Abrikosov Lattices in Superconductivity

Problem 06-006, *by* VICENŢIU RĂDULESCU[1] (Center of Nonlinear Analysis and Applications, University of Craiova, Craiova, Romania).

Let $B_1 = \{x = (x_1, x_2) \in \mathbb{R}^2;\ x_1^2 + x_2^2 = |x|^2 < 1\}$. Fix d a positive integer, and consider a configuration $S = (a_1, \ldots, a_d)$ of distinct points in B_1.

(i) Solve the linear Neumann problem

$$\Delta\Phi = 2\pi \sum_{i=1}^{d} \delta_{a_i} \quad \text{in } B_1,$$

$$\frac{\partial\Phi}{\partial\nu} = d \quad \text{on } S^1,$$

$$\int_{S^1} \Phi = 0,$$

where ν is the outward formal to S^1 and δ_b is the Dirac mass concentrated at $b \in B_1$.

(ii) Define

$$W(S) = -\pi \sum_{i \neq j} \log|a_i - a_j| + \frac{d}{2} \int_{S^1} \Phi \, d\sigma - \pi \sum_{i=1}^{d} R(a_i),$$

where

$$R(x) = \Phi(x) - \sum_{i=1}^{d} \log|x - a_i|.$$

Prove that

$$W(S) = -\pi \sum_{1 \le i < j \le d} \log \mid a_i - a_j \mid^2 - \pi \sum_{i,j=1}^{d} \log \mid 1 - a_i \overline{a}_j \mid \ .$$

(iii) Prove that if $d = 2$, then the configuration that minimizes W is unique (up to a rotation) and consists of two points that are symmetric with respect to the origin.

(iv) Prove that if $d = 3$, then the configuration that minimizes W is unique and consists of an equilateral triangle centered at the origin.

(v) Find the configuration that minimizes W in case $d \ge 4$. Establish if it is true that the minimal configuration of W "tends to the boundary" as $d \to \infty$ in the following sense: for given d, let $S = (a_1, \ldots, a_d)$ be an arbitrary configuration that minimizes W and set $x_d = \min\{|a_i|;\ 1 \le i \le d\}$. Prove or disprove that $\lim_{d\to\infty} x_d = 1$.

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Remarks. The functional W is called the *renormalized energy* and has been introduced by Bethuel, Brezis, and Hélein in their book [1].

The renormalized energy W arises in the theory of superconductors, and it can be viewed as the "finite part" of the energy of a superconductor in the following sense:

$$\frac{1}{2}\int_{B_1\setminus\cup_{i=1}^d B_\rho(a_i)} |\nabla u_\rho|^2 = \pi d \log\frac{1}{\rho} + W(S) + O(\rho) \quad \text{as } \rho \to 0,$$

where u_ρ is the solution of a certain minimization problem. (I refer the reader to [1] for more details.) The above asymptotic estimate shows that W is what remains in the energy of a superconductor after the singular *core energy* $\pi d \log\frac{1}{\rho}$ gas been removed.

The regular configurations established in (ii) and (iii) for the particular cases $d = 2$ and $d = 3$ are called "Abrikosov lattices." I mention that Alexei A. Abrikosov, Vitaly L. Ginzburg, and Anthony J. Leggett received the 2003 Nobel Prize in Physics "for pioneering contributions to the theory of superconductors and superfluids."

Status. The proposer has solutions for parts (i) through (iv). Part (v) is open.

REFERENCE

[1] F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg-Landau Vortices*, Birkhäuser, Boston, MA, 1994.