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A singular Gierer–Meinhardt system with different source terms*

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We study the existence and non-existence of classical solutions to a general Gierer–Meinhardt system with Dirichlet boundary condition. The main feature of this paper is that we are concerned with a model in which both the activator and the inhibitor have different sources given by general nonlinearities. Under some additional hypotheses and in the case of pure powers in nonlinearities, regularity and uniqueness of the solution in one dimension is also presented.

1. Introduction

The systems of nonlinear equations of Gierer–Meinhardt type have received considerable attention over the last decade. These problems arise in the study of biological pattern formation by auto- and cross-catalysis in connection with known biochemical processes and cellular properties. The general model proposed by Gierer and Meinhardt [8, 13] may be written as

$$\left.\begin{aligned} u_t &= d_1\Delta u - \alpha u + c\rho\frac{u^p}{v^q} + \rho_0\rho \quad && \text{in } \Omega\times(0,T), \\ v_t &= d_2\Delta v - \beta v + c'\rho'\frac{u^r}{v^s} && \text{in } \Omega\times(0,T), \end{aligned}\right\} \tag{1.1}$$

subject to Neumann boundary conditions. Here $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, is a bounded domain, u and v represent the concentrations of the activator and inhibitor with the source distributions ρ and ρ' respectively. Also d_1 and d_2 are diffusion coefficients with $d_1 \ll d_2$ and α, β, c, c', ρ_0 are positive constants. The exponents $p, q, r, s \geqslant 0$ verify the relation $qr > (p-1)(s+1) > 0$. The system (1.1) is of reaction–diffusion

*Dedicated to Professor Philippe G. Ciarlet on his 70th birthday.

type and involves the determination of an activator and an inhibitor concentration field. In a biological context, the Gierer–Meinhardt system (1.1) has been used to model several phenomena arising in morphogenesis and cellular differentiation.

The model presented by Gierer and Meinhardt [8] originates from Turing's model [18] for morphogenesis in the linear case and is based on the short range of activation and on the long range of inhibition. The model introduced in [8] takes into account the classification between the concentration of activators and inhibitors on the one hand and the densities of their sources on the other. A complete description of the entire dynamics of system (1.1) is given in [16], where it is shown that they exhibit various interesting behaviours such as periodic solutions, unbounded oscillating global solutions and finite-time blow-up solutions.

Many recent works have been devoted to the study of the steady-state solutions of (1.1), that is, solutions of the stationary system

$$\left.\begin{aligned} d_1\Delta u - \alpha u + c\rho\frac{u^p}{v^q} + \rho_0\rho = 0 \quad \text{in } \Omega,\\ d_2\Delta v - \beta v + c'\rho'\frac{u^r}{v^s} = 0 \quad \text{in } \Omega,\end{aligned}\right\} \tag{1.2}$$

subject to Neumann boundary conditions. Such systems are difficult to treat owing to the lack of a variational structure or *a priori* estimates. In this case it is more convenient to consider the *shadow system* associated to (1.2). More precisely, by dividing the second equation of (1.2) by d_2 and then letting $d_2 \to \infty$, we reduce the system (1.2) to a single equation. The non-constant solutions of such an equation present interior or boundary peaks or spikes, i.e. they exhibit a *point concentration* phenomenon. Among the great number of works in this direction, we refer the reader to [15, 19–21]. For the study of instability of solutions to (1.2), we also mention here the works of Miyamoto [14] and Yanagida [22].

In the case $\Omega = \mathbb{R}^N$, $N = 1, 2$, it has been shown in [3, 4] that there exist ground state solutions of (1.3) with single or multiple bumps in the activator which, after a rescaling of u, tend to a universal profile.

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be a bounded domain with smooth boundary. In this paper we consider the stationary Gierer–Meinhardt system for a wide class of nonlinearities subject to homogeneous Dirichlet boundary conditions. More precisely, we are concerned with the following elliptic system:

$$\left.\begin{aligned} \Delta u - \alpha u + \frac{f(u)}{g(v)} + \rho(x) = 0, \quad u > 0 \text{ in } \Omega,\\ \Delta v - \beta v + \frac{h(u)}{k(v)} = 0, \quad v > 0 \text{ in } \Omega,\\ u = 0, \quad v = 0 \text{ on } \partial\Omega,\end{aligned}\right\} \tag{$\mathcal{S}$}$$

where $\alpha, \beta > 0$, $\rho \in C^{0,\gamma}(\Omega)$, $0 < \gamma < 1$, $\rho \geqslant 0$, $\rho \not\equiv 0$ and $f, g, h, k \in C^{0,\gamma}[0, \infty)$ are non-negative and non-decreasing functions such that $g(0) = k(0) = 0$. This assumption on g and k, together with the Dirichlet conditions on $\partial\Omega$, makes the system singular at the boundary. Another difficulty is due to the non-cooperative (i.e. non-quasi-monotone) character of our system.

We are mainly interested in the case where the activator and inhibitor have different source terms, that is, the mappings $t \mapsto f(t)/h(t)$ and $t \mapsto g(t)/k(t)$ are not constant on $(0,\infty)$. Our study is motivated by some questions addressed by Choi and McKenna [1, 2] and Kim [11, 12] concerning existence and non-existence or even uniqueness of the classical solutions for the model system

$$\left.\begin{aligned} \Delta u - \alpha u + \frac{u^p}{v^q} + \rho(x) = 0 \quad & \text{in } \Omega, \\ \Delta v - \beta v + \frac{u^r}{v^s} = 0 \quad & \text{in } \Omega, \\ u = 0, \quad v = 0 \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.3}$$

In [1, 11] it is assumed that the activator and inhibitor have common sources and the approach relies on Schauder's fixed-point theorem. More precisely, when $p = r$ and $q = s$, we obtain a linear equation in $w = u - v$ by subtracting the two equations of (1.3). This is suitable to obtain *a priori* estimates in order to control the map whose fixed points are solutions of (1.3).

Choi and McKenna [2] obtained the existence of radially symmetric solutions of (1.3) in the cases when $\Omega = (0, 1)$ or $\Omega = B_1 \subset \mathbb{R}^2$ and $p = r > 1$, $q = 1$, $s = 0$; they obtained *a priori* bounds via sharp estimates of the associated Green function.

In § 2 we give a non-existence result for classical solutions to $(\mathcal{S})$. To the best of our knowledge, there are no results of this type in the literature. This matter relies on asymptotic behaviour of classical solutions to single singular elliptic problems in smooth bounded domains. Among the large number of works in this spirit, we refer the reader to [7] for the study of classical and weak solutions to singular elliptic problems. Special attention is paid to the case of pure powers in nonlinearities. In this sense we obtain some relations between the exponents p, q, r and s for which system (1.3) has no classical solutions.

In § 3 we give an existence result for classical solutions of $(\mathcal{S})$ under the additional hypothesis $\beta \leqslant \alpha$. In fact, this assumption is quite natural if we look at the steady-state system (1.2). We have only to divide the first equation by d_1, the second one by d_2 and to take into account the fact that $d_1 \ll d_2$. The existence in our case is obtained without assuming any growth condition on ρ near the boundary, since we are able to provide more general bounds for the regularized system associated to $(\mathcal{S})$. In particular, we obtain that (1.3) has solutions provided that $r - p = s - q \geqslant 0$ and $q > p - 1$.

The uniqueness of the solution is a delicate matter. Actually, there is only one result in the literature in this direction (see [1, theorem 1]) and which concerns the one-dimensional case of system (1.3) with $\rho \equiv 0$ and $p = q = r = s = 1$. Using the same idea as in [1], we are able to extend the uniqueness of the solution to $(\mathcal{S})$ in one dimension to the following range of exponents: $0 < q \leqslant p \leqslant 1$ and $r - p = s - q \geqslant 0$. It is worth pointing out here that the uniqueness of the solution for systems like $(\mathcal{S})$ seems to be a particular feature of the Dirichlet boundary conditions. As we can see in the above-mentioned works, in the case of Neumann boundary conditions the Gierer–Meinhardt system does not have a unique solution.

2. A non-existence result

Several times in this paper we shall use the following result. We refer the reader to [5, lemma 2.1] (see also [17, lemma 2.3]) for a complete proof.

LEMMA 2.1. *Let $\Psi : \Omega \times (0,\infty) \to \mathbb{R}$ be a Hölder continuous function such that the mapping $(0,\infty) \ni t \mapsto \Psi(x,t)/t$ is strictly decreasing for each $x \in \Omega$. Assume that there exist $v_1, v_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ such that*

(i) $\Delta v_1 + \Psi(x, v_1) \leqslant 0 \leqslant \Delta v_2 + \Psi(x, v_2)$ *in* Ω,

(ii) $v_1, v_2 > 0$ *in* Ω *and* $v_1 \geqslant v_2$ *on* $\partial\Omega$,

(iii) $\Delta v_1 \in L^1(\Omega)$ *or* $\Delta v_2 \in L^1(\Omega)$.

Then $v_1 \geqslant v_2$ *in* Ω.

Another useful tool is the following result, which is a direct consequence of the maximum principle.

LEMMA 2.2. *Let $k \in C(0,\infty)$ be a positive non-decreasing function and let $a_1, a_2 \in C(\Omega)$ with $0 < a_2 \leqslant a_1$ in Ω. Assume that there exist $\beta > 0$, $v_1, v_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $v_1, v_2 > 0$ in Ω, $v_1 \geqslant v_2$ on $\partial\Omega$ and*

$$\Delta v_1 - \beta v_1 + \frac{a_1(x)}{k(v_1)} \leqslant 0 \leqslant \Delta v_2 - \beta v_2 + \frac{a_2(x)}{k(v_2)} \quad \text{in } \Omega.$$

Then $v_1 \geqslant v_2$ *in* Ω.

Let $\Phi : [0,1) \to [0,\infty)$ be defined by

$$\Phi(t) = \int_0^t \left(2\int_\tau^1 \frac{1}{k(\theta)}\,\mathrm{d}\theta\right)^{-1/2} \mathrm{d}\tau, \quad 0 \leqslant t < 1.$$

Set $a = \lim_{t\nearrow 1} \Phi(t)$ and let $\Psi : [0,a) \to [0,1)$ be the inverse of Φ. The main result of this section is the following non-existence property.

THEOREM 2.3. *Assume that*

$$\int_0^a \frac{tf(mt)}{g(M\Psi(t))}\,\mathrm{d}t = +\infty, \tag{2.1}$$

for all $0 < m < 1 < M$. Then system ($\mathcal{S}$) has no classical solutions.

Proof. Assume by contradiction that there exists a classical solution (u, v) of system ($\mathcal{S}$) and let φ_1 be the normalized first eigenfunction of $(-\Delta)$ in $H_0^1(\Omega)$. As is well known, $\varphi_1 \in C^2(\bar{\Omega})$ and $\varphi_1 > 0$ in Ω. Let ζ be the unique solution of the problem

$$\left.\begin{aligned} \Delta\zeta - \alpha\zeta + \rho(x) &= 0 \quad \text{in } \Omega, \\ \zeta &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.2}$$

By standard elliptic arguments and strong maximum principle we deduce that $\zeta \in C^2(\bar{\Omega})$ and $\zeta > 0$ in Ω.

In view of Hopf's maximum principle and taking into account the regularity of the domain, there exist $c_1, c_2 > 0$ such that

$$c_1 d(x) \leqslant \varphi_1, \quad \zeta \leqslant c_2 d(x) \quad \text{in } \Omega, \tag{2.3}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Since

$$\begin{aligned} \Delta(u-\zeta) - \alpha(u-\zeta) \leqslant 0 \quad &\text{in } \Omega, \\ u - \zeta = 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

by the weak maximum principle [9, corollary 3.2] we have $u \geqslant \zeta$ in Ω. Hence, by (2.3) it follows that

$$u(x) \geqslant m d(x) \quad \text{in } \Omega \tag{2.4}$$

for some $m > 0$ small enough. Set $C := \max_{x\in\bar{\Omega}} h(u(x)) > 0$. Then v satisfies

$$\left.\begin{aligned} \Delta v - \beta v + \frac{C}{k(v)} \geqslant 0 \quad &\text{in } \Omega, \\ v > 0 \quad &\text{in } \Omega, \\ v = 0 \quad &\text{on } \partial\Omega. \end{aligned}\right\} \tag{2.5}$$

Let $c > 0$ be such that

$$c\varphi_1 \leqslant \min\{a, d(x)\} \quad \text{in } \Omega. \tag{2.6}$$

We need the following auxiliary result.

LEMMA 2.4. *There exists $M > 1$ large enough such that $\bar{v} := M\Psi(c\varphi_1)$ satisfies*

$$\Delta\bar{v} - \beta\bar{v} + \frac{C}{k(\bar{v})} \leqslant 0 \quad \textit{in } \Omega. \tag{2.7}$$

Proof. Since $\Phi(\Psi(t)) = t$ for all $0 \leqslant t < a$, we get $\Psi(0) = 0$ and $\Psi \in C^1(0, a)$ with

$$\Psi'(t) = \left(2\int_{\Psi(t)}^{1} \frac{1}{k(\tau)}\,\mathrm{d}\tau\right)^{-1/2} \quad \text{for all } 0 < t < a. \tag{2.8}$$

This yields

$$\left.\begin{aligned} -\Psi''(t) = \frac{1}{k(\Psi(t))} \quad &\text{for all } 0 < t < a, \\ \Psi'(t), \Psi(t) > 0 \quad &\text{for all } 0 < t < a, \\ \Psi(0) = 0. \quad & \end{aligned}\right\} \tag{2.9}$$

By Hopf's maximum principle, there exist $\omega \subset\subset \Omega$ and $\delta > 0$ such that

$$|\nabla\varphi_1| > \delta \text{ in } \Omega \setminus \omega \quad \text{and} \quad \varphi_1 > \delta \text{ in } \omega. \tag{2.10}$$

Fix $M > 1$ large enough such that

$$M(c\delta)^2 > C \quad \text{and} \quad Mc\lambda_1\delta\Psi'(c\|\varphi_1\|_\infty) > \frac{C}{\min_{x\in\omega} k(\Psi(c\varphi_1))}. \tag{2.11}$$

We have

$$-\Delta\bar{v} = \frac{Mc^2}{k(\Psi(c\varphi_1))}|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1\Psi'(c\varphi_1) \quad \text{in } \Omega.$$

By (2.10) and (2.11) we obtain

$$\begin{aligned} -\Delta\bar{v} &\geqslant Mc\lambda_1\varphi_1\Psi'(c\varphi_1) \geqslant Mc\lambda_1\delta\Psi'(c\|\varphi_1\|_\infty) \geqslant \frac{C}{k(\bar{v})} && \text{in } \omega, \\ -\Delta\bar{v} &\geqslant \frac{Mc^2}{k(\Psi(c\varphi_1))}|\nabla\varphi_1|^2 \geqslant \frac{C}{k(\Psi(c\varphi_1))} \geqslant \frac{C}{k(\bar{v})} && \text{in } \Omega\setminus\omega. \end{aligned}$$

The last two inequalities imply that $\bar{v}$ satisfies (2.7). This finishes the proof of the lemma. $\square$

By virtue of lemma 2.2, relations (2.5) and (2.7) yield $v \leqslant \bar{v}$ in Ω. Using (2.4) we find

$$\frac{f(u)}{g(v)} \geqslant \frac{f(md(x))}{g(M\Psi(c\varphi_1))} \quad \text{in } \Omega.$$

Furthermore, u satisfies

$$\left.\begin{aligned} \Delta u - \alpha u + \frac{f(md(x))}{g(M\Psi(c\varphi_1))} \leqslant 0 \quad & \text{in } \Omega, \\ u > 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.12}$$

In order to avoid the singularities near the boundary, we consider the approximated problem

$$\left.\begin{aligned} \Delta w - \alpha w + \frac{f(md(x))}{g(M\Psi(c\varphi_1)) + \varepsilon} = 0 \quad & \text{in } \Omega, \\ w = 0 \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.13}$$

Clearly, $\bar{w} := u$ is a supersolution of (2.13) while $\underline{w} := 0$ is a subsolution. By standard arguments, problem (2.13) has a unique solution $w_\varepsilon \in C^2(\bar{\Omega})$ such that $w_\varepsilon \leqslant u$ in Ω. Moreover, the maximum principle yields $w_\varepsilon > 0$ in Ω.

In order to raise a contradiction, we multiply by φ_1 in (2.13) and then we integrate over Ω. We obtain

$$(\alpha + \lambda_1)\int_\Omega w_\varepsilon\varphi_1 \,\mathrm{d}x = \int_\Omega \varphi_1 \frac{f(md(x))}{g(M\Psi(c\varphi_1)) + \varepsilon}\,\mathrm{d}x.$$

Since $w_\varepsilon \leqslant u$ in Ω we deduce

$$(\alpha + \lambda_1)\int_\Omega u\varphi_1 \,\mathrm{d}x \geqslant \int_\omega \varphi_1 \frac{f(md(x))}{g(M\Psi(c\varphi_1)) + \varepsilon}\,\mathrm{d}x \quad \text{for all } \omega \subset\subset \Omega.$$

Let

$$\tilde{C} = (\alpha + \lambda_1)\int_\Omega u\varphi_1 \,\mathrm{d}x.$$

Passing to the limit in the above inequality, we obtain

$$\int_\omega \varphi_1 \frac{f(md(x))}{g(M\Psi(c\varphi_1))}\,\mathrm{d}x \leqslant \tilde{C} < +\infty \quad \text{for all } \omega \subset\subset \Omega.$$

Hence,

$$\int_\Omega \varphi_1 \frac{f(md(x))}{g(M\Psi(c\varphi_1))}\,\mathrm{d}x \leqslant \tilde{C} < +\infty.$$

Let now $\Omega_0 = \{x \in \Omega : d(x) < a\}$. The above inequality combined with (2.6) produces

$$\int_{\Omega_0} d(x) \frac{f(md(x))}{g(M\Psi(d(x)))}\,\mathrm{d}x < +\infty,$$

but this clearly contradicts (2.1). Hence, system $(\mathcal{S})$ has no positive classical solutions. This ends the proof. □

If $k(t) = t^s$, $s > 0$, condition (2.1) can be written more explicitly by describing the asymptotic behaviour of Ψ.

Corollary 2.5. *Assume that $k(t) = t^s$, $s > 0$, and one of the following conditions holds:*

(i) $s > 1$ *and*

$$\int_0^a \frac{tf(mt)}{g(Mt^{2/(1+s)})}\,\mathrm{d}t = +\infty$$

for all $0 < m < 1 < M$*;*

(ii) $s = 1$ *and*

$$\int_0^{\min\{a,1/2\}} \frac{tf(mt)}{g(Mt\sqrt{-\ln t})}\,\mathrm{d}t = +\infty$$

for all $0 < m < 1 < M$*;*

(iii) $0 < s < 1$ *and*

$$\int_0^a \frac{tf(mt)}{g(Mt)}\,\mathrm{d}t = +\infty$$

for all $0 < m < 1 < M$*.*

Then, system $(\mathcal{S})$ has no positive classical solutions.

Proof. The main idea is to describe the asymptotic behaviour of Ψ near the origin. In our setting, the mapping $\Psi : [0, a) \to [0, 1)$ satisfies

$$\left.\begin{aligned} -\Psi''(t) &= \Psi^{-s}(t) \quad \text{for all } 0 < t < a, \\ \Psi'(t), \Psi(t) &> 0 \qquad\quad\ \text{for all } 0 < t < a, \\ \Psi(0) &= 0. \end{aligned}\right\} \tag{2.14}$$

(i) If $s > 1$, then the mapping

$$[0, a) \ni t \mapsto \left[\frac{(1+s)^2}{2(1-s)}\right]^{1/(1+s)} \cdot t^{2/(s+1)}$$

satisfies (2.14). Hence, there exist two positive constants $c_1, c_2 > 0$ such that

$$c_1 t^{2/(s+1)} \leqslant \Psi(t) \leqslant c_2 t^{2/(s+1)} \quad \text{for all } 0 < t < a.$$

Now, (i) follows directly from the above inequality.

(ii) Using the fact that Ψ is concave, we deduce that $\Psi(t) > t\Psi'(t)$ for all $0 < t < a$. From (2.14) it follows that

$$-\Psi''(t) < \frac{1}{t\Psi'(t)} \quad \text{for all } 0 < t < a.$$

We multiply by Ψ' in the last inequality and then we integrate over $[t, b]$, $0 < b < a$. We get

$$(\Psi')^2(t) - (\Psi')^2(b) \leqslant 2(\ln b - \ln t) \quad \text{for all } 0 < t \leqslant b < a.$$

Hence, there exist $c_1 > 0$ and $\delta_1 \in (0, b)$ such that $\Psi'(t) \leqslant c_1\sqrt{-\ln t}$ for all $0 < t \leqslant \delta_1$. Integrating over $[0, t]$, we obtain

$$\Psi(t) \leqslant c_1 \int_0^t \sqrt{-\ln \tau}\,\mathrm{d}\tau = c_1 t\sqrt{-\ln t} + \tfrac{1}{2}c_1 \int_0^t \frac{1}{\sqrt{-\ln \tau}}\,\mathrm{d}\tau \quad \text{for all } 0 < t \leqslant \delta_1. \tag{2.15}$$

Since the last integral in (2.15) is finite, there exist $c_2 > 0$ and $\delta_2 \in (0, \delta_1)$ such that

$$\Psi(t) \leqslant c_2 t\sqrt{-\ln t} \quad \text{for all } 0 < t \leqslant \delta_2. \tag{2.16}$$

From (2.14) and (2.16) we deduce

$$-\Psi''(t) = \frac{1}{\Psi(t)} \geqslant \frac{1}{c_2}\frac{1}{t\sqrt{-\ln t}}$$

for all $0 < t \leqslant \delta_2$.

An integration over $[t, \delta_2]$ in the last inequality yields

$$\Psi'(t) \geqslant \frac{2}{c_2}(\sqrt{-\ln t} - \sqrt{-\delta_2}) \quad \text{for all } 0 < t \leqslant \delta_2.$$

Therefore, there exist $c_3 > 0$ and $\delta_3 \in (0, \delta_2)$ such that $\Psi'(t) \geqslant c_3\sqrt{-\ln t}$ for all $0 < t \leqslant \delta_3$. Proceeding in the same manner as above, there exist $c_4 > 0$ and $\delta_4 \in (0, \delta_3)$ such that

$$\Psi(t) \geqslant c_4 t\sqrt{-\ln t} \quad \text{for all } 0 < t \leqslant \delta_4. \tag{2.17}$$

From (2.16) and (2.17) we get

$$c_3 t\sqrt{-\ln t} \leqslant \Psi(t) \leqslant c_4 t\sqrt{-\ln t} \quad \text{for all } 0 < t \leqslant \delta_4.$$

Now, (ii) follows from the above estimates.

(iii) By (2.8) we have

$$\Psi'(t) = \left(2\int_{\Psi(t)}^1 \tau^{-s}\,\mathrm{d}\tau\right)^{-1/2} = \left(\frac{2}{1-s}(1 - \Psi^{1-s}(t))\right)^{-1/2} \quad \text{for all } 0 < t < a.$$

Hence, $0 < \Psi'(0) = \sqrt{2/(1-s)} < +\infty$, which implies that $\Psi \in C^1[0, a)$ and $c_1 t \leqslant \Psi(t) \leqslant c_2 t$ in $(0, a)$ for some $c_1, c_2 > 0$. This proves (iii). □

In the case of pure powers in the nonlinearities, we have the following nonexistence result for (1.3).

Corollary 2.6. *Let $p, q, r, s > 0$ be such that one of the following conditions hold:*

(i) $s > 1$ *and* $2q \geqslant (s+1)(p+2)$;

(ii) $s = 1$ *and* $q > p + 2$;

(iii) $0 < s < 1$ *and* $q \geqslant p + 2$.

Then, the system (1.3) has no positive classical solutions.

Proof. The proofs of (i) and (iii) are simple exercises of calculus. For (ii), by corollary 2.5 we have that (1.3) has no classical solutions, provided that $s = 1$, and

$$\int_0^{1/2} t^{1+p-q}(-\ln t)^{-q/2}\,\mathrm{d}t = +\infty. \tag{2.18}$$

On the other hand, for $a, b \in \mathbb{R}$ we have

$$\int_0^{1/2} t^{a}(-\ln t)^{b}\,\mathrm{d}t < +\infty$$

if and only if $a > -1$ or $a = -1$ and $b < -1$. Now condition (2.18) reads $q > p + 2$. This concludes the proof. □

3. Existence results

For all $t_1, t_2 > 0$ we define

$$A(t_1, t_2) = \frac{f(t_1)}{h(t_1)} - \frac{g(t_2)}{k(t_2)}.$$

In this section it is assumed that A satisfies the following.

(A_1) $A(t_1, t_2) \leqslant 0$ for all $t_1 \geqslant t_2 > 0$.

(A_2) $k \in C^1(0, \infty)$ is non-negative and non-decreasing function such that

$$\lim_{t\to+\infty} \frac{K(t)}{h(t+c)} = +\infty \quad \text{for all } c > 0,$$

where $K(t) = \int_0^t k(\tau)\,\mathrm{d}\tau$.

Below there are some examples of nonlinearities that fulfil (A1) and (A2):

(i) $f(t) = t^p$, $g(t) = t^q$, $h(t) = t^r$, $k(t) = t^s$, $t \geqslant 0$, $p, q, r, s > 0$, $r - p = s - q \geqslant 0$ and $p - q < 1$;

(ii) $f(t) = \ln(1 + t^p)$, $g(t) = \mathrm{e}^{t^q} - 1$, $h(t) = t^p$ and $k(t) = t^q$, $t \geqslant 0$, $p, q > 0$, $p - q < 1$;

(iii) $f(t) = \log(1 + at)$, $g(t) = \log(1 + t)$, $h(t) = at$ and $k(t) = t$, $t \geqslant 0$, $a \geqslant 1$.

In the following we supply a general method to construct nonlinearities f, g, h, k that verify hypotheses (A1) and (A2). Let $f, g, h, k : [0, \infty) \to [0, \infty)$ be non-decreasing functions such that k and h verify (A2) and one of the following assumptions holds:

(a) $fk = gh$ and the mapping $(0, \infty) \ni t \mapsto f(t)/h(t)$ is non-increasing;

(b) there exists $m > 0$ such that $f(t)/h(t) \leqslant m \leqslant g(t)/k(t)$, for all $t > 0$.

Then the mapping A verifies (A1).

For instance, the mappings given in example (i) satisfy condition (a), while the mappings given in example (ii) verify condition (b).

The first result of this section concerns the existence of classical solutions for the general system $(\mathcal{S})$.

THEOREM 3.1. *Assume that $\alpha \geqslant \beta$ and the hypotheses (A1) and (A2) are fulfilled. Then system $(\mathcal{S})$ has at least one classical solution.*

The existence of a solution to $(\mathcal{S})$ is obtained by considering the regularized system

$$\left.\begin{aligned} \Delta u - \alpha u + \frac{f(u+\varepsilon)}{g(v+\varepsilon)} + \rho(x) &= 0 \quad \text{in } \Omega, \\ \Delta v - \beta v + \frac{h(u+\varepsilon)}{k(v+\varepsilon)} &= 0 \quad \text{in } \Omega, \\ u = 0, \quad v &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \qquad (\mathcal{S})_\varepsilon$$

LEMMA 3.2. *Let $(u_\varepsilon, v_\varepsilon)$ with $u_\varepsilon, v_\varepsilon \in C^2(\Omega) \cap C(\bar{\Omega})$ be a positive solution of $(\mathcal{S})_\varepsilon$. Then, there exists $M > 0$ which is independent of ε such that*

$$\max\{\|u_\varepsilon\|_\infty, \|v_\varepsilon\|_\infty\} \leqslant M. \tag{3.1}$$

Proof. Let $w_\varepsilon := u_\varepsilon - v_\varepsilon$ and $\omega = \{x \in \Omega : w_\varepsilon > 0\}$. In order to prove the estimate (3.1), it suffices to provide a uniform upper bound for v_ε and w_ε. From $(\mathcal{S})_\varepsilon$ we have

$$\begin{aligned} \Delta w_\varepsilon - \alpha w_\varepsilon + \rho(x) &= (\alpha - \beta) v_\varepsilon - \frac{f(u_\varepsilon + \varepsilon)}{g(v_\varepsilon + \varepsilon)} + \frac{h(u_\varepsilon + \varepsilon)}{k(v_\varepsilon + \varepsilon)} \\ &= (\alpha - \beta) v_\varepsilon - \frac{h(u_\varepsilon + \varepsilon)}{g(v_\varepsilon + \varepsilon)} A(u_\varepsilon + \varepsilon, v_\varepsilon + \varepsilon) \quad \text{in } \Omega. \end{aligned}$$

Let us note that $A(u_\varepsilon + \varepsilon, v_\varepsilon + \varepsilon) \leqslant 0$ in ω and $w_\varepsilon = 0$ on $\partial\omega$. This yields

$$\Delta w_\varepsilon - \alpha w_\varepsilon + \rho(x) \geqslant 0 \quad \text{in } \omega.$$

Let $\zeta \in C^2(\bar{\Omega})$ be the unique solution of (2.2). Then

$$\begin{aligned} \Delta(w_\varepsilon - \zeta) - \alpha(w_\varepsilon - \zeta) &\geqslant 0 \quad \text{in } \omega, \\ w_\varepsilon - \zeta &\leqslant 0 \quad \text{on } \partial\omega. \end{aligned}$$

Furthermore, by the weak maximum principle [9, corollary 3.2] we have $w_\varepsilon \leqslant \zeta$ in ω. Since $w_\varepsilon \leqslant 0$ in $\Omega \setminus \omega$, it follows that

$$w_\varepsilon \leqslant \zeta \quad \text{in } \Omega. \tag{3.2}$$

We multiply by $k(v_\varepsilon)$ in the second equation of $(\mathcal{S})_\varepsilon$ and we obtain

$$k(v_\varepsilon)\Delta v_\varepsilon - \beta v_\varepsilon k(v_\varepsilon) + \frac{k(v_\varepsilon)}{k(v_\varepsilon+\varepsilon)} h(u_\varepsilon+\varepsilon) = 0 \quad \text{in } \Omega. \tag{3.3}$$

Note that

$$k(v_\varepsilon)\Delta v_\varepsilon = \Delta K(v_\varepsilon) - k'(v_\varepsilon)|\nabla v_\varepsilon|^2 \quad \text{in } \Omega. \tag{3.4}$$

Since k is non-decreasing, we have

$$K(v_\varepsilon) = \int_0^{v_\varepsilon} k(t)\,\mathrm{d}t \leqslant v_\varepsilon k(v_\varepsilon) \quad \text{in } \Omega. \tag{3.5}$$

Using (3.3)–(3.5), we deduce

$$\Delta K(v_\varepsilon) - k'(v_\varepsilon)|\nabla v_\varepsilon|^2 - \beta K(v_\varepsilon) + \frac{k(v_\varepsilon)}{k(v_\varepsilon+\varepsilon)} h(u_\varepsilon+\varepsilon) \geqslant 0 \quad \text{in } \Omega.$$

Hence,

$$\Delta K(v_\varepsilon) - \beta K(v_\varepsilon) + h(u_\varepsilon+\varepsilon) \geqslant 0 \quad \text{in } \Omega. \tag{3.6}$$

By [9, theorem 3.7], there exists a positive constant $C>1$ depending only on Ω such that

$$\sup_{\bar\Omega} K(v_\varepsilon) \leqslant C \sup_{\bar\Omega} h(u_\varepsilon+\varepsilon) \leqslant C \sup_{\bar\Omega} h(v_\varepsilon + \|\zeta\|_\infty + 1).$$

Using assumption (A2) we deduce that $(v_\varepsilon)_\varepsilon$ is uniformly bounded, i.e. $\|v_\varepsilon\|_\infty \leqslant m$ for some $m>0$ independent on ε. This yields $u_\varepsilon = v_\varepsilon + w_\varepsilon \leqslant m + \|\zeta\|_\infty$ in Ω and the proof of lemma 3.2 is now complete. □

Lemma 3.3. *For all $0<\varepsilon<1$ there exists a solution $(u_\varepsilon, v_\varepsilon) \in C^2(\bar\Omega)\times C^2(\bar\Omega)$ of the system $(\mathcal{S})_\varepsilon$.*

Proof. We employ topological degree arguments. Consider the set

$$\mathcal{U} := \{(u,v) \in C^2(\bar\Omega)\times C^2(\bar\Omega) : \|u\|_\infty, \|v\|_\infty \leqslant M+1,\\ u, v \geqslant 0 \text{ in } \Omega,\ u\,|_{\partial\Omega} = v\,|_{\partial\Omega} = 0\},$$

where $M>0$ is the constant in (3.1). Define

$$\Phi_t : \mathcal{U} \to \mathcal{U}, \qquad \Phi_t(u,v) = (\Phi_t^1(u,v), \Phi_t^2(u,v)),$$

by

$$\Phi_t^1(u,v) = u - t(-\Delta+\alpha)^{-1}\left(\frac{f(u+\varepsilon)}{g(v+\varepsilon)} + \rho\right),$$

$$\Phi_t^2(u,v) = v - t(-\Delta+\beta)^{-1}\left(\frac{h(u+\varepsilon)}{k(v+\varepsilon)}\right).$$

Using lemma 3.2 we have $\Phi_t(u,v) \neq (0,0)$ on $\partial\mathcal{U}$, for all $0 \leqslant t \leqslant 1$. Therefore, by the invariance at homotopy of the topological degree it follows that

$$\deg(\Phi_1, \mathcal{U}, (0,0)) = \deg(\Phi_0, \mathcal{U}, (0,0)) = 1.$$

Hence, there exists $(u,v) \in \mathcal{U}$ such that $\Phi_1(u,v) = (0,0)$. This means that the system $(\mathcal{S})_\varepsilon$ has at least one classical solution. $\square$

Proof of theorem 3.1. Let $(u_\varepsilon, v_\varepsilon) \in C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ be a solution of $(\mathcal{S})_\varepsilon$. Then

$$\begin{aligned} \Delta(u_\varepsilon - \zeta) - \alpha(u_\varepsilon - \zeta) \leqslant 0 \quad &\text{in } \Omega, \\ u_\varepsilon - \zeta = 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

where ζ is the unique solution of (2.2). Hence, $\zeta \leqslant u_\varepsilon$ in Ω. By (3.2) it follows that

$$w_\varepsilon \leqslant \zeta \leqslant u_\varepsilon \quad \text{in } \Omega. \tag{3.7}$$

Let $\xi \in C^2(\bar{\Omega})$ be the unique positive solution of the boundary-value problem

$$\left.\begin{aligned} \Delta\xi - \beta\xi + \frac{h(\zeta)}{k(\xi+1)} = 0 \quad &\text{in } \Omega, \\ \xi = 0 \quad &\text{on } \partial\Omega. \end{aligned}\right\} \tag{3.8}$$

In view of lemma 2.2 we have $\xi \leqslant v_\varepsilon$ in Ω, so that, by lemma 3.2, the following estimates hold:

$$\left.\begin{aligned} \zeta(x) \leqslant u_\varepsilon(x) \leqslant M \quad &\text{in } \Omega, \\ \xi(x) \leqslant v_\varepsilon(x) \leqslant M \quad &\text{in } \Omega. \end{aligned}\right\} \tag{3.9}$$

Now, standard Hölder and Schauder estimates can be employed in order to deduce that $\{(u_\varepsilon, v_\varepsilon)\}_{0<\varepsilon<1}$ converges (up to a subsequence) in $C^2_{\mathrm{loc}}(\Omega) \times C^2_{\mathrm{loc}}(\Omega)$ to $(u,v) \in C^2(\Omega) \times C^2(\Omega)$. It remains only to obtain an upper bound near $\partial\Omega$ for $(u_\varepsilon, v_\varepsilon)$, which leads us to the continuity up to the boundary of the solution (u,v). This will be done by combining standard arguments with the estimate (3.7). First, by (3.6) we have

$$\Delta K(v_\varepsilon) + h(M+1) \geqslant 0 \quad \text{in } \Omega. \tag{3.10}$$

Fix $x_0 \in \partial\Omega$. Since $\partial\Omega$ is smooth, there exist $y \in \mathbb{R}^N \setminus \Omega$ and $R > 0$ such that $\bar{\Omega} \cap \bar{B}(y,R) = \partial\Omega \cap \bar{B}(y,R) = \{x_0\}$.

Let $\delta(x) := |x-y| - R$ and $\Omega_0 := \{x \in \Omega : 4(N-1)\delta(x) < R\}$.

Consider $\psi \in C^2(0,\infty)$ such that $\psi' > 0$ and $\psi'' < 0$ on $(0,\infty)$ and set $\phi(x) = \psi(\delta(x))$, $x \in \Omega_0$. Then

$$\begin{aligned} \Delta\phi(x) &= \psi'(\delta(x))\Delta\delta(x) + \psi''(\delta(x))|\nabla\delta(x)|^2 \\ &= \frac{N-1}{|x-y|}\psi'(\delta(x)) + \psi''(\delta(x)) \\ &\leqslant \frac{N-1}{R}\psi'(\delta(x)) + \psi''(\delta(x)) \quad \text{in } \Omega_0. \end{aligned}$$

Let us now choose $\psi(t) := C\sqrt{t}$, $t > 0$, where $C > 0$. Therefore,

$$\Delta\phi(x) \leqslant \tfrac{1}{4}C\delta^{-3/2}(x)\left[\frac{2(N-1)\delta(x)}{R} - 1\right] \leqslant -\tfrac{1}{8}C\delta^{-3/2}(x) < 0 \quad \text{in } \Omega_0.$$

We choose $C > 0$ large enough such that

$$\Delta\phi \leqslant -h(M+1) \quad \text{in } \Omega_0 \tag{3.11}$$

and

$$\phi\mid_{\partial\Omega_0\setminus\partial\Omega} > K(M) \geqslant \sup_{\bar{\Omega}_0} K(v_\varepsilon). \tag{3.12}$$

Furthermore, by (3.10)–(3.12) we obtain

$$\begin{aligned} \Delta(\phi - K(v_\varepsilon)) \leqslant 0 &\quad \text{in } \Omega_0, \\ \phi - K(v_\varepsilon) \geqslant 0 &\quad \text{on } \partial\Omega_0. \end{aligned}$$

This implies $\phi(x) \geqslant K(v_\varepsilon)$ in Ω_0, that is,

$$0 \leqslant v_\varepsilon(x) \leqslant K^{-1}(\phi(x)) \quad \text{in } \Omega_0.$$

Passing to the limit with $\varepsilon \to 0$ in the last inequality we have $0 \leqslant v(x) \leqslant K^{-1}(\phi(x))$ in Ω_0. Hence,

$$0 \leqslant \lim_{x\to x_0} v(x) \leqslant \lim_{x\to x_0} K(\phi(x)) = 0.$$

Since $x_0 \in \partial\Omega$ was arbitrarily chosen, it follows that $v \in C(\bar{\Omega})$. Using the fact that $u_\varepsilon = w_\varepsilon + v_\varepsilon \leqslant \zeta + v_\varepsilon$ in Ω, in the same manner we conclude that $u \in C(\bar{\Omega})$. This finishes the proof of theorem 3.1. □

The next result concerns the following system:

$$\left.\begin{aligned} \Delta u - \alpha u + \frac{u^p}{v^q} + \rho(x) = 0 &\quad \text{in } \Omega, \\ \Delta v - \beta v + \frac{u^{p+\sigma}}{v^{q+\sigma}} = 0 &\quad \text{in } \Omega, \\ u = v = 0 &\quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.13}$$

where $\sigma \geqslant 0$ is a non-negative real number.

THEOREM 3.4. *Assume that $p, q \geqslant 0$ satisfy $p - q < 1$.*

(i) *The system (3.13) has solutions for all $\sigma \geqslant 0$.*

(ii) *For any solution (u, v) of (3.13), there exist $c_1, c_2 > 0$ such that*

$$c_1 d(x) \leqslant u, v \leqslant c_2 d(x) \quad \text{in } \Omega. \tag{3.14}$$

Moreover, the following properties hold:

(a) *if $-1 < p - q < 0$, then $u, v \in C^2(\Omega) \cap C^{1,1+p-q}(\bar{\Omega})$;*

(b) *if $0 \leqslant p - q < 1$, then $u, v \in C^2(\bar{\Omega})$.*

Proof. Existence follows directly from theorem 3.1 since conditions (A1) and (A2) are satisfied.

(ii) Recall that from (2.3) we have $u \geqslant \zeta \geqslant \bar{c}\varphi_1$ in Ω. From the second equation in (3.13) we deduce that

$$\Delta v - \beta v + \bar{c}^{p+\sigma} \frac{\varphi_1^{p+\sigma}}{v^{q+\sigma}} \leqslant 0 \quad \text{in } \Omega.$$

Since $p-q<1$, we also get that $\underline{v}=\underline{c}\varphi_1$ satisfies

$$\Delta\underline{v}-\beta\underline{v}+\tilde{c}^{p+\sigma}\frac{\varphi_1^{p+\sigma}}{\underline{v}^{q+\sigma}}\leqslant 0\quad\text{in }\Omega,$$

provided that $\underline{c}>0$ is sufficiently small. Therefore, by virtue of lemma 2.2, we obtain $v\geqslant\underline{c}\varphi_1$ in Ω.

Let us now prove the second inequality in (3.14). To this aim, set $w=u-v$. With the same idea as in lemma 3.2 we obtain $\Delta w-\alpha w+\rho(x)\geqslant 0$ in the set $\{x\in\Omega : w(x)>0\}$. Hence,

$$w\leqslant\zeta\leqslant c\varphi_1\quad\text{in }\Omega. \tag{3.15}$$

Let $w^+=\max\{w,0\}$. Then v satisfies

$$\begin{aligned}\Delta v-\beta v+\frac{(w^+ + v)^{p+\sigma}}{v^{q+\sigma}}\geqslant 0&\quad\text{in }\Omega,\\ v=0&\quad\text{on }\partial\Omega.\end{aligned}$$

Consider the problem

$$\left.\begin{aligned}\Delta z-\beta z+2^{p+\sigma}z^{p-q}=0&\quad\text{in }\Omega,\\ z>0&\quad\text{in }\Omega,\\ z=0&\quad\text{on }\partial\Omega.\end{aligned}\right\} \tag{3.16}$$

The existence of a classical solution to (3.16) follows from [17, lemma 2.4]. Moreover, if $0\leqslant p-q<1$, then $z\in C^2(\bar{\Omega})$ and with the same arguments as in [10, theorem 1.1] we have $z\in C^2(\Omega)\cap C^{1,1+p-q}(\bar{\Omega})$ in the case $-1<p-q<0$. Furthermore, $z\leqslant m\varphi_1$ in Ω for some $m>0$. On the other hand, $\tilde{c}\varphi_1$ is a subsolution of (3.16) provided $\tilde{c}>0$ is small enough. Therefore, by lemma 2.1, we get $z\geqslant\tilde{c}\varphi_1$ in Ω. This last inequality together with (3.15) allows us to choose $M>1$ large enough such that $Mz\geqslant w^+$ in Ω. Hence,

$$\begin{aligned}\Delta(Mz)-\beta(Mz)+\frac{(w^+ + Mz)^{p+\sigma}}{(Mz)^{q+\sigma}}&\leqslant\Delta(Mz)-\beta(Mz)+2^{p+\sigma}(Mz)^{p-q}\\ &=M(\Delta z-\beta z+2^{p+\sigma}z^{p-q})=0\quad\text{in }\Omega.\end{aligned}$$

This means that $\bar{v}:=Mz$ verifies

$$\Delta\bar{v}-\beta\bar{v}+\frac{(w^+ +\bar{v})^{p+\sigma}}{\bar{v}^{q+\sigma}}\leqslant 0\text{ in }\Omega\quad\text{and}\quad\bar{v}=0\text{ on }\partial\Omega.$$

Since $p-q<1$, we can easily check that the mapping

$$\Psi(x,t)=-\beta t+\frac{(w^+(x)+t)^{p+\sigma}}{t^{q+\sigma}},\quad(x,t)\in\bar{\Omega}\times(0,\infty),$$

satisfies the hypotheses in lemma 2.1. Moreover, we have

$$\begin{gathered}\Delta\bar{v}+\Psi(x,\bar{v})\leqslant 0\leqslant\Delta v+\Psi(x,v)\quad\text{in }\Omega,\\ \bar{v},v>0\quad\text{in }\Omega,\qquad\bar{v}=v=0\quad\text{on }\partial\Omega,\qquad\Delta\bar{v}\in L^1(\Omega).\end{gathered}$$

Hence, by lemma 2.1 we obtain

$$v \leqslant \bar{v} \leqslant \tilde{c}\varphi_1 \quad \text{in } \Omega. \tag{3.17}$$

Combining (3.15) and (3.17), we deduce $u = w + v \leqslant C\varphi_1$ in Ω, for some $C > 0$. This completes the proof of (ii). As a consequence, there exists $M > 1$ such that

$$0 \leqslant \frac{u^p}{v^q}, \frac{u^{p+\sigma}}{v^{q+\sigma}} \leqslant M\varphi_1^{p-q} \quad \text{in } \Omega.$$

If $0 \leqslant p - q < 1$, then by classical regularity arguments we have $u, v \in C^2(\bar{\Omega})$. If $-1 < p - q < 0$, then the same method as in [10, theorem 1.1] can be employed in order to obtain $u, v \in C^2(\Omega) \cap C^{1,1+p-q}(\bar{\Omega})$.

This finishes the proof of theorem 3.4. □

4. Uniqueness of the solution in one dimension

In this section we are concerned with the uniqueness of the solution associated to the one-dimensional system

$$\left.\begin{aligned} u'' - \alpha u + \frac{u^p}{v^q} + \rho(x) = 0 \quad \text{in } (0,1), \\ v'' - \beta v + \frac{u^{p+\sigma}}{v^{q+\sigma}} = 0 \quad \text{in } (0,1), \\ u(0) = u(1) = 0, \qquad v(0) = v(1) = 0. \end{aligned}\right\} \tag{4.1}$$

Our approach is inspired by the methods developed in [1], where a C^2-regularity of the solution up to the boundary is needed. We shall restrict our attention to the case when $0 < q \leqslant p \leqslant 1$. Thus, by virtue of theorem 3.4, any solution of (4.1) belongs to $C^2[0,1] \times C^2[0,1]$. By Hopf's maximum principle we also have that $u'(0) > 0$, $v'(0) > 0$, $u'(1) < 0$ and $v'(1) < 0$ for any solution (u, v) of system (4.1).

The main result of this section is the following.

THEOREM 4.1. *Assume that* $0 < q \leqslant p \leqslant 1$, $\sigma \geqslant 0$. *Then system (4.1) has a unique solution* $(u, v) \in C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$.

Proof of theorem 4.1. Existence follows from theorem 3.4. We prove here only the uniqueness. Suppose that there exist $(u_1, v_1), (u_2, v_2) \in C^2[0,1] \times C^2[0,1]$ two distinct solutions of (4.1).

First we claim that we cannot have $u_2 \geqslant u_1$ or $v_2 \geqslant v_1$ in $[0,1]$. Indeed, let us assume that $u_2 \geqslant u_1$ in $[0,1]$. Then

$$v_2'' - \beta v_2 + \frac{u_2^{p+\sigma}}{v_2^{q+\sigma}} = 0 = v_1'' - \beta v_1 + \frac{u_1^{p+\sigma}}{v_1^{q+\sigma}} \quad \text{in } (0,1)$$

and, by lemma 2.2, we deduce that $v_2 \geqslant v_1$ in $[0,1]$. On the other hand,

$$u_1'' - \alpha u_1 + \frac{u_1^p}{v_2^q} + \rho(x) \leqslant 0 = u_2'' - \alpha u_2 + \frac{u_2^p}{v_2^q} + \rho(x) \quad \text{in } (0,1). \tag{4.2}$$

Note that the mapping $\Psi(x,t) := -\alpha t + t^p/v_2(x)^q + \rho(x)$, $(x,t) \in (0,1) \times (0,\infty)$ satisfies the hypotheses in lemma 2.1 since $p \leqslant 1$. Hence, $u_2 \leqslant u_1$ in $[0,1]$, that is

$u_1 \equiv u_2$. This also implies $v_1 \equiv v_2$, which is a contradiction. Replacing u_1 by u_2 and v_1 by v_2, we find that the situation $u_1 \geqslant u_2$ or $v_1 \geqslant v_2$ in $[0,1]$ is not possible.

Set $U := u_2 - u_1$ and $V := v_2 - v_1$. From the above arguments, both U and V change sign in $(0,1)$. The key result in the approach is the following.

PROPOSITION 4.2. *U and V vanish only at finitely many points in the interval* $[0,1]$.

Proof. We write the system (4.1) as

$$\begin{aligned} \boldsymbol{W}''(x) + A(x)\boldsymbol{W}(x) &= \mathbf{0} \quad \text{in } (0,1), \\ \boldsymbol{W}(0) = \boldsymbol{W}(1) &= \mathbf{0}, \end{aligned}$$

where $\boldsymbol{W} = (U,V)$ and $A(x) = (A_{ij}(x))_{1\leqslant i,j\leqslant 2}$ is a 2×2 matrix defined as

$$A_{11}(x) = -\alpha + \begin{cases} \dfrac{1}{v_2^q(x)} \cdot \dfrac{u_2^p(x) - u_1^p(x)}{u_2(x) - u_1(x)}, & u_1(x) \neq u_2(x), \\ p\dfrac{u_1^{p-1}(x)}{v_1^q(x)}, & u_1(x) = u_2(x), \end{cases}$$

$$A_{12}(x) = \begin{cases} -\dfrac{u_1^p(x)}{v_1^q(x)v_2^q(x)} \cdot \dfrac{v_2^q(x) - v_1^q(x)}{v_2(x) - v_1(x)}, & v_1(x) \neq v_2(x), \\ -q\dfrac{u_1^p(x)}{v_1^{q+1}(x)}, & v_1(x) = v_2(x), \end{cases}$$

$$A_{21}(x) = \begin{cases} \dfrac{1}{v_2^{q+\sigma}(x)} \cdot \dfrac{u_2^{p+\sigma}(x) - u_1^{p+\sigma}(x)}{u_2(x) - u_1(x)}, & u_1(x) \neq u_2(x), \\ (p+\sigma)\dfrac{u_1^{p+\sigma-1}(x)}{v_1^{q+\sigma}(x)}, & u_1(x) = u_2(x), \end{cases}$$

$$A_{22}(x) = -\beta - \begin{cases} \dfrac{u_1^{p+\sigma}(x)}{v_1^{q+\sigma}(x)v_2^{q+\sigma}(x)} \cdot \dfrac{v_2^{q+\sigma}(x) - v_1^{q+\sigma}(x)}{v_2(x) - v_1(x)}, & v_1(x) \neq v_2(x), \\ (q+\sigma)\dfrac{u_1^{p+\sigma}(x)}{v_1^{q+\sigma+1}(x)}, & v_1(x) = v_2(x). \end{cases}$$

Therefore, $A \in C(0,1)$ and $A_{12}(x) \neq 0$, $A_{21}(x) \neq 0$ for all $x \in (0,1)$. Moreover, $xA(x)$, $(1-x)A(x)$ are bounded in $L^\infty(0,1)$. Indeed, let us note first that, by (3.14) in theorem 3.4, there exist $c_1, c_2 > 0$ such that

$$c_1 \leqslant \frac{u_i}{\min\{x, 1-x\}}, \frac{v_i}{\min\{x, 1-x\}} \leqslant c_2, \quad i = 1,2 \quad \text{in } (0,1).$$

Then, by the mean-value theorem, we have

$$\begin{aligned} x|A_{12}(x)| &\leqslant qx\frac{u_1^p(x)}{v_1^q(x)v_2^q(x)} \max\{v_1^{q-1}(x), v_2^{q-1}(x)\} \\ &\leqslant qx^{p-q}\left(\frac{u_1(x)}{x}\right)^p \max\left\{\left(\frac{x}{v_1(x)}\right)^{q+1}, \left(\frac{x}{v_2(x)}\right)^{q+1}\right\} \\ &\leqslant cx^{p-q} \quad \text{for all } 0 < x \leqslant \tfrac{1}{2}. \end{aligned}$$

We obtain similar estimates for xA_{11}, xA_{21} and xA_{22}. This allows us to employ [1, lemmas 7 and 8]. Note that condition $xA(x) \in L^\infty(0,1)$ suffices in order to obtain the same conclusion as in [1, lemma 8]. In particular, we get that U and V vanish only at finitely many points in any compact interval $[a,b] \subset (0,1)$.

It remains to show that U and V cannot have infinitely many zeros in the neighbourhood of $x=0$ and $x=1$. We shall consider only the case where U and V have infinitely many zeros near $x=0$, the case where this situation occurs near $x=1$ being similar.

Without loss of generality, we may assume that U has infinitely many zeros in a neighbourhood of $x=0$. Since $U \in C^2[0,1]$, Rolle's theorem implies that both U' and U'' have infinitely many zeros near $x=0$. As a consequence, we obtain $U'(0)=0$, that is, $u_1'(0)=u_2'(0)$.

If $V'(0)=0$, then $\boldsymbol{W}(0)=\boldsymbol{W}'(0)=\boldsymbol{0}$ and by [1, lemma 8] we deduce $\boldsymbol{W}\equiv\boldsymbol{0}$ in $[0,\frac{1}{2}]$, which is a contradiction. Hence, $V'(0)\neq 0$. Subtracting the first equation in (4.1) corresponding to u_1 and u_2, we have

$$\begin{aligned} U''(x) &= \alpha U(x) + \frac{u_1^p(x)}{v_1^q(x)} - \frac{u_2^p(x)}{v_2^q(x)} \\ &= x^{p-q}\left\{\alpha\frac{U(x)}{x^{p-q}} + \left(\frac{u_1(x)}{x}\right)^p\left(\frac{x}{v_1(x)}\right)^q - \left(\frac{u_2(x)}{x}\right)^p\left(\frac{x}{v_2(x)}\right)^q\right\}. \end{aligned}$$

Since $0 \leqslant p-q<1$, $u_1'(0)=u_2'(0)$ and $v_1'(0)\neq v_2'(0)$ we find

$$\begin{aligned} \lim_{x\searrow 0}&\left\{\alpha\frac{U(x)}{x^{p-q}} + \left(\frac{u_1(x)}{x}\right)^p\left(\frac{x}{v_1(x)}\right)^q - \left(\frac{u_2(x)}{x}\right)^p\left(\frac{x}{v_2(x)}\right)^q\right\} \\ &= u_1'^p(0)\left(\frac{1}{v_1'^q(0)} - \frac{1}{v_1'^q(0)}\right) \neq 0. \end{aligned}$$

Therefore, U'' has constant sign in a small neighbourhood of $x=0$, which contradicts our assumption. The proof of proposition 4.2 is now complete. □

Set

$$\begin{aligned} \mathcal{I}^+ &= \{x\in[0,1] : U(x)\geqslant 0\}, & \mathcal{I}^- &= \{x\in[0,1] : U(x)\leqslant 0\}, \\ \mathcal{J}^+ &= \{x\in[0,1] : V(x)\geqslant 0\}, & \mathcal{J}^- &= \{x\in[0,1] : V(x)\leqslant 0\}. \end{aligned}$$

According to proposition 4.2, the above sets consist of finitely many disjoint closed intervals. Therefore, $\mathcal{I}^+ = \bigcup_{i=1}^m I_i^+$. For simplicity, let I^+ denote any interval I_i^+ and we use similar notations for I^-, J^+ and J^-.

Lemma 4.3. *For any intervals I^+, I^-, J^+ and J^- defined above, the following situations cannot occur:*

(i) $I^+ \subset J^+$;

(ii) $I^- \subset J^-$;

(iii) $J^+ \subset I^-$;

(iv) $J^- \subset I^+$.

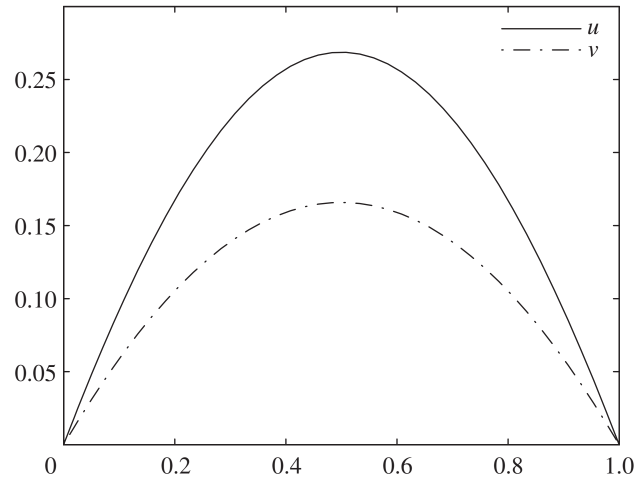

Figure 1. The solution (u, v) of system $(\mathcal{S})_\varepsilon$ with $\alpha = 1$, $\beta = 0.5$, $p = q = 1$, $\varepsilon = 10^{-2}$, $\sigma = 0$ and $\rho(x) = \sin(\pi x)$.

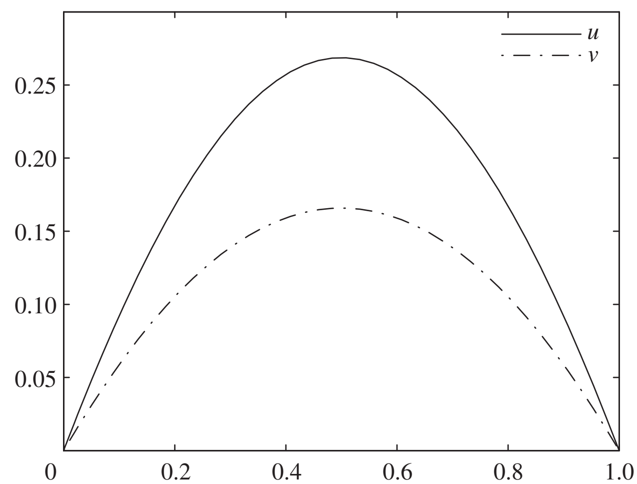

Figure 2. The solution (u, v) of system $(\mathcal{S})_\varepsilon$ with $\alpha = 1$, $\beta = 0.5$, $p = q = 1$, $\varepsilon = 10^{-2}$, $\sigma = 2$ and $\rho(x) = \sin(\pi x)$.

Proof. (i) Assume that $I^+ \subset J^+$. Since $v_2 \geqslant v_1$ in I^+ we deduce that the inequality (4.2) holds in I^+. Using the fact that $u_2 = u_1$ on ∂I^+, by virtue of lemma 2.1 we obtain $u_2 \leqslant u_1$ in I^+. Hence, $u_2 \equiv u_1$ in I^+, which contradicts proposition 4.2. Similarly, we can prove (ii).

(iii) Assume that $J^+ \subset I^-$. Then $u_1^{p+\sigma}/v_1^{q+\sigma} \geqslant u_2^{p+\sigma}/v_2^{q+\sigma}$ in J^+. Note that $V = v_2 - v_1$ verifies

$$\begin{aligned} V'' - \beta V &= \frac{u_1^{p+\sigma}}{v_1^{q+\sigma}} - \frac{u_2^{p+\sigma}}{v_2^{q+\sigma}} \geqslant 0 \quad &&\text{in } J^+, \\ V &= 0 &&\text{on } \partial J^+. \end{aligned}$$

By the maximum principle, it follows that $V \leqslant 0$ in J^+, i.e. $v_2 \leqslant v_1$ in J^+. This yields $v_2 \equiv v_1$ in J^+ which again contradicts proposition 4.2. The proof of (iv) follows in the same manner. □

From now on, the proof of theorem 4.1 is the same as in [1, theorem 6]. □

REMARK 4.4. As a consequence of theorem 3.1, the unique solution (u, v) of system (3.13) can be approximated by the solutions of $(\mathcal{S})_\varepsilon$. Furthermore, the shooting method, combined with the Broyden method in order to avoid the derivatives, is appropriate to numerical approximation of the solution of (3.13). We have considered $\alpha = 1$, $\beta = 0.5$, $p = q = 1$, $\varepsilon = 10^{-2}$ and $\rho(x) = \varphi_1(x) = \sin(\pi x)$. The solution (u, v) of $(\mathcal{S})_\varepsilon$ is plotted for $\sigma = 0$ (figure 1) and $\sigma = 2$ (figure 2), respectively.

Some of the results obtained in this paper have been communicated in [6].

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