j = 1, \cdots, k-1)$, $A_k(z) \equiv 1$. □

3. Proof of Theorem 1.3

Proof. In the case when $d_1(z) \equiv 0$ or $d_2(z) \equiv 0$, then the conclusions of Theorem 1.3 are trivial. Suppose that f_1 and f_2 are two nontrivial linearly independent solutions of (1) such that $d_i(z) \not\equiv 0$ $(i = 1, 2)$ and let

$$g_2 = d_1 f_1 + d_2 f_2. \tag{9}$$

Then, by Lemma 2.3 we have $\rho(f_j) = \infty$ $(j = 1, 2)$ and $\rho_2(f_j) = \rho(A_0)$ $(j = 1, 2)$. Suppose that $d_1 = cd_2$, where c is a complex number. Then, by (9) we obtain

$$g_2 = cd_2 f_1 + d_2 f_2 = (cf_1 + f_2)\, d_2.$$

Since $f = cf_1 + f_2$ is a solution of (1) and $\rho(d_2) < \rho(A_0)$, then we have

$$\rho(g_2) = \rho(cf_1 + f_2) = \infty$$

and

$$\rho_2(g_2) = \rho_2(cf_1 + f_2) = \rho(A_0).$$

Suppose now that $d_1 \not\equiv cd_2$ where c is a complex number. Dividing both sides of (9) by d_2, we obtain

$$F_2 = \frac{g_2}{d_2} = f_2 + \frac{d_1}{d_2} f_1. \tag{10}$$

Differentiating both sides of equation (10), k times for all integers $j = 1, \cdots, k$, we get

$$F_2^{(j)} = f_2^{(j)} + \sum_{i=0}^{j} C_j^i f_1^{(i)} \left(\frac{d_1}{d_2}\right)^{(j-i)}. \tag{11}$$

Equations (10) and (11) are equivalent to

$$\begin{cases} F_2 = f_2 + \frac{d_1}{d_2} f_1, \\ F_2' = f_2' + \left(\frac{d_1}{d_2}\right) f_1' + \left(\frac{d_1}{d_2}\right)' f_1, \\ F_2'' = f_2'' + \left(\frac{d_1}{d_2}\right) f_1'' + 2\left(\frac{d_1}{d_2}\right)' f_1' + \left(\frac{d_1}{d_2}\right)'' f_1, \\ \cdots \\ F_2^{(k-1)} = f_2^{(k-1)} + \sum\limits_{i=0}^{k-1} C_{k-1}^i \left(\frac{d_1}{d_2}\right)^{(k-1-i)} f_1^{(i)}, \\ F_2^{(k)} = f_2^{(k)} + \sum\limits_{i=0}^{k} C_k^i \left(\frac{d_1}{d_2}\right)^{(k-i)} f_1^{(i)} \end{cases}$$

which is also equivalent to

$$\begin{cases} A_0F_2 = A_0f_2 + A_0\frac{d_1}{d_2}f_1, \\ A_1F_2' = A_1f_2' + A_1\left(\left(\frac{d_1}{d_2}\right)f_1' + \left(\frac{d_1}{d_2}\right)'f_1\right), \\ A_2F_2'' = A_2f_2'' + A_2\left(\left(\frac{d_1}{d_2}\right)f_1'' + 2\left(\frac{d_1}{d_2}\right)'f_1' + \left(\frac{d_1}{d_2}\right)''f_1\right), \\ \cdots \\ A_{k-1}F_2^{(k-1)} = A_{k-1}f_2^{(k-1)} + A_{k-1}\sum\limits_{i=0}^{k-1}C_{k-1}^i\left(\frac{d_1}{d_2}\right)^{(k-1-i)}f_1^{(i)}, \\ F_2^{(k)} = f_2^{(k)} + \sum\limits_{i=0}^{k}C_k^i\left(\frac{d_1}{d_2}\right)^{(k-i)}f_1^{(i)}. \end{cases} \tag{12}$$

By (12) we can obtain

$$F_2^{(k)} + A_{k-1}(z)F_2^{(k-1)} + \cdots + A_0(z)F_2 = \left(f_2^{(k)} + A_{k-1}(z)f_2^{(k-1)} + \cdots + A_0(z)f_2\right)$$

$$+\sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right) = \sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right), \tag{13}$$

where $A_k(z) \equiv 1$. By using Lemma 2.6, we have

$$F_2^{(k)} + A_{k-1}(z)F_2^{(k-1)} + \cdots + A_0(z)F_2 = \sum_{j=1}^{k}A_j\left(\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right). \tag{14}$$

By Lemma 2.7, we get

$$\sum_{j=1}^{k}A_j\left(\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right) = k\frac{\left(d_1'd_2 - d_2'd_1\right)d_2^{\sum_{n=0}^{k-1}2^n-1}}{d_2^{2^k}}f_1^{(k-1)} + \frac{1}{d_2^{2^k}}\sum_{i=0}^{k-2}D_if_1^{(i)}, \tag{15}$$

where D_i $(i = 0, \cdots, k-2)$ are entire functions depending on d_1, d_2 and A_j $(j = 1, \cdots, k-1)$, $A_k(z) \equiv 1$. By using (14) and (15), we obtain

$$F_2^{(k)} + A_{k-1}(z)F_2^{(k-1)} + \cdots + A_0(z)F_2 = \frac{L_{k-1}(f_1)}{d_2^{2^k}},$$

where

$$L_{k-1}(f_1) = \sum_{i=0}^{k-1}D_if_1^{(i)}$$

is differential polynomial with entire coefficients D_i $(i = 0, \cdots, k-1)$ of order $\rho(D_i) < \rho(A_0)$ $(i = 0, \cdots, k-1)$ and $D_{k-1} = k\frac{\left(d_1'd_2 - d_2'd_1\right)d_2^{\sum_{n=0}^{k-1}2^n-1}}{d_2^{2^k}} \not\equiv 0$ because $d_1 \not\equiv cd_2$. By Lemma 2.1 (i), we have

$$\rho_2\left(F_2^{(k)} + A_{k-1}(z)F_2^{(k-1)} + \cdots + A_0(z)F_2\right) = \rho_2\left(L_{k-1}(f_1)\right) = \rho_2(f_1).$$

Since

$$\rho_2(f_1) = \rho_2\left(F_2^{(k)} + A_{k-1}(z)F_2^{(k-1)} + \cdots + A_0(z)F_2\right) \le \rho_2(F_2) = \rho_2(g_2) \le \rho_2(f_1),$$

then

$$\rho_2(g_2) = \rho_2(f_1).$$

□

4. Proof of Theorem 1.4

Proof. By using a similar reasoning as in the proof of Theorem 1.3, Lemma 2.4 and Lemma 2.1 (ii) we obtain Theorem 1.4. □

5. Proof of Theorem 1.5

Proof. Without loss of generality, by using Lemma 2.2, we suppose that

$$\max\{\rho(f_j),\ j=1,\cdots,k\}=\rho(f_1)=\lambda=1+\max_{0\le j\le k-1}\frac{\deg p_j}{k-j}$$

and there exist at least two integers p and q such that $d_p\not\equiv cd_q$ where c is a complex number and $1\le p\le q\le k$. By the same proof as Theorem 1.3 we obtain

$$F_2^{(k)}+p_{k-1}(z)F_2^{(k-1)}+\cdots+p_0(z)F_2=\sum_{j=1}^{k}\left(p_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right) \tag{16}$$

and by Lemma 2.7, we get

$$\sum_{j=1}^{k}\left(p_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right)=k\frac{(d_1'd_2-d_2'd_1)\,d_2^{\sum_{n=0}^{k-1}2^n-1}}{d_2^{2^k}}f_1^{(k-1)}+\frac{1}{d_2^{2^k}}\sum_{i=0}^{k-2}D_if_1^{(i)}, \tag{17}$$

where $p_k(z)\equiv 1$ and D_i $(i=0,\cdots,k-2)$ are entire functions. By using (16) and (17), we have

$$F_2^{(k)}+A_{k-1}(z)F_2^{(k-1)}+\cdots+A_0(z)F_2=\frac{L_{k-1}(f_1)}{d_2^{2^k}},$$

where

$$L_{k-1}(f_1)=\sum_{i=0}^{k-1}D_if_1^{(i)}$$

is differential polynomial with entire coefficients D_i $(i=0,\cdots,k-1)$ of order $\rho(D_i)<\lambda$ $(i=0,\cdots,k-1)$ and there exists $0\le i\le k-1$ such that $D_i\not\equiv 0$. By Lemma 2.1 (ii), we have

$$\rho\left(F_2^{(k)}+p_{k-1}(z)F_2^{(k-1)}+\cdots+p_0(z)F_2\right)=\rho(L_{k-1}(f_1))=\rho(f_1).$$

Since

$$\rho(f_1)=\rho\left(F_2^{(k)}+p_{k-1}(z)F_2^{(k-1)}+\cdots+p_0(z)F_2\right)\le\rho(F_2)=\rho(g_2)\le\rho(f_1),$$

then

$$\rho(g_2)=\rho(f_1).$$

Now, we suppose that

$$\rho(g_n)=\rho(f_1)$$

is true for all $n=1,\cdots,k-1$ and we show that

$$\rho(g_k)=\rho(f_1).$$

We have

$$g_k=d_1f_1+d_2f_2+\cdots+d_kf_k=g_{k-1}+d_kf_k. \tag{18}$$

Suppose that $d_k \not\equiv 0$, and dividing both sides of (18) by d_k, we get

$$F_k = \frac{g_k}{d_k} = \frac{g_{k-1}}{d_k} + f_k.$$

By the same reasoning as before, we obtain

$$\begin{cases} p_0 F_k = p_0 f_k + p_0 \frac{1}{d_k} g_{k-1}, \\ p_1 F_k' = p_1 f_k' + p_1 \left(\left(\frac{1}{d_k} \right) g_{k-1}' + \left(\frac{1}{d_k} \right)' g_{k-1} \right), \\ p_2 F_k'' = p_2 f_k'' + p_2 \left(\left(\frac{1}{d_k} \right) g_{k-1}'' + 2 \left(\frac{1}{d_k} \right)' g_{k-1}' + \left(\frac{1}{d_k} \right)'' g_{k-1} \right), \\ \cdots \\ p_{k-1} F_k^{(k-1)} = p_{k-1} f_k^{(k-1)} + p_{k-1} \sum\limits_{i=0}^{k-1} C_{k-1}^i \left(\frac{1}{d_k} \right)^{(k-1-i)} g_{k-1}^{(i)}, \\ F_k^{(k)} = f_k^{(k)} + \sum\limits_{i=0}^{k} C_k^i \left(\frac{1}{d_k} \right)^{(k-i)} g_{k-1}^{(i)}. \end{cases} \tag{19}$$

By (19) we can deduce

$$F_k^{(k)} + p_{k-1}(z) F_k^{(k-1)} + \cdots + p_0(z) F_k = \left(f_k^{(k)} + p_{k-1}(z) f_k^{(k-1)} + \cdots + p_0(z) f_k \right)$$

$$+ \sum_{j=0}^{k} \left(p_j \sum_{i=0}^{j} C_j^i \left(\frac{1}{d_k} \right)^{(i)} g_{k-1}^{(j-i)} \right) = \sum_{j=0}^{k} \left(p_j \sum_{i=0}^{j} C_j^i \left(\frac{1}{d_k} \right)^{(i)} g_{k-1}^{(j-i)} \right). \tag{20}$$

By Lemma 2.6, we have

$$\sum_{j=0}^{k} \left(p_j \sum_{i=0}^{j} C_j^i \left(\frac{1}{d_k} \right)^{(i)} g_{k-1}^{(j-i)} \right) = \sum_{j=1}^{k} \left(p_j \sum_{i=1}^{j} C_j^i \left(\frac{1}{d_k} \right)^{(i)} g_{k-1}^{(j-i)} \right)$$

$$= -k \frac{d_k' d_k^{\sum_{n=0}^{k-1} 2^n - 1}}{d_k^{2^k}} g_{k-1}^{(k-1)} + \frac{1}{d_k^{2^k}} \sum_{i=0}^{k-2} B_i g_{k-1}^{(i)}, \tag{21}$$

where $p_k(z) \equiv 1$ and B_i $(i = 0, \cdots, k-1)$ are entire functions. By using (20) and (21), we obtain

$$F_k^{(k)} + A_{k-1}(z) F_k^{(k-1)} + \cdots + A_0(z) F_k = \frac{M_{k-1}(g_{k-1})}{d_k^{2^k}},$$

where

$$M_{k-1}(g_{k-1}) = \sum_{i=0}^{k-1} B_i g_{k-1}^{(i)}$$

is differential polynomial with entire coefficients B_i $(i = 0, \cdots, k-1)$ of order $\rho(B_i) < \lambda$ $(i = 0, \cdots, k-1)$. By Lemma 2.1 (ii), we have

$$\rho\left(F_k^{(k)} + p_{k-1}(z) F_k^{(k-1)} + \cdots + p_0(z) F_k \right) = \rho\left(M_{k-1}(g_{k-1}) \right) = \rho(f_1).$$

Since

$$\rho(f_1) \le \rho\left(F_k^{(k)} + p_{k-1}(z) F_k^{(k-1)} + \cdots + p_0(z) F_k \right) \le \rho(F_k) = \rho(g_{k-1}) \le \rho(f_1),$$

then

$$\rho(F_k) = \rho(g_{k-1}) = \rho(f_1),$$

which implies that

$$\rho(g_k) = \rho(g_{k-1}) = \rho(f_1) = \lambda.$$

This completes the proof of Theorem 1.5. □

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On fuzzy real valued asymptotically equivalent sequences and lacunary ideal convergence

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Abstract. In this paper we present some definitions which are the natural combination of the definition of asymptotic equivalence, statistical convergence, lacunary statistical convergence of fuzzy real numbers and ideal. In addition, we also present asymptotically ideal equivalent sequences of fuzzy real numbers and establish some relations related to this concept. Finally we introduce the notion of Cesaro Orlicz asymptotically equivalent sequences of fuzzy real numbers and establish their relationship with other classes.

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1. Introduction

The concept of fuzzy set and fuzzy set operations were first introduced by Zadeh [30] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [20] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [4], Mursaleen and Basarir [21], Altin et al. [1], Nanda [22] and many others.

Actually the idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [31]. The concept was formally introduced by Fast [6], Steinhaus [28] and later on it was reintroduced by Schoenberg [27]. A lot of developments have been made in this areas after the works of Šalát [26] and Fridy [8]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Fridy and Orhan [9] introduced the concept of lacunary statistical convergence which is a generalization of statistical convergence. Mursaleen and Mohiuddine [23], introduced the concept of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. For details related to lacunary statistical convergence, we refer to [3, 10].

Marouf [19] peresented definitions for asymptotically equivalent and asymptotic regular matrices. Pobyvancts [25] introduced the concept of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative numbers sequences. Patterson [24] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for

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nonnegative summability matrices. Esi [5] introduced the concept of an asymptotically lacunary statistical equivalent sequences of interval numbers.

Kostyrko et. al [15] introduced the notion of I-convergence with the help of an admissible ideal I denote the ideal of subsets of $\mathbb{N}$, which is a generalization of statistical convergence. Kumar and Sharma [18] introduced the lacunary equivalent sequences of real numbers using ideals and studied some basic properties of this notion. More applications of ideals can be found in the works of [2, 11, 12, 13, 14, 29].

A family of sets $I \subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is called an *ideal* if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is a *filter* on $\mathbb{N}$ if and only if $\phi \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$ and each $A \in \mathcal{F}$ and each $B \supset A$, we have $B \in \mathcal{F}$. An ideal I is called non-trivial ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A non-trivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{x\} : x \in \mathbb{N}\} \subset I$. A non-trivial ideal I is *maximal* if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Recall from [15] that, a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to a real number L if $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$. In this case we write $I - \lim x_k = L$.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}$. For $X = [x_1, x_2]$ and $Y = [y_1, y_2]$ in D, we define

$$X \leq Y \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

Define a metric $\bar{d}$ on D by

$$\bar{d}(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

It can be easily proved that $\bar{d}$ is a metric on D and $(D, \bar{d})$ is a complete metric space. Also the relation $\leq$ is a partial order on D.

A *fuzzy number* X is a fuzzy subset of the real line $\mathbb{R}$ i.e. a mapping $X : \mathbb{R} \to J(= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

Let $L(\mathbb{R})$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R})$ induces the addition $X + Y$ and the scalar multiplication λX in terms of α-level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda [X]^\alpha \text{ for each } 0 \leq \alpha \leq 1.$$

The set $\mathbb{R}$ of real numbers can be embedded in $L(\mathbb{R})$ if we define $\overline{r} \in L(\mathbb{R})$ by

$$\overline{r}(t) = \begin{cases} 1, & \text{if } t = r; \\ 0, & \text{if } t \neq r \end{cases}$$

The additive identity and multiplicative identity of $L(\mathbb{R})$ are denoted by $\overline{0}$ and $\overline{1}$, respectively.

For r in $\mathbb{R}$ and X in $L(\mathbb{R})$, the product rX is defined as follows:

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \neq 0; \\ 0, & \text{if } r = 0 \end{cases}$$

Define a map $d : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$$d(X, Y) = \sup_{0 \leq \alpha \leq 1} \bar{d}(X^\alpha, Y^\alpha).$$

For $X, Y \in L(\mathbb{R})$ define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0, 1]$. It is known that $(L(\mathbb{R}), d)$ is complete metric space (see [20]).

A sequence $u = (u_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(\mathbb{R})$. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ (see [20]). We denote by w^F the set of all sequences $u = (u_k)$ of fuzzy numbers.

A sequence $u = (u_k)$ of fuzzy numbers is said to be bounded if the set $\{u_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded. We denote by ℓ_∞^F the set of all bounded sequences $u = (u_k)$ of fuzzy numbers.

A sequence $u = (u_k)$ of fuzzy numbers is said to be convergent to a fuzzy number u_0 if for every $\varepsilon > 0$ there is a positive integer k_0 such that $d(u_k, u_0) < \varepsilon$ for $k > k_0$. We denote by c^F the set of all convergent sequences $u = (u_k)$ of fuzzy numbers. It is straightforward to see that $c^F \subset \ell_\infty^F \subset w^F$.

2. Definitions and Notations

Throughout the paper, we denote I is an admissible ideal of subsets of $\mathbb{N}$, unless otherwise stated.

Now we recall the definitions which are using throughout the article.

Definition 2.1. A real or complex number sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.

A lacunary sequence $\theta = (k_r)$ is an increasing sequence of non-negative integers where $k_0 = 0$, $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The space of lacunary strongly convergent sequences $\mathcal{N}_\theta$ was defined by Freedman et al., [7] as follows.

$$\mathcal{N}_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

Definition 2.2. [9] A sequence $x = (x_k)$ is said to be *lacunary statistically convergent* to the number L if for every $\varepsilon > 0$

$$\lim_{r \to \infty} \frac{1}{h_r} |\{k \in J_r : \ |x_k - L| \geq \varepsilon\}| = 0.$$

Let S_θ denote the set of all lacunary statistically convergent sequences. If $\theta = (2^r)$, then S_θ is the same as S.

Definition 2.3. [19] Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically equivalent* if

$$\lim_k \frac{x_k}{y_k} = 1,$$

denoted by $x \sim y$.

Definition 2.4. [24] Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0,$$

denoted by $x \overset{S^L}{\sim} y$ and *simply asymptotically statistical equivalent* if $L = 1$.

Definition 2.5. Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically lacunary statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in J_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0,$$

denoted by $x \overset{S_\theta^L}{\sim} y$ and *simply asymptotically lacunary statistical equivalent* if $L = 1$. If we take $\theta = (2^r)$, then we get the definition 4.

Definition 2.6. [18] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *strongly asymptotically I-equivalent* of multiple L provided that for each $\varepsilon > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in I$$

denoted by $x \overset{I[C_1]^L}{\sim} y$ and *simply strongly asymptotically I-equivalent* if $L = 1$.

Definition 2.7. [18] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically I-lacunary statistical equivalent of multiple L provided that for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in J_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in I$$

denoted by $x \overset{I(S_\theta)^L}{\sim} y$ and simply asymptotically I-lacunary statistical equivalent if $L = 1$.

3. Asymptotically lacunary statistical equivalent sequences using ideals

In this section, we define I-statistical convergence, asymptotically I-equivalent, asymptotically I-statistical equivalent and asymptotically I-lacunary statistical equivalent sequences of fuzzy real numbers and obtain some analogous results from these new definitions point of views.

Definition 3.1. Two sequences $u = (u_k)$ and $v = (v_k)$ of fuzzy real numbers are said to be *asymptotically statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : d\left(\frac{u_k}{v_k}, L \right) \geq \varepsilon \right\} \right| = 0,$$

denoted by $x \overset{\mathcal{S}^L}{\sim} y$ and simply asymptotically statistical equivalent if $L = 1$.

Definition 3.2. [17] A sequence $u = (u_k)$ of fuzzy numbers is said to be *I-convergent* to a fuzzy number u_0 if for each $\epsilon > 0$

$$A = \{ k \in \mathbb{N} : d(u_k, u_0) \geq \varepsilon \} \in I.$$

Definition 3.3. A sequence (u_k) of fuzzy real numbers is said to be *I-statistically convergent* to a fuzzy real number u_0 if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : d(u_k, u_0) \geq \varepsilon \} \right| \geq \delta \right\} \in I.$$

In this case we write $I(\mathcal{S}) - \lim u_k = u_0$.

Definition 3.4. Two nonnegative sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically I-equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\left\{k \in \mathbb{N} : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\} \in I,$$

denoted by $(u_k) \overset{I^L}{\sim} (v_k)$ and *simply asymptotically I-equivalent* if $L = 1$.

Lemma 3.1. *Let $I \subset P(\mathbb{N})$ be an admissible ideal. Let $(u_k), (v_k) \in \ell_\infty^F$ with $I - \lim_k u_k = \bar{0} = I - \lim_k v_k$ such that $(u_k) \overset{I^L}{\sim} (v_k)$. Then there exists a sequence $(w_k) \in \ell_\infty^F$ with $I - \lim_k w_k = \bar{0}$ such that $(u_k) \overset{I^L}{\sim} (w_k) \overset{I^L}{\sim} (v_k)$.*

Definition 3.5. Two sequences (u_k) and (v_k) of fuzzy numbers are said to be *asymptotically I-statistical equivalent* of multiple L provided that for every $\varepsilon > 0$ and for every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I,$$

denoted by $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$ and *simply asymptotically I-statistical equivalent* if $L = 1$.

Definition 3.6. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *strongly asymptotically Cesàro I-equivalent* (or *$I([C_1])$-equivalent*) of multiple L provided that for every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k}, L\right) \geq \delta\right\} \in I$$

denoted by $(u_k) \overset{I([C_1])^L}{\sim} (v_k)$ and *simply strongly asymptotically Cesàro I-equivalent* if $L = 1$.

Theorem 3.2. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. If $(u_k), (v_k) \in \ell_\infty^F$ and $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$. Then $(u_k) \overset{I([C_1])^L}{\sim} (v_k)$.*

Proof. Suppose that $(u_k), (v_k) \in \ell_\infty^F$ and $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$. Then we can assume that

$$d\left(\frac{u_k}{v_k}, L\right) \leq M \text{ for almost all } k.$$

Let $\varepsilon > 0$. Then we have

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k}, L\right)\right| &\leq \frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k}, L\right) \\
&\leq \frac{1}{n}\sum_{\substack{k=1 \\ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon}}^{n} d\left(\frac{u_k}{v_k}, L\right) + \frac{1}{n}\sum_{\substack{k=1 \\ d\left(\frac{u_k}{v_k}, L\right) < \varepsilon}}^{n} d\left(\frac{u_k}{v_k}, L\right) \\
&\leq M.\frac{1}{n}\left|\left\{k \leq n : \ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| + \frac{1}{n}.n.\varepsilon.
\end{aligned}$$

Consequently for any $\delta > \varepsilon > 0$, δ and ε are independent, put $\delta_1 = \delta - \varepsilon > 0$ we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k}, L\right) \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \frac{\delta_1}{M}\right\} \in I.$$

This shows that $(u_k) \overset{I([C_1])^L}{\sim} (v_k)$. □

Definition 3.7. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *strongly asymptotically I-lacunary equivalent* (or $I([N_\theta])$-*equivalent*) of multiple L provided that for every $\delta > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \delta \right\} \in I$$

denoted by $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k)$ and *simply strongly asymptotically I-lacunary equivalent* if $L = 1$.

Definition 3.8. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically I-lacunary statistical equivalent* (or $I(\mathcal{S}_\theta)$-*equivalent*) of multiple L provided that for every $\varepsilon > 0$, for every $\delta > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \right\} \right| \geq \delta \right\} \in I$$

denoted by $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$ and *simply asymptotically I-lacunary statistical equivalent* if $L = 1$.

Remark 3.1. If we take $I = I_{fin} = \{A \subset \mathbb{N} : A \text{ is finite set}\}$, then the asymptotically I-statistical equivalent, $I([N_\theta])$-equivalent and $I(\mathcal{S}_\theta)$-equivalent of sequences, respectively coincides with their statistically equivalent, lacunary-equivalent and lacunary statistically equivalent.

Theorem 3.3. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Then*

(a) $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k) \Rightarrow (u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$.

(b) *Let $(u_k) \in \ell_\infty^F$ and $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$, then $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k)$.*

(c) $I(\mathcal{S}_\theta)^L \cap \ell_\infty^F = I([N_\theta])^L \cap \ell_\infty^F$.

Proof. (a) Let $\varepsilon > 0$ and $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k)$. Then we can write

$$\sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \sum_{\substack{k \in J_r \\ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon}} d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \left| \left\{ k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \right\} \right|$$

$$\Rightarrow \frac{1}{\varepsilon . h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \frac{1}{h_r} \left| \left\{ k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \right\} \right|.$$

Thus for any $\delta > 0$,

$$\frac{1}{h_r} \left| \left\{ k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \right\} \right| \geq \delta$$

implies that

$$\frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\delta.$$

Therefore we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \right\} \right| \geq \delta \right\} \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\delta \right\}.$$

Since $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k)$, so that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\delta\right\} \in I$$

which implies that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I.$$

This shows that $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$.

(b) Suppose that $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$ and $(u_k), (v_k) \in \ell_\infty^F$. We assume that $d\left(\frac{u_k}{v_k}, L\right) \leq M$ for all $k \in \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\frac{1}{h_h} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) = \frac{1}{h_r} \sum_{\substack{k \in J_r \\ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon}} d\left(\frac{u_k}{v_k}, L\right) + \frac{1}{h_r} \sum_{\substack{k \in J_r \\ d\left(\frac{u_k}{v_k}, L\right) < \varepsilon}} d\left(\frac{u_k}{v_k}, L\right)$$

$$\leq \frac{M}{h_r} \left|\left\{k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| + \varepsilon.$$

If we put

$$A(\varepsilon) = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}$$

and

$$B(\varepsilon_1) = \left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \frac{\varepsilon_1}{M}\right\},$$

where $\varepsilon_1 = \delta - \varepsilon > 0$ (δ and ε are independent), then we have $A(\varepsilon) \subset B(\varepsilon_1)$ and so $A(\varepsilon) \in I$. This shows that $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k)$.

(c) It follows from (a)and (b).

□

If we let $\theta = (2^r)$ in Theorem 3.3, then we have the following corollary.

Corollary 3.4. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Then*

(a) $(u_k) \overset{I([C_1])^L}{\sim} (v_k) \Rightarrow (u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$.

(b) *Let $(u_k) \in \ell_\infty^F$ and $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$, then $(u_k) \overset{I([C_1])^L}{\sim} (v_k)$.*

(c) $I(\mathcal{S})^L \cap \ell_\infty^F = I([C_1])^L \cap \ell_\infty^F$.

Theorem 3.5. *Let I be a non-trivial admissible ideal. Suppose for given $\delta > 0$ and every $\varepsilon > 0$*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{0 \leq k \leq n-1 : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| < \delta\right\} \in \mathcal{F}$$

then $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$.

Proof. Let $\delta > 0$ be given. For every $\varepsilon > 0$, choose n_1 such that

$$\frac{1}{n} \left|\left\{0 \leq k \leq n-1 : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| < \frac{\delta}{2}, \text{ for all } n \geq n_1. \tag{1}$$

It is sufficient to show that there exists n_2 such that for $n \geq n_2$

$$\frac{1}{n}\left|\left\{0\leq k\leq n-1: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|<\frac{\delta}{2}.$$

Let $n_0=\max\{n_1,n_2\}$. The relation (3.1) will be true for $n>n_0$. If m_0 chosen fixed, then we get

$$\left|\left\{0\leq k\leq m_0-1: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|=M.$$

Now for $n>m_0$ we have

$$\begin{aligned}\frac{1}{n}\left|\left\{0\leq k\leq n-1: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right| &\leq \frac{1}{n}\left|\left\{0\leq k\leq m_0-1: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|\\ &\quad+\frac{1}{n}\left|\left\{m_0\leq k\leq n-1: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|\\ &\leq \frac{M}{n}+\frac{1}{n}\left|\left\{m_0\leq k\leq n-1: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|\leq\frac{M}{n}+\frac{\delta}{2}.\end{aligned}$$

Thus for sufficiently large n

$$\frac{1}{n}\left|\left\{m_0\leq k\leq n-1: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|\leq\frac{M}{n}+\frac{\delta}{2}<\delta.$$

This established the result. □

Theorem 3.6. *Let (u_k) and (v_k) be two sequences of fuzzy real numbers. Let $\theta=(k_r)$ be a lacunary sequence with $\liminf_r q_r>1$. Then $(u_k)\overset{I(\mathcal{S})^L}{\sim}(v_k)\Rightarrow(u_k)\overset{I(\mathcal{S}_\theta)^L}{\sim}(v_k)$.*

Proof. Suppose that $\liminf_r q_r>1$ then there exists an $a>0$ such that $q_r\geq 1+a$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r}\geq\frac{a}{1+a}.$$

Suppose that $(u_k)\overset{I(\mathcal{S})^L}{\sim}(v_k)$. For a given $\varepsilon>0$ and sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r}\left|\left\{k\leq k_r: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right| &\geq \frac{1}{k_r}\left|\left\{k\in J_r: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|\\ &\geq\left(\frac{a}{1+a}\right).\frac{1}{h_r}\left|\left\{k\in J_r: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|.\end{aligned}$$

and for any $\delta>0$ we have

$$\begin{aligned}&\left\{r\in\mathbb{N}:\frac{1}{k_r}\left|\left\{k\in J_r: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|\geq\delta\right\}\\ &\qquad\subseteq\left\{r\in\mathbb{N}:\frac{1}{k_r}\left|\left\{k\leq k_r: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|\geq\frac{a\delta}{1+a}\right\}\in I.\end{aligned}$$

This shows that $(u_k)\overset{I(\mathcal{S}_\theta)^L}{\sim}(v_k)$. □

Theorem 3.7. *Let $I=I_{fin}=\{A\subset\mathbb{N}: A \text{ is a finite set}\}$ be a non trivial ideal. Let $(u_k),(v_k)$ be two sequences of fuzzy real numbers and $\theta=(k_r)$ be a lacunary sequence with $\limsup_r q_r<\infty$, then $(u_k)\overset{I(\mathcal{S}_\theta)^L}{\sim}(v_k)\Rightarrow(u_k)\overset{I(\mathcal{S})^L}{\sim}(v_k)$.*

Proof. Suppose that $\limsup_r q_r < \infty$, then there exists a $H > 0$ such that $q_r < H$ for all r. Suppose that $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$ and for every $\varepsilon > 0$, we put

$$N_r = \left|\left\{k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right|.$$

Since $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$ it follws that for every $\varepsilon > 0$ and $\delta > 0$

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} = \left\{r \in \mathbb{N} : \frac{N_r}{h_r} \geq \delta\right\} \in I$$

and therefore it is a finite set. We can choose an integer $r_0 \in \mathbb{N}$ such that

$$\frac{N_r}{h_r} < \delta \text{ for all } r > r_0. \tag{2}$$

Let $M = \max\{N_r : 1 \leq r \leq r_0\}$ and n be any integer satisfying $k_{r-1} < n \leq k_r$, then we have

$$\begin{aligned}
\frac{1}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| &\leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \\
&= \frac{1}{k_{r-1}}\left\{N_1 + N_2 + ... + N_{r_0} + N_{r_0+1} + ... + N_r\right\} \\
&\leq \frac{M}{k_{r-1}}.r_0 + \frac{1}{k_{r-1}}\left\{h_{r_0+1}\left(\frac{N_{r_0+1}}{h_{r_0+1}}\right) + ... + h_r\left(\frac{N_r}{h_r}\right)\right\} \\
&\leq \frac{M}{k_{r-1}}.r_0 + \frac{1}{k_{r-1}}\left(\sup_{r>r_0}\left(\frac{N_r}{h_r}\right)\right)\left\{H_{r_0+1} + ... + h_r\right\} \\
&\leq \frac{M}{k_{r-1}}.r_0 + \delta\left(\frac{k_r - k_{r_0}}{k_{r-1}}\right) \\
&\leq \frac{M}{k_{r-1}}.r_0 + \delta q_r \leq \frac{M}{k_{r-1}}.r_0 + \delta.H
\end{aligned}$$

This completes the proof of the theorem. □

Definition 3.9. Let $p \in (0, \infty)$. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically lacunary p-equivalent* provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right)^p = 0$$

denoted by $(u_k) \overset{[N_{\theta_p}]^L}{\sim} (v_k)$ and *simply asymptotically lacunary p-equivalent* if $L = 1$.

Definition 3.10. Let $p \in (0, \infty)$. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically lacunary statistical p-equivalent* provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in J_r : d\left(\frac{u_k}{v_k}, L\right)^p \geq \varepsilon\right\}\right| = 0$$

denoted by $(u_k) \overset{\mathcal{S}^L_{\theta_p}}{\sim} (v_k)$ and *simply asymptotically lacunary statistical p-equivalent* if $L = 1$.

The proof of the following theorem is similar to Theorem 3.3 for $I = I_{fin}$.

Theorem 3.8. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Then*

(a) $(u_k) \overset{[N_{\theta_p}]^L}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathcal{S}^L_{\theta_p}}{\sim} (v_k)$.

(b) *Let* $(u_k) \in \ell_\infty^F$ *and* $(u_k) \overset{\mathcal{S}_{\theta_p}^L}{\sim} (v_k)$, *then* $(u_k) \sim^{[N_{\theta_p}]^L} (v_k)$.
(c) $\mathcal{S}_{\theta_p}^L \cap \ell_\infty^F = [N_{\theta_p}]^L \cap \ell_\infty^F$.

Definition 3.11. Let $p \in (0, \infty)$. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically I-lacunary p-equivalent* (or $I([N_{\theta_p}])$-*equivalent*) provided that for every $\varepsilon > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L \right)^p \geq \varepsilon \right\} \in I$$

denoted by $(u_k) \overset{I([N_{\theta_p}])^L}{\sim} (v_k)$ and simply asymptotically I-lacunary p-equivalent if $L = 1$.

Definition 3.12. Let $p \in (0, \infty)$. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically I-lacunary statistical p-equivalent* provided that for every $\varepsilon > 0$, for every $\delta > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \leq n : d\left(\frac{u_k}{v_k}, L \right)^p \geq \varepsilon \right\} \right| \geq \delta \right\} \in I$$

denoted by $(u_k) \overset{I(\mathcal{S}_{\theta_p})^L}{\sim} (v_k)$ and *simply asymptotically I-statistical p-equivalent* if $L = 1$.

The proof of the following theorem follows from Theorems 3.3 and 3.8.

Theorem 3.9. *Let* $(u_k), (v_k)$ *be two sequences of fuzzy real numbers. Then*
(a) $(u_k) \overset{I([N_{\theta_p}])^L}{\sim} (v_k) \Rightarrow (u_k) \overset{I(\mathcal{S}_{\theta_p})^L}{\sim} (v_k)$.
(b) *Let* $(u_k) \in \ell_\infty^F$ *and* $(u_k) \overset{I(\mathcal{S}_{\theta_p})^L}{\sim} (v_k)$, *then* $(u_k) \overset{I([N_{\theta_p}])^L}{\sim} (v_k)$.
(c) $I(\mathcal{S}_{\theta_p})^L \cap \ell_\infty^F = I([N_{\theta_p}])^L \cap \ell_\infty^F$.

4. Cesàro Orlicz asymptotically ϕ-statistical equivalent sequences

In this section we define the notion of Cesàro Orlicz asymptotically ϕ-statistical equivalent sequences of fuzzy real numbers.

Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $p(\sigma)$ the sequence $\{p_n(\sigma)\}$ such that $p_n(\sigma) = 1$ for $n \in \sigma$ and $p_n(\sigma) = 0$ otherwise. Further

$$P_s = \left\{ \sigma \in P : \sum_{n=1}^{\infty} p_n(\sigma) \leq s \right\},$$

i.e. P_s is the set of those σ whose support has cardinality at most s, and we get

$$\Phi = \{ \phi = (\phi_n) : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1} \text{ and } n\phi_{n+1} \leq (n+1)\phi_n \}.$$

We define

$$\tau_s = \frac{1}{\phi_s} \sum_{k \in \sigma, \sigma \in P_s} x_k.$$

Now we give the following definitions.

Definition 4.1. A sequence $x = (x_k)$ is said to be ϕ-*summable* to ℓ if $\lim_s \tau_s = \ell$.

Definition 4.2. A sequence $x = (x_k)$ is said to be *strongly ϕ-summable* to ℓ if

$$\lim_{s\to\infty} \frac{1}{\phi_s} \sum_{k\in\sigma,\sigma\in P_s} |x_k - \ell| = 0.$$

In this case we write $x_k \overset{[\phi]}{\to} \ell$ and $[\phi]$ denote the set of all strongly ϕ-summable sequences.

Definition 4.3. Let $E \subseteq \mathbb{N}$. The number

$$\delta_\phi(E) = \lim_{s\to\infty} \frac{1}{\phi_s} |\{k \in \sigma, \sigma \in P_s : k \in E\}|$$

is said to be the *ϕ-density* of E.

Definition 4.4. A sequence $x = (x_k)$ is said to be *ϕ-statistical convergent* to $\ell \in \mathbb{R}$ if for each $\varepsilon > 0$

$$\lim_{s\to\infty} \frac{1}{\phi_s} |\{k \in \sigma, \sigma \in P_s : |x_k - \ell| \geq \varepsilon\}| = 0.$$

In this case we write $S_\phi - \lim_k x_k = \ell$ or $x_k \overset{S_\phi}{\to} \ell$ and S_ϕ denote the set of all ϕ-statisticallly convergent sequences.

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function M is said to satisfy the $\Delta_2 - condition$ for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$. Note that, if $0 < \lambda < 1$,then $M(\lambda x) \leq \lambda M(x)$, for all $x \geq 0$(see [16]).

Now we define the following asymptotic ϕ-statistical equivalence sequences of fuzzy real numbers.

Definition 4.5. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *Cesàro Orlicz asymptotically equivalent* of multiple L provided that

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) = 0$$

denoted by $(u_k) \overset{[C_1]^L(M)}{\sim} (v_k)$ and *simply Cesàro Orlicz asymptotically equivalent* if $L = 1$.

Definition 4.6. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *Cesàro Orlicz asymptotically I-equivalent* of multiple L provided that for every $\delta > 0$

$$\left\{n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \delta\right\} \in I$$

denoted by $(u_k) \overset{I[C_1]^L(M)}{\sim} (v_k)$ and simply *Cesàro Orlicz asymptotically I-equivalent* if $L = 1$.

Definition 4.7. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *Orlicz asymptotically ϕ-equivalent* of multiple L provided that

$$\lim_s \frac{1}{\phi_s} \sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) = 0$$

denoted by $(u_k) \overset{[\phi]^L(M)}{\sim} (v_k)$ and *simply Orlicz asymptotically ϕ-equivalent* if $L = 1$.

Definition 4.8. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically ϕ-statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\lim_s \frac{1}{\phi_s} \left| \left\{ k \in \sigma, \sigma \in P_s : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \right\} \right| = 0$$

denoted by $(u_k) \overset{\mathcal{S}_\phi^L}{\sim} (v_k)$ and *simply asymptotically ϕ-statistical equivalent* if $L = 1$.

Definition 4.9. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *Orlicz asymptotically ϕ-statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\lim_s \frac{1}{\phi_s} \left| \left\{ k \in \sigma, \sigma \in P_s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \varepsilon \right\} \right| = 0$$

denoted by $(u_k) \overset{\mathcal{S}_\phi^L(M)}{\sim} (v_k)$ and *simply Orlicz asymptotically ϕ-statistical equivalent* if $L = 1$.

Theorem 4.1. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers and M be an Orlicz function. Then*

(a) $(u_k) \overset{I[C_1]^L(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$.

(b) $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$ *implies* $(u_k) \overset{I[C_1]^L(M)}{\sim} (v_k)$, *if M is bounded.*

Proof. (a) Suppose that $(u_k) \overset{I[C_1]^L(M)}{\sim} (v_k)$ and let $\varepsilon > 0$ be given, then we can write

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) &\geq \frac{1}{n} \sum_{\substack{k=1 \\ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon}}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \\ &\geq \frac{M(\varepsilon)}{n} \left| \left\{ k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \right\} \right|. \end{aligned}$$

Consequently for any $\eta > 0$, we have

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \right\} \right| \geq \frac{\eta}{M(\varepsilon)} \right\} \\ &\qquad \subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \eta \right\} \in I. \end{aligned}$$

Hence $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$.

(b) Suppose that M is bounded and $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$. Since M is bounded then there exists a real number $K > 0$ such that $\sup_t M(t) \leq K$. Moreover for any $\varepsilon > 0$ we can write

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) &= \frac{1}{n} \left[\sum_{\substack{k=1 \\ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon}}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) + \sum_{\substack{k=1 \\ d\left(\frac{u_k}{v_k}, L\right) < \varepsilon}}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \right] \\ &\leq \frac{K}{n} \left| \left\{ k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \right\} \right| + M(\varepsilon). \end{aligned}$$

Now applying $\varepsilon \to 0$, then the result follows. □

Theorem 4.2. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers and (ϕ_s) be a nondecreasing sequence of positive real numbers such that $\phi_s \to \infty$ as $s \to \infty$ and $\phi_s \leq s$ for every $s \in \mathbb{N}$. Then $(u_k) \overset{\mathcal{S}^L}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathcal{S}^L_\phi}{\sim} (v_k)$.*

Proof. By the definition of the sequences ϕ_s it follows that $\inf_s \frac{s}{s-\phi_s} \geq 1$. Then there exists a $a > 0$ such that

$$\frac{s}{\phi_s} \leq \frac{1+a}{a}.$$

Suppose that $(u_k) \overset{\mathcal{S}^L}{\sim} (v_k)$, then for every $\varepsilon > 0$ and sufficiently large s we have

$$\begin{aligned}
\frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| &= \frac{1}{s}.\frac{s}{\phi_s}\left|\left\{k \leq s : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \\
&\quad - \frac{1}{\phi_s}\left|\left\{k \in \{1,2,...s\} - \sigma, \sigma \in P_s : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \\
&\leq \frac{1+a}{a}\frac{1}{s}\left|\left\{k \leq s : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \\
&\quad - \frac{1}{\phi_s}\left|\left\{k_0 \in \{1,2,...s\} - \sigma, \sigma \in P_s : d\left(\frac{u_{k_0}}{v_{k_0}}, L\right) \geq \varepsilon\right\}\right|.
\end{aligned}$$

This completes the proof of the theorem. □

Theorem 4.3. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers and let M be an Orlicz function satifies the Δ_2-conditions. Then $(u_k) \overset{\mathcal{S}^L}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathcal{S}^L_\phi(M)}{\sim} (v_k)$.*

Proof. By the definition of the sequences ϕ_s it follows that $\inf_s \frac{s}{s-\phi_s} \geq 1$. Then there exists an $a > 0$ such that

$$\frac{s}{\phi_s} \leq \frac{1+a}{a}.$$

Suppose that $(u_k) \overset{\mathcal{S}^L}{\sim} (v_k)$, then for every $\varepsilon > 0$ and sufficiently large s we have

$$\begin{aligned}
\frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \varepsilon\right\}\right| &= \frac{1}{s}.\frac{s}{\phi_s}\left|\left\{k \leq s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \varepsilon\right\}\right| \\
&\quad - \frac{1}{\phi_s}\left|\left\{k \in \{1,2,...s\} - \sigma, \sigma \in P_s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \varepsilon\right\}\right| \\
&\leq \frac{1+a}{a}\frac{1}{s}\left|\left\{k \leq s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \varepsilon\right\}\right| \qquad (3) \\
&\quad - \frac{1}{\phi_s}\left|\left\{k_0 \in \{1,2,...s\} - \sigma, \sigma \in P_s : M\left(d\left(\frac{u_{k_0}}{v_{k_0}}, L\right)\right) \geq \varepsilon\right\}\right|.
\end{aligned}$$

Since M satisfies the Δ_2-conditions, it follows that

$$M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \leq K.d\left(\frac{u_k}{v_k}, L\right)$$

for some constant $K > 0$ in both the cases where $d\left(\frac{u_k}{v_k}, L\right) \leq 1$ and $d\left(\frac{u_k}{v_k}, L\right) \geq 1$.

In first case it follows from the definition of Orlicz function and for the second case we have

$$d\left(\frac{u_k}{v_k}, L\right) = 2.L^{(1)} = 2^2.L^{(2)} = ... = 2^s.L^{(s)}$$

such that $L^{(s)} \leq 1$. Using the Δ_2-conditions of Orlicz functions we get the following estimation

$$M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \leq T.L^{(s)}.M(1) = K.d\left(\frac{u_k}{v_k}, L\right), \tag{4}$$

where K and T are constants. The proof of the theorem follows from the relations (4.1) and (4.2). □

Remark 4.1. From the Theorems 4.2 and 4.3, we can concluded that $(u_k) \overset{\mathcal{S}^L}{\sim} (v_k) \Leftrightarrow (u_k) \overset{\mathcal{S}^L_\phi(M)}{\sim} (v_k)$.

Theorem 4.4. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Let M be an Orlicz function and $k \in \mathbb{Z}$ such that $\phi_s \leq [\phi_s] + k$, $\sup_s \frac{[\phi_s]+k}{\phi_{s-1}} < \infty$. Then $(u_k) \overset{\mathcal{S}^L_\phi(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathcal{S}^L}{\sim} (v_k)$.*

Proof. If $\sup_s \frac{[\phi_s]+k}{\phi_{s-1}} < \infty$, then there exists $K > 0$ such that $\frac{[\phi_s]+k}{\phi_{s-1}} < K$ for all $s \geq 1$. Let n be an integer such that $\phi_{s-1} < n \leq \phi_s$. Then for every $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| &\leq \frac{1}{n}\left|\left\{k \leq n : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq M(\varepsilon)\right\}\right| \\
&\leq \frac{1}{[\phi_s]+k}.\frac{[\phi_s]+k}{\phi_{s-1}}\left|\left\{k \leq \phi_s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq M(\varepsilon)\right\}\right| \\
&\leq \frac{1}{[\phi_s]+k}.\frac{[\phi_s]+k}{\phi_{s-1}}\left|\left\{k \in \sigma, \sigma \in P_{[\phi_s]+k} : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq M(\varepsilon)\right\}\right| \\
&\leq \frac{K}{[\phi_s]+k}\left|\left\{k \in \sigma, \sigma \in P_{[\phi_s]+k} : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq M(\varepsilon)\right\}\right|.
\end{aligned}$$

This established the result. □

Theorem 4.5. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Let M be an Orlicz function. Then*

(a) $(u_k) \overset{[C_1]^L(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{[\phi]^L(M)}{\sim} (v_k)$.

(b) $\sup_s \frac{\phi_s}{\phi_{s-1}} < \infty$ *for every* $s \in \mathbb{N}$, *then* $(u_k) \overset{[\phi]^L(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{[C_1]^L(M)}{\sim} (v_k)$.

Proof. (a) From definition of the sequence (ϕ_s) it follows that $\inf_s \frac{s}{s-\phi_s} \geq 1$. Then there exists $a > 0$ such that

$$\frac{s}{\phi_s} \leq \frac{1+a}{a}.$$

Then we get the following relation

$$\begin{aligned}
&\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \\
&\quad = \frac{s}{\phi_s}.\frac{1}{s}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) - \frac{1}{\phi_s}\sum_{k\in\{1,2,\ldots s\}-\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \\
&\quad \leq \frac{1+a}{a}\frac{1}{s}\sum_{k=1}^{s} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) - \frac{1}{\phi_s}\sum_{k_0\in\{1,2,\ldots s\}-\sigma,\sigma\in P_s} M\left(d\left(\frac{u_{k_0}}{v_{k_0}}, L\right)\right).
\end{aligned}$$

Since $(u_k) \overset{[C_1]^L(M)}{\sim} (v_k)$ and M is continuous, letting $s \to \infty$ on the last relation we get

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\to 0.$$

Hence $(u_k)\overset{[\phi]^L(M)}{\sim}(v_k)$.

(b) Suppose that $\sup_s \frac{\phi_s}{\phi_{s-1}}<\infty$ then there exists $A>0$ such that $\frac{\phi_s}{\phi_{s-1}}<A$ for all $s\geq 1$. Suppose $(u_k)\overset{[\phi]^L(M)}{\sim}(v_k)$. Then for every $\varepsilon>0$ there exists $R>0$ such that for every $s\geq R$

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k},L\right)\right)<\varepsilon.$$

We can also find a constant $K>0$ such that

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k},L\right)\right)<K \text{ for all } s\in\mathbb{N}.$$

Let n be any integer with $\phi_{s-1}<n\leq[\phi_s]$ for every $s>R$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\leq\frac{1}{\phi_{s-1}}\sum_{k=1}^{[\phi_s]} M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$

$$=\frac{1}{\phi_{s-1}}\left(\sum_{k=1}^{[\phi_1]} M\left(d\left(\frac{u_k}{v_k},L\right)\right)+\sum_{[\phi_1]}^{[\phi_2]} M\left(d\left(\frac{u_k}{v_k},L\right)\right)+...+\sum_{[\phi_{s-1}]}^{[\phi_s]} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)$$

$$\leq\frac{\phi_1}{\phi_{s-1}}\left(\frac{1}{\phi_1}\sum_{k\in\sigma,\sigma\in P^{(1)}} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)+\frac{\phi_2}{\phi_{s-1}}\left(\frac{1}{\phi_2}\sum_{k\in\sigma,\sigma\in P^{(2)}} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)+...$$

$$+\frac{\phi_R}{\phi_{s-1}}\left(\frac{1}{\phi_R}\sum_{k\in\sigma,\sigma\in P^{(R)}} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)+...+\frac{\phi_s}{\phi_{s-1}}\left(\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P^{(s)}} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right),$$

where $P^{(t)}$ are sets of integer which have more than $[\phi_t]$ elements for $t\in\{1,2,...s\}$. By taking limit as $n\to\infty$ on the last relation we get

$$\frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\to 0.$$

It follows that $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$. □

Theorem 4.6. *Let $(u_k),(v_k)$ be two sequences of fuzzy real numbers. Let M be an Orlicz function. Then*

(a) $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)\Rightarrow(u_k)\overset{\mathcal{S}^L}{\sim}(v_k)$.

(b) *If M satisfies the Δ_2-condition and $(u_k)\in\ell_\infty^F(M)$ such that $(u_k)\overset{\mathcal{S}^L}{\sim}(v_k)$ then $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$.*

(c) *If M satisfies the Δ_2-condition, then $[C_1]^L(M)\cap\ell_\infty^F(M)=\mathcal{S}^L\cap\ell_\infty^F(M)$, where $\ell_\infty^F(M)=\{(u_k)\in w^F : M(u_k)\in\ell_\infty^F\}$.*

Proof. (a) Suppose that $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$. Then for every $\varepsilon>0$ we have

$$\frac{1}{n}\left|\left\{k\leq n : M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq M(\varepsilon)\right\}\right|$$

$$\leq \frac{1}{n} \sum_{\substack{k=1 \\ M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq M(\varepsilon)}}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right) \leq \frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right).$$

This established the result.
(b) The proof of this part follows from the same techniques used in the proofs of the Theorems 3.3 and 4.3.
(c) It follows from (a) and (b). □

Theorem 4.7. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Let M be an Orlicz function. Then*

(a) $(u_k) \overset{[\phi]^L(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathcal{S}_\phi^L}{\sim} (v_k)$.

(b) *If M satisfies the Δ_2-condition and $(u_k) \in \ell_\infty^F(M)$ such that $(u_k) \overset{\mathcal{S}_\phi^L}{\sim} (v_k)$ then $(u_k) \overset{[\phi]^L(M)}{\sim} (v_k)$.*

(c) *If M satisfies the Δ_2-condition, then $[\phi]^L(M) \cap \ell_\infty^F(M) = \mathcal{S}_\phi^L \cap \ell_\infty^F(M)$.*

Proof. The proof of this theorem follows from the same techniques used in the proofs of the Theorems 3.3, 4.3 and 4.6. □

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Generalized Hermite-Hadamard type integral inequalities for s-convex functions via fractional integrals

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Abstract. In this paper, we have established Hermite-Hadamard type inequalities for the class of functions whose derivatives in absolute value at certain powers are s-convex functions by using fractional integrals depending on a parameter.

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1. Introduction

Definition 1.1. The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be *convex* if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[18, p.137], [12]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 12, 13, 18]) and the references cited therein.

Definition 1.2. [4] Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be $s-$*convex* (*in the second sense*), or that f belongs to the class K_s^2, if f

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in [0, \infty)$ and $\lambda \in [0, 1]$.

An $s-$convex function was introduced in Breckner's paper [4] and a number of properties and connections with $s-$convexity in the first sense are discussed in paper [11]. Of course, $s-$convexity means just convexity when $s = 1$.

In [10], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense.

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Theorem 1.1. [10] *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1)$, and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^1([a,b])$, then the following inequalities hold:*

$$2^{s-1}f(\frac{a+b}{2})\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{s+1}. \tag{2}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (2).

Meanwhile, Sarikaya et al.[21] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha>0$.

Lemma 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \\ &=\frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)\,dt.\end{aligned} \tag{3}$$

It is remarkable that Sarikaya et al.[21] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{4}$$

with $\alpha>0$.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [14, 15, 17, 19].

Definition 1.3. Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\geq0$ are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\quad x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

For some recent results connected with fractional integral inequalities see ([3, 5, 6, 7, 8, 9, 16, 20, 22, 23, 24]).

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for the class of functions whose derivatives in absolute value at certain powers are s-convex functions by using Riemann-Liouville fractional integral and some other integral inequalities. The results presented in this paper provide extensions of those given in earlier works.

2. Main Results

For our results, we give the following important fractional integral identity [22]:

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}-\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}&+\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\\&\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right] \quad (5)\\&=\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\end{aligned}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$ and $\alpha>0$.

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$. If $|f'|^{q}$, $q\geq 1$ is s-convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\&\quad\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\right|\\&\leq\left(\frac{2}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\right)^{1-\frac{1}{q}}\left[|f'(\lambda a+(1-\lambda)b)|^{q}+|f'(\lambda b+(1-\lambda)a)|^{q}\right]^{\frac{1}{q}} \quad (6)\\&\quad\times\left[B_{1\setminus 2}(s+1,\alpha+1)-\frac{1}{2^{\alpha+s}(\alpha+s+1)}+\frac{1}{\alpha+s+1}-B_{1/2}(\alpha+1,s+1)\right]^{\frac{1}{q}}\end{aligned}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$, $\alpha>0$, and B_x is the incomplete beta function defined as follows

$$B_x(m,n)=\int_0^x t^{m-1}(1-t)^{n-1},\quad m,n>0,\ 0<x\leq 1. \quad (7)$$

Proof. Firstly, we suppose that $q=1$. Using Lemma 2.1 and s-convexity of $|f'|^{q}$, we find that

$$\begin{aligned}&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\&\qquad\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\right|\\&\leq\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt\\&\leq\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left[t^{s}\left|f'(\lambda a+(1-\lambda)b)\right|+(1-t)^{s}\left|f'(\lambda b+(1-\lambda)a)\right|\right]dt\\&=\left|f'(\lambda a+(1-\lambda)b)\right|\left[\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}.t^{s}-t^{\alpha+s}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha+s}-(1-t)^{\alpha}.t^{s}\right]dt\right]\end{aligned}$$

$$+\left|f'(\lambda b+(1-\lambda)a)\right|\left[\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha+s}-t^{\alpha}(1-t)^{s}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}(1-t)^{s}-(1-t)^{\alpha+s}\right]dt\right]$$

$$=\left[B_{1/2}(s+1,\alpha+1)-B_{1/2}(\alpha+1,s+1)-\frac{1}{2^{\alpha+s}(\alpha+s+1)}+\frac{1}{\alpha+s+1}\right]$$
$$\times\left[\left|f'(\lambda a+(1-\lambda)b)\right|+\left|f'(\lambda b+(1-\lambda)a)\right|\right]$$

Secondly, we suppose that $q>1$. Using Lemma 2.1 and power mean inequality, we obtain

$$\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt$$
$$\leq\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right)^{1-\frac{1}{q}} \tag{8}$$
$$\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^{q}dt\right)^{\frac{1}{q}}.$$

Hence, using s-convexity of $|f'|^q$ and (8) we obtain

$$\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.$$
$$\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^{q}dt\right)^{\frac{1}{q}}$$
$$\leq\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left[t^{s}\left|f'(\lambda a+(1-\lambda)b)\right|^{q}+(1-t)^{s}\left|f'(\lambda b+(1-\lambda)a)\right|^{q}\right]dt\right)^{\frac{1}{q}}$$
$$\leq\left(\frac{2}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\right)^{1-\frac{1}{q}}\left[\left|f'(\lambda a+(1-\lambda)b)\right|^{q}+\left|f'(\lambda b+(1-\lambda)a)\right|^{q}\right]^{\frac{1}{q}}$$
$$\left[B_{1/2}(s+1,\alpha+1)-B_{1/2}(\alpha+1,s+1)-\frac{1}{2^{\alpha+s}(\alpha+s+1)}+\frac{1}{\alpha+s+1}\right]^{\frac{1}{q}}.$$

This completes the proof. □

Remark 2.1. If we take $s = 1$ in Theorem 2.2, then Theorem 2.2 reduces to Theorem 3 which is proved by Sarikaya and Budak in [22].

Remark 2.2 (Trapezoid Inequality)**.** If we take $s = 1,\ \alpha = 1$ and $\lambda = 0$ (or $\lambda = 1$) in Theorem 2.2, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}2^{\frac{q-1}{q}}\left[|f'(a)|^q + |f'(b)|^q\right]^{\frac{1}{q}}$$

where $q \geq 1$. Choosing $q = 1$ in last inequality, it follows that

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[|f'(a)| + |f'(b)|\right]$$

which are proved by Dragomir and Agarwal in [13].

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $|f'|^q$ s-convex on $[a, b]$ for same fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\
&\qquad\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b) + J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\
&\leq \left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q + |f'(\lambda b+(1-\lambda)a)|^q}{s+1}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\dfrac{1}{p}+\dfrac{1}{q} = 1$, $\alpha > 0$ *and* $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$.

Proof. Using Lemma 2.1, s-convexity of $|f|^q$ and well-known Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\
&\qquad\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b) + J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\
&\leq \int_0^1 |(1-t)^{\alpha} - t^{\alpha}|\,|f'\left[t(\lambda a+(1-\lambda)b) + (1-t)(\lambda b+(1-\lambda)a)\right]|\,dt\\
&\leq \left(\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'\left[t(\lambda a+(1-\lambda)b) + (1-t)(\lambda b+(1-\lambda)a)\right]|^q\,dt\right)^{\frac{1}{q}}\\
&\leq \left(\int_0^{\frac{1}{2}} [(1-t)^{\alpha} - t^{\alpha}]^p\,dt + \int_{\frac{1}{2}}^1 [t^{\alpha} - (1-t)^{\alpha}]^p\,dt\right)^{\frac{1}{p}}\\
&\qquad\times\left(\int_0^1 t^s\,|f'(\lambda a+(1-\lambda)b)|^q + (1-t)^s\,|f'(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$= \left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q+|f'(\lambda b+(1-\lambda)a)|^q}{s+1}\right)^{\frac{1}{q}}.$$

Here, we use

$$(c-d)^p \leq c^p - d^p,$$

for any $c > d \geq 0$ and $p \geq 1$. □

Remark 2.3. If we take $s = 1$ in Theorem 2.3, then Theorem 2.3 reduces to Theorem 4 which is proved by Sarikaya and Budak in [22].

Remark 2.4 (Trapezoid Inequality)**.** If we take $s = 1$, $\alpha = 1$ and $\lambda = 0$ (or $\lambda = 1$) in Theorem 2.3, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left[\frac{2}{p+1}\left(1-\frac{1}{2^p}\right)\right]^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

which are proved by Dragomir and Agarwal in [13].

Theorem 2.4. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $|f'|^q$ is a s-convex on $[a, b]$ for same fixed $q \geq 1$,then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\
&\qquad\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\
&\leq \left[B_{1/2}\left(\alpha q+1,s+1\right)-B_{1\backslash 2}\left(s+1,\alpha q+1\right)+\frac{1}{\alpha q+s+1}\left(1-\frac{1}{2^{\alpha q+s}}\right)\right]^{\frac{1}{q}}\\
&\qquad\times\left(|f'(\lambda a+(1-\lambda)b)|^q+|f'(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}}
\end{aligned}$$

where $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$ and $\alpha > 0$.

Proof. Using Lemma 2.1, s-convexity of $|f'|^q$, and well-known Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\
&\qquad\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\
&\leq \int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]|\,dt\\
&\leq \left(\int_0^1 1^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|^q\,|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]^q|\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg(\int_0^{\frac{1}{2}} \left[(1-t)^{\alpha} - t^{\alpha}\right]^q \left| f'\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right]\right|^q dt \\
&\quad + \int_0^{\frac{1}{2}} \left[t^{\alpha} - (1-t)^{\alpha}\right]^q \left| f'\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right]\right|^q dt \Bigg)^{\frac{1}{q}} \\
&\leq \Bigg(|f'(\lambda a + (1-\lambda)b)|^q \int_0^{\frac{1}{2}} \left[(1-t)^{q\alpha} t^s - t^{q\alpha+s}\right] dt \\
&\quad + |f'(\lambda b + (1-\lambda)a)|^q \int_0^{\frac{1}{2}} \left[(1-t)^{q\alpha+s} - t^{q\alpha}(1-t)^s\right] dt \\
&\quad + |f'(\lambda a + (1-\lambda)b)|^q \int_{\frac{1}{2}}^1 \left[t^{q\alpha+s} - (1-t)^{q\alpha} t^s\right] dt \\
&\quad + |f'(\lambda b + (1-\lambda)a)|^q \int_{\frac{1}{2}}^1 \left[t^{q\alpha}(1-t)^s - (1-t)^{q\alpha+s}\right] dt \Bigg)^{\frac{1}{q}} \\
&= \left[B_{1/2}(\alpha q + 1, s+1) - B_{1\backslash 2}(s+1, \alpha q + 1) + \frac{1}{\alpha q + s + 1} - \frac{1}{2^{\alpha q + s}(\alpha q + s + 1)} \right]^{\frac{1}{q}} \\
&\quad \times \left[|f'(\lambda a + (1-\lambda)b)| + |f'(\lambda b + (1-\lambda)a)| \right]^{\frac{1}{q}}.
\end{aligned}
$$

Here, we use $(A-B)^p \leq A^p - B^p$, for any $A > B \geq 0$ and $q \geq 1$. □

Remark 2.5. If we take $s = 1$ in Theorem 2.4, the Theorem 2.4 reduces the Theorem 5 which is proved by Sarikaya and Budak in [22].

Remark 2.6 (Trapezoid Inequality)**.** If we take $s = 1$, $\alpha = 1$ and $\lambda = 0$ (or $\lambda = 1$) in Theorem 2.4, we have

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left[\frac{1}{q+1} \left(1 - \frac{1}{2^{q+1}} \right) \right]^{\frac{1}{q}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.
$$

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Sinc-Legendre collocation method for the non-linear Burgers' fractional equation

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Abstract. This paper deals with the numerical solution of the nonlinear fractional Burgers' equation. The fractional derivatives are described based on the Caputo sense. We construct the solution using different approach, that is based on using collocation techniques. The solution is based on using the Sinc method, which builds an approximate solution valid on the entire spatial domain, and in the time domain, we use the shifted Legendre polynomials to replace the time fractional derivatives. The error in the approximation is shown to converge to the exact solution at an exponential rate. Illustrative examples are given with an applications from traffic flow, and the numerical results are shown to demonstrate the efficiency of the newly proposed method.

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1. Introduction

Nonlinear partial differential equations appear in many branches of chemistry, physics, engineering and applied mathematics. The Burgers equation [6, 7], which is a nonlinear partial differential equation of second order, is used in disciplines as a simplified model for turbulence, boundary layer behavior, shock wave formation and mass transport. Due to the recent development of new modeling approaches, reaction-diffusion equations are the subject of new mathematical interest concerning chemical reactions and electro-chemistry of corrosion, it has turned that many phenomena in engineering, chemistry and other sciences can be described very successfully by models using mathematical tools from fractional calculus. For better understanding the phenomena that a given nonlinear fractional partial differential equation describes, the solutions of differential equations of fractional order is much involved. The fractional Burgers equation [11] describes the physical processes of unidirectional propagation of weakly nonlinear acoustic wave through a gas-filled pipe. Fractional derivatives provide more accurate models of real world problems than integer order derivatives do.

In recent years, there has been a growing interest in the field of fractional calculus. Oldham and Spanier [14], Miller and Ross [10], Momani [12, 13] and Podlubny [15] provide the history and a comprehensive treatment of this subject. Fractional calculus is the field of mathematical analysis, which deals with the investigation and applications of integrals and derivatives of arbitrary order, which can be real or complex. The idea appeared in a letter by Leibniz to L'Hospital in 1695. The subject of

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fractional calculus has gained importance during the past three decades due mainly to its demonstrated applications in different areas of physics and engineering. Several fields of applications of fractional differentiation and fractional integration are already well established, some others just started. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical systems, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, for more see [18] and the references therein. Indeed, it provides several potentially useful tools for solving differential equations. It is important to solve time fractional partial differential equations. It was found that fractional time derivatives arise generally as infinitesimal generators of the time evolution when taking along time scaling limit. Hence, the importance of investigating fractional equations arises from the necessity to sharpen the concepts of equilibrium, stability states, and time evolution in the long time limit. In general, there exists no method that yields an exact solution for nonlinear fractional partial differential equations. There has been some attempt to solve linear problems with multiple fractional derivatives. In [1], an approximate solution based on the decomposition method is given for the generalized fractional diffusion-wave equation. In [30], the authors used the Sinc-Legendre collocation method to a numerical solution for a class of fractional convection-diffusion equation. The survey paper [22] discusses the application of Sinc methods to fractional differential equations. In the present paper, we consider the fractional Burgers' equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \epsilon \frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t) \in (a,b) \times (0,T) \tag{1}$$

with the following initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \tag{2}$$

$$u(a,t) = \gamma(t), \quad u(b,t) = \delta(t), \quad t \geq 0 \tag{3}$$

where ϵ is the coefficient of the kinematic viscosity, T is the total time, and $u_0(x), \gamma(t)$, and $\delta(t)$ are given functions of the variables. We construct the solution of Equations (1)-(3) using a new approach. This approach is based on the use of collocation techniques. The main idea consists of reducing the solution of the problem to a set of algebraic equations by expanding the required solution as the elements of Legendre polynomials in time direction, and the Sinc function basis in the space direction. To the best of the author's knowledge, such approach has not been used for solving non-linear fractional partial differential equations. But, it has been used for linear fractional PDEs [30]. One important application of equation (1) when $\alpha = 1$, which was used as a simple model of turbulence in an extensive study by Burgers'. Regarding the velocity field of a fluid, the essential ingredient is the competition between the dissipative term, ϵu_{xx} the coefficient of which is the kinematic viscosity, and the nonlinear term uu_x. Equation (1), which appears as a mathematical model for many physical events, such as gas dynamics, turbulence and shock wave theory [28]. Many researchers have used various numerical methods to solve Burgers' equation [26, 25, 32, 31]. In [19], numerical solutions of Burgers equation defined by a new generalized time-fractional derivative are discussed. The approximate solution of time and/or space fractional Burgers equations are obtained by several methods, such as Adomian decomposition [11], variational iteration [9], homotopoy perturbation analysis [20]. Lund [27], uses Sinc-Galerkin method to find a numerical solution of the nonlinear advection-diffusion equation (Burgers' equation). The method results in an iterative scheme of an error of order $O(\exp(-c/h))$ for some positive constants c, h.

In [29], the Burgers' equation is transformed into an equivalent integral equation, and a Sinc-collocation procedure is developed for the integral equation. In [33] a comparison between Cole-Hopf transformation and the decomposition method is made for solving Burgers' equation. Fractional calculus has been used as a model for many physical processes, for this reason a reliable and efficient technique for the solution of nonlinear fractional differential equations is sought. We construct a solution of Equations (1)-(3) using a new approach. This approach is based on the use of collocation techniques. The main idea consists of reducing the solution of the problem to a set of algebraic equations by expanding the required solution as the elements of Legendre polynomials in time direction, and the Sinc function basis in the space direction. Many definitions and studies for the fractional calculus may be found in the literature. These definitions include, Riemman-Liouville, Weyl, Reize, Campos, Caputa, and Nishimoto fractional operator. The Riemann-Liouville definition of fractional derivative operator J_a^α which is defined in [23, 14, 16]. The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differentiation operator D^α proposed by Caputo's (see, [23]). Sinc function that will be used in this paper, are discussed in Stenger [29] and by Lund [27]. The paper is organized as follows: In section 2, we recall notations and definitions of the Sinc function, and derive some formulas that will be needed for developing our method. In section 3, we introduce some necessary definitions of the fractional calculus theory. In section 4, the fractional-order shifted Legendre functions and their properties are obtained. Section 5 is devoted to the solution of the fractional Burgers equation using the proposed method. In section 6, we apply the newly method to specific problems, compare the results, and the accuracy of the proposed schemes is demonstrated. Also, a conclusion is given in the last section.

2. Sinc function properties

The goal of this section is to recall notations and definitions of the Sinc function that will be used in this paper. These are discussed in [29, 27, 4]. Let f be a function defined on $\mathbb{R}$ and $h > 0$ a step size. Then Whittaker cardinal function is defined by the series $C(f, h, x) = \sum_{k=-\infty}^{\infty} f(kh)S(k, h)(x)$, whenever this series converges, and where the k-th Sinc function is defined as

$$S(k, h)(x) = \operatorname{sinc}\left[(x - kh)/h\right] = \frac{\sin[\pi(x - kh)/h]}{\pi(x - kh)/h}$$

The properties of Sinc functions have been extensively studied in [29, 27]. A comprehensive survey of these approximation properties is found in [2, 3]. Now, for positive integer N, define

$$C_N(f, h, x) = \sum_{k=-N}^{N} f(kh)S(k, h)(x) \tag{4}$$

Definition 2.1. Let $d > 0$, and let D_d denote the region $\{z = x + iy|y| < d\}$ in the complex plane $\mathbb{C}$, and ϕ the conformal map of a simply connected domain D in the complex plane domain onto D_d such that $\phi(a) = -\infty$ and $\phi(b) = \infty$, where a and b are the boundary points of D. Let ψ denote the inverse map of ϕ, and let the arc Γ, with end points a and b $(a, b \notin \Gamma)$, be given by $\Gamma = \psi(-\infty, \infty)$. For $h > 0$, let the points x_k on Γ be given by $x_k = \psi(kh), z \in Z$ and $\rho(z) = \exp(\phi(z))$.

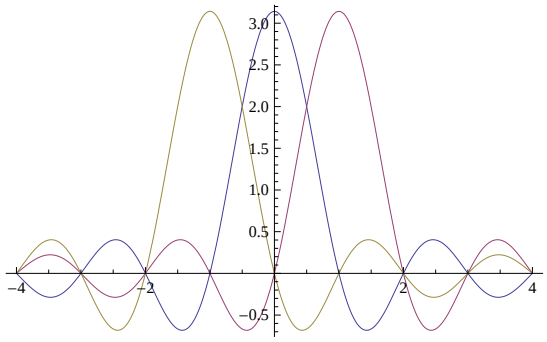

FIGURE 1. The $k-th$ Sinc function $S(k,h)(x), k=-1,0,1$ and $h=1$.

Hence, the numerical process developed in the domain containing the whole real line can be carried over to infinite interval by the inverse map.

Definition 2.2. Let $\mathbf{B}(D)$ be the class of functions f that are analytic in D and satisfy $\int_{\psi(L+u)}|F(z)dz|\to 0,\ u\to\mp\infty$ where $L=\{iy:|y|<d\le\pi/2\}$, and on the boundary ∂D satisfy $T(F)=\int_{\partial D}|f(z)dz|<\infty$. Corresponding to the number α, let $\mathbf{L}_\alpha(D)$ denote the family of all functions f that are analytic for which there exists a constant C_0 such that

$$|f(z)|\le C_0\frac{|\rho(z)|^\alpha}{[1+|\rho(z)|]^{2\alpha}},\ \forall z\in D.$$

By introducing the conformal map ϕ, and a "nullifier" function g the following theorem gives a formula for approximating the $m-$th derivatives of f on Γ. Let g be analytic function on D, and for $k\in\mathbb{Z}$, set

$$S_j(z)=g(z)\text{sinc}\left[\frac{\phi(z)-jh}{h}\right]=g(z)S(j,h)\circ\phi(z),\ z\in D$$

Theorem 2.1. [29, p. 208] *Let $\phi' f/g\in\mathbf{B}(D)$,*

$$\sup_{-\pi/h\le t\le\pi/h}\left|\left(\frac{d}{dx}\right)^n g(x)\exp(it\phi(x))\right|\le C_1h^{-n},\ x\in\Gamma,$$

for $n=0,1,2,...,m$, with C_1 a constant depending only on m,ϕ and g. If $f/g\in\mathbf{L}_\alpha(D)$, α a positive constant, then taking $h=\sqrt{\pi d/(\alpha N)}$ it follows that

$$\sup_{x\in\Gamma}\left|f^{(n)}(x)-\sum_{j=-N}^{N}\frac{f(x_j)}{g(x_j)}S_j^{(n)}(x)\right|\le C_2N^{\frac{n+1}{2}}\exp(-\sqrt{\pi d\alpha N})$$

for $n=0,1,...,m$, with C_2 a constant depending only on m,ϕ,g,d,α and f.

The approximation of the $m-$th derivative of f in Theorem 2.1 is simply an $m-$th derivative of of $C_N(f/g,h,x)$ in equation (4). The weight function g is chosen relative to the order of the derivative that is to be approximated. For instance, to approximate the $m-$th derivative, the choice $g(x)=1/(\phi'(x))^m$ is often suffice. The Sinc method requires that the derivatives of Sinc functions be evaluated at the nodes. Technical calculations provide the following results that will be useful in formulating the discrete system [29, 27], and these quantities are delineated by

$$\delta_{jk}^{(q)}=h^q\frac{d^q}{d\phi^q}[S_j\circ\phi(x)]\Big|_{x=x_k},q=0,1,2.$$

In particular, the following convenient notation will be useful in formulating the discrete system

$$\delta_{jk}^{(0)} = [S(j,h)\circ\phi(x)]\Big|_{x=x_k} = \begin{cases} 1, & j=k \\ 0, & j\neq k, \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]\Big|_{x=x_k} = \begin{cases} 0, & j=k \\ \frac{(-1)^{k-j}}{(k-j)}, & j\neq k \end{cases}$$

and,

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]\Big|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j=k \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j\neq k \end{cases}$$

So the approximation of a function $f(x)$ by Sinc expansion is given by

$$f(x)\approx f_N(x) = \sum_{j=-N}^{N}\frac{f(x_j)}{g(x_j)}S_j(x). \tag{5}$$

To approximate the $k-$th derivative of $f(x)$, we solve the linear system of equations

$$\int_a^b f_N^{(k)}(x)\frac{S(k,h)\circ\phi(x)}{\phi'(x)}dx = 0,\ k=-N,...,N$$

Integration by parts to change integrals involving derivatives of f_N into integrals involving f_N, so the approximation of the first and second derivatives at the Sinc nodes x_k takes the form

$$f'(x_k) = \sum_{j=-N}^{N}\left\{\frac{\delta_{jk}^{(1)}}{h}+\delta_{jk}^{(0)}g'(x_j)\right\}\frac{f(x_j)}{g(x_j)}+E_1 \tag{6}$$

and,

$$f''(x_k) = \sum_{j=-N}^{N}\left\{\frac{\delta_{jk}^{(2)}}{h^2}+h\delta_{jk}^{(1)}\Big[\frac{\phi''(x_k)}{(\phi'(x_k))^2}\Big]\right\}\frac{f(x_j)}{g(x_j)}+E_2 \tag{7}$$

where $E_1 = O(N\exp(-\sqrt{\pi d\alpha N}))$ and $E_2 = O(N^{3/2}\exp(-\sqrt{\pi d\alpha N}))$. The approximations (6), (7) are more conveniently recorded by defining the vector $\vec{f} = (f_{-N},....,f_0,....,f_N)^T$. Then define the $m\times m, (m=2N+1)$ Toeplitz matrices $I_m^{(q)} = [\delta_{jk}^{(q)}], q=0,1,2.$ i.e., the matrix whose $jk-$ entry is given by $\delta_{jk}^{(q)}, q=0,1,2.$ Also define the diagonal matrix $D(g) = \text{diag}\,[g(x_{-N}),...,g(x_N)]$. Note that the matrix $I^{(2)}$ is a symmetric matrix, i.e., $I_{jk}^{(2)} = I_{kj}^{(2)}$. The matrix $I^{(1)}$ is skew-symmetric matrix, i.e., $I_{jk}^{(1)} = -I_{kj}^{(1)}$ and they take the form

$$I_m^{(2)} = \begin{pmatrix} \frac{-\pi^2}{3} & 2 & \vdots & \frac{(-1)^{m-1}}{m-1} \\ 2 & \cdot & \cdot & \cdot \\ \vdots & & \cdot & \vdots \\ \frac{(-1)^{m-1}}{m-1} & \vdots & 2 & \frac{-\pi^2}{3} \end{pmatrix},\quad I_m^{(1)} = \begin{pmatrix} 0 & -1 & \dots & \frac{(-1)^{m-1}}{m-1} \\ 1 & 0 & & \vdots \\ \vdots & & & \vdots \\ \frac{(-1)^{m-1}}{m-1} & \dots & 1 & 0 \end{pmatrix} \tag{8}$$

While the matrix $I^{(0)}$ is an identity matrix. For the present paper the interval Γ in Theorem 2.1 is (a,b). Therefore, to approximate the first derivative, we take

$\phi(x) = \ln(\frac{x-a}{b-x})$, and $g(x) = 1/\phi'(x)$. Then the approximation of the first derivative evaluated at the vector nodes x_j can be written as

$$\vec{f'}(x_j) \approx \Big[\frac{-1}{h} I_m^{(1)} D(\phi') + I_m^{(0)} D(\phi''/\phi')\Big]\vec{f}(x_j) \equiv A\vec{f}(x_j) \tag{9}$$

and for the second derivative takes the form

$$\vec{f''}(x_j) \approx \Big[\frac{1}{h^2} I_m^{(2)} + h I_m^{(1)} D(\phi'/\phi'^2)\Big]\vec{f}(x_j) \equiv B\vec{f}(x_j) \tag{10}$$

3. Basic Definition of Fractional Calculus

This section is devoted to a description of the operational properties of the purpose of acquainting with sufficient fractional calculus theory, to enable us to follow the solution of the fractional Burgers equation. Many definitions and studies of fractional calculus have been proposed in the last two centuries. These definitions include, Riemman-Liouville, Weyl, Reize, Campos, Caputa, and Nishimoto fractional operator. Mainly, in this paper, we will re-introduce section 2 of [1]. The Riemann-Liouville definition of fractional derivative operator J_a^α is defined as follows:

Definition 3.1. Let $\alpha \in \mathbb{R}_+$. The operator J^α, defined on the usual Lebesque space $L_1[a,b]$ by

$$J_a^\alpha f(x) = \tfrac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

$$J_a^0 f(x) = f(x)$$

for $a \le x \le b$, is called the *Riemann-Liouville fractional integral operator of order* α.

Properties of the operator J^α can be found in [16], we mention the following: For $f \in L_1[a,b], \alpha, \beta \ge 0$ and $\gamma > -1$

(1) $J_a^\alpha f(x)$ exists for almost every $x \in [a,b]$
(2) $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x)$
(3) $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x)$
(4) $J_a^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}$.

As mentioned in [11], the Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce now a modified fractional differentiation operator D^α proposed by Caputo in his work on the theory of viscoelasticity [23].

Definition 3.2. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{11}$$

$m-1 < \alpha \le m, m \in \mathbb{N}, x > 0$.

Also, we need here two of its basic properties.

Lemma 3.1. *If* $m-1 < \alpha \le m$, *and* $f \in L_1[a,b]$, *then* $D_a^\alpha J_a^\alpha f(x) = f(x)$, *and*

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^-) \frac{(x-a)^k}{k!},\ x > 0.$$

The Caputo fractional derivative is considered in the Caputo sense. The reason for adopting the Caputo definition is as follows [11]. To solve differential equations, we need to specify additional conditions in order to produce an unique solution. For the

case of Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are taken to those of classical differential equations, and are therefore familiar to us. In contrast, for Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives of the unknown solution at the initial point $x = 0$, which are functions of x. The unknown function $u = u(x, t)$ is assumed to be a causal function of time, i.e., vanishing for $t < 0$. Also, the initial conditions are not physical; furthermore, it is not clear how much quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details on the geometric and physical interpretation for fractional derivatives of both Riemann-Liouville and Caputo types see [23, 11].

Definition 3.3. For m to be the smallest integer that exceeds α, the Caputo fractional derivatives of order $\alpha > 0$ are defined as

$$D^{\alpha}u(x,t) = \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial \tau^m}d\tau, & m-1<\alpha<m \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in \mathbb{N} \end{cases}$$

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

4. Shifted-Legendre Polynomials

Legendre polynomials $\ell_k(t), k = 0, 1, 2, ...$ are the eigenfunctions for the Sturm-Liouville problem

$$\frac{d}{dt}[(1-t^2)\ell_k'(t)] + k(k+1)\ell_k(t) = 0,\ t \in [-1, 1]$$

that are orthogonal in $[-1, 1]$, and satisfy the orthogonality property

$$\int_{-1}^{1} \ell_i(t)\ell_j(t)dt = \frac{2}{2i+1}\delta_{ij} = \begin{cases} \frac{2}{2i+1}, & i = j \\ 0, & i \neq j, \end{cases}$$

and the difference equation

$$\ell_{i+1}(t) = \frac{2i+1}{i+1}t\ell_k(t) - \frac{i}{i+1}\ell_{i-1}(t),\ i \geq 1,$$

where $\ell_0(t) = 1$, and $\ell_1(t) = t$. In order to use Legendre polynomials in the interval $[0, T]$, we define the shifted Legendre polynomials, $P_i(z) = \ell(\frac{2t}{T} - 1)$, so that the new Legendre polynomials $P_i(z)$ satisfy the orthogonality condition

$$\int_0^T P_i(z)P_j(z)dz = \frac{T}{2i+1}\delta_{ij} = \begin{cases} \frac{T}{2i+1}, & i = j \\ 0, & i \neq j, \end{cases}$$

The analytic closed form of the shifted Legendre polynomials of degree i is given by

$$P_i(t) = \sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{t^k}{(k!)^2T^k}.$$

Note that $P_i(T) = 1$ and $P_i(0) = (-1)^i$. To approximate a function $u(t)$, that is square integrable in $[0, T]$, using the first $(i+1)-$terms shifted Legendre polynomials,

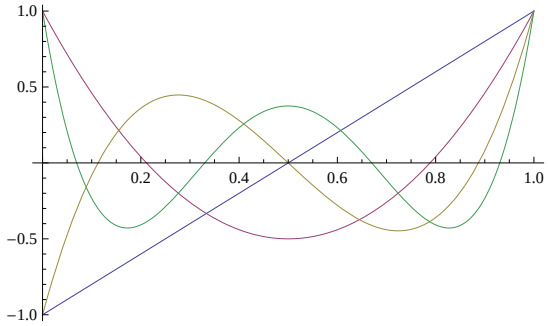

FIGURE 2. Shifted Lengendre Polynomials $P_1(t)$ till $P_4(t)$. when $T = 1$

we may use

$$u_i(t) = \sum_{j=0}^{i} c_j P_j(t) = C^T \Phi(t) \tag{12}$$

where the shifted Legendre coefficient vector C is given by $C^T = [c_0, c_1, ..., c_i]$. While the shifted Legendre vector $\Phi(t)$ is $\Phi(t) = [P_0(t), P_1(t), ..., P_i(t)]^T$. In equation (12), the coefficients c_j can be calculated by

$$c_j = \frac{2j+1}{T} \int_0^T u(t) P_j(t) dt, \ j = 1, 2, ...$$

One of the common and efficient methods for solving fractional partial differential equations of order $\alpha > 0$ is to use the shifted Legendre polynomials. The derivative of order $\alpha > 0$ is given by

$$D^\alpha u_i(t) = D^\alpha \sum_{j=0}^{i} c_j P_j(t)$$

Using the fact that Caputo's fractional differentiation is a linear operator, we get

$$D^\alpha u_i(t) = \sum_{j=0}^{i} c_j D^\alpha P_j(t) = \sum_{j=0}^{\lceil \alpha \rceil - 1} c_j D^\alpha P_j(t) + \sum_{j=\lceil \alpha \rceil}^{i} c_j D^\alpha P_j(t) \tag{13}$$

with $\lceil \alpha \rceil$ denoting the integer part of α. Recalling that, for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order. Therefore, the first sum in equation (13) vanishes, while in the second sum, we use property 4 in Definition 3.1 to arrive to the formula that give us the derivative of order α in Caputo sense for $u_i(t)$ is

$$D^\alpha(u_i(t)) = \sum_{j=\lceil \alpha \rceil}^{i} \sum_{k=\lceil \alpha \rceil}^{j} c_j \ b_{j,k}^{(\alpha)} \ t^{k-\alpha}, \ \text{ where } \ b_{j,k}^{(\alpha)} = \frac{(-1)^{j+k}(j+k)!}{(j-k)!(k!)\Gamma(k-\alpha+1)T^k}. \tag{14}$$

Example 4.1. Consider the case when $i = 2$, and $\alpha = 1/2$, for the function $u(t) = t^2$ Using the shifted Legendre series (12), we have $t^2 = \frac{1}{3}P_0(t) + \frac{1}{2}P_1(t) + \frac{1}{6}P_2(t)$. Using equation (14), we obtain

$$D^{\frac{1}{2}}(t^2) = \sum_{j=1}^{2} \sum_{k=1}^{j} c_j \ b_{j,k}^{(\frac{1}{2})} \ t^{k-\frac{1}{2}} = c_1 \ b_{1,1}^{(\frac{1}{2})} t^{\frac{1}{2}} + c_2 \ b_{2,1}^{(\frac{1}{2})} t^{\frac{1}{2}} + c_2 \ b_{2,2}^{(\frac{1}{2})} t^{\frac{3}{2}}$$

$$= \frac{1}{2}\frac{2}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} + \frac{1}{6}\frac{-6}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} + \frac{1}{6}\frac{12}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} = \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}.$$

It has been proved in [21] that the fractional derivative for the shifted Legendre polynomials can be approximated by $D^{(\alpha)}\Phi(t)$, where $D^{(\alpha)}$ is the $(i+1)\times(i+1)$ matrix of fractional derivative of order $\alpha > 0$ in the Caputo sense, and is defined as

$$D^{(\alpha)} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,1,k} & \dots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,i,k} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,1,k} & \dots & \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,i,k} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,1,k} & \dots & \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,i,k} \end{pmatrix} \tag{15}$$

where $\theta_{m,j,k}$ is given by

$$\theta_{m,j,k} = (2j+1)\sum_{\ell=0}^{j}\frac{(-1)^{m+j+k+\ell}(m+k)!(\ell+j)!}{(m-k)!\Gamma(k-\alpha+1)(\ell!)^2(k+\ell-\alpha+1)}.$$

For example, if $i = 4$, the operational matrix derivative of order $\alpha = 1/2$ in the Caputo sense is given by

$$D^{(\frac{1}{2})} = \sqrt{\pi}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -3+\frac{6}{\pi} & \frac{6}{\pi} & 0 & 0 & 0 \\ 11-\frac{30}{\pi} & \frac{15}{2}-\frac{30}{\pi} & \frac{5}{2} & 0 & 0 \\ -45+\frac{410}{3\pi} & \frac{-105}{2}+\frac{174}{\pi} & \frac{-35}{2}+\frac{140}{3\pi} & \frac{28}{3\pi} & 0 \end{pmatrix}$$

We may convert the fractional Burgers' equation to an integral equation. The following Theorem has been proved in [5], and will be used to solve the obtained integral equation.

Theorem 4.1. [5] *Let $\Phi(t)$ be a shifted Legendre polynomial, then $I^{\nu}\Phi(t) \approx A^{\nu}\Phi(t)$, where A^{ν} is the $(i+1)\times(i+1)$ operational matrix of integration of order ν Riemann-Liouville sense and is defined as follows*

$$A^{\nu} = \begin{pmatrix} \sum_{k=0}^{0} \theta_{0,0,k} & \sum_{k=0}^{0} \theta_{0,1,k} & \cdots & \sum_{k=0}^{0} \theta_{0,i,k} \\ \sum_{k=0}^{1} \theta_{1,0,k} & \sum_{k=0}^{1} \theta_{1,1,k} & \cdots & \sum_{k=0}^{1} \theta_{1,i,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=0}^{m} \theta_{m,0,k} & \sum_{k=0}^{m} \theta_{m,1,k} & \cdots & \sum_{k=0}^{m} \theta_{m,i,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=0}^{i} \theta_{i,0,k} & \sum_{k=0}^{i} \theta_{i,1,k} & \cdots & \sum_{k=0}^{i} \theta_{i,i,k} \end{pmatrix} \tag{16}$$

where $\theta_{m,j,k}$ is given by

$$\theta_{m,j,k} = (2j+1)\sum_{\ell=0}^{j} \frac{(-1)^{m+j+k+\ell}(m+k)!(\ell+j)!}{(m-k)!\; k!\; (k+\alpha+1)(j-\ell)!\; (\ell!)^2(k+\ell+\alpha+1)}.$$

5. Analysis of the Method

The concern of the existence and uniqueness of the solution to fractional Burgers' equations has been discussed in [17, 24] by using Banach fixed point theorem. To solve the problem in equation (1), we use our approximate solution

$$u_{N,n}(x,t) = \sum_{i=-N}^{N}\sum_{j=0}^{n} c_{ij} S_i(x) P_j(t), \tag{17}$$

where $S_i(x)$ is the Sinc basis function, and $P_j(t)$ are the $(n+1)$ shifted Legendre polynomials. The unknown coefficients $\{c_{ij}\}$ are determined by collocation scheme. For sake of simplicity, we assume that $\gamma(t) = \delta(t) = 0$ in equation (3). To solve equation (1), substituting equation (17) into equation (1), we obtain

$$\frac{\partial^{\alpha} u_{N,n}(x,t)}{\partial t^{\alpha}} + u_{N,n}(x,t)\frac{\partial u_{N,n}(x,t)}{\partial x} = \epsilon \frac{\partial^2 u_{N,n}(x,t)}{\partial x^2} \tag{18}$$

A collocation scheme can be defined by evaluating equation (18) at the Sinc nodes, $x_k = \phi^{-1}(kh),\ k = 0, \mp 1, \mp 2, ...$ and the shifted Legendre roots $t_{\ell},\ \ell = 1, 2, ..., n+1$ of $P_{n+1}(t)$. So, for $0 < \alpha \leq 1$, we have

$$\frac{\partial^{\alpha} u_{N,n}(x_k,t)}{\partial t^{\alpha}} = \sum_{i=-N}^{N}\sum_{j=0}^{n} c_{ij} S_i(x_k) D^{\alpha} P_j(t)$$

Using equation (14), together with the fact $\delta_{ik}^{(0)} = 0$, if $i \neq k$, we arrive at

$$\frac{\partial^{\alpha} u_{N,n}(x_k,t)}{\partial t^{\alpha}} = \sum_{i=-N}^{N}\sum_{j=1}^{n}\sum_{s=1}^{j} c_{ij}\; b_{j,s}^{(\alpha)} S_i(x_k) t^{s-\alpha} = \sum_{i=-N}^{N}\sum_{j=1}^{n}\sum_{s=1}^{j} c_{ij}\; b_{j,s}^{(\alpha)} \delta_{ik}^{(0)} t^{s-\alpha} \tag{19}$$

$$= \sum_{j=1}^{n} \sum_{s=1}^{j} c_{kj}\ b_{j,s}^{(\alpha)} t^{s-\alpha}.$$

While,

$$\frac{\partial^2 u_{N,n}(x_k,t)}{\partial x^2} = \sum_{i=-N}^{N} \sum_{j=0}^{n} c_{ij}\ \frac{d^2 S_i(x)}{dx^2}\Big|_{x=x_k} P_j(t)$$

Employing equation (7), we get

$$\frac{\partial^2 u_{N,n}(x_k,t)}{\partial x^2} = \sum_{i=-N}^{N} \sum_{j=0}^{n} c_{ij}\ \left\{ \frac{\delta_{jk}^{(2)}}{h^2} + h\delta_{jk}^{(1)} \Big[\frac{\phi''(x_k)}{(\phi'(x_k))^2} \Big] \right\} \frac{1}{g(x_j)} P_j(t) \tag{20}$$

For the first derivative u_x, employing equation (6) we obtain

$$\frac{\partial u_{N,n}(x_k,t)}{\partial x} = -\sum_{i=-N}^{N} \sum_{j=0}^{n} c_{ij}\ \left\{ \frac{\delta_{jk}^{(1)}}{h} + \delta_{jk}^{(0)} g'(x_k) \right\} \frac{1}{g(x_k)}\ P_j(t) \tag{21}$$

Finally, for the non-linear term uu_x, we have

$$u_{N,n}(x_k,t) \frac{\partial u_{N,n}(x_k,t)}{\partial x} = \frac{1}{2} \sum_{i=-N}^{N} \sum_{j=0}^{n} c_{ij}\ \left\{ \frac{\delta_{jk}^{(1)}}{h} + \delta_{jk}^{(0)} g'(x_k) \right\}^2 \frac{1}{g(x_k)}\ P_j(t) \tag{22}$$

Also, from the initial condition in equation (2), we obtain

$$\sum_{i=-N}^{N} \sum_{j=0}^{n} c_{ij} S_i(x) P_j(0) = u_0(x). \tag{23}$$

To obtain a matrix representation of the above equations, and using the matrices in (9), (10) and (15), with the notation $U = [u(x_i,t_j)], U^0 = [u_0(x_i,0)]$, we arrive at the following discrete system

$$D^{(\alpha)} U + U \circ AU = \epsilon BU \tag{24}$$

where the symbol " $\circ$" means the Hadamard matrix multiplication. Then the number of unknown coefficients c_{ij} is equal to $(n+1)(2m+1)$ and can be obtained by solving equations (24) and (23) by a fixed point iteration, consequently, the approximate solution $u_{N,n}(x,t)$ can be calculated. An alternative solution is to convert the fractional Burgers' equation to an integral equation by operating with J^α on both sides of equation (1), and using Lemma 3.1, we get

$$u(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+) \frac{t^k}{k!} + J^\alpha \Big(\epsilon \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) \frac{\partial u(x,t)}{\partial x} \Big) \tag{25}$$

Since $0 < \alpha \leq 1$, we choose $m = 1$ in equation (25). Using the operational matrix of integration for the shifted Legendre polynomials A^ν in equation (16), we arrive at the discrete system

$$U = [\epsilon U \circ B - \frac{1}{2} U \circ AU](A^\nu)^{-1} + U^0.$$

Note that the approximation for $u_x(x,t)$ and $u_{xx}(x,t)$ in matrix form has an exponential error E_1 and E_2 respectively. Thus the approximate solution will converge to the exact solution exponentially. The convergence proof of the solution for the discrete system can be done using fixed point Theory.

Remark 5.1. In equation (3), if $\gamma(t), \delta(t)$ are not both zero, then we may reformulate the problem to a homogeneous boundary conditions via the transformation $w(x,t) = u(x,t) + \frac{x-b}{b-a}\gamma(t) + \frac{a-x}{b-a}\delta(t)$.

6. Numerical Results

Here, we obtain some numerical results for the solutions of Burgers' equation. We use the parameters, $d = \pi/2, N = 8, n = 6$ to check the performance for the solution of the fractional Burgers' equation. The computations associated with the examples were performed using Mathematica and Maple.

Example 6.1. Consider the non-linear fractional Burgers' equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \epsilon \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 \le x \le 1,\ t > 0 \tag{26}$$

subject to the initial condition, $u(x,0) = c\Big[1 - \tanh(\frac{cx}{2\epsilon})\Big]$, and to the boundary conditions $u(0,t) = 0 = u(1,t)$. It is be noted that the value of $\alpha = 1$ is the only case for which we know the exact solution and has the closed form

$$u(x,t) = \frac{1}{10}\Big[1 - \tanh\Big(\frac{x - 0.1t}{20\epsilon}\Big)\Big] \tag{27}$$

In order to illustrate the approximate solution is efficient and accurate, we will give explicit values of the parameters t and α for fixed $x = 0.2$. The comparison of the

t	$\alpha = 1$	$\alpha = 0.99$	$\alpha = 0.1$	$\alpha = 0.001$
0.1	0.0905307	0.0905444	0.0941872	0.0949997
0.2	0.0910282	0.0910486	0.0944853	0.0950032
0.3	0.0915257	0.0915502	0.0946695	0.0950052
0.4	0.0920232	0.0920500	0.0948048	0.0950066
0.5	0.0925208	0.0925486	0.0949125	0.0950077

TABLE 1. The results obtained by equation (27) when $\alpha = 1$, and by equation (24) for various values of α when $x = 0.2, \epsilon = 0.1$.

t	$\alpha = 1$	$\alpha = 0.99$	$\alpha = 0.1$	$\alpha = 0.001$
0.1	0.0270809	0.0271703	0.0508952	0.0561874
0.2	0.0303211	0.0304540	0.0528370	0.0562098
0.3	0.0335614	0.0337205	0.0540368	0.0562230
0.4	0.0368017	0.0369760	0.0549181	0.0562325
0.5	0.0400419	0.0402232	0.0556194	0.0562395

TABLE 2. The results obtained by equation (27) when $\alpha = 1$, and by equation (24) for various values of α when $x = 0.2, \epsilon = 0.01$.

numerical solutions using the present method, and those obtained by using Equation (27) when $\alpha = 1$ are shown in Tables 1, 2, and also are depicted in Figures 3, 4. From the numerical solutions in Tables 1, 2, it can be seen that the exact solution ($\alpha = 1$) is quite close to the approximate solution when $\alpha = 0.99$. Also, it is observed that the values of the approximate solution at different α's have the same behavior as

those obtained using equation (27) for which $\alpha = 1$. This shows the approximate solution is efficient. In the theory of fractional calculus, it is obvious that when the fractional derivative $\alpha(m-1 < \alpha \leq m)$ tends to positive integer m, then the approximate solution *continuously* tends to the exact solution of the problem with derivative $m = 1$. A closer look at the values in Tables 1 and 2, we observe that our approach do have this characteristic. It can be seen from Figure 4 that the approximate solution when $\alpha = 0.999$ by the present method is nearly identical with the exact solution when $\alpha = 1$. In Figure 5, the comparison shows that as $\alpha \to 1$, the approximate solution tends to the exact solution in the case of $\alpha = 1$.

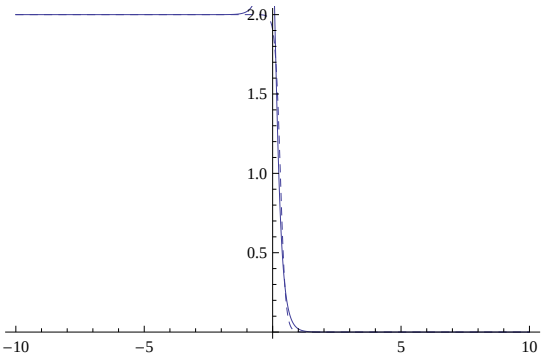

FIGURE 3. Dashed line $\alpha = 0.999$, Solid line $\alpha = 1$, and $\epsilon = 0.2$.

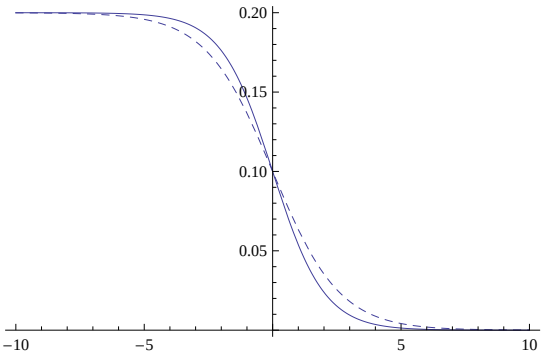

FIGURE 4. Dashed line $\alpha = 0.999$, Solid line $\alpha = 1$, and $\epsilon = 1$.

Figures 6 and 7 show the approximate solution for $\alpha = 0.5$ and $\alpha = 0.1$ respectively. Comparison of Figures 8 and 9 shows that the solution continuously depends on the fractional derivatives. Clear conclusion can be drawn from Figures $4 - 9$ that the solution tends to a finite number as $|x|$ approaches infinity for all values of α, which is in full agreement with the results in [11].

Example 6.2. As a second example, the classical Burgers equation often appears in traffic flow and gas dynamics [8]. The flow through porous media can be better described by fractional models than the classical ones, since they include inherently memory effects caused by obstacles in the structures. We consider once again the

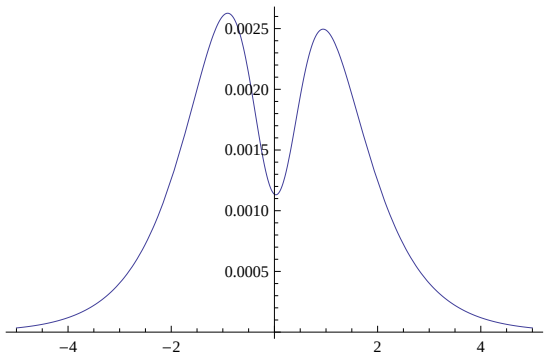

FIGURE 5. The difference between the exact solution ($\alpha = 1$), and the approximate solution for $t = 0.1, \epsilon = 0.1$, and $\alpha = 0.95$.

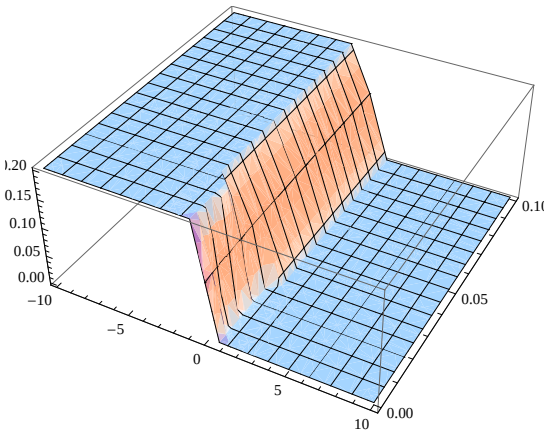

FIGURE 6. Approximate numerical solution $u(x,t)$ with $\alpha = 0.5$,and $\epsilon = 0.05$.

time-fractional Burgers equation (26). However, in this case we consider different initial conditions, where $u(x,0) = \sin(6\pi x) + 2x(1-x) - x\sin(6\pi)$, where $u(x,t)$ is the flow's velocity, and ϵ is the viscosity coefficient. It is revealed that the effect of the fractional derivative accumulates slowly to give rise to a significant dissipation. Numerical solutions for $u(x,t)$ are depicted in Figures 10 and 11 for two different values of α and $\epsilon = 0.3$. Comparing Figure 10 to Figure 11, we observe that as the fractional order increases, the rate of diffusion at the beginning time becomes slow.

Discussion and Conclusions

The Legendre Sinc-Collocation method appears to be very promising for solving the fractional Burgers' equation. An important advantage to be gained from the use of this method is the ability to produce very accurate results on a reasonably coarse mesh. For the assumptions considered, the resulting nonlinear system of algebraic equations was solved efficiently by fixed-point iteration. The example presented demonstrate the accuracy of the method, which is an improvement over current methods such as finite elements and finite difference methods. This feature shows the method to be an

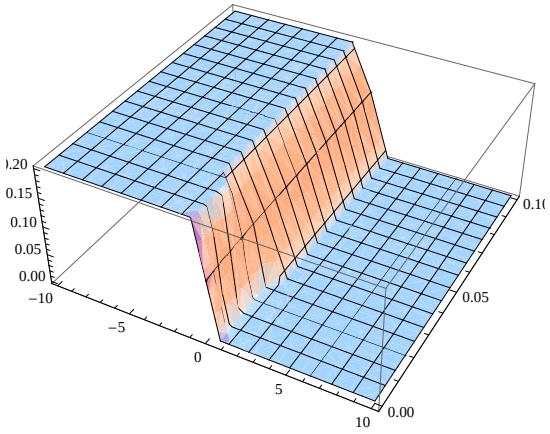

FIGURE 7. Approximate numerical solution $u(x,t)$ with $\alpha = 0.1$, and $\epsilon = 0.05$.

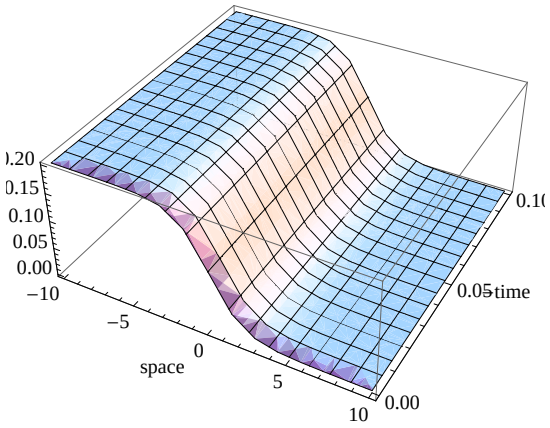

FIGURE 8. Approximate numerical solution $u(x,t)$ with $\alpha = 0.01$, and $\epsilon = 0.2$.

attractive for numerical solutions to the fractional Burgers' equation. We conclude, with confidence, that the collocation using Sinc basis can be considered as a beneficial method for solving a broad class of fractional nonlinear partial differential equations. The study of these equations will be the matter of furthers investigations.

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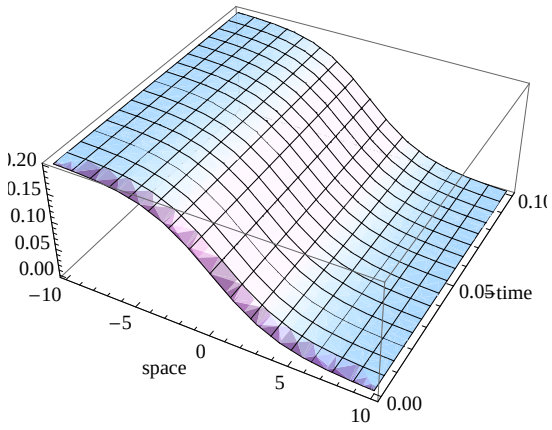

FIGURE 9. Approximate numerical solution $u(x,t)$ with $\alpha = 0.1$, and $\epsilon = 0.2$.

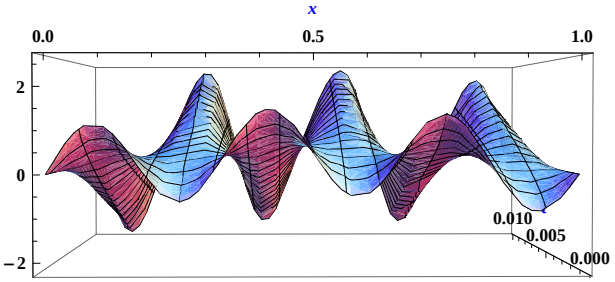

FIGURE 10. Approximate numerical solution $u(x,t)$ of Example 6.2 with $\alpha = 0.5$, and $\epsilon = 0.3$.

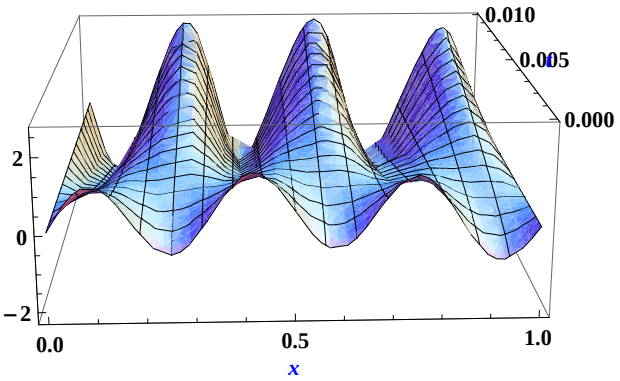

FIGURE 11. Approximate numerical solution $u(x,t)$ of Example 6.2 with $\alpha = 0.9$, and $\epsilon = 0.3$.

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Survey on distributed approaches to swarm intelligence for graph search problems

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Abstract. In this paper a survey on existing distributed approaches to swarm intelligence approaches for graph search problems is presented. In particular we reviewed papers on Ant Colony Optimization (ACO) and Bee Colony Optimization (BCO). The comparison criteria that have been used are computational efficiency and speedup. The conclusion of this study is that the coarse grained master-slave model is the most studied. However, we found a large amount of papers sustaining that the island approach offers better solution quality. The added flexibility of the communication strategy between the islands makes this model preferred by the most recent papers.

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Key words and phrases. swarm intelligence, distributed computing, ACO, BCO, parallel computing.

1. Introduction

Before going into the subject of this paper, we first introduce a few recurrent terms that will be used throughout the following sections.

SwarmIntelligence(*SI*) is the emergence of coherent functional global patterns from the collective behaviors of entities interacting locally [48]. Two families of SI algorithms are Ant Colony Optimization(ACO) and Bee Colony Optimization (BCO).

Entity is a general term used to refer to a unit of the Swarm Intelligence population, such as an ant, bee, particle, bird, fish etc.

Agraph is a set of objects and the pairwise relations between them. In [28] the objects are named vertices (singular: vertex) while the relations are called edges. Edges are actually considered to be connections between the vertices and are graphically represented as lines while vertices are represented as dots or ovals.

We define the term *computingnode* as an abstraction of a computer or machine, physical or virtual, having its own or allocated processor and memory. This term is used in the subject of distributed applications [65] and adopted by distributed computing support systems [106].

SI is inspired by natural bio systems consisting of populations of simple beings such as: ants, bees, birds, fish etc. The main characteristic of these is the inherently distributed way they solve problems. This suggests SI algorithms should allow straightforward mapping directly onto distributed software systems.

Based on our literature review we found that existing approaches rely on running multiple instances of sequential SI algorithms using parallel and highperformance

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computing architectures. SI computational approaches [48] are used to solve high complexity problems, such as NP-hard problems. These approaches are heuristic [47] by nature. Basically they provide an optimal solution, or one that is close to an optimum, to a high complexity problem in reasonable time. This is accomplished by using a population of entities that interact locally with each other and/or with the environment. There are two approaches to the interaction model:

1) the entities interact in the problem environment in order to guide each other to better solutions. For example, such a SI approach is Ant Colony Optimization [45], Bee Colony Optimization [116]

2) the entities interact in the solution space of the problem in order to improve existing initial solutions given as an output of an additional algorithm. For example, such a SI approach is Particle Swarm Optimization (PSO) [27], Cat Swarm Optimization (CSO) [24]. This type of SI is generally designed for continuous solution space that is given as input. Discrete adaptations exist [83], however, they require an additional algorithm, especially tailored to each problem to be solved, that outputs the vicinity of a solution which can be modeled as a graph vertex. Therefore this type of SI algorithm is not well suited for "graph search", the type of problem we are trying to solve.

There is a certain level of abstraction at which SI systems can be modeled as distributed computational systems composed of interacting artificial entities. Thus, we would expect distributed computing, including multiagent middleware, to have a lot of potential for the application of SI approaches.

The rest of this paper presents a survey on the subject of distributing SI and in particular ACO and BCO but also introduces the background knowledge required for the proper understanding of the said survey.

2. Graph Search Problems Formalization

SI algorithms using the first interaction model, mentioned previously in this paper, are very well suited for graph search problems. Consequently, these algorithms are usually tested on the benchmark problem of the NP-hard Traveling Salesman Problem (TSP), which falls in this category of problems. We now present a formal mathematical description of the problem type we are addressing. Let $G = (V, E)$ be a directed graph, where V is a set of vertices and E is a set of edges [59]. An edge is defined as an ordered pair of vertices (x, y). The size of the set of vertices is labeled $n = |V|$.

A path $p = (v1, v2, ..., vn), k > 0$ in the graph G is an ordered sequence of vertices such that for any two subsequent vertices vi and v_{i+1} there is an edge $(v_i, v_{i+1}) \in E$. We label the totality of paths in a graph with P. Any graph search problem can be formalized as minimizing or maximizing a function $f : P' \to \mathbb{R}$ where $P' \subseteq P$, i.e. P' is P restricted by some constraints. As an example, we now formalize the Traveling Salesman Problem (TSP). We first define a weight function w that assigns a real number to an edge of the graph. TSP can be formulated mathematically as the minimization of the function:

$$f : P' \to \mathbb{R}; f(p) = \sum_{i=1}^{n-1} w_{1,1+1} + w_{n,1} \tag{1}$$

where p is a path that is evaluated; $w_{i,i+1}$ is the weight of the edge $(i, i + 1)$; P' is the totality of Hamiltonian cycles.

Other examples of problems that can be described in this manner include: graph flow problems [59], generalized TSP [3], pickup and delivery problem [90] etc. Therefore each of these problems can be defined as the minimization of a function f over a subset of paths $P' \subseteq P$. An approach that can solve one of them can be adapted to solve all graph search problems.

2.1. Ant Colony Optimization(ACO). ACO [45] refers to a family of SI optimization algorithms that get their inspiration from the metaphor of real ants searching for food. When ants search for food, they secrete pheromone on their way back to the anthill. Other colony members sense the pheromone and become attracted by marked paths; the more pheromone is deposited on a path, the more attractive that path becomes.

The pheromone is volatile, i.e. disappears over time. Evaporation erases pheromone on longer paths as well as on the paths that are not of interest anymore. However, shorter paths are more quickly refreshed, thus having the chance of being more frequently explored. Intuitively, ants will converge towards the most efficient path, as that path gets the strongest concentration of pheromone.

In the SI approach called ACO, artificial ants are programmed to mimic the behavior of real ants while searching for food. The ants' environment is modeled as a graph and the path to the food becomes the solution to a given graph search problem. Artificial ants originate from the anthills that are vertices in the environment and travel between vertices to find optimal solutions, following ACO rules. When a solution is found, ants mark the solution with pheromone by retracing their path.

At the core of ACO algorithms there are rules that determine the amount of pheromone deposited on edges traversed by ants, the edge chosen by each ant on its way, and how fast the deposited pheromone evaporates. Ants randomly choose to travel across edges with a probability proportional to the pheromone-weight ratio. Ant a located at vertex i decides to move to vertex j with the probability $p_{i,j}$ computed as follows:

$$p_{i,j} = \frac{(\tau_{i,j})^{\alpha}(\eta_{i,j})^{\beta}}{\Sigma_j(\tau_{i,j})^{\alpha}(\eta_{i,j})^{\beta}} \tag{2}$$

where:

- α is a parameter to control the influence of $\tau_{i,j}$
- β is a parameter to control the influence of $\eta_{i,j}$
- j represents a node reachable from node i that was not visited yet
- $\tau_{i,j}$ is the amount of pheromone deposited on edge (i, j)
- $\eta_{i,j}$ is the desirability of edge (i, j) computed as the inverse of the edge weight, i.e. $1/w_{i,j}$
- N_i represents the set of neighbors of node i

Better solutions need to be marked with more pheromone. So whenever an ant k determines a new tour V_k of cost L_k the ant will increase pheromone strength on each edge of the tour with a value that is inversely proportional to the cost of the solution.

$$\Delta\tau_{i,j}^{k} = \begin{cases} 1/L_k \& \text{if edge } (i,j) \text{ belongs to found tour } V_k \\ 0 \& \text{otherwise} \end{cases} \tag{3}$$

where L_k is the cost of the k-th ant's tour.

As pheromone is volatile, if a real ant travels more, pheromone will have more time to evaporate, thus favoring better solutions to be discovered in the future. When an ant completes a solution it will retrace its steps marking the edges on the way with pheromone. The update will also take into account pheromone evaporation. Both

evaporation and pheromone updates are implemented as follows:

$$\tau_{i,j} = (1-\rho)\tau_{i,j} + \rho\Delta\tau_{i,j}^{k} \tag{4}$$

where ρ is the evaporation rate $0 \leq \rho < 1$.

All ants use formula 2 to probabilistically determine their next step. Therefore they will often choose the edge with the highest pheromone, i.e. the exploration of less probable edges is lower. The solution is to decrease the pheromone on edges chosen by ants, i.e. apply a local evaporation process. This has the effect of making them less desirable, increasing the exploration of the edges that have not been picked yet. Whenever an ant traverses an edge it applies local evaporation by updating pheromone as follows:

$$\tau_{i,j} = (1-\xi)\tau_{i,j} + \xi\tau_0 \tag{5}$$

where:

- ξ is the local evaporation rate $0 \leq \xi < 1$.
- τ_0 is the initial amount of pheromone on each edge

A good heuristics to initialize pheromone trails is to set them to a value slightly higher than the expected amount of pheromone deposited by the ants on a solution; a rough estimate of this value can be obtained by setting $\tau_0 = 1/(nC)$, where n is the number of nodes, and C is the tour cost generated by a reasonable solution approximation procedure [45]. For example we can set $C = nw_{avg}$ where w_{avg} is the average edge cost.

2.2. Bee Colony Optimization (BCO). Bee Colony refers to a family of SI optimization algorithms that get their inspiration from the metaphor of real bees searching for food. Bee colonies forage for pollen sources by moving randomly in the physical environment. When a bee finds a rich food source, starts to perform the so called "waggle dance" [46] upon its return to the hive. The purpose of this dance is to inform the other bees about the direction and distance to the food source [12]. The other bees from the swarm will then be inclined to explore the indicated location.

In the SI approach Bee Colony, artificial bees are programmed to mimic the behavior of real bees while searching for food. The bees environment is modeled as a graph while the path to the food becomes the solution to a given graph search problem. In [101] the authors present a survey of Bee Colony algorithms among them the basic "Bee Colony Optimization" (BCO) algorithm and its mathematical model which we will now detail. In BCO a bee randomly moves across edge (i,j) according to the transition probability formally defined in the following equation.

$$P_{i,j} = \frac{(p_{i,j})^{\alpha}(\eta_{i,j})^{\beta}}{\Sigma_j (p_{i,j})^{\alpha}(\eta_{i,j})^{\beta}} \tag{6}$$

where:

- p it the fitness function of edge (i,j)
- α is a parameter to control the influence of $p_{i,j}$
- β is a parameter to control the influence of $\eta_{i,j}$
- j represents a node reachable from node i that was not visited yet

The fitness of a particular edge is calculated according to the "preferred path" suggested by other bees that found a solution. The edge (μ,σ) in the "preferred path" that originates in the current vertex μ where the bee is located has the fitness $p_{\mu,\sigma} = \lambda$. The other edges originating have the fitness $p_{\mu,\sigma} = (1-\lambda)/\phi$ where ϕ is the number of unvisited vertices apart from the "preferred path" next vertex σ. Concretely, if the bee has to choose between the unvisited vertices $(\nu_1, \nu_2, ..., \nu_k, \sigma)$ the probabilities of choosing each vertex will be $(\frac{1-\lambda}{\phi}, \frac{1-\lambda}{\phi}, ..., \frac{1-\lambda}{\phi}, \lambda)$ respectively.

However, if the preferred vertex σ has already been visited or a "preferred path" does not exist yet, the fitness of the edge (μ, σ) automatically becomes null, i.e. $p_{\mu,\sigma} = 0$. Intuitively, the bees will explore variations of the best solutions while notifying the other bees about the improved solutions found using "waggle dance".

2.3. Using Local Search To Improve Solutions. "Local Search" [62] is an algorithm used to improve suboptimal solutions to graph search problems found by heuristic approaches. This algorithm chooses k distinct edges of the path representing the solution and tries to reorganize them in such a way that the quality of the solution improves. In this algorithm k is a natural number smaller than the solution size that is given as a parameter.

Depending on the number k, Local Search algorithms are called $2 - opt$, $3 - opt$ and so on up to $k - opt$. Basically, k edges from the solution are replaced such that a lower cost solution is generated. We will now present a very general version of the k-opt local search algorithm LOCAL-SEARCH(solution, k) where:

- "solution" is the path that the algorithm operates on,
- k is the number of considered edges,
- CHOOSE-EDGES(solution, k) randomly chooses the k distinct edges,
- REORGANIZE(edges) generates a new solution by reorganizing the k edges,
- SOL-ACCEPTABLE(solution) returns true if the solution is improved by an acceptable amount
- SOL-IMPROVED(solution) returns true when the solution is improved by reorganizing the edges,
- MAX-ITERATIONS() returns true is a predefined number of iterations have been executed with no improvement.

The complexity of this algorithm depends on the parameter k and how fast an improvement is achieved, however it has been experimentally shown that 2-opt local search performs just as well as 3-opt or higher values for k, while using a lot less CPU time [62]. Therefore there is little or no incentive to use a value higher than $k = 2$.

```
LOCAL-SEARCH(solution, k)
1.   initialSolution=solution
2.   for all groups of k edges of solution
3.          edges=CHOOSE-EDGES(solution, k)
4.          for all solution=REORGANIZE(edges)
5.                if SOL-ACCEPTABLE(solution)return solution
6.                if SOL-IMPROVED(solution)
7.                       initialSolution=solution
8.          if MAX-ITERATIONS() return initialSolution
```

Using local search has been shown to improve the performance of SI algorithms [45, 48, 111]. Therefore, SI approaches sometimes use 2-opt local search on solutions before "advertising" it by using "waggle dance", pheromone deposits etc.

2.4. Distributed Approach Performance Measures. During our review of the literature on the topic we discovered several distributing models (independent runs, master-slave, island and hybrid), some general purpose distributed frameworks designed to distribute any heuristic algorithm, inherently distributed agent-based approaches and several isolated combinations of existing models..

The reviewed papers present various scalability experiments solving multitude of graph search problems. Performing a fair comparison of the different approaches to distributing various SI algorithms requires some common normalized measures to put side by side. For this reason we introduce here the notions of speedup and

computational efficiency. The speedup metric of a distributed application measures how much faster the application runs when more computing nodes are used as opposed to a single computing node. The computational efficiency metric normalizes the value of the speedup to the number of computing nodes. The equations for computing the two metrics are:

$$S_c = \frac{T_1}{T_c} \tag{7}$$

$$e_c = \frac{S_c}{c} \tag{8}$$

where

c is the number of computing nodes used
S_c is the speedup of the application
T_1 is the execution time of the application on a single computing node
T_c is the execution time of the application on c computing nodes
e_c is the efficiency of running the application on c computing nodes

3. State of the Art in Distributing ACO and BCO

A very recent overview and classification of parallel computing approaches to Ant Colony Optimization (ACO) was reported by [87]. The authors propose an interesting and novel classification scheme for parallel ACO algorithms. The classification includes: master-slave model, cellular model, independent runs model, island model and hybrid models. However, the authors do not include agent-based approaches which, they consider, are not specifically designed to take advantage of multi processor architectures.

The cellular model mentioned in [87] is actually the author's own work [86] which proposes splitting ACO search space into overlapping neighborhoods each one with its own pheromone matrix. The good solutions gradually spread from one neighborhood to another through diffusion. This approach requires the search space to be partitioned in such a way that each set contains a continuous part of the optimal solution. The problem with this is that if the search space is not partitioned in this manner, an optimal solution can never be reached. The authors are the only ones that have ever implemented and tested this model based on the cell diffusion approach.

In the case of BCO we could find four papers [88, 89, 102, 103] that collectively offer a good classification of BCO parallel approaches. Interestingly, these papers identify the same general taxonomy of approaches for BCO as [87] does for ACO: master-slave model, independent runs model, island model and hybrid models. Therefore in the following subsections we identify and describe the research done using master-slave model, independent runs model, island model, hybrid models and the agent-based models. But because these terms are not standardized we must first define what we mean by them.

3.1. The Basic Taxonomy of Distributed SI Approaches. The master-slave model of distributing SI requires a central "master" computing node that manages a global best-so-far solution while multiple "slave" computing nodes find candidate solutions.

This model can be further divided into:

- fine grained master-slave model, where the slaves do atomic actions such as move one ant or find one solution. This requires a large amount of communication with the master.

- coarse grained master-slave model, in which the slaves do complex actions such as find multiple solutions and communicate the best one to the master.

In the fine grained version the slaves have to communicate with the master after each tiny task they are asked to execute and then they wait for new requests from the master. This overloads the master computing node and introduces large wait times when many slaves are used. In the coarse grained version the slaves find whole solutions or even sets of solutions before reporting to the master. Since in most implementations that is all the slaves do, they actually do not need to be explicitly told what to do next. Evidently the coarse grained master-slave model puts less stress on the master and therefore is a lot more scalable.

The independent runs model uses many instances of the SI algorithm that solve the problem independently and the best solution is chosen. SI approaches are inherently heuristic therefore they cannot guarantee finding the optimal solution, however, running an SI algorithm multiple times increases the chances of getting a very good quality solution. The fact that no communication is used but more than one instance of the algorithm is executed entails three consequences:

- faster execution time and implicitly better efficiency than any other implementation that requires communication, but only if the same number of solutions are explored in all considered implementations,
- better quality of solutions than a single sequential run,
- less qualitative solutions than implementations that use communication.

The island model of distributing SI requires running a separate group of entities, referred to as an "island", on each available computing node. Each group has its own map of the search space, therefore can be considered a sequential implementation of the SI algorithm. At predetermined points in time, with fixed frequency, solutions are communicated and the best solution over all is marked as required by the SI algorithm. Solutions can also be communicated asynchronously. This type of collaboration between the islands sets it apart from the master-slave model and the independent runs model.

When implementing the island model there are at least two issues to consider: i) the communication topology of the islands where the unidirectional ring (on the left) and full mesh topology (on the right) are depicted, ii) the frequency with which the islands communicate solutions to each other and whether this should be done synchronously or asynchronously. When the ring communication topology is used, each island sends a single message containing a solution to a predefined island. When using the mesh topology, any given island can broadcast its solution to all other islands.

This model is also called the "multi-colony model" in the case of ACO and the "multi-hive model" for BCO. This model is even more scalable than the coarse-grained master-slave since there is no master computing node to cause a "bottleneck". The existence of a master decreases performance when too many slaves try to communicate at once.

The hybrid model is a combination of the island with the master-slave model. An example of the hybrid model could be an island model where each island is actually a master-slave model instead of a simple sequential implementation of the SI algorithm.

Before going into each model separately, we present a quick overview of the taxonomy papers that will be referred multiple times in the current subsection. The work introduced in [88] and continued in [89] differentiates between the master-slave model, hybrid model and the island model of BCO. Experiments were done with each model

implemented using the message passing interface (MPI) library [82], and tested on numerical benchmark functions: Rastrigin, Griewank and generalized Schaffer. The implementations were executed using 11 computing nodes for the master-slave approach and 4 computing nodes for the island and hybrid model. The authors concluded that the island approach offers better quality solutions in less time, closely followed by the master-slave approach. No scalability tests were done on any of the models, the network delay was not taken into account. Furthermore, in papers [102, 103] the authors identify the model of independent runs additionally to the master-slave model, hybrid model and the island model of BCO. The authors also identify two variants on the island model: i) communicating just one solution and ii) the complete solution matrix. Their experiments were run in a simulated environment. The conclusion was that the independent serial runs approach was the fastest, however, the models that used communication offered the best quality of solutions.

3.2. Fine Grained Master-Slave Model. Implementing a scalable fine grained master-slave model is not trivial, as the first attempts show. Paper [79] analyzes the amount of communication needed by the fine grained model of ACO, concluding that it is not feasible. The first successful attempt at achieving scalability by using a hierarchy of master computing nodes was published in [13], however, their approach did not scale very well. In paper [96] a fine grained master-slave model of ACO is analyzed and the maximum efficiency obtained, of 0.8, was when using only two processors. Bullnheiner et al. in [15] made an attempt at the same model. The authors concluded that "acceptable execution time can be achieved" for TSP problems larger than 200 vertices. In the papers [36, 37] the distributed approach from [15] is revisited using a shared memory machine in the experiments reaching the maximum efficiency when employing 8 nodes. In such an architecture the access to the memory is concurrent, i.e. there is a shared memory and each process uses it one at a time in order to centralize its knowledge. No other modifications have been made to the architecture. The authors continue their work in [38] by comparing their approach from [36] to implementations that use synchronized messages instead of a shared memory. The access to the shared memory is much faster than messaging, which also requires broadcasting the current solution. This allows the ants to benefit sooner from the improvements found by the other population members. The conclusions were that their approach offers better solutions in less time. The authors obtained a speedup of 5.45 using 16 nodes, therefore resulting in an efficiency of merely 0.34. The most recent approach we could find on the fine-grained master-slave model is [55]. The authors of this paper used up to 240 GPUs to solve TSP using ACO and they succeeded in obtaining speedups of up to 30. The small amount of research using this model and the conflicting opinions about its effectiveness suggest that it is being abandoned as a viable model and is not, in fact, the state of the art approach to distributing SI.

3.3. Coarse Grained Master-Slave Model. The coarse grained master-slave model, unlike the fine grained one, is not plagued by conflicting opinions about its effectiveness. [74] implemented the coarse grained master-slave model for BCO and obtained near optimal solutions surpassing the quality of the results offered by the sequential approach. In paper [104] the authors present their approach to the coarse grained master-slave model called "ANTabu" which offers high quality solutions to the Quadratic Assignment Problem (QAP) [14]. Peng et al. presented their experiments with the coarse grained model [91, 92] that offered better quality solutions than evolutionary algorithms in the case of two different problems. In paper [75] groups of ants are

used on each slave resulting in better performance than sequential ACO. However, no comparison was made with the case of using only one ant per slave. Although these experiments used more than one computing node, these papers did not study the computational efficiency of their approaches.

We found several papers describing scalable implementations of the coarse grained master-slave model. In the work started in [9] and continued in [10] the authors implement the coarse master-slave and the hybrid models for BCO and test their implementations on a cluster of computers. The best execution time was achieved by the master-slave while the hybrid model obtained better quality of solutions. In [69] the authors present experiments showing that the coarse grained master-slave model is more efficient than the independent runs model. Their approach obtained efficiency of 0.8 using up to 16 processors. Papers [70] and [56] present experimental results where this model offers a speedup of up to 1.72 on dual core computers when running the slaves on separate threads, which means 0.86 efficiency.

Very good results were obtained using graphical processing units (GPU processors) for the slaves. One example is the already mentioned paper [55]. Another would be [16] that implemented a coarse grained master-slave where the master executed on the CPU and the slaves executed on the 8 pixel processors of a graphics card. The execution time depended heavily on the number of ACO entities used.

We also researched the problem of synchronous versus asynchronous centralization to the master computing node. The authors of [6] have implemented their version of this model for BCO and concluded that asynchronous centralization more efficient. Paper [16] compares the two approaches to coarse grained master-slave model for ACO and found them to offer similar speedup. On the other hand papers [110] and [63] found the asynchronous approach to be much faster for their respective implementations of ACO. The synchronous implementation from [45] offers an efficiency of 0.7 using 8 computing nodes, while the asynchronous approach from [21] boasts 0.9 efficiency on 8 nodes. We can conclude that in most cases an asynchronous implementation has better performance.

3.4. Independent Runs Model. The independent runs model is found to be used rarely. Experiments in [95] confirm that the independent runs model is more efficient than the coarse grained approach and offers slightly better solution quality than sequential implementations. Papers [1] and [2] also concluded that it is more efficient than coarse grained master-slave, synchronous and asynchronous island model. However, at the same time they confirm the fact that the independent runs model offers less qualitative solutions than implementations that use communication. These results are confirmed in the case of BCO by the authors of [102, 103], described earlier in this subsection. Therefore the implementations that require communication on top of constructing and validating solutions will automatically have longer execution times. However, the models that use communication offer better solutions sooner than the independent runs model. Thus, it is our conclusion that better alternatives to the independent runs model have already been developed.

3.5. The Hybrid Model. The hybrid model was implemented and studied in just a few publications. In papers [37, 71] each colony becomes the master of multiple slaves. This type of approach was tested in [37] on a cluster of 9 heterogeneous machines. The papers reported modest speedup values and slight improvements in quality of solutions when compared to other models. On the other hand, papers such as [64, 99] propose that the solutions of an island model should be collected and processed by a master computing node. Both papers lack efficiency analysis.

3.6. The Island Mode. The island model is a very popular research topic. We will first present the research conducted on the main features to be considered in this model: (i) the communication topology and (ii) the synchronization intervals.

On the subject of the communication topology to be used in the island model, the authors of [111] thoroughly analyze multi-colony ACO algorithms applied on TSP, experimenting with different communication policies and variable synchronization frequency, used as input parameters. They tested the communication topologies in terms of execution time and solution quality, with focus on maximizing the latter. The authors of paper [111] found that a ring topology outperforms the sequential approach to ACO and other tested topologies, confirming through independent experimentation the results of [76, 94, 79, 25]. Although, [23] sustains that the star topology offers the best results, their conclusions are, by far, less rigorously supported by experimental data than those in paper [111]. In all of these approaches the colonies distribute their solutions synchronously.

Experiments concerning point (ii), the synchronization intervals for the island model can also be found in the literature. In papers [34, 35] the authors analyze implementations of BCO using: the independent run model, a fine grained, coarse grained master-slave model and an island asynchronous implementation, using a ring communication topology. The experimental results show that the island asynchronous implementation outperformed all other models in both execution time and solution quality. [18, 20] found that the island model can remain scalable up to 25 computing nodes with self-adaptation but efficiency is worse than without self-adaptation. Experiments in [73] present that asynchronous solution updating between the colonies offers better scalability. The authors experimented on a cluster of 72 dualcore machines using one thread per island. We can conclude that in most cases an asynchronous implementation and larger intervals between synchronizations offer better performance.

Scalability and effectiveness of the island model was also extensively studied. A distributed version of the island model is presented and compares it with the sequential version of their BCO algorithm in [6]. The conclusions drawn by the authors were that the distributed approach is superior in both the quality of solutions and the execution time. In paper [84] the island model for BCO is implemented. The authors use shared memory to synchronize the solutions obtained by the separate hives and boast execution time decrease when increasing the number of computing nodes. [120] presents experimental data on island model ACO tests on up to 64 computing nodes, however, they do show a dramatic decrease in speedup for more than 8 nodes. In papers [17, 19] and [20], although the results were less than ideal, good execution time was still obtained when increasing the number of CPUs up to 25. In [68] the island model experiments show better execution times and higher quality of solutions than genetic algorithms (GA) [58] and sequential models of ACO. In paper [25] the authors claim that multi-colony is faster than the master-slave model. Their approach was tested on up to 5 computing nodes. Paper [121] confirms that multi-colony is more efficient than sequential ACO when using up to 8 computing nodes. The experiments show that for given TSP problem sizes there is always a number of colonies above which efficiency decreases.

Many authors noted that island model offers an increase of solution quality compared to sequential implementations. In [111] it is stated that the island model consistently offers better solutions than the sequential implementations of the same

ACO algorithm. On the other hand, [122] found that the island model offers "competitive" solutions when compared with other approaches. In paper [100] the authors experiment with exchanging full pheromone matrices instead of just one best-so-far solution. Their conclusion was that this greatly affected the execution time of the approach but slightly improved the quality of the solutions. Given the large number of papers written on the subject and the outstanding results, it is clear that the island approach is the current state of the art model of distributing SI.

3.7. Variations of the Master-slave Model. Some approaches to the master-slave model use the slaves only to perform local search on the solutions generated by the master. In [72] this idea is implemented for ACO using one thread per slave on two dualcore machines. The authors obtained speedup in their experiments, however, the hardware used did not facilitate detailed testing of their approach. Paper [108] tested a multi-threading implementation of ACO on two quad-core computing nodes. The experiments showed significant improvement in execution time. Continuing their work in [109] using two dual core machines, they obtained better results than a synchronous communication island model and an independent parallel runs model. [115] experimented with this model of ACO on 47 non-dedicated machines, they concluded that larger problems generate better speedup values. In paper [29] the authors used their parallel framework for master-slave approaches to implement similarly asynchronous ACO. They achieved very good speedup values up to 25 computing nodes, however, efficiency decreased directly proportional to the resources used.

3.8. Agent-based Approaches to Distributing SI. Papers [112, 113, 114] present and improve an agent based approach to the Vehicle Routing Problem (VRP) [57] inspired by BCO called "BeeJamA". The authors use two types of bee agents to disseminate routing cost information using BCO rules. Their approach is evaluated only in terms of quality using a simulation in MATSim [77], no scalability or efficiency study was performed.

Paper [105] presents a potential application of BCO in road traffic implemented as a multi-agent system. The authors propose a reactive agent system that leads to good results through the nature of the BCO algorithm. This is a practical application with a high interest in quality of solutions and adaptability, however, not efficiency during distributed execution. The work in [101] describes a purely theoretical overview on multi-agent optimization based on BCO. The authors describe the bees formally as reactive agents. The paper consists of detailed descriptions on how to program the bee agents to respect different BCO algorithm rules. No real implementation is mentioned.

The authors of [98] present a multi-agent system [8] implementation of ACO applying their solution to TSP. In their approach, graph vertices and ants are implemented as JADE [8] agents. Ants centralize the information about pheromone deposits and vertices' best tour cost through a single message exchange per vertex. This procedure adds up to $2n$ messages per tour for each ant, where n is the number of vertices. Each ant has to notify the vertex about its next hop and the cost of its tour for the vertex to be able to update its pheromone levels. This generates other n messages. When an ant completes a tour, it compares the tour cost with the collected best tours from the vertices. A best tour synchronization is triggered for all the vertices if a better tour has been found. This brings an additional overhead of n messages. Hence this approach requires $4n$ messages per tour per ant. Unfortunately, the authors provide no experimental data to support their claim of "good results".

In [50] the authors compare a distributed form of ACS with the flooding algorithm applied to resource discovery problem using ns2 network simulation tool [85]. They show that ACS is the better approach in terms of: best success rate, least number of hops and least traffic. The detailed algorithm and ACO parameters are not presented in order to duplicate their approach for a realistic comparison. The main characteristics of their approach are: (i) resource queries are handled centrally at a single computing node, thus introducing single point of failure, (ii) they do not take edge weights into account as they are trying to solve the resource discovery problem, and (iii) ants are implemented as ns2 mobile agents. Moreover, in practice very often there is no need for code mobility as every ant is governed by the same behavioral rules.

In the article [33] the authors present a fully distributed approach to ACO that models both vertices and ants as agents. Their proposal uses multiple vertices that play the role of anthills, generating colored ants modeled as mobile agents. They test their proposal on the problem of load balancing in distributed systems and compare it with the workstealing approach. The article concludes that the proposed architecture is more efficient in terms of the number of busy vertices and the elapsed time until a load distribution of 50% is reached. No efficiency study is made on the approach and in this case too there is no need for code mobility either, since all ants are governed by the same rules.

In the article [22] the authors present an agent-based distributed approach similar to the fine grained master-slave model from [37] but implemented with agents with learning capabilities. This papers main goal is not to use the distributed architecture to improve the efficiency of [37] but to improve other performance measures by the use of a knowledge base. As a consequence, this approach inherits the efficiency shortcomings of its predecessor.

In paper [78] the authors present a hybrid ACO/PSO (Particle Swarm Optimization) control algorithm for distributed swarm robots applied on the resource finding problem. A virtual pheromone is used in the form of messages. The messages contain the two-dimensional coordinates of the resource, its size and quality reflected in the quantity of pheromone. All robots have to manage their own map and pheromone deposits. PSO is used to avoid convergence to a local optima. The experiments are done in a virtual sequential environment. No scalability studies have been made.

Paper [119] proposes a JADE-based [8] multi-agent environment for dynamic manufacturing scheduling that combines intelligent techniques of ACO and multi-agent coordination. There is no globally accessible map, hence each agent needs to manage their own map and pheromone deposits. However, the focus in [119] is to evaluate the impact of ACO intelligence on multi-agent coordination, rather than to utilize multi-agent middleware for improving ACO. Therefore ACO intelligence is embedded into job and machine agents that have the role of ants.

We are aware of only one agent-based approach that was studied in terms of computational efficiency: the ACODA approach from [123]. The authors model subsets of the search graph vertices as agents and execute them on separate computing nodes. Ants are modeled are software objects passed from one vertex to another via a local queue or agent message. The authors boast an e=0.98 efficiency value on 11 computing nodes.

3.9. General-purpose Distributed Frameworks. During our review of the work done on distributing Si we have also found a few general-purpose distributed frameworks designed for any algorithms .

Paper [32] proposes JABAT (JADE-Based A-Team) - a JADE [8] based middleware for collaborative problem solving using asynchronous teams known as A-teams. JABAT supports distributed implementation and collaboration of population-based optimization algorithms. JABAT agents represent sequential improvement algorithms that cooperate in order to solve a given problem in a distributed manner. Unfortunately, we could not find scalability studies of the JABAT distributed architecture. Earlier, a similar approach was introduced in [30, 31] using an object oriented approach [118] to run several sequential algorithms in parallel. In this case the authors impose a master-slave organization system between the processes. The framework is tested for image processing using ACO.

The scalable distributed constraint architecture called DisChoco is described in the paper [49]. In this approach, agents manage parts of the problem domain as well as the constraints specific to their partition. The agents propose, propagate or refuse partial solutions to each other.

Paper [61] proposes a purely theoretical framework for multi-agent systems. No experiments or implementations are mentioned in this work. The authors present a distributed form of ACO based on so called smart messages approach to multi-agent systems. Agent mobility is used to implement complex communication over dynamic networks. They use delegate multi-agent systems to manage these smart messages in order to design a multi-agent approach for ACO. No ways of modeling the environment, determining convergence or stopping condition of ACO experiments are presented.

The frameworks presented in papers [31, 7, 32] are designed for the coarse grained master-slave approach. The [61] approach is much better suited for the distribution of the search space then migrating entities in the form of messages, according to the particular SI rules. However, this does not motivate the use of code mobility since all entities are governed by the same behavioral rules.

3.10. Partitioned Search Space Approaches. Briefly mentioned at the beginning of this subsection, the cellular approach from [86] proposes splitting ACO search space into overlapping neighborhoods, each one with its own pheromone matrix. The good solutions gradually spread from one neighborhood to another through diffusion. The authors obtained the efficiency of 0.9 on 4 nodes.

A similarly interesting approach [81] to distributed ACO divides the search space between the available and merges the solutions offered by the slaves at a central computing node. The authors boast exponential decrease of execution time. Another approach that divides the search space called "D-ant" is introduced in [41] and continued in [42] which was reported to outperform a multi-colony and a coarse grained implementation of ACO. The efficiency of the approach peaked at 0.7 when using 8 computing nodes. However these approaches would need to be inherently synchronized, which has negative effects on performance of implementations with a central computing node. Most research conducted on the matter agrees that asynchronous messages offer better execution time than synchronous ones. Another criticism that can be brought to these implementations is the fact that they are not purely ACO since they are using a different algorithm to merge the partial solutions.

4. Conclusions

The independent runs model is rarely used by the scientific community. When many instances of the SI algorithm solve the problem independently and the best

solution is chosen in the end increases the chances of getting a higher quality solution. We draw two conclusions from the work on the subject: i) this model is faster, however, it offers worse solutions than any other distribution model and ii) it provides better quality of solutions than a single sequential run.

The hybrid models and other variations of the master-slave model are isolated attempts to improve the execution time of the existing approaches. Although they sometimes boast better results taking advantage of hardware architecture such as multiple GPUs they are not widely accepted as best approach to distributing SI.

In the case of fine grained master-slave model the communication overhead that is necessary to exchange solutions is too large. The small amount of research using this model and the conflicting opinions about its effectiveness suggests that it is being abandoned as a viable model and is not, in fact, the state of the art approach to distributing SI.

Most authors focus on the coarse grained master-slave model or the island approach. Obtaining top efficiency when distributing their implementations on 8 to 25 nodes distribution. The approaches that were tested on TSP generally used graphs from the benchmark library TSPLIB [97] consisting of up to 500 vertices. The exception is the work presented in [120], where an island model was tested on a 15 000 vertex TSP instance of TSPLIB. However, efficiency peaked at 8 nodes. The most scalable approach in all the considered papers was presented in [20], an island model that reached peak efficiency at 25 nodes. This was tested on a 318 vertex TSP instance. However, the peak efficiency was inferior to maximum efficiency we encountered of 0.9 , obtained in paper [21] using 8 computing nodes. We found significant research sustaining the superior asynchronous communication for the island model. It is our conclusion supported by the large number of papers written on the subject and the outstanding results, that the island approach is the current state of the art model of distributing SI. These distributing models of SI may be criticized for not being especially designed for SI approaches. Therefore they do not take advantage of the inherently distributed nature of the SI approaches they are using.

In the case of agent-based approaches to distributed SI, the authors model entities and vertices as agents. Although this type of modeling is fully distributed, the separation of SI entities and search environment results in a large amount of messaging. All the necessary actions are done through messages: visiting a vertex, current solutions update any other SI specific action. Agents can run on the same computing node or on separate nodes. In the latter messaging between agents is slower due to network connection delay.

The frameworks presented in papers [7, 32, 31] are designed for the coarse grained master-slave approach. On the other hand, [61] is much better suited for the implementation of a distributed environment that migrates entities in the form of "smart messages", according to the particular SI rules. However, again, this does not motivate the use of code mobility since all entities are governed by the same behavioral rules. [61] presents a purely theoretical design, on the other hand, the somewhat similar distributed constraint architecture called "DisChoco" from paper [49] has been tested and proved to be scalable.

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Socket extensions for esoteric languages

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Abstract. In this paper we have advanced the first interpreter for the brainfuck (BF) esoteric programming language, entirely written in awk, a fast text processing programming language. However, the main objective remained introducing socket communication to brainfuck programming language. In order to achieve that goal, we have improved the brainfuck language with a byte long instruction through which it is allowed socket communication. For that, we have advanced a series of procedures in order to test the compatibility of our brainfuck interpreter.

Moreover, we have maintained brainfucks minimalism, which is one of the special characteristics of this programming language. In the end, we succeeded to map a minimalistic programming language to the client-server paradigm.

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Key words and phrases. brainfuck, socket, interpreter, esoteric languages.

1. Introduction

This paper describes a simple interpreter for the brainfuck (BF) esoteric programming language, written in AWK. It is compatible with almost any version of AWK that is supplied within UNIX systems. The interpreter illustrates the use of AWK in implementing small programs using BF language, mainly for proof of concept reasons.

1.1. Motivation. Even if BF is known for its extreme minimalism and it is designed to challenge programmers, it is not usually suitable for practical use. However, the motivation of developing such an application comes from the usage of a simple and easy syntax which can help programmers to deeply understand other programming languages and besides that, other programming paradigms.

The idea of improving BF language came from the necessity of implementing one of the most fundamental technologies of computer networking: socket communication.

Most computer programming languages implement socket communication. Including functional programming languages such as Lisp, Haskell or Erlang and logical programming languages such as Prolog ([1], [2], [3], [4]).

1.2. Objectives. The main objective is to introduce socket communication to BF programming language. In order to maintain BF's minimalistic language properties, we will define a single one byte new BF instruction.

In order to add socket support and to accommodate the new instruction, a BF interpreter needed to be modified. However, we decided not to modify an existing version, but to implement a new one using AWK programming language. Therefore,

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we will show how to map a minimalistic programming language to the client-server paradigm.

2. Related Work

2.1. Esoteric languages. An esoteric programming language is a programming language best suited for testing the boundaries of computer programming language design, as a proof of concept, as software art, or as a joke. Even if their goal is not the usability of programming, the main purpose is to remove or replace conventional language features while still maintaining a language that is Turing-complete.

A few examples of esoteric programming languages are: Intercal, Befunge, P", Brainfuck [5].

P" is a primitive computer programming language created in 1964 to describe a family of Turing machines. This language is formally defined as a set of words on the four-instruction alphabet R, λ, (,).

P" was the first "GOTO-less" imperative structured programming language to be proven Turing-complete. The BF language (apart from its I/O commands) is a minor informal variation of P".

2.2. Brainfuck. BF esoteric programming language is, as we previously stated, a minimalistic language, which consists in eight (one byte long) simple commands. This programming language was created as a Turing-complete language [6], meaning that, theoretically speaking, having access to an unlimited amount of memory, BF is capable of computing any function or simulating any mathematical model. All BF commands are sequentially executed and generally all other characters are ignored. The execution of the program ends at the end of the instructions set. Bellow there are few examples of code written in BF:

2.2.1. *Hello World - version 1.* This is the original "Hello World!" program where the ASCII codes for each printable character in the text are generated by running the code.

Listing 1. Hello World - version 1

```
++++++++[>++++[>++>+++>+++>+<<<<-]>+>+>->>+[<]<-]>>.>---.
+++++++..+++.>>.<-.<.+++.------.--------.>>+.>++.
```

2.2.2. *Hello World - version 2.* This is a more lisible code example that will output "hello" if you input letter "h". It will use the ASCII code of character "h" to obtain the ASCII code of "e", "l" and "o" by decrementing or incrementing the value at the current memory location.

Listing 2. Hello World - version2

```
,.---.+++++++..+++.
```

2.2.3. *Comparing numbers.* In this example we read from stdin 2 numbers (a and b) and the character "0". The program will output "0" if $a \leq b$ and "1" if $a > b$.

The algorithm works by setting a number of a memory locations to value 1. Then, b of them are set to 0. If there remains locations set to 1, it means that $a > b$.

TABLE 1. BF Commands

Character	Meaning
>	increment the data pointer which will point to the next cell to the right;
<	decrement the data pointer which will point to the next cell to the left;
+	increment the byte at the data pointer;
-	decrement the byte at the data pointer;
.	output the byte at the data pointer;
,	accept one byte of input and store its value in the byte at the data pointer;
[	if the byte at the data pointer is zero, then instead of moving the instruction pointer forward to the next command, jump it forward to the command after the matching] command.
]	if the byte at the data pointer is nonzero, then instead of moving the instruction pointer forward to the next command, jump it back to the command after the matching [command.

LISTING 3. Comparing numbers

```
>>,                    #move to location 2 and read value of a
[-[>+<-]+>]            #fill a locations with 1
<[<]                   #go back to location 1
,                      #read value of b
[>[-]<-[>+<-]>]        #fill b locations with 0
                       #if location b plus 1 is 0 then a was
,>[<+.[-]>[-]]<.       #less than b and print "0"
                       #else print "1"
```

2.3. Related BF implementations. There is a large collection of BF implementations, most of them listed in [7].

The BF implementations include:

- interpreters that use other programming languages to take BF programs as input and execute commands in that specific programming language. There are even some BF interpreters written in BF;
- compilers that take BF source code and generate executable code;
- hardware implementations including emultated and real CPU-s that can run BF code directly.

Most of those implementations are pure proofs of concept just to show that BF can be implemented using some particular technology. Some of them are optimizing implementations created to run BF code as fast as possible. Some of them are notable intentions to implement support for network communications.

The BF++ project [8] aims to extend BF with file I/O and TCP network communication. Some specifications are given but the project was discontinued and no final implementation was provided.

NetFuck [9] introduces communication between two NetFuck programs over TCP. The remote host and port are given to the interpreter as command line arguments and cannot be specified within the BF code.

Another step forward toward network communication using BF was taken by another NetFuck interpreter written in C# [10] which redirects standard BF I/O commands (.,) to read/write into a TCP socket instead of standard I/O. The developer also provides an IRCbot client written in NetFuck. However, the interpreter requires the IP and remote port to be specified in the command line as arguments, and by redirecting I/O to the connected TCP socket, no further interaction with the user is possible.

Two distributed applications using BF are presented below.

In [11] is presented a BF module for a particular webserver implemented in PHP. The module allows a developer to write server-side BF code and the result will be displayed on the clients browser. The communication is performed through the web server, so no socket facilities are provided for BF.

In [12], a NodeJS webserver implementation is extended to support writting BF code as scripting language for client side web programming in web browsers.

Therefore, we choose to implement our own BF interpreter with general purpose networking support using AWK programming language since there are none known BF implementations using AWK, and because AWK has a simple to use and understand socket abstractization layer.

3. AWK

The main purpose of AWK programming language was to allow users to write short programs intended for fast text processing and generating reports. AWK code is intended to be executed for each line of each processed file after applying some input regex filters. The processing speed of various AWK implementations versus alternatives such as Perl, Python or Ruby is beyond the scope of this document. However, AWK is present as a main programming/scripting language in most UNIX-like operating systems and therefore, it is studied in computer science curriculums all around the word. While a lot of students study AWK programming and most system administrators use it on a daily basis, few of them are aware of it's networking capabilities.

Our main concern was to write an efficient program through which we can access network services. But AWK was not meant initially to be used for networking purposes and it does not introduce special functions for socket access, like other languages do. However, it treats network connections like files [13]. The special file name for network access is made up of several mandatory fields, such as:

/inet/protocol/localport/hostname/remoteport

Network communication in AWK is implemented by creating a pipeline between the main AWK process and a process over the network specified as a special filename.

So, it was necessary to introduce a new operator '|&' to make possible communication over a network.

An example of reading a line from a tcp server (i.e. the ssh server on localhost listening on port 22 and presenting the ssh server welcome message) is presented below:

$echo | AWK '{"/inet/tcp/0/localhost/22" |& getline a; print a}'
SSH-2.0-OpenSSH_5.9p1 Debian-5ubuntu1.3

Another example is how to write a message ("hello") to a UDP server using a simple AWK command:

$echo | AWK 'print "hello" |& "/inet/udp/0/localhost/2222"'

4. AWK INTERPRETER FOR BF

BF is a minimalistic esoteric programming language. Hence, as shown in the previous section, there are many interpreters and compilers for BF in most programming languages, including a BF interpreter written in BF. However, there is none written in AWK.

Our first task was to write a BF interpreter in AWK that can run any BF program. There are several procedures for testing the compatibility and the performance of BF interpreters, some of them including recursive execution of BF code using the BF interpreter written in BF [14]. It is our main goal to write a compatible AWK interpreter for BF, while the performance is not our concern for the moment.

Along with a lot of trivial BF programs that we tried out in order to test our implementation, we also run a BF program that generates an ASCII Mandelbrot set. Although it runs slower than the default Ubuntu BF interpreter, it runs successfully as it can be seen in the Figure 1.

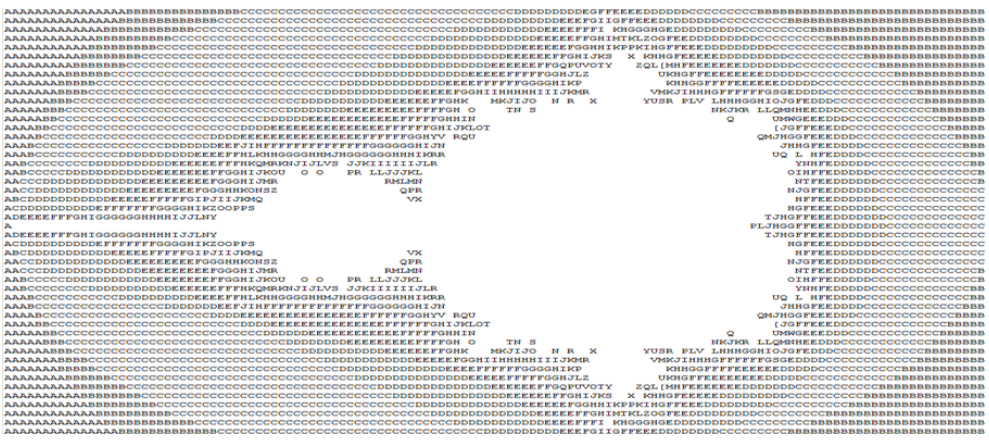

FIGURE 1. Output of the BF Mandelbrot set generator

Although AWK is a fast text processing programming language [15], we will piggyback here it's network communication abilities.

At first, the AWK interpreter reads all commands from the BF source file, then, they are parsed sequentially or repetitive for the "[","]" pair of instructions. The pseudocode is presented in Listing 4.
To maintain the minimalism of BF, we introduce only one new instruction, called @.

Depending on the interpreter which is usually used, one may consider this new instruction (i.e. our AWK interpreter) or ignore it, considering that it is a comment (as regular interpreters do). In order to make possible the client-server communication, we need to define some rules which specify the protocols that should be followed.

Listing 4. Pseudocode for interpreting network capable BF

```
// array m used for storing 30000 bytes long virtual BF tape
// c indicates current tape position
READ source code into array v
FOR i in v
    CASE v[i]
        ",":    READ a byte into m[c]
        ".":    WRITE m[c]
        "+":    Increment m[c]
        "-":    Decrement m[c]
        ">":      Increment tape pointer c
        "<":      Decrement tape pointer c
        "[":    IF m[c] not 0
                          THEN execute instructions up to
                               corresponding "]"
                          ELSE GOTO corresponding "]"
                ENDIF
        "]":    IF m[c] not 0
                          THEN GOTO corresponding "["
                ENDIF
        "@":    CASE m[c+1]
                    0:  Listen TCP on port m[c+2]*1000+m[c+3]
                        and IP address m[c+4].m[c+5].m[c+6].m[c+7]
                        and WRITE m[c] when client READS
                    1:  Listen TCP on port m[c+2]*1000+m[c+3]
                        and IP address m[c+4].m[c+5].m[c+6].m[c+7]
                        and READ m[c] when client WRITES
                    2:  Connect to TCP on port m[c+2]*1000+m[c+3]
                        and IP address m[c+4].m[c+5].m[c+6].m[c+7]
                        and WRITE m[c] when server READS
                    3:  Connect to TCP on port m[c+2]*1000+m[c+3]
                        and IP address m[c+4].m[c+5].m[c+6].m[c+7]
                        and READ m[c] when server WRITES
                    4:  //not used because UDP server cannot WRITE
                        //before READ
                    5:  Listen UDP on port m[c+2]*1000+m[c+3]
                        and IP address m[c+4].m[c+5].m[c+6].m[c+7]
                        and READ m[c] when client WRITES
                    6:  Connect to UDP on port m[c+2]*1000+m[c+3]
                        and IP address m[c+4].m[c+5].m[c+6].m[c+7]
                        and WRITE m[c] when server READS
                    7:  //not used because UDP client cannot READ
                        //before WRITE
                ENDCASE
    ENDCASE
ENDFOR
```

Memory pointer	Meaning
c+0	current memory location
c+1	case byte
c+2	port high byte
c+3	port low byte
c+4	IP byte 1
c+5	IP byte 2
c+6	IP byte 3
c+7	IP byte 4

TABLE 2.
Specifications for implementing @ command

Values m[c+1]
0 = (server, send, TCP)
1 = (server, receive, TCP)
2 = (client, send, TCP)
3 = (client, receive, TCP)
4 = (server, send, UDP)
5 = (server, receive, UDP)
6 = (client, send, UDP)
7 = (client, receive, UDP)

TABLE 3.
Cases for values of byte m[c+1]

4.1. Specifications for instruction @. This new instruction will allow the user to send/receive a single byte through a socket. The instruction will send/receive the byte referred by the current memory pointer (c). The type of the socket is defined by the value of c+1 as described in Table 2. The byte values at c+2 and c+3 are used to define the local or remote port foc communication, and bytes c+4 to c+7 represent the remote IPvP address (as described in Table 2).

The second byte (i.e. m[c+1]) can have only eight possible values, otherwise it will be ignored, and the communication will not be able to take place.

It is well known that two processes can be either client or server, they can either send or receive messages following one of the TCP or UDP protocols. Depending on the chosen combination from the set obtained by making the cartesian product between the above named sets: {client, server} x {send, receive} x {TCP, UDP}, the second byte should have only the values described in Table 3.

4.2. Examples of BF network communication.

4.2.1. *TCP client.* The first example is a TCP client that will read the first byte of a SMTP server welcome message running on localhost (127.0.0.1) on port 25, and outputs on the screen the value received in the current memory location:

LISTING 5. TCP client

```
>+++>                                  #value of m(c1) set to 3
                                       #for TCP client receive
+++++[>+++++<-]>>                      #setting m(c3) port to
                                       #25 multiplying 5 by 5
>>+++++[<++++[<++++++++>-]>-]<<-       #setting m(c4) first
                                       #octet of IP address
                                       #to 127=4*4*8 minus 1
>>>+                                   #setting octets m(c5) to
                                       #m(c7) to 0 and 0 and 1
                                       #move back to m(c)
<<<<<<<@.                              #execute socket command
                                       #and output the byte
                                       #read
```

The program prints on the screen the character "2" which was the first byte read from the server welcome message:

"220 athena.ubbcluj.ro ESMTP Postfix (Ubuntu)"

4.2.2. *UDP client - server.* The second example is a pair of BF programs that communicate with each other using UDP port 2000:

Listing 6. BF UDP server

```
>+++++>                                 #value of m(c1) set to 5
                                        #for UDP server receive
++>>                                    #setting m(c3) port to
                                        #2000 (high octet 2,
                                        #low octet 000)
>>++++[<++++[<++++++++>-]>-]<<- #setting m(c4) first
                                        #octet of IP address
                                        #to 127=4*4*8 minus 1
>>>+                                    #setting octets m(c5)
                                        #to m(c7) to 0 and 0
                                        #and 1
                                        #move back to m(c)
<<<<<<<@.                               #execute socket command
                                        #and output the byte
                                        #read
```

Listing 7. BF UDP client

```
,                                       #read a byte from
                                        #standard input
>++++++>                                #value of m(c1) set to 6
                                        #for UDP client send
++>>                                    #setting m(c3) port to
                                        #2000 (high octet 2,
                                        #low octet 000)
>>++++[<++++[<++++++++>-]>-]<<- #setting m(c4) first
                                        #octet of IP address
                                        #to 127=4*4*8 minus 1
>>>+                                    #setting octets m(c5) to
                                        #m(c7) to 0 and 0 and 1
<<<<<<<@                                #move back to m(c)
                                        #execute socket command
```

4.3. Limitations. Although, a lot can be accomplished by using an application layer protocol that transmits only byte sized payload packets, it is not an efficient use of resources. This implementation can only transmit/receive maximum one byte of information for every @ instruction used. This was our choice for this proof of concept implementation in order to keep the BF extension minimalistic.

The interpreter could be easily changed to send/receive variable sized data packets by using another byte (or two) in the memory to specify packet sizes up to 265 (65535) bytes. But this would greatly increase the already high complexity and readability of BF code.

5. CONCLUSIONS AND FUTURE WORK

The goal of writing a BF interpreter and extending BF for client-server communication has been realised in the most minimalistic possible way.

We hope that what we realised was indeed a proof of concept and that our ideas will inspire others to learn how to program in AWK.

As future directions for research and development we are considering implementing (and testing) variable packet size network communication in BF in such a way so to

maintain the minimalistic and esoteric principles of BF. Also, based on the research done for this paper we could also extend BF for running code in parallel so it could be used as a didactic tool for understanding the non-sequential programming paradigm.

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Hilbert-Pachpatte-type inequality due to fractional differential inequalities

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Abstract. The main objective of this paper is a study of some generalizations of Hilbert-Pachpatte-type inequality. We apply our general results to homogeneous functions. Also, this paper presents improvements and weighted versions of Hilbert-Pachpatte type inequalities involving the fractional derivatives.

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1. Introduction

Although classical, Hilbert's inequality is still of interest to numerous mathematicians. Through the years, Hilbert-type inequalities were discussed by numerous authors, who either reproved them using various techniques, or applied and generalized them in many different ways. For more details as regards Hilbert's inequality the reader is referred to [4], [7] and [8]. In particular, in [9], Pachpatte proved some new inequalities similar to Hilbert's inequality. In this paper, we establish some new integral Hilbert-Pachpatte type inequalities.

We start with the following result of Zhongxue Lü from [11]: let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $s > 2 - \min\{p, q\}$, and $f(x)$, $g(y)$ be real-valued continuous functions defined on $[0, \infty)$, respectively, and let $f(0) = g(0) = 0$, and

$$0 < \int_0^\infty \int_0^x x^{1-s} |f'(\tau)|^p d\tau dx < \infty, \qquad 0 < \int_0^\infty \int_0^y y^{1-s} |g'(\delta)|^q d\delta dy < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{|f(x)||g(y)|}{(qx^{p-1} + py^{q-1})(x+y)^s} dxdy \tag{1}$$
$$\leq \frac{B\left(\frac{q+s-2}{q}, \frac{p+s-2}{p}\right)}{pq} \left(\int_0^\infty \int_0^x x^{1-s} |f'(\tau)|^p d\tau dx\right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^y y^{1-s} |g'(\delta)|^q d\delta dy\right)^{\frac{1}{q}}.$$

Here, $B(\cdot, \cdot)$ denotes the usual Beta function. In this paper we shall consider more general form of inequality (1). Moreover, the main objective of this paper is to deduce Hilbert-Pachpatte type inequalities using the Taylor series of function and refinement of arithmetic-geometric inequality from [5]. Also, this paper presents improvements and weighted versions of Hilbert-Pachpatte type inequalities involving the fractional derivatives. Our results will be based on the following result of Krnić and Pečarić

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(see [6]) for conjugate exponents p and q. More precisely, they obtained the following two equivalent inequalities:

$$\int_{\Omega\times\Omega} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) \tag{2}$$
$$\leq \left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}} \left[\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}}$$

and

$$\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y) \leq \int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x), \tag{3}$$

where $p>1$, μ_1,μ_2 are positive σ-finite measures, $K:\Omega\times\Omega\to\mathbb{R}$, $f,g,\varphi,\psi:\Omega\to\mathbb{R}$ are measurable, non-negative functions and

$$F(x)=\int_\Omega \frac{K(x,y)}{\psi^p(y)}d\mu_2(y) \quad \text{and} \quad G(y)=\int_\Omega \frac{K(x,y)}{\varphi^q(x)}d\mu_1(x). \tag{4}$$

On the other hand, here we also refer to a paper of Brnetić *et al*, [10], where a general Hilbert-type inequality was obtained for $l\geq 2$ conjugate exponents, that is, real parameters $p_1,\ldots,p_l>1$, such that $\sum_{i=1}^l \frac{1}{p_i}=1$. Namely, let $K:\Omega^l\to\mathbb{R}$ and $\phi_{ij}:\Omega\to\mathbb{R}$, $i,j=1,\ldots,l$, be non-negative measurable functions. If $\prod_{i,j=1}^l \phi_{ij}(x_j)=1$, then the inequality

$$\int_{\Omega^l} K(x_1,\ldots,x_l)\prod_{i=1}^l f_i(x_i)\,dx_1\ldots dx_l \leq \prod_{i=1}^l \left(\int_\Omega F_i(x_i)(\phi_{ii}f_i)^{p_i}(x_i)\,dx_i\right)^{\frac{1}{p_i}}, \tag{5}$$

holds for all non-negative measurable functions $f_1,\ldots,f_l:\Omega\to\mathbb{R}$, where

$$F_i(x_i)=\int_{\Omega^{l-1}} K(x_1,\ldots,x_l)\prod_{j=1,j\neq i}^l \phi_{ij}^{p_i}(x_j)dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_l, \tag{6}$$

for $i=1,\ldots,l$.

2. Main results

In this section we shall state our main results. We suppose that all integrals converge and shall omit these types of conditions. To obtain the main result we need some lemmas.

Lemma 2.1. *For $f\in C^n[a,b]$, $n\in\mathbb{N}$, the Taylor series of function f is given by*

$$f(x)=\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f^{(n)}(t)dt+\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{7}$$

Define the subspace $C_a^n[a,b]$ of $C^n[a,b]$ as

$$C_a^n[a,b]=\{f\in C^n[a,b]: f^{(k)}(a)=0, k=0,1,\ldots,n-1\}.$$

Obviously, if $f\in C_a^n[a,b]$, then the right-hand side of (7) can be written as

$$f(x)=\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f^{(n)}(t)dt. \tag{8}$$

M. Krnić et *al.* in [5] proved the following refinements and converses of Young's inequality in quotient and difference form. For that sake, if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, we denote $P_n = \sum_{i=1}^n p_i$,

$$A_n(\mathbf{x}) = \frac{\sum_{i=1}^n x_i}{n}, \quad G_n(\mathbf{x}) = \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}},$$

and

$$M_r(\mathbf{x}, \mathbf{p}) = \begin{cases} \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0 \\ \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}, & r = 0 \end{cases}.$$

Lemma 2.2. ([5]) *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ *and* $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ *be positive* $n-$*tuples such that* $\sum_{i=1}^n \frac{1}{p_i} = 1$, *and*

$$\mathbf{x}^{\mathbf{p}} = (x_1^{p_1}, x_2^{p_2}, \ldots, x_n^{p_n}), \quad \boldsymbol{p}^{-1} = \left(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right).$$

Then
(i)

$$\left[\frac{A_n(\mathbf{x}^{\mathbf{p}})}{G_n(\mathbf{x}^{\mathbf{p}})}\right]^{n\min_{1\le i\le n}\{\frac{1}{p_i}\}} \le \frac{M_1(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1})}{M_0(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1})} \le \left[\frac{A_n(\mathbf{x}^{\mathbf{p}})}{G_n(\mathbf{x}^{\mathbf{p}})}\right]^{n\max_{1\le i\le n}\{\frac{1}{p_i}\}},$$

and
(ii)

$$n \min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}[A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}})] \le M_1(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1}) - M_0(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1}) \le n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}[A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}})].$$

We start with the refinement of Hilbert-Pachpatte type inequalities with the general kernel.

Theorem 2.3. *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p, q > 1$, *and* $0 \le a < b \le \infty$. *If* $K : [a, b] \times [a, b] \to \mathbb{R}$ *is non-negative function,* $\varphi(x)$, $\psi(y)$ *are non-negative functions on* $[a, b]$ *and* $f, g \in C_a^n[a, b]$, *then the following inequalities hold*

$$\begin{aligned} &\int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}} + (y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}} dxdy \\ &\le \frac{1}{4^{M-m}} \int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}} dxdy \qquad (9) \\ &\le \frac{1}{4^{M-m}[(n-1)!]^2} \left(\int_a^b \int_a^x (x-t)^{p(n-1)} \varphi^p(x) F(x) |f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}} \\ &\quad \times \left(\int_a^b \int_a^y (y-t)^{q(n-1)} \psi^q(y) G(y) |g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}}, \end{aligned}$$

and

$$\int_a^b G^{1-p}(y)\psi^{-p}(y)\left(\int_a^b K(x,y)\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}} dx\right)^p dy$$
$$\leq \int_a^b \int_a^x (x-t)^{p(n-1)}\varphi^p(x)F(x)|f^{(n)}(t)|^p dt dx, \tag{10}$$

where $m = \min\{\frac{1}{p}, \frac{1}{q}\}$, $M = \max\{\frac{1}{p}, \frac{1}{q}\}$, *and* $F(x)$ *and* $G(y)$ *are defined as in (4).*

Proof. By using (8) and Hölder's inequality, we have

$$\begin{aligned} |f(x)| &= \frac{1}{(n-1)!}\left|\int_a^x (x-t)^{n-1} f^{(n)}(t)dt\right| \\ &\leq \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}|f^{(n)}(t)|\cdot 1 dt \\ &\leq \frac{1}{(n-1)!}\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}\left(\int_a^x 1^q dt\right)^{\frac{1}{q}} \\ &= \frac{(x-a)^{\frac{1}{q}}}{(n-1)!}\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}, \end{aligned} \tag{11}$$

and similarly

$$|g(x)| \leq \frac{(y-a)^{\frac{1}{p}}}{(n-1)!}\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}}. \tag{12}$$

Now, from (11) and (12) we get

$$\begin{aligned} |f(x)|\,|g(y)| \leq\ & \frac{1}{[(n-1)!]^2}(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}} \times \left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}} \\ & \times \left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{13}$$

Applying Lemma 2.2(i) (see also [5]), we have

$$4^{M-m}(x^p y^q)^{M-m} \leq (x^p + y^q)^{2(M-m)}, \quad x \geq 0,\ y \geq 0, \tag{14}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, and $m = \min\{\frac{1}{p}, \frac{1}{q}\}$, $M = \max\{\frac{1}{p}, \frac{1}{q}\}$. From (13) and (14) we observe that

$$\begin{aligned} &\frac{4^{M-m}|f(x)|\,|g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}} + (y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}} \leq \frac{|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}} \\ &\leq \frac{1}{[(n-1)!]^2}\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}}, \end{aligned}$$

and therefore

$$4^{M-m}\int_a^b\int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}}+(y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy$$
$$\le \int_a^b\int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}}dxdy \tag{15}$$
$$\le \frac{1}{[(n-1)!]^2}\int_a^b\int_a^b K(x,y)\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}} dxdy.$$

Applying the substitutions

$$f_1(x)=\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}},\quad g_1(y)=\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}}$$

and (2), we have

$$\int_a^b\int_a^b K(x,y)f_1(x)g_1(y)dxdy \le \left(\int_a^b \varphi^p(x)F(x)f_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b \psi^q(y)G(y)g_1^q(y)dy\right)^{\frac{1}{q}}$$
$$=\left(\int_a^b\int_a^x (x-t)^{p(n-1)}\varphi^p(x)F(x)|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}$$
$$\times\left(\int_a^b\int_a^y (y-t)^{q(n-1)}\psi^q(y)G(y)|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}}. \tag{16}$$

By using (15) and (16) we obtain (9). The second inequality (10) can be proved by applying (3). □

Now we can apply our main result on non-negative homogeneous functions. Recall that for homogeneous function of degree $-s$, $s>0$, the equality $K(tx,ty)=t^{-s}K(x,y)$ is satisfied. Further, we define

$$k(\alpha):=\int_0^\infty K(1,u)u^{-\alpha}du$$

and suppose that $k(\alpha)<\infty$ for $1-s<\alpha<1$. To prove first application of our main results we need the following lemma.

Lemma 2.4. *If $\lambda>0$, $1-\lambda<\alpha<1$ and $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, then*

$$\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^\alpha dy = x^{1-\lambda}k(\alpha), \tag{17}$$

and

$$\int_0^\infty K(x,y)\left(\frac{y}{x}\right)^\alpha dx = y^{1-\lambda}k(2-\lambda-\alpha). \tag{18}$$

Proof. We use the substitution $y=ux$. The proof follows easily from homogeneity of the function $K(x,y)$. □

Corollary 2.5. *Let $\frac{1}{p}+\frac{1}{q}=1$, with $p,q>1$. If $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative and homogeneous function of degree $-\lambda$, $\lambda>0$, and $f,g\in C_0^n[0,\infty]$, then the following inequalities hold*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{K(x,y)|f(x)|\,|g(y)|}{\left(x^{\frac{1}{q(M-m)}}+y^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy \qquad (19)\\
&\le \frac{pq}{4^{M-m}}\int_0^\infty\int_0^\infty K(x,y)|f(x)|\,|g(y)|d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})\\
&\le \frac{L}{4^{M-m}[(n-1)!]^2}\left(\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^\infty\int_0^y y^{q(A_2-A_1+n-1)+1-\lambda}|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty y^{(p-1)(\lambda-1)+p(A_1-A_2)}\left(\int_0^\infty K(x,y)\left(\int_0^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}dx\right)^p dy\\
&\le L^p\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx, \qquad (20)
\end{aligned}$$

where $A_1\in(\frac{1-\lambda}{q},\frac{1}{q})$, $A_2\in(\frac{1-\lambda}{p},\frac{1}{p})$, $L=k(pA_2)^{\frac{1}{p}}k(2-\lambda-qA_1)^{\frac{1}{q}}$, and M,m are defined as in Theorem 2.3.

Proof. Let $F(x)$, $G(y)$ be the functions defined by (4). Setting $\varphi(x)=x^{A_1}$ and $\psi(y)=y^{A_2}$ in (9), using the fact $(x-t)^{p(n-1)}\le x^{p(n-1)}$, for $x\ge 0$ and $t\in[0,x]$, and Lemma 2.4, we get

$$\begin{aligned}
&\int_0^\infty\int_0^x (x-t)^{p(n-1)}\varphi^p(x)F(x)|f^{(n)}(t)|^p dtdx\\
&\le\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)}\left(\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^{pA_2}dy\right)|f^{(n)}(t)|^p dtdx\\
&= k(pA_2)\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx, \qquad (21)
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_0^\infty\int_0^y (y-t)^{q(n-1)}\psi^q(y)G(y)|g^{(n)}(t)|^q dtdy \qquad (22)\\
&\le k(2-\lambda-qA_1)\int_0^\infty\int_0^y y^{p(A_2-A_1+n-1)+1-\lambda}|g^{(n)}(t)|^q dtdy.
\end{aligned}$$

From (9), (21) and (22), we get (19). □

We proceed with some special homogeneous functions. First, by putting $K(x,y)=\frac{\ln\frac{y}{x}}{y-x}$ in Corollary 2.5, we get the following result.

Corollary 2.6. *Let $\frac{1}{p}+\frac{1}{q}=1$, with $p,q>1$. Let M,m,f,g be defined as in Corollary 2.5. Then the following inequalities hold*

$$\int_0^\infty \int_0^\infty \frac{\ln \frac{y}{x}|f(x)|\,|g(y)|}{(y-x)\left(x^{\frac{1}{q(M-m)}}+y^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy$$
$$\le \frac{pq}{4^{M-m}}\int_0^\infty\int_0^\infty \frac{\ln \frac{y}{x}|f(x)|\,|g(y)|}{y-x}d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})$$
$$\le \frac{L_1}{4^{M-m}[(n-1)!]^2}\left(\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)}|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty\int_0^y y^{q(A_2-A_1+n-1)}|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}},$$

and

$$\int_0^\infty y^{p(A_1-A_2)}\left(\int_0^\infty \frac{\ln \frac{y}{x}}{y-x}\left(\int_0^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}dx\right)^p dy$$
$$\le L_1^p\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)}|f^{(n)}(t)|^p dtdx,$$

where $A_1\in(0,\frac{1}{q})$, $A_2\in(0,\frac{1}{p})$, *and*

$$L_1=\pi^2(\sin pA_2\pi)^{-\frac{2}{p}}(\sin qA_1\pi)^{-\frac{2}{q}}.$$

Similarly, for the homogeneous function of degree $-\lambda$, $\lambda>0$, $K(x,y)=(\max\{x,y\})^{-\lambda}$, $A_1=A_2=\frac{2-\lambda}{pq}$, with the condition $\lambda>2-\min\{p,q\}$, we have:

Corollary 2.7. *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p,q>1$. *Let* M,m,f,g *be defined as in Corollary 2.5. Then the following inequalities hold*

$$\int_0^\infty\int_0^\infty \frac{(\max\{x,y\})^{-\lambda}|f(x)|\,|g(y)|}{\left(x^{\frac{1}{q(M-m)}}+y^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy \le \frac{pq}{4^{M-m}}\int_0^\infty\int_0^\infty \frac{|f(x)|\,|g(y)|}{(\max\{x,y\})^\lambda}d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})$$
$$\le \frac{L_2}{4^{M-m}[(n-1)!]^2}\left(\int_0^\infty\int_0^x x^{p(n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty\int_0^y y^{q(n-1)+1-\lambda}|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}},$$

and

$$\int_0^\infty y^{(p-1)(\lambda-1)}\left(\int_0^\infty (\max\{x,y\})^{-\lambda}\left(\int_0^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}dx\right)^p dy$$
$$\le L_2^p\int_0^\infty\int_0^x x^{p(n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx,$$

where $L_2=k(\frac{2-\lambda}{q})$ *and* $k(\alpha)=\frac{\lambda}{(1-\alpha)(\lambda+\alpha-1)}$.

In the proof of the following result we used a general Hilbert-type inequality (5) of Brnetić et al, [10].

Theorem 2.8. *Let* $n,l\in\mathbb{N}$, $l\ge 2$, $\sum_{i=1}^l\frac{1}{p_i}=1$ *with* $p_i>1$, $i=1,\dots,l$. *Let* α_i, $i=1,\dots,l$, *is defined by* $\alpha_i=\prod_{j=1,j\ne i}^l p_j$. *If* $K:[a,b]^l\to\mathbb{R}$ *is non-negative function,*

$\phi_{ij}(x_j)$, $i,j=1,\dots,l$, are non-negative functions on $[a,b]$, such that $\prod_{i,j=1}^{l}\phi_{ij}(x_j)=1$, and $f_i\in C_a^n[a,b]$, $i=1,\dots,l$, then the following inequality holds

$$\begin{aligned}
&\int_{(a,b)^l}\frac{K(x_1,\dots,x_l)\prod_{i=1}^{l}|f_i(x_i)|}{\left(\sum_{i=1}^{l}(x_i-a)^{\frac{1}{\alpha_i(M-m)}}\right)^{l(M-m)}}dx_1\dots dx_l\\
&\le\frac{1}{l^{(M-m)l}}\int_{(a,b)^l}\frac{K(x_1,\dots,x_l)\prod_{i=1}^{l}|f_i(x_i)|}{\prod_{i=1}^{l}(x_i-a)^{\frac{1}{\alpha_i}}}dx_1\dots dx_l\\
&\le\frac{1}{l^{(M-m)l}[(n-1)!]^l}\prod_{i=1}^{l}\left(\int_a^b\int_a^{x_i}(x_i-t)^{p_i(n-1)}\phi_{ii}^{p_i}(x_i)F_i(x_i)|f_i^{(n)}(t)|^{p_i}dtdx_i\right)^{\frac{1}{p_i}},
\end{aligned}$$

where $m=\min_{1\le i\le l}\{\frac{1}{p_i}\}$, and $M=\max_{1\le i\le l}\{\frac{1}{p_i}\}$, and $F_i(x_i)$, $i=1,\dots,l$ is defined by (6).

Obviously, Theorem 2.8 is a generalization of Theorem 2.3.

Remark 2.1. Applying the second refinement of arithmetic-geometric inequality (see Lemma 2.2(ii)) we obtain

$$x^py^q\le\left(\frac{x^p+y^q}{2}-\frac{1}{M-m}\right)^2,\quad x\ge0,\ y\ge0,\tag{23}$$

where $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, and $m=\min\{\frac{1}{p},\frac{1}{q}\}$, $M=\max\{\frac{1}{p},\frac{1}{q}\}$. If we take (23) and proceed as in the proof of Theorem 2.3, then

$$\begin{aligned}
\int_a^b\int_a^b\frac{K(x,y)|f(x)|\,|g(y)|}{\left(\frac{1}{2}[(x-a)^{\frac{1}{q}}+(y-a)^{\frac{1}{p}}]-\frac{1}{M-m}\right)^2}dxdy&\le\int_a^b\int_a^b\frac{K(x,y)|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}}dxdy\\
&\le\frac{1}{[(n-1)!]^2}\left(\int_a^b\int_a^x(x-t)^{p(n-1)}\varphi^p(x)F(x)|f^{(n)}(t)|^pdtdx\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_a^b\int_a^y(y-t)^{q(n-1)}\psi^q(y)G(y)|g^{(n)}(t)|^qdtdy\right)^{\frac{1}{q}},
\end{aligned}$$

where $F(x)$ and $G(y)$ are defined by (4).

3. The fractional derivatives and applications to Hilbert-Pachpatte type inequalities

First, we introduce some facts about fractional derivatives (see [3]). Let $[a,b]$, $-\infty<a<b<\infty$, be a finite interval on real axis $\mathbb{R}$. By $L_p[a,b]$, $1\le p<\infty$, we denote the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a,b]$. For $f\in L_1[a,b]$ the left-sided and right-sided *the Riemann-Liouville integral* of f of order α are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt,\quad x>a,$$

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\quad x<b.$$

For $f:[a,b]\to\mathbb{R}$ the left-sided *the Riemann-Liouville derivative* of f of order α is defined by

$$D^{\alpha}_{a+}f(x)=\frac{d^n}{dx^n}J^{n-\alpha}_{a+}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x(x-t)^{n-\alpha-1}f(t)dt.$$

Our result with the Riemann-Liouville fractional derivative is based on the following lemma (see [1]). By $AC^m[a,b]$ we denote the space of all functions $g\in C^{m-1}[a,b]$ with $g^{(m-1)}\in AC[a,b]$, where $AC[a,b]$ is the space of all absolutely continuous functions fnctions on $[a,b]$. For $\alpha>0$, $[\alpha]$ denotes the integral part of α.

Lemma 3.1. ([1]) *Let $\beta>\alpha\geq 0$, $m=[\beta]+1$, $n=[\alpha]+1$. The composition identity*

$$D^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\beta-\alpha)}\int_0^x(x-t)^{\beta-\alpha-1}D^{\beta}_{a+}f(t)dt,\quad x\in[a,b],$$

is valid if one of the following conditions holds:

(i) $f\in J^{\beta}_{a+}(L_1[a,b])=\{f:f=J^{\beta}_{a+}\varphi,\varphi\in L_1[a,b]\}$.

(ii) $J^{m-\beta}_{a+}f\in AC^m[a,b]$ *and* $D^{\beta-k}_{a+}f(a)=0$ *for* $k=1,\dots,m$.

(iii) $D^{\beta-1}_{a+}f\in AC[a,b]$, $D^{\beta-k}_{a+}f\in C[a,b]$ *and* $D^{\beta-k}_{a+}f(a)=0$ *for* $k=1,\dots,m$.

(iv) $f\in AC^m[a,b]$, $D^{\beta}_{a+}f$, $D^{\alpha}_{a+}f\in L_1[a,b]$, $\beta-\alpha\notin\mathbb{N}$, $D^{\beta-k}_{a+}f(a)=0$ *for* $k=1,\dots,m$ *and* $D^{\alpha-k}_{a+}f(a)=0$ *for* $k=1,\dots,n$.

(v) $f\in AC^m[a,b]$, $D^{\beta}_{a+}f$, $D^{\alpha}_{a+}f\in L_1[a,b]$, $\beta-\alpha=l\in\mathbb{N}$, $D^{\beta-k}_{a+}f(a)=0$ *for* $k=1,\dots,l$.

(vi) $f\in AC^m[a,b]$, $D^{\beta}_{a+}f$, $D^{\alpha}_{a+}f\in L_1[a,b]$, *and* $f^{(k)}(a)=0$ *for* $k=0,\dots,m-2$.

(vii) $f\in AC^m[a,b]$, $D^{\beta}_{a+}f$, $D^{\alpha}_{a+}f\in L_1[a,b]$, $\beta\notin\mathbb{N}$ *and* $D^{\beta-1}_{a+}f$ *is bounded in a neighborhood of* $m=a$.

By using Lemma 2.2 (see also Remark 2.1) and Lemma 3.1 we obtain our first result with the fractional derivative.

Theorem 3.2. *Let α, β, f, g be defined as in Theorem 3.1. If $K:[a,b]^2\to\mathbb{R}$ is non-negative function, $\varphi(x)$, $\psi(y)$ are non-negative functions on $[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\int_a^b\int_a^b\frac{K(x,y)|D^{\alpha}_{a+}f(x)|\,|D^{\alpha}_{a+}g(y)|}{\left(\frac{1}{2}[(x-a)^{\frac{1}{q}}+(y-a)^{\frac{1}{p}}]-\frac{1}{M-m}\right)^2}dxdy\\
&\leq\int_a^b\int_a^b\frac{K(x,y)|D^{\alpha}_{a+}f(x)|\,|D^{\alpha}_{a+}g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}}dxdy\\
&\leq\frac{1}{[\Gamma(\beta-\alpha)]^2}\left(\int_a^b\int_a^x(x-t)^{p(\beta-\alpha-1)}\varphi^p(x)F(x)|D^{\alpha}_{a+}f(t)|^pdtdx\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_a^b\int_a^y(y-t)^{q(\beta-\alpha-1)}\psi^q(y)G(y)|D^{\alpha}_{a+}g(t)|^qdtdy\right)^{\frac{1}{q}},
\end{aligned}$$

where $m,M,F(x),G(y)$ are defined as in Theorem 2.3.

Proof. The proof is similar to the proof of Theorem 2.3. □

Let $\nu>0$, $n=[\nu]$, and $\overline{\nu}=\nu-n$, $0\leq\overline{\nu}<1$. Let $[a,b]\subseteq\mathbb{R}$ and $x_0,x\in[a,b]$ such that $x\geq x_0$ where x_0 is fixed. For $f\in C[a,b]$ *the generalized Riemann-Liouville fractional integral* of f of order ν is given by

$$(J^{x_0}_{\nu}f)(x)=\frac{1}{\Gamma(\nu)}\int_{x_0}^x(x-t)^{\nu-1}f(t)dt,\quad x\in[x_0,b].$$

Further, define the subspace $C^{\nu}_{x_0}[a,b]$ of $C^n[a,b]$ as

$$C^{\nu}_{x_0}[a,b] = \{f \in C^n[a,b] : J^{x_0}_{1-\overline{\nu}} f^{(n)} \in C^1[x_0,b]\}.$$

For $f \in C^{\nu}_{x_0}[a,b]$ *the generalized Canavati $\nu-$fractional derivative* of f over $[x_0,b]$ is given by

$$D^{\nu}_{x_0} f = D\, J^{x_0}_{1-\overline{\nu}} f^{(n)},$$

where $D = d/dx$. Notice that

$$(J^{x_0}_{1-\overline{\nu}} f^{(n)})(x) = \frac{1}{\Gamma(1-\overline{\nu})} \int_{x_0}^{x} (x-t)^{-\overline{\nu}} f^{(n)}(t)dt$$

exists for $f \in C^{\nu}_{x_0}[a,b]$.
To obtain the result with generalized Canavati $\nu-$fractional derivative of f we need the following lemma.

Lemma 3.3. ([3]) *Let $f \in C^{\nu}_{x_0}[a,b]$, $\nu > 0$ and $f^{(i)}(x_0) = 0$, $i = 0,1,\ldots,n-1$, $n = [\nu]$. Then*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} (D^{\nu}_{x_0} f)(t)dt,$$

for all $x \in [a,b]$ with $x \geq x_0$.

Theorem 3.4. *Let $\nu > 0$ and $x_0, y_0 \in [a,b]$. Let $K : [a,b]^2 \to \mathbb{R}$ is non-negative function, $\varphi(x)$, $\psi(y)$ are non-negative functions on $[a,b]$. If $f \in C^{\nu}_{x_0}[a,b]$ and $g \in C^{\nu}_{y_0}[a,b]$ such that $f^{(i)}(x_0) = g^{(i)}(y_0) = 0$, $i = 0,1,\ldots,n-1$, $n = [\nu]$, then the following inequalities hold*

$$\begin{aligned}
&\int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{\left((x-x_0)^{\frac{1}{q(M-m)}} + (y-y_0)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}} dxdy \\
&\leq \frac{1}{4^{M-m}} \int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{(x-x_0)^{\frac{1}{q}}(y-y_0)^{\frac{1}{p}}} dxdy \qquad (24)\\
&\leq \frac{1}{4^{M-m}[\Gamma(\nu)]^2} \left(\int_a^b \int_{x_0}^{x} (x-t)^{p(\nu-1)} \varphi^p(x) F(x) |(D^{\nu}_{x_0} f)(t)|^p dtdx\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_a^b \int_{y_0}^{y} (y-t)^{q(\nu-1)} \psi^q(y) G(y) |(D^{\nu}_{y_0} g)(t)|^q dtdy\right)^{\frac{1}{q}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b G^{1-p}(y)\psi^{-p}(y) \left(\int_a^b K(x,y) \left(\int_{x_0}^{x} (x-t)^{p(\nu-1)} |(D^{\nu}_{x_0} f)(t)|^p dt\right)^{\frac{1}{p}} dx\right)^p dy \\
&\leq \int_a^b \int_{x_0}^{x} (x-t)^{p(\nu-1)} \varphi^p(x) F(x) |(D^{\nu}_{x_0} f)(t)|^p dtdx, \qquad (25)
\end{aligned}$$

where $m = \min\{\frac{1}{p}, \frac{1}{q}\}$, $M = \max\{\frac{1}{p}, \frac{1}{q}\}$, and $F(x)$ and $G(y)$ are defined by (4).

Proof. To prove the inequalities (24) and (25) we follow the same procedure as in the proof of Theorem 2.3, except we use Lemma 3.3 instead Lemma 2.1. □

In a similar manner as in the previous section, using the inequality (5) we obtain a generalization of Theorem 3.4.

Theorem 3.5. *Let $\nu > 0$ and α_i, $i = 1, \dots, l$, is defined by $\alpha_i = \prod_{j=1, j\neq i}^{l} p_j$, where $\sum_{i=1}^{l} \frac{1}{p_i} = 1$ with $p_i > 1$, $i = 1, \dots, l$. Let $K(x_1, \dots, x_l)$, ϕ_{ij}, $i, j = 1, \dots, l$, are defined as in Theorem 2.8. If $f_i \in C^{\nu}_{x_0^{(i)}}[a, b]$ ($x_0^{(i)} \in [a, b]$), $i = 1, \dots, l$, such that $f_i^{(j)}(x_0^{(i)}) = 0$, $j = 0, 1, \dots, n-1$, $n = [\nu]$, then the following inequality holds*

$$\begin{aligned}
&\int_{(a,b)^l} \frac{K(x_1, \dots, x_l) \prod_{i=1}^{l} |f_i(x_i)|}{\left(\sum_{i=1}^{l} (x_i - x_0^{(i)})^{\frac{1}{\alpha_i (M-m)}} \right)^{l(M-m)}} dx_1 \dots dx_l \\
&\leq \frac{1}{l^{(M-m)l}} \int_{(a,b)^l} \frac{K(x_1, \dots, x_l) \prod_{i=1}^{l} |f_i(x_i)|}{\prod_{i=1}^{l} (x_i - x_0^{(i)})^{\frac{1}{\alpha_i}}} dx_1 \dots dx_l \\
&\leq \frac{1}{l^{(M-m)l} [\Gamma(\nu)]^l} \prod_{i=1}^{l} \left(\int_a^b \int_{x_0^{(i)}}^{x_i} (x_i - t)^{p_i(\nu-1)} \phi_{ii}^{p_i}(x_i) F_i(x_i) |(D^{\nu}_{x_0^{(i)}} f_i)(t)|^{p_i} dt dx_i \right)^{\frac{1}{p_i}}
\end{aligned}$$

where $m = \min_{1 \leq i \leq l} \{\frac{1}{p_i}\}$, and $M = \max_{1 \leq i \leq l} \{\frac{1}{p_i}\}$, and $F_i(x_i)$, $i = 1, \dots, l$ is defined by (6).

For $\alpha > 0$, $f \in AC^n[a, b]$, where $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ if $\alpha \in \mathbb{N}_0$, *the Caputo fractional derivative* of f of order α ${}^cD^{\alpha}_{a+} f$ (left-sided) and ${}^cD^{\alpha}_{b-} f$ (right-sided) are defined by

$$\begin{aligned}
{}^cD^{\alpha}_{a+} f(x) &= D^{\alpha}_{a+} \left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1)} (x-a)^k \right], \\
{}^cD^{\alpha}_{b-} f(x) &= D^{\alpha}_{b-} \left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k+1)} (b-x)^k \right],
\end{aligned}$$

where D^{α}_{a+}, D^{α}_{b-} denote the left-hand sided and the right-hand sided Riemann-Liouville derivatives.

Very recently, Andrić et al [2] proved the following result.

Theorem 3.6. *Let $\nu > \gamma \geq 0$, $n = [\nu] + 1$, $m = [\gamma] + 1$ and $f \in AC^k[a, b]$, $k = n$ if $\nu \notin \mathbb{N}_0$ and $k = n - 1$ if $\nu \in \mathbb{N}_0$. Let ${}^cD^{\nu}_{a+} f$, ${}^cD^{\gamma}_{a+} f \in L^1[a, b]$. Suppose that one of the following conditions holds:*
(a) *$\nu, \gamma \notin \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m, \dots, n-1$.*
(b) *$\nu \in \mathbb{N}$, $\gamma \notin \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m, \dots, n-2$.*
(c) *$\nu \notin \mathbb{N}$, $\gamma \in \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m-1, \dots, n-1$.*
(d) *$\nu \in \mathbb{N}$, $\gamma \in \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m-1, \dots, n-2$.*
Then

$${}^cD^{\gamma}_{a+} f(x) = \frac{1}{\Gamma(\nu - \gamma)} \int_a^x (x-t)^{\nu-\gamma-1} D^{\nu}_{a+} f(t) dt.$$

Applying Lemma 2.2(i) and Theorem 3.6 (see also [2]) we obtain the following result.

Theorem 3.7. *Let ν, γ, f, g be defined as in Theorem 3.6. If $K : [a, b]^2 \to \mathbb{R}$ is non-negative function, $\varphi(x)$, $\psi(y)$ are non-negative functions on $[a, b]$, then the following inequality holds*

$$\int_a^b \int_a^b \frac{K(x,y)|{}^cD_{a+}^{\gamma} f(x)|\,|{}^cD_{a+}^{\gamma} g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}} + (y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}} dxdy$$
$$\leq \frac{1}{4^{M-m}} \int_a^b \int_a^b \frac{K(x,y)|{}^cD_{a+}^{\gamma} f(x)|\,|{}^cD_{a+}^{\gamma} g(y)|}{(x-x_0)^{\frac{1}{q}}(y-y_0)^{\frac{1}{p}}} dxdy$$
$$\leq \frac{1}{4^{M-m}[\Gamma(\nu-\gamma)]^2} \left(\int_a^b \int_a^x (x-t)^{p(\nu-\gamma-1)} \varphi^p(x) F(x) |{}^cD_{a+}^{\nu} f(t)|^p dtdx \right)^{\frac{1}{p}}$$
$$\times \left(\int_a^b \int_a^y (y-t)^{q(\nu-\gamma-1)} \psi^q(y) G(y) |{}^cD_{a+}^{\nu} g(t)|^q dtdy \right)^{\frac{1}{q}},$$

where $m, M, F(x), G(y)$ are defined as in Theorem 2.3.

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Nonlinear analysis on elliptic curves subspaces with cryptographic applications

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Abstract. Defining an adequate and well secured system is the ideal solution to protect the data against online attacks. With respect for this, one of the methods which are used is the double authentication in order to reduce the likelihood that a user presents false evidence of identity. In this article, we present cryptographic systems based on elliptic curves defined over finite space F_{p_t}. The security of the cryptosystems is given by a set of numbers, chosen from an elliptic curve and the calculation difficulty for the attack on the encryption system, taking an effective approach by establishing a common secret key required for group authentication. Also we present a complete description of the necessary system for the double authentication scheme.

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1. Introduction

The only defense against online attacks and combating online identity theft is by securing access to data where authentication occurs. Authentication represents the process of verifying a user identity that is to accessed the database.

Accessing data can be done either through a virtual private network (VPN), remote desktop connection, e-mail application (Outlook Anywhere) or an online portal that are authorized to access the data. Traditional authentication systems includes a lower security level authentication such as a password.

In this article we propose to improve the process of authentication for access to the database by double authentication, key exchange security on both sides. Two-factor authentication adds stronger security for users to authenticate with additional accreditations in addition to a password. Double authentication requires two of the following factors: something he knows (password, PIN), something he has (identity card, keys), and something that uniquely represents the user (fingerprint, retinal image).

However, the choice of an adequate and well secured two-factor authentication system can be difficult and expensive, compared with an unified authentication system, so it is easier to use and computational cost-effective. Two-factor authentication is a security process in which the user has two means of identification, one of which is usually physical, such as a card, and the second is something memorized, such as a security code. The two factors involved in the authentication process are something that the user knows and something he owns. Integrity is the assurance that the data which a user refers to can be accessed and modified only by those authorized to do so.

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Double authentication is used to reduce the likelihood that the user provides false evidence of identity. To reduce the number of stored keys and broadcasted messages, the researchers inserted the group key distribution protocol [13, 20] which is based on DLP (Discrete Logarithm Problem) assuming several public-key cryptosystems.

For a group double authentication, elliptic curve cryptosystems can be used to perform digital signature schemes and encryption schemes. Cryptosystems based on elliptic curve encryption systems refer to use numbers generated by an elliptic curve. It has been demonstrated over time that the cryptographic systems that are encrypted with the numbers generated by an elliptic curve are safer.

Cryptographic systems based on elliptic curves are defined equally well in any finite group (such as a group of points on a elliptic curve) and have stronger cryptographic security per bit than of any other public key encryption system known at the time. Memory requirements and bandwidth are substantially lower. Systems based on elliptic curves are easier to implement, being more efficient than any other public key system.

These systems are ideal for small hardware implementations such as cards, furthermore, encryption and signing can be performed in separate steps with an significant speed compared to other systems, which simplifies the security problem at login.

The security of cryptosystems based on elliptic curves consists in the difficulty of calculating discrete logarithms in discrete fields (DLP). The group key protocol has a major role in securing data and is divided into two categories: session key protocol and broadcast management key between the group members.

A group transfer key protocol is KGC, reliable key generation center (trusted key generation center) [2, 18, 23] is responsible for the generation and transport of the key to each involved member in the authentication process. For each distribution there is a registration process for users to subscribe group key, randomly selecting a private key.

The first step to a secured group authentication is to establish a secret code that is sent to group members. It establishes a protocol that ensures data security against active attacks. Of course, the cryptographic security levels can be achieved in various ways.

To transmit the confidentiality data, an effective approach is to establish a common secret key, obtained only by group members, required by the authentication. Recoding the group key is required whenever there is a change in the group, for data confidentiality. The confidentiality of the group key distribution is theoretically secure information and the security group key transfer depends on the members actions not an calculation hypothesis.

2. State of the art

For a secure communication between two or more users in an insecure network, it is necessary to establish common session keys. For communication between two users, things are somewhat simpler than in the case of multiple users, in which is more difficult to establish a rigorous communication with no problems.

The best known protocol for establishing a common session key using a hostile communication channel is Diffie-Hellman protocol. It also aims for exponential key exchange, the protocol is resistant to passive attacks, any active malevolent person can intervene and establish a session key with both involved parties in the communication.

The algorithm that has as input a prime number p big enough and a generator g.

- U_1 picks randomly a number, a number such that $0 \leq a \leq p-2$;
- U_1 computes $c = g^b$ and sends c to U_2;
- U_2 picks randomly a number b, such hat $a \leq b \leq p-2$;
- U_2 computes $d = g^b$ and sends d to U_1;
- U_1 computes the key $k = d^a = (g^b)^a$;
- U_2 computes the key $k = c^a = (g^b)^b$.

Starting from this algorithm, A. Jaux introduced another protocol called "A one round protocol for tripartite Diffie-Hellman", this protocol establishes a shared session key for three users.

With the growth of networks, the need for communications between more than three users becomes evident. There have been many attempts to establish a shared secret key for communication between n users [13, 20].

The protocol for establishing the session keys is divided into distributed commune keys establishment protocols and centralized key establishment protocols. The distributed mode have a higher cost in case of a large number of users, because each must participate in establishing process of the common communication key. As computer networks are increasing in terms of user growth, management protocols of centralized communication keys seems as an adequate choice.

Among the latest and ones of the most used protocols are presented in [3, 5, 12, 14, 16, 19] protocol that is to satisfy the three most important properties of a communication key:

- key authentication (assures the users that they use the correct key);
- key freshness (the key used in that moment wasn't used in the past);
- key confidentiality (protects the current key that isn't alienated to other users).

The advantage of these centralized protocols is that they use a common key generation center. At every entrance or exit of a user from a group, the generation center reconstructs this session key based on the new user or the removed user.

3. Background on Elliptic Curves Cryptography (ECC)

Let E_ϑ be an elliptic curve over a finite field K. E_ϑ is defined by Weierstrass equation, as follows:

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6 \tag{1}$$

where $a_1, a_2, a_3, a_4, a_6 \in K$ and $\Delta \neq 0$. This request, $\Delta \neq 0$, ensure the existence of the differential coefficients in every point of the elliptic curve, for detailed information, see [8], [9], [11]. For E_ϑ, Δ will be computed as:

$$\begin{cases} b_2 = a_1^2 + 4a_2 \\ b_4 = a_1a_3 + 2a_4 \\ b_6 = a_3^2 + 4a_6 \\ b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \\ \Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \end{cases}$$

According with [15], the curve points set together with infinity point $\mathcal{O}$ and an addition operation (as defined below) construct an abelian group.

Let $P, Q \neq 0$ be two points defined over an elliptic curve E_ϑ over a finite field K, and k a natural number. Then, for nonzero points with values $P = (X_1, Y_1)$, $Q = (X_2, Y_2)$, $R = (X_3, Y_3)$, we have:

(1) Inverse of a point:

$$-P = (X_1, -Y_1 - a_1X_1 - a_3)$$

(2) Two points sum: $R = P + Q$

$$\begin{cases} X_3 = \lambda^2 + a_1\lambda - a_2 - X_1 - X_2 \\ Y_3 = -(\lambda + a_1)X_3 - \nu - a_3 \end{cases}$$

where

$$\lambda = \begin{cases} \dfrac{Y_2 - Y_1}{X_2 - X_1} & \text{if } P \neq Q \\ \dfrac{3X_1^2 + 2a_2X_1 + a_4 - a_1Y_1}{2Y_1 + a_1X_1 + a_3} & \text{if } P = Q \end{cases}$$

$$\nu = \begin{cases} \dfrac{Y_1X_2 - Y_2X_1}{X_2 - X_1} & \text{if } P \neq Q \\ \dfrac{-X_1^3 + a_4X_1 + 2a_6 - a_3Y_1}{2Y_1 + a_1X_1 + a_3} & \text{if } P = Q \end{cases}$$

(3) Scalar multiplication:

$$Q = k \cdot P = \underbrace{P + P + \ldots + P}_{k \text{ times}}$$

For big values of k, from cryptographically point of view is usefully to define an algorithm for exponentiation, with applications for [1], [10], [17]. In the particular case of $k = \sum_{\theta=0}^{\kappa} \lambda_\theta 2^\theta, \lambda_\theta \in \{0, 1\}$, then

$$kP = \sum_{\theta=1}^{\kappa} \lambda_\theta (2^\theta P)$$

where doubling of it is necessary to obtain $2P, 2^2P, 2^3P, \ldots, 2^\kappa P$, and the most of complementary values κ depends by $\lambda_\theta = 1$.

3.1. ECC for message authentication. Elliptic Curves Cryptosystems for certain subspaces has some advantages, like key dimension and generating time, beside classical ones, because if the elliptic curve subspace is carefully chosen, ECDLP (Elliptic Curve Discrete Logarithm Problem) will define the attack complexity according with:

Let E be an elliptic curve defined over the F_{p_t}, where $F_{p_t} \in F_p$ (F_p a finite space), a point $P \in E$ (F_{p_t}) of order n, and a point $Q \in E$ (F_{p_t}), we have to found a point $k \in [0, n-1]$ such that $Q = kP$. The integer k is named the discrete logarithm of Q with P base, noted with $k = log_P Q$.

Encryption/Decryption

Let A and B be two entities which should communicate by sending the message P_{m_ϖ}. The entity A will send the message P_{m_ϖ} to the entity B by choosing first a pseudoaleator number k and private key χ_A. A will generate a public key $P_A = \chi_A \times \beta$, from that will compute the encrypted message $'C'_{m_A}$ which consist of points pair $C_{m_A} = \{k\beta, P_{m_\varpi} + k_{P_B}\}$, where β is the basic point, selected from the elliptic curve. In the next step, the entity B will choose a public key $P_B = \chi_B \times \beta$, respectively a private one, χ_B.

In order to decrypt the message, B must extract the result of the next equation:

$$P_{m_\varpi} + kP_B - \chi_B(k\beta) = P_{m_\varpi} k(\chi_B\beta) - \chi_B(k\beta) = P_{m_\varpi}$$

In encryption and decryption elliptic curves model, and for Digital Declaration, the basic parameters will define the elliptic curve subspace type. The strength of the cryptographic system is determined by the $D = (q, \alpha_a, \alpha_b, G, \beta, \tau)$, in the next algorithm.

- Let be the subspaces of parameter p.
- The parameters α_a, α_b defines the elliptic curve E over E_q, and the equations

$$Y^2 = X^3 + AX + B(p > 3) \text{ or } Y^2 + XY = X^3 + AX + B(p = 2).$$

- G - a generator point.
- β elliptic curve order.
- τ - co-factor, where $\tau = \#E(F_p)/\beta.\#E(F_p)$ represents the points number from the elliptic curve.

In order to implement the mathematical model, the previous enunciated parameters should achieved the next conditions, named general parameters:

- $q = p$ or $q = 2^{\sigma_x}$, where σ_x is a prime number.
- The elliptic curve should be non-singular.
- β should be $q^{\partial} - 1(1 \leq \partial \leq \pi)$, where π usually is 20.
- The elliptic curve should be non-anomalous, which means $|E(F_q)| \neq q$.

The main security parameter is β, and the key length is β.

4. F_{p_t} subspaces for ECC

Starting from [21] studies, below we present a method to transfer information by encrypting it with a public key known only to those who are authorized to access information through the communication channel. This system of information security requires each group participant to expose a parameter that will be part of the session key construction.

The group may be formed of 2, 3, 4, ..., t participants. Each participant selects an unique parameter, as an ID, known to himself and the key generator, which will be used to provide secure communication. Parameters will form a function $f(2,\ 3,\ 4,\ ...,\ t)$ resulting a public key, noted as k_t.

To extract the key, the user personal parameter is used and an using control key will communicate the private key to the participants. The private key will be generated by the public key generator at the request of a participant authenticated with his personal parameter, the ID.

Using an algorithm for transforming a sequence (ST) in a public key$k_t \in \{0,1\}^*$ will encrypt a plain text with that key and the encrypted text will be transferred through a secure channel. The message decryption will be done by an user using the regular private key obtained from the public key generator. However, the encrypted message must be intended to him so he can decrypt it.

For such a cryptographic system the following steps are defined:

- *configuration* - the unique parameters are generated for each user/member of the group for group identification and to create the common private key;
- *extraction* - according to the corresponding sting of the ID (ST) the master key is used to generate the private key corresponding to the public key needed for group authentication
- *encryption* - the encryption algorithm is described that will encrypt the message using the public key;
- *decryption* - using the private key the message will be decrypted using the decryption algorithm;

The algorithm used to obtain the private key from the public key generator is a type of challenge-response system. Because it is part of this category, for construction we will use, in the same way as in [6, 7] the following steps:

Definitions:

The expression x^3+1 is a permutation for F_{p_t}, where E is an elliptic curve given by the equation $y^2 = x^3 + 1$, defined over F_{p_t}, and p_t is a number through which will satisfy $p_t = 6m - 1$, where m is a prime number, and $m > 3$.

Let $G \in E/F_{p_t}$ be a group generator of points of order $m = (p_t + 1)/6$.

Selection stage:

Let $y_0 \in F_{p_t}$ be a point than there is a unique point x_0, so that $(x_0, y_0) \in E/F_{p_t}$.

First transformation stage:

Let $1+\delta \in F_{p_t^2}$ be a solution for the equation $x^3 - 1 = 0 \bmod p_t$. A transformation of the equation $\gamma(x, y) = (\delta x, y)$ is an automorphism of groups on elliptic curve E. If $G = (x, y) \in E/F_{p_t}$ then $\gamma(G) \in E/F_{p_t}$.

Determination stage:

The points $G, \gamma(G)$ generates a group morphism $Z_m \times Z_m$. We note this group of points as being $E[m]$. These points can be computed as in [6, 7].

Second transformation stage:

In order to encrypt the plain text will be made a concatenation between the ID of whom will receive the message and the string obtained from the encryption function. To transform the key K_T we need a string of code. For this the beginning sequence of the code (according to ASCII code) will be $ST^{(1)} \in \{0,1\}^*$. Using a cryptographic function, noted with $Q : \{0,1\}^* \to F_{p_t^*}$ we construct the point $A : y_0$ which will be equal to $Q(ST^{(1)})$ and $x_0 = (y_0^2 - 1)^{1/3} = (y_0^2 - 1)^{(2p_t-1)/3} \bmod p_t$. So $A = (x_0, y_0) \in E/F_{p_t}$, and we compute $A_{ST^{(1)}} = 6A$ in order to get the necessary order for A.

4.1. Transmission and validity of the key. Following the ideas from the previous paragraphs, ST is made by concatenating the full name of a participant. This means that the key exchange is required at a certain time and when a participant leaves the group to deny unauthorized access to information. The validity period for a key pair (public key and private key) is very small if we want as result a high-level security.

4.2. Functional scheme. Next we want to describe the four steps of the algorithm which ensure privacy and data integrity.

Setup

As we previous described, we chose an elliptic curve E generated by the instrumentality of an prime number $p_t, p_t = 6m - 1$, where the prime number is $m, m > 3$. We chose $G \in E/F_{p_t}$ in accordance with the key transformation $ST - K_T$. Also, we select $s \in Z_*^{p_t}$ and $G_{pub} = sG$. The master key is $s \in Z_*^{p_t}$.

The extraction

As we saw, the public key generator will construct a private key, through the expression: $Q(ST^{(1)}) = ST^{(2)} \in \{0,1\}^*$. From this follows $A_{ST^{(2)}} \in E/F_{p_t}$ of order m. The private key will be $d_{ST^{(2)}} = sA_{ST^{(2)}}$, where s is the master key.

Encryption

Let M be a simple text used by an sender, participant to the encryption process. The encrypted text C will be obtained as being $(rG, M \oplus Q(A'))$, where $A = (A_{ST^{(2)}}, G_{pub}) \in F_{p_t}$ and r is chosen randomly, $r \in Z_{p_t}$.

Decryption

We have $B = (U, V)$, an encrypted text with a public key $A_{ST^{(2)}}$. We will again obtain the simple text from the encrypted text by the following method: $M = V + Q(J)$, where $J = (d_{ST^{(2)}}, U) \in F_{p_t^2}$.

4.3. The system's security. It has been shown that the security of the system is given by the security of the public key system based on elliptic curves. These aspects are described in detail and demonstrated in [4, 22]. The method used to create and generate the public key and the private key does not offer the advantage that attackers cannot break the security of various systems, but, given the fact that it is based on numbers generated by elliptic curves it will be an rather secured system. It has been shown that security systems based on numbers generated by an elliptic curve are currently the most secure cryptographic systems.

5. Conclusion

In this paper we presented the advantages of using the particular subspaces of elliptic curves, instead of the entire ones, in cryptographic applications, with the description of an encryption system based on elliptic curves for a particular space F_{p_t}. This system represented a higher level method for ensuring data confidentiality in a group communication by using a subspace which requires a larger volume of calculations in order to penetrate the encryption system.

Future studies will be made in the direction of defining the computation time for the keys in such subspace, for a certain nonsupersingular elliptic curve, which has direct application in Digital Declaration. The main point of the research is based on figure the nonlinearity of subspaces, from cryptographic point of view.

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