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Global Existence and Finite Time Blow-up for the m-Laplacian Parabolic Problem

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Abstract In this paper, we consider an initial boundary value problem of m-Laplacian parabolic equation arising in various physical models. We tackle this problem at three different initial energy levels. For the sub-critical initial energy, we obtain the blow-up result and estimate the lower and upper bounds of the blow-up time. For the critical initial energy, we show the global existence, asymptotic behavior, finite time blow-up and the lower bound of the blow-up time. For the sup-critical initial energy, we prove the finite time blow-up and estimate the lower and upper bounds of the blow-up time.

Keywords m-Laplacian parabolic equation, blow-up, blow-up time, global existence

MR(2010) Subject Classification 35K05, 35B44, 35A01

1 Introduction

In this paper we consider the following initial boundary value problem of the m-Laplacian parabolic equation

$$u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) = |u|^{p-2}u, \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.3}$$

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where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, the nonlinear term $\mathrm{div}(|\nabla u|^{m-2} \nabla u) = \nabla \cdot (|\nabla u|^{m-2} \nabla u) = \Delta_m u$ is called m-Laplace operator, where

$$|\nabla u|^{m-2} = \left\{ \left(\frac{\partial u}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n} \right)^2 \right\}^{\frac{m-2}{2}}$$

and

$$2 \leq m < p < \infty, \tag{1.4}$$

p also satisfies (H) as follows

$$\text{(H)} \quad 2 < p < \infty, \ \text{if } n \leq m; \qquad m < p < \frac{nm}{n-m}, \quad \text{if } n > m.$$

The prototypical model of the reaction-diffusion equation can be written as [7]

$$\frac{\partial u}{\partial t} = \mathrm{div}(D(u, \nabla u)\nabla u) + F(u), \quad u(x,0) = u_0(x). \tag{1.5}$$

Here D is a coefficient, that could be a constant as the simplest case of linear diffusion [13, 14, 26, 31], a function depending on the domain of definition [4–6], a matric function $D(u) = A(x,t)$ to model various ecological and evolutionary processes in spatio-temporally varying environments [25], a function depending on u, i.e. $D(u) = |u|^{m-1}$ to mean porous medium equation [15, 28, 33, 37], a function $D(u, \nabla u) = |\nabla u|^{m-2}$ as the celebrated m-Laplacian parabolic equation [9, 19, 22, 42], or in general a function $D = D(u, \nabla u)$ [12, 32], which includes the 3D incompressible micropolar equations with fractional dissipations [38] and the possibility of fractional diffusion associated with nonlocal quantum mechanics [27], and the function $F(u)$ represents the reaction term. These above diverse model equations contained in (1.5) have corresponding various physical backgrounds and received enormous attentions.

In the present paper, we focus on the m-Laplacian parabolic equation in form of (1.1) which, in mathematical form can be regarded as a comparison between the dynamical behavior of the model with nonlinear diffusion and the evolution property associated with the linear diffusion model, and also the mathematical description of the corresponding physical phenomena introduced as follows. Besides the mathematical formal extension in (1.5), the m-Laplacian parabolic equation seems to be first introduced in [30] with the name n-diffusion equation as a generalized form of diffusion related to the unsteady vertical heat transfer from a horizontal surfaces by turbulent free convection, and unsteady turbulent flow of a liquid with a free surface over a plane. The heat conduction in a uniform temperature-dependent medium suggests the equation $S(u)\frac{\partial u}{\partial t} = \mathrm{div}(K(u)\nabla u)$, where u is the temperature, $S(u)$ is the volumetric heat capacity and $K(u)$ is the thermal conductivity. As $S(u)$ is always supposed to be a constant, $\frac{K(u)}{s}$ is the thermal diffusivity [2]. It is customary to say that (1.5) with $F(u) = 0$ is a parabolic equation with implicit degeneracy, which takes the equation of Newtonian polytropic filtration $u_t = \Delta(|u|^{m-1}u)$, $m > 1$ as an important example, and parabolic for $u \neq 0$ and degenerates for $u = 0$ to describe the non-stationary flow of a compressible Newtonian fluid in a porous medium (filtration) under polytropic conditions [3, 10, 11]. If the flow is not polytropic, the above model should be replaced by the so-called non-Newtonian elastic filtration model equation $u_t = \mathrm{div}(|\nabla u|^{m-2}\nabla u)$, $m > 2$ to describe the non-stationary flow in a porous medium of fluids with a power dependence of the tangential stress on the velocity of the displacement under elastic condition [17, 23, 39]. If the medium has heat sources or sinks whose power depends on the temperature, then we need to consider the special case of (1.5) like $u_t = \mathrm{div}(|\nabla u|^{m-2}\nabla u)+F(u)$, $m > 2$. If, moreover, $F'(u) < 0$ for $u > 0$, then we shall call (1.5) the nonlinear heat equation with absorption. If $F'(u) > 0$ at least on some interval $(0, u_0)$, we shall call (1.5) the nonlinear heat equation with sources [19]. Next we shall recall some results about the above model equations.

For the Cauchy problem of

$$u_t = \mathrm{div}(|\nabla u|^{m-2}\nabla u), \quad m > 2, \tag{1.6}$$

based on the existence of global weak solution, the large time behavior of the global solution was discussed in [42]. Choe and Kim [9] considered some interesting issues involving the interface given by the Cauchy problem of (1.6), and showed that its interface is Lipschitz continuous for large time, and the interface is globally Lipschitz continuous for some special initial data. The regularity of such interface was also focused on by Ko in [22], and some geometric properties for the long time behavior of the solution were studied in [24]. The potential theoretic aspects of certain model equation were studied in [18] and [21].

Differently from the above problem without nonlinear source term, the following equation with nonlinear inhomogeneous source term

$$u_t - \mathrm{div}(|\nabla u|^{m-2}\nabla u) = f(u) \tag{1.7}$$

will lose many good properties belonging to homogeneous model, insteadly, a lot of interesting new phenomena can be observed. Tsutsumi [36] studied the initial boundary value problem of (1.7) for $f(u) = u^{1+\alpha}$, $m < \alpha + 2$ (α is a nonnegative real number) and he proved the global existence and uniqueness of solution with sub-critical initial energy, i.e., $J(u_0) < d$ and blow-up in finite time with negative initial energy, i.e., $J(u_0) < 0$, where d is the so-called potential well depth or mountain pass level which is characterized by $d = \min_{u\in W_0^{1,m}(\Omega)\setminus\{0\}} \max_{s\geq 0} J(su)$, where $J(u)$ is the so-called potential energy functional and will be defined later. Five years later, the above conclusions were improved in [16] by extending the negative initial energy blow-up to the positive initial energy blow-up with $J(u_0) < d_1$, where d_1 is smaller than the depth of potential well d, and additionally obtaining the decay of the global solution when $J(u_0) < d$ (not d_1). During the following two decades, a lot of efforts have been devoted to searching the relations between the initial data and the behavior of the corresponding solution to such problems, like the local existence with large initial datum [1], blow-up solution to the problem with critical Sobolev exponent [35], the behavior of the solution from the initial data near the flat hats [38]. Most recently, the initial boundary value problem of (1.7) with the power-type growth conditions on the nonlinearity $f(u)$ was considered in [8], and the nonlinear term in (1.1), i.e. $|u|^{p-2}u$ satisfies certain growth conditions in [8]. They obtained the finite time blow-up for both the classical solution ($m = 2$) and the weak solution ($m > 2$) under some conditions on the initial data, which indicate $J(u_0) < -\gamma$ ($\gamma > 0$), the so-called negative initial energy blow-up. Hence the positive initial energy case, i.e. $J(u_0) > 0$ becomes an interesting unsolved issue. In the present paper, we aim to tackle this issue by employing the variational method, i.e. the so-called potential well method, hence the case $J(u_0) > 0$ will be divided into two cases: $0 < J(u_0) < d$ and $J(u_0) > 0$, where d is the depth of potential well, also called mountain pass level, which will be defined later. We divide this issue in this way not only due to such parameter d, but also considering the structures of the main conclusions for each cases, which will be explained in the end of the paper. It is worth mentioning that the steady-state problems related to parabolic problems play an important role in the potential well theory. The usability of the potential well theory to problem (1.1) was provided in [29], like its applications to the semilinear pseudo-parabolic equations [41] and the coupled parabolic systems [40].

In order to give a clear picture of the already well-established conclusions, the main conclusions obtained in the present paper and the still unsolved problems, we use Table 1 to show the relations among all of above.

Initial data	Global existence	Asymptotic behavior of the global solution	Blow-up	Lower bound of the maximum existence time T	Upper bound of the maximum existence time T
$J(u_0) < -\gamma$	\	\	[8]	*	[8]
$J(u_0) < 0$	\	\	[36]	*	*
$0 < J(u_0) < d$	[36]	[16]	*	*	*
$J(u_0) = d$	*	*	*	*	?
$J(u_0) > 0$	?	?	*	*	*

Table 1 Research background: the symbol "\" denotes the case that does not hold under this condition; reference number denotes the literature announcing the corresponding results; the symbol "*" denotes the results obtained in the present paper; and "?" denotes the still unsolved problems. The detailed descriptions of $J(u_0) > 0$ is in Remark 5.4.

This article is structured as follows:

(i) Sub-critical initial energy case ($J(u_0) < d$) in Section 3: by introducing two different auxiliary functions, we obtain the finite time blow-up of solution for problem (1.1)–(1.3) and estimate the lower and upper bounds of the blow-up time, which also hold for $J(u_0) < -\gamma$ and $J(u_0) < 0$ as shown in Table 1.

(ii) Critical initial energy case ($J(u_0) = d$) in Section 4: we get the global existence, asymptotic behavior, finite time blow-up of solutions and estimate the lower bound of the blow-up time.

(iii) Sup-critical initial energy case ($J(u_0) > 0$) in Section 5: by using the concave function method instead of comparison principle, we get the finite time blow-up of solution, and estimate the lower and upper bounds of the blow-up time.

2 Preliminaries

First, we denote by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$ and $\|\nabla\cdot\|_m$ the Dirichlet norm in $W_0^{1,m}(\Omega)$. Moreover, from now on, C denotes various positives constants depending on the known constants and may be different at each appearance.

Next, we introduce some functionals and sets as follows

$$J(u) = \frac{1}{m}\|\nabla u\|_m^m - \frac{1}{p}\|u\|_p^p,$$

$$I(u) = \|\nabla u\|_m^m - \|u\|_p^p,$$
$$W = \{u \in W_0^{1,m}(\Omega) \mid I(u) > 0, J(u) < d\} \cup \{0\}, \tag{2.1}$$
$$V = \{u \in W_0^{1,m}(\Omega) \mid I(u) < 0, J(u) < d\} \tag{2.2}$$

and the depth of potential well d is defined as [29]

$$d = \inf_{u \in \mathcal{N}} J(u),$$

where the Nehari manifold is defined by

$$\mathcal{N} = \{u \in W_0^{1,m}(\Omega) \mid I(u) = 0, \|\nabla u\|_m \neq 0\},$$

and $\mathcal{N}$ separates two unbounded sets

$$\mathcal{N}_+ = \{u \in W_0^{1,m}(\Omega) \mid I(u) > 0\} \cup \{0\}$$

and

$$\mathcal{N}_- = \{u \in W_0^{1,m}(\Omega) \mid I(u) < 0\}.$$

Then, we introduce the definition of weak solution for problem (1.1)–(1.3).

Definition 2.1 (Weak solution) *A function $u = u(x,t)$ is called a weak solution of problem* (1.1)–(1.3) *on $\Omega \times (0,T)$, if $u \in L^\infty(0,T; W_0^{1,m}(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ satisfying the following conditions*:

(i) *for any $\nu \in W_0^{1,m}(\Omega)$, $t \in (0,T)$,*

$$(u_t, \nu) + (|\nabla u|^{m-2}\nabla u, \nabla \nu) = (|u|^{p-2}u, \nu); \tag{2.3}$$

(ii) $u(x,0) = u_0(x)$ *in* $W_0^{1,m}(\Omega)$;

(iii) *for* $0 \leq t < T$,

$$\int_0^t \|u_\tau\|_2^2 d\tau + J(u) \leq J(u_0). \tag{2.4}$$

Then, we give the existence theorem of the local solution of problem (1.1)–(1.3) established in [36].

Theorem 2.2 (Local solution) *Let $u_0 \in W_0^{1,m}(\Omega)\backslash\{0\}$ and p satisfy* (H). *Then there exist a $T > 0$ and a unique weak solution u of* (1.1)–(1.3) *satisfying $u \in C(0,T; W_0^{1,m}(\Omega))$, and the energy inequality*

$$\int_0^t \|u_\tau\|_2^2 d\tau + J(u(t)) \leq J(u_0), \quad 0 \leq t \leq T,$$

where T is the maximum existence time of solution $u(t)$. Moreover,

(i) *If $T < \infty$, then*

$$\lim_{t \to T} \|u\|_q = \infty \text{ for all } q > 1 \text{ such that } q > \frac{n(p-m)}{m};$$

(ii) *If $T = \infty$, then $u(t)$ is a global solution of problem* (1.1)–(1.3).

Here, we have the following qualitative analysis about $J(u)$ and $I(u)$.

Lemma 2.3 *Let $u \in W_0^{1,m}(\Omega)$ and $\|\nabla u\|_m \neq 0$. Then*

(i) $\lim_{\lambda \to 0} J(\lambda u) = 0$, $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$;

(ii) *There exists a unique* $\lambda^* = \lambda^*(u)$ *on the interval* $0 < \lambda < \infty$ *such that*

$$\frac{d}{d\lambda} J(\lambda u)|_{\lambda=\lambda^*} = 0;$$

(iii) $J(\lambda u)$ *is increasing on* $0 \le \lambda \le \lambda^*$, *decreasing on* $\lambda^* \le \lambda < +\infty$ *and takes the maximum at* $\lambda = \lambda^*$;

(iv) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ *for* $\lambda^* < \lambda < \infty$ *and* $I(\lambda^* u) = 0$.

Proof (i) By the definition of $J(u)$ and (1.4), we get

$$J(\lambda u) = \frac{\lambda^m}{m} \|\nabla u\|_m^m - \frac{\lambda^p}{p} \|u\|_p^p,$$

which gives

$$\lim_{\lambda \to 0} J(\lambda u) = 0$$

and

$$\lim_{\lambda \to +\infty} J(\lambda u) = -\infty.$$

(ii) According to the definition of $J(\lambda u)$, we know

$$\begin{aligned} \frac{d}{d\lambda} J(\lambda u) &= \lambda^{m-1} \|\nabla u\|_m^m - \lambda^{p-1} \|u\|_p^p \\ &= \lambda^{m-1} (\|\nabla u\|_m^m - \lambda^{p-m} \|u\|_p^p), \end{aligned} \tag{2.5}$$

which implies that $\frac{d}{d\lambda} J(\lambda u)|_{\lambda=\lambda^*} = 0$ for

$$\lambda = \left(\frac{\|\nabla u\|_m^m}{\|u\|_p^p} \right)^{\frac{1}{p-m}} := \lambda^*(u).$$

(iii) In view of (ii), we obtain

$$\frac{d}{d\lambda} J(\lambda u) > 0 \quad \text{for } 0 < \lambda < \lambda^*,$$
$$\frac{d}{d\lambda} J(\lambda u) < 0 \quad \text{for } \lambda^* < \lambda < \infty,$$

and the maximum of $J(\lambda u)$ is achieved at $\lambda = \lambda^*$.

(iv) By the definition of $I(u)$ and (2.5), we have

$$I(\lambda u) = \lambda^m \|\nabla u\|_m^m - \lambda^p \|u\|_p^p = \lambda \frac{d}{d\lambda} J(\lambda u),$$

then we can get the conclusion by (iii). □

Next, we find a ball in $W_0^{1,m}(\Omega)$ space with a radius as $\|\nabla u\|_m$ to reveal the relations between $I(u)$, $\|\nabla u\|_m$ and the depth of potential well d.

Lemma 2.4 *Let* $u \in W_0^{1,m}(\Omega)$ *and assume that* (1.4), (H) *and* $J(u) \le d$ *hold.*

(i) *If* $0 < \|\nabla u\|_m < r$, *then* $I(u) > 0$ *and*

$$\|\nabla u\|_m^m < \frac{mp}{p-m} d.$$

(ii) *If*

$$\|\nabla u\|_m^m > \frac{mp}{p-m} d,$$

then $I(u) < 0$ *and* $\|\nabla u\|_m > r$.

(iii) *If* $I(u) = 0$, *then* $\|\nabla u\|_m = 0$ *or*

$$r^m \le \|\nabla u\|_m^m \le \frac{mp}{p-m} d,$$

where $r = (\frac{1}{C_*^p})^{\frac{1}{p-m}}$ *and* C_* *is the best embedding constant from* $W_0^{1,m}(\Omega)$ *to* $L^p(\Omega)$.

Proof (i) From (H), (1.4) and $0 < \|\nabla u\|_m < r$, we have

$$\|u\|_p^p \le C_*^p \|\nabla u\|_m^p = C_*^p \|\nabla u\|_m^{p-m} \|\nabla u\|_m^m < \|\nabla u\|_m^m,$$

which gives $I(u) > 0$. According to the definition of $J(u)$, (1.4) and $I(u) > 0$, we compute

$$\begin{aligned} J(u) &= \frac{1}{m}\|\nabla u\|_m^m - \frac{1}{p}\|u\|_p^p \\ &= \left(\frac{1}{m} - \frac{1}{p}\right)\|\nabla u\|_m^m + \frac{1}{p}(\|\nabla u\|_m^m - \|u\|_p^p) \\ &= \frac{p-m}{mp}\|\nabla u\|_m^m + \frac{1}{p} I(u) \\ &> \frac{p-m}{mp}\|\nabla u\|_m^m, \end{aligned} \tag{2.6}$$

then $J(u) \le d$ gives

$$\frac{p-m}{mp}\|\nabla u\|_m^m < d,$$

i.e.

$$\|\nabla u\|_m^m < \frac{mp}{p-m} d.$$

(ii) By (2.6) and $\|\nabla u\|_m^m > \frac{mp}{p-m} d$, we have

$$\begin{aligned} J(u) &= \frac{p-m}{mp}\|\nabla u\|_m^m + \frac{1}{p} I(u) \\ &> d + \frac{1}{p} I(u), \end{aligned}$$

then $J(u) \le d$ shows

$$I(u) < 0,$$

which also means $\|\nabla u\|_m \ne 0$ due to Sobolev inequality. Then $I(u) < 0$ gives

$$\|\nabla u\|_m^m < \|u\|_p^p \le C_*^p \|\nabla u\|_m^{p-m} \|\nabla u\|_m^m,$$

that is $\|\nabla u\|_m > r$.

(iii) As $I(u) = \|\nabla u\|_m^m - \|u\|_p^p = 0$. If $\|\nabla u\|_m \ne 0$, then by

$$\|\nabla u\|_m^m = \|u\|_p^p \le C_*^p \|\nabla u\|_m^{p-m} \|\nabla u\|_m^m,$$

we get $\|\nabla u\|_m \ge r$. By (2.6) and $I(u) = 0$, we see

$$J(u) = \frac{p-m}{mp}\|\nabla u\|_m^m,$$

combining $J(u) \le d$, which yields

$$\|\nabla u\|_m^m \le \frac{mp}{p-m} d. \qquad \square$$

In the following lemma, we give the expression of d in term of r, prove the nonincreasing of $J(u)$ and show a relation between $J(u)$, $I(u)$ and d.

Lemma 2.5 (i) *If r is defined in Lemma 2.4, we have*

$$d = \frac{p-m}{mp} r^m. \tag{2.7}$$

(ii) *The potential energy $J(u)$ is nonincreasing.*

(iii) *If $u \in W_0^{1,m}(\Omega)$ and $I(u) < 0$, then we have the relation between $J(u)$, $I(u)$ and the potential well depth d as follows*

$$I(u) < p(J(u) - d). \tag{2.8}$$

Proof (i) For all $u \in \mathcal{N}$, by (iii) of Lemma 2.4 we know $\|\nabla u\|_m \geq r$, which combining (2.6) gives

$$\begin{aligned} J(u) &= \frac{p-m}{mp}\|\nabla u\|_m^m + \frac{1}{p} I(u) \\ &= \frac{p-m}{mp}\|\nabla u\|_m^m \\ &\geq \frac{p-m}{mp} r^m. \end{aligned}$$

Therefore, by the definition of d, we get (2.7).

(ii) Let $\nu = u_t$ in (2.3), then we have

$$\int_\Omega |u_t|^2 dx + \frac{d}{dt}\int_\Omega \frac{1}{m}|\nabla u|^m dx = \frac{d}{dt}\int_\Omega \frac{1}{p}|u|^p dx,$$

which says

$$\frac{d}{dt}\left(\int_\Omega \left(\frac{1}{m}|\nabla u|^m - \frac{1}{p}|u|^p\right) dx\right) = -\int_\Omega |u_t|^2 dx,$$

that is

$$J'(t) = \frac{d}{dt} J(u) = -\int_\Omega |u_t|^2 dx \leq 0.$$

(iii) According to (iv) of Lemma 2.3 and $I(u) < 0$, we know that there exists a $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$. Set

$$h(\lambda) := pJ(\lambda u) - I(\lambda u), \quad \lambda > 0.$$

By the definition of $J(u)$, $I(u)$, (1.4) and (ii) in Lemma 2.4, we derive

$$\begin{aligned} h'(\lambda) &= p\frac{dJ(\lambda u)}{d\lambda} - \frac{dI(\lambda u)}{d\lambda} \\ &= p(\lambda^{m-1}\|\nabla u\|_m^m - \lambda^{p-1}\|u\|_p^p) - m\lambda^{m-1}\|\nabla u\|_m^m + p\lambda^{p-1}\|u\|_p^p \\ &= (p-m)\lambda^{m-1}\|\nabla u\|_m^m \\ &> (p-m)\lambda^{m-1} r^m > 0. \end{aligned}$$

Hence $h(\lambda)$ is strictly increasing for $\lambda > 0$, then $h(1) > h(\lambda^*)$ for $\lambda^* \in (0,1)$. By the definition of d and the fact $I(\lambda^* u) = 0$, we get

$$pJ(u) - I(u) > pJ(\lambda^* u) - I(\lambda^* u) = pJ(\lambda^* u) \geq pd,$$

which gives (2.8) immediately. □

3 Global Existence, Asymptotic Behavior and Blow-up in Finite Time with $J(u_0) < d$

In this section, we prove the finite time blow-up and estimate the upper and lower bounds of the blow-up time of solution for problem (1.1)–(1.3) with $J(u_0) < d$. The global existence and asymptotic behavior of solution for problem (1.1)–(1.3) with $J(u_0) < d$ has been obtained by Tsutsumi [36] and Ishii [16]. So we omit the proof and only mention it in order to show the systematical conclusions.

Theorem 3.1 ([16, 36]) *Let p satisfy* (H) *and $u_0 \in W_0^{1,m}(\Omega)$. Assume that $J(u_0) < d$ and $I(u_0) > 0$. Then problem* (1.1)–(1.3) *admits a global weak solution $u(t) \in L^\infty(0,\infty;W_0^{1,m}(\Omega))$ with $u_t \in L^2(0,\infty;L^2(\Omega))$ and $u(t) \in W$ for $0 \le t < \infty$. Further, there exists a constant $\kappa > 0$ such that*

$$\|u\|_2 \le (\|u_0\|_2^{2-m} + (m-2)\kappa t)^{\frac{1}{2-m}}.$$

Next, in order to prove the blow-up in finite time of solution to problem (1.1)–(1.3) for $J(u_0) < d$, we first introduce the invariant set V in Lemma 3.2. In Theorem 3.3, we prove the blow-up in finite time of solution and give a sufficient condition by introducing a simple auxiliary function. In Theorem 3.5, we introduce another auxiliary function to prove the blow-up in finite time and estimate the upper bound of the blow-up time. For the finite time blow-up results, Theorems 3.3 and 3.5 give two different proofs of the same conclusion. We observe that the auxiliary function introduced in the proof of Theorem 3.5 is more effective as it not only helps to prove the finite time blow-up of solution, but also estimates the upper bound of the blow-up time. In Theorem 3.6, based on the conclusion of Theorems 3.3 and 3.5, i.e. the solution blows up in finite time, we estimate the lower bound of the blow-up time with the help of a differential inequality. As Theorem 3.6 does not prove the finite time blow-up but relies on the finite time blow-up results, for other initial data leading to the finite time blow-up, the estimate of lower bound of the blow-up time in Theorem 3.6 is still valid.

Lemma 3.2 (Invariant set for $J(u_0) < d$) *Let p satisfy* (H), *$u_0 \in W_0^{1,m}(\Omega)$, T be the maximal existence time. Then the weak solution u of problem* (1.1)–(1.3) *with $J(u_0) < d$ belongs to V for $0 \le t < T$, provided $I(u_0) < 0$.*

Proof Since $J(u_0) < d$ and $I(u_0) < 0$, we get $u_0 \in V$. We prove $u(t) \in V$ for $0 < t < T$. Arguing by contradiction, by the continuity of $J(u)$ and $I(u)$ in t, we suppose that $t_0 \in (0,T)$ is the first time such that $J(u(t_0)) = d$ or $I(u(t_0)) = 0$ and $\|\nabla u(t_0)\|_m \ne 0$. By Definition 2.1 (iii) and $J(u_0) < d$, we have

$$\int_0^t \|u_\tau\|_2^2 d\tau + J(u) \le J(u_0) < d, \quad 0 \le t < T, \tag{3.1}$$

which means $J(u(t_0)) \ne d$. If $I(u(t_0)) = 0$ and $\|\nabla u(t_0)\|_m \ne 0$, then by the definition of d we have $J(u(t_0)) \ge d$, which contradicts (3.1). The proof is completed. □

Theorem 3.3 (Blow-up for $J(u_0) < d$) *Let p satisfy* (H) *and $u_0 \in W_0^{1,m}(\Omega)$. Assume that $J(u_0) < d$ and $I(u_0) < 0$. Then the weak solution $u(t)$ of problem* (1.1)–(1.3) *blows up in finite time.*

Proof According to Theorem 2.2, we know problem (1.1)–(1.3) admits a unique local weak solution $u \in C(0,T;W_0^{1,m}(\Omega))$, where T is the maximal existence time of $u(t)$. We prove the existence time is finite. Arguing by contradiction, we suppose that the existence time $T=+\infty$.

We define

$$M(t) := \int_0^t \|u\|_2^2 d\tau, \quad t \in [0,+\infty), \tag{3.2}$$

then

$$M'(t) = \|u\|_2^2.$$

Further, letting $\nu = u$ in (2.3), we obtain

$$\begin{aligned} M''(t) &= 2(u,u_t) \\ &= 2(|u|^{p-2}u,u) - 2(|\nabla u|^{m-2}\nabla u, \nabla u) \\ &= 2\|u\|_p^p - 2\|\nabla u\|_m^m \\ &= -2I(u). \end{aligned} \tag{3.3}$$

Combining (2.4) and (2.6), we get

$$\begin{aligned} J(u_0) &\geq J(u) + \int_0^t \|u_\tau\|_2^2 d\tau \\ &\geq \frac{p-m}{mp}\|\nabla u\|_m^m + \frac{1}{p}I(u) + \int_0^t \|u_\tau\|_2^2 d\tau, \end{aligned}$$

which is

$$\frac{1}{p}I(u) \leq J(u_0) - \frac{p-m}{mp}\|\nabla u\|_m^m - \int_0^t \|u_\tau\|_2^2 d\tau,$$

i.e.,

$$I(u) \leq pJ(u_0) - \frac{p-m}{m}\|\nabla u\|_m^m - p\int_0^t \|u_\tau\|_2^2 d\tau. \tag{3.4}$$

Substituting (3.4) into (3.3), we derive

$$M''(t) \geq -2pJ(u_0) + \frac{2(p-m)}{m}\|\nabla u\|_m^m + 2p\int_0^t \|u_\tau\|_2^2 d\tau. \tag{3.5}$$

Due to

$$\int_0^t (u_\tau, u) d\tau = \frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|u_0\|_2^2,$$

we derive

$$\begin{aligned} \left(\int_0^t (u_\tau,u)d\tau\right)^2 &= \left(\frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|u_0\|_2^2\right)^2 \\ &= \frac{1}{4}(\|u\|_2^4 - 2\|u_0\|_2^2\|u\|_2^2 + \|u_0\|_2^4) \\ &= \frac{1}{4}((M'(t))^2 - 2\|u_0\|_2^2 M'(t) + \|u_0\|_2^4), \end{aligned}$$

then

$$(M'(t))^2 = 4\left(\int_0^t (u_\tau,u)d\tau\right)^2 + 2\|u_0\|_2^2 M'(t) - \|u_0\|_2^4. \tag{3.6}$$

Hence, combining (3.5) and (3.6) we observe that

$$\begin{aligned}
&M(t)M''(t) - \frac{p}{2}(M'(t))^2 \\
&\geq M(t)\biggl(-2pJ(u_0) + \frac{2(p-m)}{m}\|\nabla u\|_m^m + 2p\int_0^t \|u_\tau\|_2^2 d\tau \biggr) \\
&\quad - \frac{p}{2}\biggl(4\biggl(\int_0^t (u_\tau, u) d\tau \biggr)^2 + 2\|u_0\|_2^2 M'(t) - \|u_0\|_2^4 \biggr) \\
&= -2pJ(u_0)M(t) + \frac{2(p-m)}{m}\|\nabla u\|_m^m M(t) \\
&\quad + 2p\biggl(\int_0^t \|u\|_2^2 d\tau \int_0^t \|u_\tau\|_2^2 d\tau - \biggl(\int_0^t (u_\tau, u) d\tau \biggr)^2 \biggr) \\
&\quad - p\|u_0\|_2^2 M'(t) + \frac{p}{2}\|u_0\|_2^4 \\
&> -2pJ(u_0)M(t) + \frac{2(p-m)}{m}\|\nabla u\|_m^m M(t) \\
&\quad + 2p\biggl(\int_0^t \|u\|_2^2 d\tau \int_0^t \|u_\tau\|_2^2 d\tau - \biggl(\int_0^t (u_\tau, u) d\tau \biggr)^2 \biggr) - p\|u_0\|_2^2 M'(t). \qquad (3.7)
\end{aligned}$$

By Cauchy–Schwarz inequality, we get

$$\biggl(\int_0^t (u_\tau, u) d\tau \biggr)^2 \leq \int_0^t \|u_\tau\|_2^2 d\tau \int_0^t \|u\|_2^2 d\tau, \qquad (3.8)$$

which makes (3.7) to be

$$M(t)M''(t) - \frac{p}{2}(M'(t))^2 > \frac{2(p-m)}{m}\|\nabla u\|_m^m M(t) - 2pJ(u_0)M(t) - p\|u_0\|_2^2 M'(t). \qquad (3.9)$$

For $2 \leq m < p$, we have the embedding inequality from $W_0^{1,m}(\Omega)$ to $W_0^{1,2}(\Omega)$ as

$$C_1\|\nabla u\|_2 \leq \|\nabla u\|_m \qquad (3.10)$$

and the Poincaré inequality

$$C_2\|u\|_2 \leq \|\nabla u\|_2. \qquad (3.11)$$

According to (1.4), (3.10) and (3.11), we have

$$\begin{aligned}
\frac{2(p-m)}{m}\|\nabla u\|_m^m M(t) &\geq \frac{2C_1^m C_2^m (p-m)}{m}\|u\|_2^m M(t) \\
&= \frac{2C_1^m C_2^m (p-m)}{m}\|u\|_2^{m-2}\|u\|_2^2 M(t),
\end{aligned}$$

then (3.9) becomes

$$\begin{aligned}
&M(t)M''(t) - \frac{p}{2}(M'(t))^2 \\
&> \frac{2C_1^m C_2^m (p-m)}{m}\|u\|_2^{m-2} M'(t)M(t) - 2pJ(u_0)M(t) - p\|u_0\|_2^2 M'(t). \qquad (3.12)
\end{aligned}$$

Next, we discuss the following two cases, i.e. $J(u_0) \leq 0$ and $0 < J(u_0) < d$.

(i) If $J(u_0) \leq 0$, then (3.12) gives

$$M(t)M''(t) - \frac{p}{2}(M'(t))^2 > \frac{2C_1^m C_2^m (p-m)}{m}\|u\|_2^{m-2} M'(t)M(t) - p\|u_0\|_2^2 M'(t). \qquad (3.13)$$

Combining (2.3), (2.6) and $J(u_0) \le 0$, we get

$$0 \ge J(u_0) > J(u) = \frac{p-m}{mp}\|\nabla u\|_m^m + \frac{1}{p}I(u),$$

that is $I(u) < 0$. From this and (3.3), we get $M''(t) > 0$ for $t \ge 0$, then $M'(t) = \|u\|_2^2$ is increasing with $t \in (0, \infty)$. Noting that $M'(0) = \|u_0\|_2^2 > 0$ and $M''(t) > 0$, we get $M'(t) > M'(0) > 0$ for $t > 0$, which means $M(t)$ is increasing over $[0, \infty)$, then we obtain $M(t) > M(0) = 0$. Thereby we have

$$M(t) - M(0) = \int_0^t M'(\tau)d\tau > \int_0^t M'(0)d\tau = M'(0)t,$$

that is

$$M(t) > M'(0)t, \quad t > 0.$$

Therefore, for $M'(t) > M'(0) > 0$ and sufficiently large t, we find

$$\frac{2C_1^m C_2^m(p-m)}{m}\|u\|_2^{m-2}M(t) > \frac{2C_1^m C_2^m(p-m)}{m}\|u_0\|_2^{m-2}M(t) > p\|u_0\|_2^2,$$

which makes (3.13) to be

$$M(t)M''(t) - \frac{p}{2}(M'(t))^2 > M'(t)\left(\frac{2C_1^m C_2^m(p-m)}{m}\|u_0\|_2^{m-2}M(t) - p\|u_0\|_2^2\right) > 0.$$

(ii) If $0 < J(u_0) < d$, then by Lemma 3.2 we have $u(t) \in V$ for $t \ge 0$. By (2.8), (2.4) and $0 < J(u_0) < d$, (3.3) becomes

$$\begin{aligned} M''(t) &= -2I(u) \\ &> 2p(d - J(u)) \\ &\ge 2p\left(d - J(u_0) + \int_0^t \|u_\tau\|_2^2 d\tau\right) \\ &> 2p(d - J(u_0)) \\ &=: C_M > 0. \end{aligned} \tag{3.14}$$

Then by (3.14) and $M'(0) = \|u_0\|_2^2 > 0$, we get

$$M'(t) - M'(0) = \int_0^t M''(\tau)d\tau > C_M t, \quad 0 < t < \infty,$$

that is

$$M'(t) > C_M t + M'(0) > C_M t. \tag{3.15}$$

Similarly, by $M''(t) > 0$, $M(0) = 0$ and (3.15), for $t \in (0, \infty)$ we obtain

$$M(t) - M(0) = \int_0^t M'(\tau)d\tau > \int_0^t C_M \tau d\tau = \frac{1}{2}C_M t^2,$$

i.e.,

$$M(t) > \frac{1}{2}C_M t^2 + M(0) = \frac{1}{2}C_M t^2. \tag{3.16}$$

Therefore, for sufficiently large t, the fact $M'(t) > M'(0) > 0$, (3.15) and (3.16) gives

$$\frac{C_1^m C_2^m(p-m)}{m}\|u\|_2^{m-2}M(t) > \frac{C_1^m C_2^m(p-m)}{m}\|u_0\|_2^{m-2}M(t) > p\|u_0\|_2^2 \tag{3.17}$$

and

$$\frac{C_1^m C_2^m (p-m)}{m}\|u\|_2^{m-2}M'(t) > \frac{C_1^m C_2^m (p-m)}{m}\|u_0\|_2^{m-2}M'(t) > 2pJ(u_0). \tag{3.18}$$

Then, by (3.17) and (3.18), (3.12) becomes

$$\begin{aligned} M(t)M''(t) - \frac{p}{2}(M'(t))^2 \geq &\left(\frac{C_1^m C_2^m (p-m)}{m}\|u_0\|_2^{m-2}M(t) - p\|u_0\|_2^2\right)M'(t) \\ &+ \left(\frac{C_1^m C_2^m (p-m)}{m}\|u_0\|_2^{m-2}M'(t) - 2pJ(u_0)\right)M(t) > 0 \end{aligned} \tag{3.19}$$

for sufficiently large t. In view of $M(t)$, $M'(t)$ and $M''(t)$ are all positive for sufficiently large $\overline{t}$, then (3.19) gives

$$\frac{M''(t)}{M'(t)} > \frac{pM'(t)}{2M(t)}, \quad t \in [\overline{t}, \infty).$$

Integrating above inequality from $\overline{t}$ to t with respect to t, we have

$$\int_{\overline{t}}^{t} \frac{dM'(\tau)}{M'(\tau)} > \frac{p}{2}\int_{\overline{t}}^{t} \frac{dM(\tau)}{M(\tau)},$$

i.e.,

$$\ln\frac{M'(t)}{M'(\overline{t})} > \frac{p}{2}\ln\frac{M(t)}{M(\overline{t})} = \ln\left(\frac{M(t)}{M(\overline{t})}\right)^{\frac{p}{2}},$$

which means

$$\frac{M'(t)}{M'(\overline{t})} > \left(\frac{M(t)}{M(\overline{t})}\right)^{\frac{p}{2}},$$

i.e.,

$$\frac{M'(t)}{(M(t))^{\frac{p}{2}}} > \frac{M'(\overline{t})}{(M(\overline{t}))^{\frac{p}{2}}}.$$

Integrating above inequality again from $\overline{t}$ to t with respect to t gives

$$\int_{\overline{t}}^{t} \frac{dM(\tau)}{(M(\tau))^{\frac{p}{2}}} > \frac{M'(\overline{t})}{(M(\overline{t}))^{\frac{p}{2}}}(t-\overline{t}),$$

which says

$$M(t)^{-\frac{p-2}{2}}(t) < M(\overline{t})^{-\frac{p-2}{2}}\left(1 - \frac{(p-2)M'(\overline{t})}{2M(\overline{t})}(t-\overline{t})\right),$$

i.e.,

$$M(t) > M(\overline{t})\left(1 - \frac{(p-2)M'(\overline{t})}{2M(\overline{t})}(t-\overline{t})\right)^{-\frac{2}{p-2}}. \tag{3.20}$$

Since we have assumed that the solution is global, i.e. the existence time $T = +\infty$, we only need to discuss the possibility of finite time blow-up solution for some finite time t^*. Next we shall make the blow-up happen and see if there exists such finite time t^*. In order to treat (3.20), we set

$$G(t) := 1 - \frac{(p-2)M'(\overline{t})}{2M(\overline{t})}(t-\overline{t})$$

for any $t \in [\overline{t}, +\infty)$. In the following, in order to find a finite time t^* such that the solution blows up, we solve the equation $G(t) = 0$ and get a unique root as $\overline{t} + \frac{2M(\overline{t})}{(p-2)M'(\overline{t})}$. Therefore, for

$$0 < t^* \leq \bar{t} + \frac{2M(\bar{t})}{(p-2)M'(\bar{t})}$$

we have

$$\lim_{t \to t^*} M(t) = +\infty,$$

which contradicts $T = +\infty$. □

In order to estimate the upper bound of the blow-up time, we introduce the following lemma.

Lemma 3.4 ([20]) *Suppose that a positive, twice-differentiable function $\varphi(t)$ satisfies the inequality*

$$\varphi''(t)\varphi(t) - (1+\theta)(\varphi'(t))^2 \geq 0, \qquad t > 0,$$

where $\theta > 0$ is some constant. If $\varphi(0) > 0$ and $\varphi'(0) > 0$, then there exists $0 < t_1 \leq \frac{\varphi(0)}{\theta\varphi'(0)}$ such that $\varphi(t)$ tends to infinity as $t \to t_1$.

Next, we introduce a different auxiliary function from Theorem 3.3 to prove that the solution blows up in finite time. Further, we also estimate the upper bound of the blow-up time.

Theorem 3.5 *Let p satisfy* (H) *and $u_0 \in W_0^{1,m}(\Omega)$. Assume that $J(u_0) < d$ and $I(u_0) < 0$. Then the weak solution $u(t)$ of problem* (1.1)–(1.3) *blows up in finite time. And we estimate the upper bound of the blow-up time as*

$$0 < T \leq \frac{4\|u_0\|_2^2}{(p-2)^2\beta},$$

where $0 < \beta < \frac{p(d-J(u_0))}{p-1}$ is a constant.

Proof According to Theorem 2.2, we know that problem (1.1)–(1.3) admits a unique local weak solution $u \in C(0,T; W_0^{1,m}(\Omega))$, where T is the maximal existence time of $u(t)$. Next we shall prove that the existence time is finite. Arguing by contradiction, we suppose that the existence time $T = +\infty$.

For any $T > 0$, we define

$$F(t) := \frac{1}{2}\int_0^t \|u\|_2^2 d\tau + \frac{1}{2}(T-t)\|u_0\|_2^2 + \frac{1}{2}\beta(t+\sigma)^2 \quad \text{for } t \in [0,T), \tag{3.21}$$

where σ is a positive constant which will be determined later. It is easy to verify that $F(t) > 0$ for any $t \in [0,T)$. By the definition of $J(u)$, $I(u)$ and (2.6), we get

$$J(u) = \frac{1}{m}\|\nabla u\|_m^m - \frac{1}{p}\|u\|_p^p = \frac{p-m}{mp}\|\nabla u\|_m^m + \frac{1}{p}I(u),$$

that is

$$I(u) = pJ(u) - \frac{p-m}{m}\|\nabla u\|_m^m. \tag{3.22}$$

Let $\nu = u$ in (2.3). We obtain

$$(u, u_\tau) = -I(u). \tag{3.23}$$

By (3.21)–(3.23) and (2.4), we obtain for any $t \in [0,T)$ that

$$F'(t) = \frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|u_0\|_2^2 + \beta(t+\sigma)$$

$$= \int_0^t (u, u_\tau)d\tau + \beta(t+\sigma) \tag{3.24}$$

and

$$\begin{aligned} F''(t) &= (u, u_\tau) + \beta \\ &= \frac{p-m}{m}\|\nabla u\|_m^m - pJ(u) + \beta \\ &\geq \frac{p-m}{m}\|\nabla u\|_m^m - p\bigg(J(u_0) - \int_0^t \|u_\tau\|_2^2 d\tau\bigg) + \beta \\ &= \frac{p-m}{m}\|\nabla u\|_m^m - pJ(u_0) + p\int_0^t \|u_\tau\|_2^2 d\tau + \beta. \end{aligned} \tag{3.25}$$

Thus, by (3.21) and (3.24), it follows that

$$\begin{aligned} FF'' - \alpha(F')^2 &= FF'' - \alpha\bigg(\int_0^t (u, u_\tau)d\tau + \beta(t+\sigma)\bigg)^2 \\ &= FF'' - \alpha\bigg(\int_0^t (u, u_\tau)d\tau + \beta(t+\sigma)\bigg)^2 \\ &\quad + \alpha\bigg(\int_0^t \|u\|_2^2 d\tau + \beta(t+\sigma)^2\bigg)\bigg(\int_0^t \|u_\tau\|_2^2 d\tau + \beta\bigg) \\ &\quad - \alpha(2F - (T-t)\|u_0\|_2^2)\bigg(\int_0^t \|u_\tau\|_2^2 d\tau + \beta\bigg). \end{aligned} \tag{3.26}$$

By (3.8) and Young's inequality, we obtain for any $t \in [0, T)$ that

$$\begin{aligned} &\bigg(\int_0^t \|u\|_2^2 d\tau + \beta(t+\sigma)^2\bigg)\bigg(\int_0^t \|u_\tau\|_2^2 d\tau + \beta\bigg) - \bigg(\int_0^t (u, u_\tau)d\tau + \beta(t+\sigma)\bigg)^2 \\ &= \bigg(\int_0^t \|u\|_2^2 d\tau \int_0^t \|u_\tau\|_2^2 d\tau - \bigg(\int_0^t (u_\tau, u)d\tau\bigg)^2\bigg) \\ &\quad + \bigg(\beta\int_0^t \|u\|_2^2 d\tau + \beta(t+\sigma)^2 \int_0^t \|u_\tau\|_2^2 d\tau - 2\beta(t+\sigma)\int_0^t (u, u_\tau)d\tau\bigg) \\ &\geq 2\beta(t+\sigma)\bigg(\int_0^t \|u\|_2^2 d\tau\bigg)^{\frac{1}{2}}\bigg(\int_0^t \|u_\tau\|_2^2 d\tau\bigg)^{\frac{1}{2}} - 2\beta(t+\sigma)\int_0^t (u, u_\tau)d\tau \\ &= 0. \end{aligned} \tag{3.27}$$

Then by (3.27) and (3.25), (3.26) becomes

$$\begin{aligned} FF'' - \alpha(F')^2 &\geq FF'' - \alpha(2F - (T-t)\|u_0\|_2^2)\bigg(\int_0^t \|u_\tau\|_2^2 d\tau + \beta\bigg) \\ &\geq F\bigg(F'' - 2\alpha\bigg(\int_0^t \|u_\tau\|_2^2 d\tau + \beta\bigg)\bigg) \\ &\geq F\bigg(\frac{p-m}{m}\|\nabla u\|_m^m - pJ(u_0) + p\int_0^t \|u_\tau\|_2^2 d\tau + \beta - 2\alpha\int_0^t \|u_\tau\|_2^2 d\tau - 2\alpha\beta\bigg) \\ &\geq F\bigg(\frac{p-m}{m}\|\nabla u\|_m^m - pJ(u_0) + (p-2\alpha)\int_0^t \|u_\tau\|_2^2 d\tau - (2\alpha-1)\beta\bigg). \end{aligned} \tag{3.28}$$

Let $\alpha := \frac{p}{2}$. (3.28) gives

$$FF'' - \frac{p}{2}(F')^2 \geq F\bigg(\frac{p-m}{m}\|\nabla u\|_m^m - pJ(u_0) - (p-1)\beta\bigg), \quad t \in [0, T).$$

Next, by (ii) of Lemma 2.4 and (2.7), we know

$$\frac{p-m}{m}\|\nabla u\|_m^m > \frac{p-m}{m}r^m = pd,$$

and letting $0<\beta<\frac{p(d-J(u_0))}{p-1}$, we discover

$$FF''-\frac{p}{2}(F')^2 > F(p(d-J(u_0))-(p-1)\beta)>0. \tag{3.29}$$

Hence, by Lemma 3.4, we have

$$\lim_{t\to T}F(t)=+\infty$$

and

$$0<T\leq \frac{2F(0)}{(p-2)F'(0)}. \tag{3.30}$$

In order to get the exact estimates of the blow-up time and verify it, we shall go on with the above inequality. By (3.21) and (3.24), (3.30) gives

$$\frac{2F(0)}{(p-2)F'(0)}=\frac{2(\frac{1}{2}\|u_0\|_2^2T+\frac{1}{2}\beta\sigma^2)}{(p-2)\sigma\beta}=\frac{\|u_0\|_2^2}{(p-2)\sigma\beta}T+\frac{\sigma}{p-2} \tag{3.31}$$

for any $\sigma>0$ and $0<\beta<\frac{p(d-J(u_0))}{p-1}$. Combining with (3.30) and (3.31), we get

$$\left(1-\frac{\|u_0\|_2^2}{(p-2)\sigma\beta}\right)T\leq\frac{\sigma}{p-2}.$$

In order to ensure $1-\frac{\|u_0\|_2^2}{(p-2)\sigma\beta}>0$, we choose $\sigma\in(\frac{\|u_0\|_2^2}{(p-2)\beta},+\infty)$ such that $0<\frac{\|u_0\|_2^2}{(p-2)\sigma\beta}<1$, then

$$T\leq\frac{\sigma}{p-2}\left(1-\frac{\|u_0\|_2^2}{(p-2)\sigma\beta}\right)^{-1}=\frac{\beta\sigma^2}{(p-2)\sigma\beta-\|u_0\|_2^2}:=T_\beta(\sigma),$$

and $T_\beta(\sigma)$ takes its minimum at $\sigma=\frac{2\|u_0\|_2^2}{(p-2)\beta}$. Therefore, we get

$$0<T\leq T_\beta\left(\frac{2\|u_0\|_2^2}{(p-2)\beta}\right)=\frac{4\|u_0\|_2^2}{(p-2)^2\beta}.$$ □

Next we shall estimate the lower bound of the blow-up time without proving the finite time blow-up results, hence we use the sufficient conditions in Theorem 3.3 and Theorem 3.5 to make the finite time blow-up happen. And Theorem 3.6 also works for the other initial data leading to the finite time blow-up.

Theorem 3.6 (Lower bound of blow-up time) *Assume that* $m<p<m+\frac{2m}{n}$, $J(u_0)<d$ *and* $I(u_0)<0$. *We have the estimate of the lower bound of the blow-up time of solution for problem* (1.1)–(1.3) *as follows*

$$T\geq\frac{\|u_0\|_2^{2-p\eta}}{(p\eta-2)C_G^{\frac{p}{1-\frac{p\theta}{m}}}},$$

where C_G *is the constant of Gagliardo–Nirenberg's inequality*

$$\|u\|_p\leq C_G\|\nabla u\|_m^\theta\|u\|_2^{1-\theta},$$

$\theta=\frac{(p-2)nm}{p(mn+2m-2n)}\in(0,1)$ *and* $\eta=\frac{1-\theta}{1-\frac{p\theta}{m}}>1$.

Proof By Theorem 3.3, the solution to problem (1.1)–(1.3) blows up in finite time, that is $\lim_{t\to T}\int_0^t \|u\|_2^2 d\tau = +\infty$, i.e.

$$\lim_{t\to T}\|u\|_2^2 = +\infty. \tag{3.32}$$

By Lemma 3.2, we get $I(u) < 0$, i.e. $\|\nabla u\|_m^m < \|u\|_p^p$. Then combining the Gagliardo–Nirenberg's inequality, we get

$$\|u\|_p \le C_G\|\nabla u\|_m^\theta \|u\|_2^{1-\theta} < C_G\|u\|_p^{\frac{p\theta}{m}}\|u\|_2^{1-\theta},$$

which yields

$$\|u\|_p < C_G^{\frac{1}{1-\frac{p\theta}{m}}}\|u\|_2^\eta, \tag{3.33}$$

where $\eta = \frac{1-\theta}{1-\frac{p\theta}{m}} > 1$, $\theta = \frac{(p-2)nm}{p(mn+2m-2n)} \in (0,1)$ and $\frac{p\theta}{m} < 1$ due to $m < p < m + \frac{2m}{n}$. Substituting (3.33) into (3.3), we find

$$\begin{aligned}\frac{d}{dt}\|u\|_2^2 \le& -2I(u) = 2\|u\|_p^p - 2\|\nabla u\|_m^m < 2\|u\|_p^p\\ <&2\Big(C_G^{\frac{1}{1-\frac{p\theta}{m}}}\|u\|_2^\eta\Big)^p = 2C_G^{\frac{p}{1-\frac{p\theta}{m}}}\|u\|_2^{p\eta} = 2C_G^{\frac{p}{1-\frac{p\theta}{m}}}(\|u\|_2^2)^{\frac{p\eta}{2}}.\end{aligned}$$

Solving the differential inequality above, we get

$$\|u\|_2^{2-p\eta} - \|u_0\|_2^{2-p\eta} > (2-p\eta)C_G^{\frac{p}{1-\frac{p\theta}{m}}}t,$$

i.e.

$$\|u\|_2^{2-p\eta} + (p\eta-2)C_G^{\frac{p}{1-\frac{p\theta}{m}}}t > \|u_0\|_2^{2-p\eta}.$$

Since (3.32) and $p\eta > 2$, letting $t \to T$, we have

$$T > \frac{\|u_0\|_2^{2-p\eta}}{(p\eta-2)C_G^{\frac{p}{1-\frac{p\theta}{m}}}} > 0.$$ □

4 Global Existence, Asymptotic Behavior and Blow-up in Finite Time with $J(u_0) = d$

In this section, we extend all obtained results for the low initial energy $J(u_0) < d$ to the case of critical initial energy $J(u_0) = d$. In other words, we shall prove the global existence, asymptotic behavior and blow-up in finite time of solution for problem (1.1)–(1.3) with the critical initial energy $J(u_0) = d$. Furthermore, we estimate the upper and lower bounds of the blow-up time.

Theorem 4.1 (Global existence for $J(u_0) = d$) *Let p satisfy* (H), $u_0 \in W_0^{1,m}(\Omega)$. *Assume that $J(u_0) = d$ and $I(u_0) \ge 0$. Then problem* (1.1)–(1.3) *admits a global weak solution $u(t) \in L^\infty(0,\infty; W_0^{1,m}(\Omega))$ with $u_t \in L^2(0,\infty; L^2(\Omega))$.*

Proof First the condition $J(u_0) = d$ implies that $\|\nabla u_0\|_m \ne 0$. For $s = 2, 3, \ldots$, we define $k_s := 1 - \frac{1}{s}$ and $u_{0s}(x) := k_s u_0(x)$, then $0 < k_s < 1$ and $k_s \to 1$ as $s \to \infty$. Now we consider problem (1.1) and (1.3) corresponding to the initial condition

$$u(x,0) = u_{0s}(x), \quad s = 2, 3, \ldots. \tag{4.1}$$

From $I(u_0) = \|\nabla u_0\|_m^m - \|u_0\|_p^p \geq 0$ and (ii) of Lemma 2.3, we get

$$\lambda^*(u_0) = \left(\frac{\|\nabla u_0\|_m^m}{\|u_0\|_p^p} \right)^{\frac{1}{p-m}} \geq 1,$$

then $0 < k_s < 1 \leq \lambda^*(u_0)$ for $s = 2, 3, \ldots$, which combines (iii) and (iv) of Lemma 2.3 to give $J(u_{0s}) = J(k_s u_0) < J(u_0) = d$ and $I(u_{0s}) = I(k_s u_0) > 0$. From [36, Theorem 2], it follows that for each s problem (1.1), (1.3), (4.1) admits a global weak solution $u_s(t) \in L^\infty(0, \infty; W_0^{1,m}(\Omega))$ with $u_{st}(t) \in L^2(0, \infty; L^2(\Omega))$ and $u_s(t) \in W$ for any $\nu \in W_0^{1,m}(\Omega)$ satisfying

$$(u_{st}, \nu) + (|\nabla u_s|^{m-2} \nabla u_s, \nabla \nu) = (|u_s|^{p-2} u_s, \nu), \quad 0 \leq t < \infty$$

and

$$\int_0^t \|u_{s\tau}\|_2^2 d\tau + J(u_s) \leq J(u_{0s}) < d.$$

The remainder of this proof is similar to that of [36, Theorem 2]. □

In the following, we shall extend the asymptotic behavior and blow-up results of the subcritical initial energy $J(u_0) < d$ to the case of critical initial energy $J(u_0) = d$. It is well known that in order to use the potential well method the first step is to prove the invariance of the stable set W and the unstable set V. But we can not directly derive this conclusion under the case of $J(u_0) = d$. Hence, we need to find a way to get the invariance of W and V under $J(u_0) = d$ with the help of the case of $J(u_0) < d$. In fact, according to the local existence theorem (Theorem 2.2) we know that for the problem (1.1)–(1.3), there exists a local solution $u(t)$ with the initial value u_0, and if the initial value $u_0 \in W$ or $u_0 \in V$ then the solution $u(t) \in W$ or $u(t) \in V$. Based on this, if the time goes a little forward, which says for sufficiently small time $t_1 > 0$, the solution $u(t)$ also belongs to W or V at this moment t_1. Inspired by above, we choose a sufficiently small time t_1 as the new initial time to complete all proof of the case of $J(u_0) = d$. In Theorem 4.3, we choose a new initial time $t_1 > 0$ to prove the blow-up in finite time for problem (1.1)–(1.3) by the similar auxiliary function as Theorem 3.3. In Theorem 4.4, we estimate the lower bound of the blow-up time by adding a new condition on p and using the same method as Theorem 3.6.

Next, based on Theorem 4.1, we show that the global solution decays in polynomial form.

Theorem 4.2 (Asymptotic behavior of solution for $J(u_0) = d$) *Let p satisfy* (H), $u_0 \in W_0^{1,m}(\Omega)$. *Assume that $J(u_0) = d$ and $I(u_0) > 0$. Then for the global weak solution u of problem* (1.1)–(1.3), *there exists a constant $\kappa > 0$ such that*

$$\|u\|_2 \leq (\|u_0\|_2^{2-m} + (m-2)\kappa t)^{\frac{1}{2-m}}.$$

Proof First, according to Theorem 4.1 we have proved that the solution $u(t)$ is global. Next, we prove $I(u) > 0$ for any $t > 0$. Arguing by contradiction, let $t_0 > 0$ be the first time such that $I(u(t_0)) = 0$, $\|\nabla u(t_0)\|_m \neq 0$ and $I(u) > 0$ for $t \in [0, t_0)$, then by the definition of d, we have

$$J(u(t_0)) \geq d. \tag{4.2}$$

Meanwhile, (2.4) indicates

$$J(u(t_0)) \leq d - \int_0^{t_0} \|u_\tau\|_2^2 d\tau \leq d. \tag{4.3}$$

Then combining (4.2) and (4.3) gives $\int_0^{t_0}\|u_\tau\|_2^2 d\tau = 0$, that is $u_t \equiv 0$ for $t \in [0, t_0)$, which contradicts $(u, u_t) = -I(u) < 0$ for $0 \le t < t_0$ due to (3.23). Hence we get $I(u) > 0$ for $0 \le t < \infty$. By the continuity of both $J(u)$ and $I(u)$ in t, we redefine the initial time by taking a sufficiently small $t_1 > 0$ such that $0 < J(u(t_1)) < d$ and $I(u(t_1)) > 0$. Therefore, by [16, Theorem 5.3] we get the conclusion. □

Theorem 4.3 (Blow-up for $J(u_0) = d$) *Let p satisfy* (H), $u_0 \in W_0^{1,m}(\Omega)$. *Assume that $J(u_0) = d$ and $I(u_0) < 0$. Then the existence time of weak solution for problem* (1.1)–(1.3) *is finite.*

Proof The proof is similar to Theorem 3.3. First of all by (3.2)–(3.12) and $J(u_0) = d$, we get

$$M(t)M''(t) - \frac{p}{2}(M'(t))^2 \ge \left(\frac{C_1^m C_2^m (p-m)}{m}\|u\|_m^{m-2} M(t) - p\|u_0\|_2^2\right) M'(t) + \left(\frac{C_1^m C_2^m (p-m)}{m}\|u\|_m^{m-2} M'(t) - 2pd\right) M(t).$$

On the other hand, from $J(u_0) = d > 0$, $I(u_0) < 0$ and the continuity of both $J(u)$ and $I(u)$ in t, it follows that there exists a sufficiently small $t_1 > 0$ such that $J(u(t_1)) > 0$ and $I(u) < 0$ for $t \in [0, t_1)$. Combining (3.23) we get $(u, u_t) = -I(u) > 0$ for $t \in [0, t_1]$, i.e. $u_t \ne 0$. Then by (2.4) we get

$$0 < J(u(t_1)) \le d - \int_0^{t_1} \|u_\tau\|_2^2 d\tau = d_1 < d.$$

Thus taking $t = t_1$ as the new initial time, then we have $u \in V$ for $0 < t < \infty$. The remainder proof is similar to Theorem 3.3. □

Here, we should estimate the upper bound of the blow-up time with $J(u_0) = d$ based on Theorem 3.5 and Theorem 4.3. Although the invariance of V in case of $J(u_0) = d$ has been proved in Theorem 4.3, we still can not estimate the upper bound of the blow-up time at this moment. In fact, similar to Theorem 3.5, by (3.29), for $\beta > 0$ we directly get

$$FF'' - \frac{p}{2}(F')^2 > F(p(d - J(u_0)) - (p-1)\beta) = -F(p-1)\beta,$$

which implies that $FF'' - \frac{p}{2}(F')^2 > 0$ cannot be obtain, then Lemma 3.4 is invalid to estimate the upper bound of the blow-up time.

Theorem 4.4 *Assume that $m < p < m + \frac{2m}{n}$, $J(u_0) = d$ and $I(u_0) < 0$. We have the lower bound estimate of the blow-up time of solution for problem* (1.1)–(1.3) *as follows*

$$T > \frac{\|u_0\|_2^{2-p\eta}}{(p\eta - 2)C_G^{\frac{p}{1-\frac{p\theta}{m}}}} > 0,$$

where C_G, η and θ are defined in Theorem 3.6.

Proof Based on Theorem 4.3, we know that the solution of problem (1.1)–(1.3) blows up in finite time $T > 0$ and $I(u) < 0$ for $0 < t < T$. The remainder proof is similar to Theorem 3.6. □

5 Blow-up and Blow-up Time with High (sup-critical) Initial Energy $J(u_0) > 0$

In this section, we prove the finite time blow-up of solution to problem (1.1)–(1.3) and estimate the upper bound of the blow-up time of blow-up solution with high initial energy by using the concave function method. In order to prove the main results, we need the following lemma.

Lemma 5.1 *Assume that $u_0 \in W_0^{1,m}(\Omega)$ satisfies*

$$J(u_0) < A\|u_0\|_2^m, \tag{5.1}$$

where $A = \frac{C_1^m C_2^m (p-m)}{mp}$, C_1, C_2 are defined in (3.10) *and* (3.11). *Then $u \in \mathcal{N}_- = \{u \in W_0^{1,m}(\Omega) \,|\, I(u) < 0\}$.*

Proof Let $u(t)$ be the weak solution of problem (1.1)–(1.3). By the definition of $J(u)$, (2.6), (3.10) and (3.11), we deduce

$$\begin{aligned} J(u_0) &= \frac{1}{m}\|\nabla u_0\|_m^m - \frac{1}{p}\|u_0\|_p^p \\ &= \frac{p-m}{mp}\|\nabla u_0\|_m^m + \frac{1}{p}I(u_0) \\ &\geq \frac{C_1^m C_2^m (p-m)}{mp}\|u_0\|_2^m + \frac{1}{p}I(u_0) \\ &=: A\|u_0\|_2^m + \frac{1}{p}I(u_0), \end{aligned}$$

then $I(u_0) < 0$ due to (5.1).

Next, we prove $u(t) \in \mathcal{N}_-$ for all $t \in [0, T)$. Arguing by contradiction, by the continuity of $I(u)$ in t, we assume that there exists an $s \in (0, T)$ such that $u(t) \in \mathcal{N}_-$ for $0 \leq t < s$ and $u(s) \in \mathcal{N}$, then (3.23) indicates

$$\frac{d}{dt}\|u(t)\|_2^2 = -2I(u) > 0 \quad \text{for } t \in [0, s),$$

which implies that

$$\|u_0\|_2^2 < \|u(s)\|_2^2. \tag{5.2}$$

By (ii) of Lemma 2.5, we know that

$$J(u(s)) < J(u_0). \tag{5.3}$$

From the definition of $J(u)$, $u(s) \in \mathcal{N}$, (3.10), (3.11) and (5.2), we derive

$$\begin{aligned} J(u(s)) &= \frac{1}{m}\|\nabla u(s)\|_m^m - \frac{1}{p}\|u(s)\|_p^p \\ &= \frac{p-m}{mp}\|\nabla u(s)\|_m^m + \frac{1}{p}I(u(s)) \\ &= \frac{p-m}{mp}\|\nabla u(s)\|_m^m \\ &\geq \frac{C_1^m C_2^m (p-m)}{mp}\|u(s)\|_2^m \\ &= A\|u(s)\|_2^m, \end{aligned}$$

then further combining (5.1) and (5.3), we obtain

$$A\|u(s)\|_2^m \leq J(u(s)) < J(u_0) < A\|u_0\|_2^m,$$

which contradicts (5.2). □

Next, based on Lemma 5.1, we prove the finite time blow-up of solution under $J(u_0) > 0$. In addition, we also estimate the both upper and lower bounds of the blow-up time with the help of Lemma 3.4 and Theorem 3.6.

Theorem 5.2 *Let p satisfy* (H), *if $u_0 \in W_0^{1,m}(\Omega)$, $J(u_0) > 0$ and* (5.1) *hold, then the solution $u(x,t)$ of problem* (1.1)–(1.3) *blows up in finite time. Moreover, we estimate the upper bound of the blow-up time as follows*

$$0 < t_* \leq \frac{c}{(\alpha-1)\varepsilon^{-1}\|u_0\|_2^4},$$

where $1 < \alpha < \frac{A\|u_0\|_2^m}{J(u_0)}$, $\varepsilon < \frac{2(A\|u_0\|_2^m - \alpha J(u_0))}{\alpha\|u_0\|_2^2}$ *and* $c > \frac{1}{4}\varepsilon^{-2}\|u_0\|_2^4$.

Proof According to Theorem 2.2, we know that problem (1.1)–(1.3) has a unique local weak solution in time $t \in [0,T]$, where T is the maximum existence time of $u(t)$. We claim that the maximum existence time of $u(t)$ is finite with the condition (5.1). Arguing by contradiction, we assume the existence time of solution $T = \infty$.

Now we define $y(t) := \int_0^t \|u\|_2^2 d\tau$. Since we have assumed that the solution $u(x,t)$ is global, thus the function $y(t)$ is bounded for all $t \geq 0$. Then we have

$$y'(t) = \|u\|_2^2 \quad \text{for all } t \in [0,\infty).$$

Combining the definition of $J(u)$, $I(u)$ and (3.23), we have

$$\begin{aligned} y''(t) &= \frac{d}{dt}\|u\|_2^2 = -2I(u) \\ &= -2(\|\nabla u\|_m^m - \|u\|_p^p) \\ &= -2\left(\|\nabla u\|_m^m - \frac{m}{p}\|u\|_p^p + \left(\frac{m}{p} - 1\right)\|u\|_p^p\right) \\ &= -2m\left(\frac{1}{m}\|\nabla u\|_m^m - \frac{1}{p}\|u\|_p^p + \left(\frac{1}{p} - \frac{1}{m}\right)\|u\|_p^p\right) \\ &= -2m\left(\frac{1}{m}\|\nabla u\|_m^m - \frac{1}{p}\|u\|_p^p\right) + \frac{2(p-m)}{p}\|u\|_p^p \\ &= -2mJ(u) + \frac{2(p-m)}{p}\|u\|_p^p. \end{aligned} \tag{5.4}$$

In the rest of the proof, we consider the following two cases.

Case I $J(u) \geq 0$ for all $t > 0$. From (5.1), we let

$$1 < \alpha < \frac{A\|u_0\|_2^m}{J(u_0)}. \tag{5.5}$$

Substituting (2.4) into (5.4), we get

$$\begin{aligned} y''(t) &= 2m(\alpha-1)J(u) - 2m\alpha J(u) + \frac{2(p-m)}{p}\|u\|_p^p \\ &> -2m\alpha J(u) + \frac{2(p-m)}{p}\|u\|_p^p \\ &\geq -2m\alpha J(u_0) + 2m\alpha\int_0^t \|u_\tau\|_2^2 d\tau + \frac{2(p-m)}{p}\|u\|_p^p. \end{aligned} \tag{5.6}$$

Combining Lemma 5.1, i.e., $I(u) < 0$ and (3.23), we derive

$$y''(t) = \frac{d}{dt}\|u\|_2^2 > 0. \tag{5.7}$$

From (3.10), (3.11) and (5.7), (5.6) becomes

$$y''(t) > -2m\alpha J(u_0) + 2m\alpha\int_0^t \|u_\tau\|_2^2 d\tau + \frac{2(p-m)}{p}\|\nabla u\|_m^m$$

$$
\begin{aligned}
&\geq -2m\alpha J(u_0) + 2m\alpha \int_0^t \|u_\tau\|_2^2 d\tau + \frac{2C_1^m C_2^m (p-m)}{p} \|u\|_2^m \\
&= -2m\alpha J(u_0) + 2m\alpha \int_0^t \|u_\tau\|_2^2 d\tau + \frac{2C_1^m C_2^m (p-m)}{p} \|u\|_2^{m-2} \|u\|_2^2 \\
&> -2m\alpha J(u_0) + 2m\alpha \int_0^t \|u_\tau\|_2^2 d\tau + \frac{2C_1^m C_2^m (p-m)}{p} \|u_0\|_2^{m-2} \|u\|_2^2 \\
&= -2m\alpha J(u_0) + 2m\alpha \int_0^t \|u_\tau\|_2^2 d\tau + 2mA \|u_0\|_2^{m-2} \|u\|_2^2 \\
&= -2m\alpha J(u_0) + 2m\alpha \int_0^t \|u_\tau\|_2^2 d\tau + 2mB \|u\|_2^2, \qquad (5.8)
\end{aligned}
$$

where $B := A\|u_0\|_2^{m-2}$. In view of $2m\alpha \int_0^t \|u_\tau\|_2^2 d\tau \geq 0$, (5.8) gives

$$
\frac{d}{dt}\|u\|_2^2 - 2mB\|u\|_2^2 > -2m\alpha J(u_0).
$$

Solving the differential inequality above, we get

$$
\|u\|_2^2 > \|u_0\|_2^2 \mathrm{e}^{2mBt} + \frac{\alpha}{B} J(u_0)(1 - \mathrm{e}^{2mBt}). \qquad (5.9)
$$

Substituting (5.9) into (5.8) shows

$$
\begin{aligned}
y''(t) &> -2m\alpha J(u_0) + 2m\alpha \int_0^t \|u_\tau\|_2^2 d\tau + 2mB\|u_0\|_2^2 \mathrm{e}^{2mBt} \\
&\quad + 2m\alpha J(u_0)(1 - \mathrm{e}^{2mBt}) \\
&= 2m\mathrm{e}^{2mBt}(B\|u_0\|_2^2 - \alpha J(u_0)) + 2m\alpha \int_0^t \|u_\tau\|_2^2 d\tau. \qquad (5.10)
\end{aligned}
$$

In view of (5.5), we take $\varepsilon > 0$ such that

$$
\varepsilon < \frac{2(B\|u_0\|_2^2 - \alpha J(u_0))}{\alpha \|u_0\|_2^2},
$$

which combining (5.10) gives

$$
y''(t) > m\varepsilon\alpha \|u_0\|_2^2 \mathrm{e}^{2mBt} + 2m\alpha \int_0^t \|u_\tau\|_2^2 d\tau. \qquad (5.11)
$$

Here, for any $c > 0$ and $t \geq 0$ we introduce the second auxiliary function

$$
\phi(t) := y^2(t) + \varepsilon^{-1}\|u_0\|_2^2 y(t) + c,
$$

then

$$
\phi'(t) = \left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right) y'(t) \qquad (5.12)
$$

and

$$
\phi''(t) = \left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right) y''(t) + 2(y'(t))^2. \qquad (5.13)
$$

Now, from (5.12) we can write

$$
\begin{aligned}
(\phi'(t))^2 &= (2y(t) + \varepsilon^{-1}\|u_0\|_2^2)^2 (y'(t))^2 \\
&= (4y^2(t) + 4\varepsilon^{-1}\|u_0\|_2^2 y(t) + \varepsilon^{-2}\|u_0\|_2^4)(y'(t))^2.
\end{aligned}
$$

In order to establish a connection between $\phi'(t)$ and $\phi(t)$, we pick $c > 0$ such that

$$c > \frac{1}{4}\varepsilon^{-2}\|u_0\|_2^4,$$

and let $\delta := 4c - \varepsilon^{-2}\|u_0\|_2^4 > 0$, then

$$\begin{aligned}(\phi'(t))^2 &= (4y^2(t) + 4\varepsilon^{-1}\|u_0\|_2^2 y(t) + 4c - \delta)(y'(t))^2 \\ &= (4\phi(t) - \delta)(y'(t))^2, \end{aligned} \tag{5.14}$$

i.e.,

$$4\phi(t)(y'(t))^2 = (\phi'(t))^2 + \delta(y'(t))^2. \tag{5.15}$$

By the fact that

$$\frac{1}{2}(\|u\|_2^2 - \|u_0\|_2^2) = \int_0^t (u, u_\tau)d\tau,$$

i.e.,

$$\|u\|_2^2 = \|u_0\|_2^2 + 2\int_0^t (u, u_\tau)d\tau,$$

combining the Hölder and Young's inequalities, we get

$$\begin{aligned}(y'(t))^2 &= \|u\|_2^4 \\ &= \left(\|u_0\|_2^2 + 2\int_0^t (u, u_\tau)d\tau\right)^2 \\ &\le \left(\|u_0\|_2^2 + 2\left(\int_0^t \|u\|_2^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_\tau\|_2^2 d\tau\right)^{\frac{1}{2}}\right)^2 \\ &= \|u_0\|_2^4 + 4\|u_0\|_2^2 (y(t))^{\frac{1}{2}}\left(\int_0^t \|u_\tau\|_2^2 d\tau\right)^{\frac{1}{2}} + 4y(t)\int_0^t \|u_\tau\|_2^2 d\tau \\ &\le \|u_0\|_2^4 + 4y(t)\int_0^t \|u_\tau\|_2^2 d\tau + 2\varepsilon\|u_0\|_2^2 y(t) + 2\varepsilon^{-1}\|u_0\|_2^2\int_0^t \|u_\tau\|_2^2 d\tau. \end{aligned} \tag{5.16}$$

From (5.13) and (5.15), we observe that

$$\begin{aligned}2\phi(t)\phi''(t) &= 2((2y(t) + \varepsilon^{-1}\|u_0\|_2^2)y''(t) + 2(y'(t))^2)\phi(t) \\ &= 2(2y(t) + \varepsilon^{-1}\|u_0\|_2^2)y''(t)\phi(t) + 4(y'(t))^2\phi(t) \\ &= 2(2y(t) + \varepsilon^{-1}\|u_0\|_2^2)y''(t)\phi(t) + (\phi'(t))^2 + \delta(y'(t))^2. \end{aligned} \tag{5.17}$$

Now, combining (5.17), (5.14) and the definition of δ, we obtain

$$\begin{aligned}&2\phi(t)\phi''(t) - (1+\alpha)(\phi'(t))^2 \\ &\quad = 2(2y(t) + \varepsilon^{-1}\|u_0\|_2^2)y''(t)\phi(t) + (\phi'(t))^2 + \delta(y'(t))^2 - (1+\alpha)(\phi'(t))^2 \\ &\quad = 2(2y(t) + \varepsilon^{-1}\|u_0\|_2^2)y''(t)\phi(t) - \alpha(\phi'(t))^2 + \delta(y'(t))^2 \\ &\quad = 2(2y(t) + \varepsilon^{-1}\|u_0\|_2^2)y''(t)\phi(t) - \alpha(4\phi(t) - \delta)(y'(t))^2 + \delta(y'(t))^2 \\ &\quad = 2(2y(t) + \varepsilon^{-1}\|u_0\|_2^2)y''(t)\phi(t) - 4\alpha\phi(t)(y'(t))^2 + \delta(1+\alpha)(y'(t))^2 \\ &\quad > 2\phi(t)(2y(t) + \varepsilon^{-1}\|u_0\|_2^2)y''(t) - 4\alpha\phi(t)(y'(t))^2 \\ &\quad = 2\phi(t)((2y(t) + \varepsilon^{-1}\|u_0\|_2^2)y''(t) - 2\alpha(y'(t))^2). \end{aligned}$$

By (5.11), the fact that $e^{2mBt} > 1$, (5.16) and (1.4), we compute

$$\begin{aligned}
&(2y(t) + \varepsilon^{-1}\|u_0\|_2^2)y''(t) - 2\alpha(y'(t))^2 \\
&\quad > (2y(t) + \varepsilon^{-1}\|u_0\|_2^2)\left(m\varepsilon\alpha\|u_0\|_2^2 e^{2mBt} + 2m\alpha\int_0^t \|u_\tau\|_2^2 d\tau\right) - 2\alpha(y'(t))^2 \\
&\quad > m\alpha(2y(t) + \varepsilon^{-1}\|u_0\|_2^2)\left(\varepsilon\|u_0\|_2^2 + 2\int_0^t \|u_\tau\|_2^2 d\tau\right) - 2\alpha(y'(t))^2 \\
&\quad = m\alpha\left(2\varepsilon\|u_0\|_2^2 y(t) + \|u_0\|_2^4 + 4y(t)\int_0^t \|u_\tau\|_2^2 d\tau + 2\varepsilon^{-1}\|u_0\|_2^2\int_0^t \|u_\tau\|_2^2 d\tau\right) - 2\alpha(y'(t))^2 \\
&\quad \geq (m\alpha - 2\alpha)(y'(t))^2 \geq 0,
\end{aligned}$$

that is

$$\phi(t)\phi''(t) - \frac{1+\alpha}{2}(\phi'(t))^2 > 0,$$

which implies that

$$(\phi^{-\beta}(t))'' = -\frac{\beta}{\phi^{\beta+2}}(\phi''(t)\phi(t) - (\beta+1)(\phi'(t))^2) < 0, \quad \beta = \frac{\alpha-1}{2} > 0.$$

Since $\phi(0) > 0$ and $\phi'(0) > 0$, by Lemma 3.4, it follows that there exists a

$$0 < t_* \leq \frac{2\phi(0)}{(\alpha-1)\phi'(0)} = \frac{c}{(\alpha-1)\varepsilon^{-1}\|u_0\|_2^4}$$

such that

$$\lim_{t\to t_*} \phi^{-\beta}(t) = 0$$

and

$$\lim_{t\to t_*} \phi(t) = +\infty,$$

which contradicts $T = +\infty$. Now, by considering the continuity of ϕ with respect to y, we can conclude that $y(t)$ tends to infinity at some finite time.

Case II $J(u(t)) < 0$ for some $t > 0$.

Since $J(u)$ is continuous with respect to t, for $J(u_0) > 0$ and (3.23) there must exist a time $t_1 > 0$ such that $J(u) < 0$ for $t > t_1$ and $J(u(t_1)) = 0$. We choose $u(t_1)$ as a new initial datum of problem (1.1)–(1.3), then Lemma 5.1 gives $u \in \mathcal{N}_-$ for $t > t_1$. Similar to the proof of Theorem 3.3, we obtain the blow-up of solution in finite time.

Combining Case I and Case II, we conclude the blow-up of solution in finite time. □

Since $J(u_0) < A\|u_0\|_2^m$ indicates $I(u) < 0$, we can get the same lower bound of blow-up time as $J(u_0) \leq d$.

Theorem 5.3 *Assume that* $m < p < m + \frac{2m}{n}$, $d < J(u_0) < A\|u_0\|_2^m$. *We estimate the lower bounded of blow-up time of solution for problem* (1.1)–(1.3) *as follows*

$$T \geq \frac{\|u_0\|_2^{2-p\eta}}{(p\eta-2)C_G^{\frac{p}{1-\frac{p\theta}{m}}}},$$

where C_G, η *and* θ *are defined in Theorem* 3.6.

Proof By Lemma 5.1, we know $I(u) < 0$. Then the remainder proof is similar to Theorem 3.6. □

To verify the validity of the conditions on the initial data required in Theorem 5.2, and also the confusion caused by the restriction $J(u_0) > 0$ and $J(u_0) < d$, we give the following two remarks to clarify above two issues.

Remark 5.4 (Validity of conditions $J(u_0) > 0$ and (5.1)) Here we claim that there exist some data which satisfy the conditions $J(u_0) < A\|u_0\|_2^m$ and $J(u_0) > 0$, where $A = \frac{C_1^m C_2^m (p-m)}{mp}$, C_1 and C_2 are defined by (3.10) and (3.11) respectively, required in Theorem 5.2. Let $u_0 = a\phi$, where $a > 0$ is some positive constant, ϕ is non-zero function in $W_0^{1,p}(\Omega)$ that will be defined later. First, fix $\phi \in W_0^{1,p}(\Omega)$ such that for $a > 0$, which will also be defined later, we have

$$\|u_0\|_2^m = a^m \|\phi\|_2^m > 0. \tag{5.18}$$

For this fixed ϕ and $m < p$, we pick $a^{p-m} < \frac{p\|\nabla\phi\|_m^m}{m\|\phi\|_p^p}$ to ensure that

$$\begin{aligned} J(u_0) &= \frac{1}{m}\|\nabla u_0\|_m^m - \frac{1}{p}\|u_0\|_p^p \\ &= \frac{a^m}{m}\|\nabla\phi\|_m^m - \frac{a^p}{p}\|\phi\|_p^p \\ &= a^m\left(\frac{1}{m}\|\nabla\phi\|_m^m - \frac{a^{p-m}}{p}\|\phi\|_p^p\right) > 0. \end{aligned} \tag{5.19}$$

Next, we verify the condition (5.1), i.e., $J(u_0) < A\|u_0\|_2^m$. By comparing (5.18) and (5.19), we only need to verify

$$\|\phi\|_2^m > \frac{\frac{1}{m}\|\nabla\phi\|_m^m - \frac{a^{p-m}}{p}\|\phi\|_p^p}{A}.$$

A simple calculation shows that we need

$$a^{p-m} > \frac{p\|\nabla\phi\|_m^m}{m\|\phi\|_p^p} - \frac{Ap\|\phi\|_2^m}{\|\phi\|_p^p},$$

also

$$\begin{cases} \dfrac{p\|\nabla\phi\|_m^m}{m\|\phi\|_p^p} - \dfrac{Ap\|\phi\|_2^m}{\|\phi\|_p^p} < a^{p-m} < \dfrac{p\|\nabla\phi\|_m^m}{m\|\phi\|_p^p}, & \text{if } \dfrac{p\|\nabla\phi\|_m^m}{m\|\phi\|_p^p} - \dfrac{Ap\|\phi\|_2^m}{\|\phi\|_p^p} > 0; \\ 0 < a^{p-m} < \dfrac{p\|\nabla\phi\|_m^m}{m\|\phi\|_p^p}, & \text{if } \dfrac{p\|\nabla\phi\|_m^m}{m\|\phi\|_p^p} - \dfrac{Ap\|\phi\|_2^m}{\|\phi\|_p^p} < 0. \end{cases}$$

Hence, there exists an initial value $u_0 = a\phi$ to satisfy $J(u_0) > 0$ and (5.1).

Remark 5.5 (Some comments on $J(u_0) < d$ and $J(u_0) > 0$) In the studies of the relations between the initial data and the global well-posedness of the solution to problem (1.1)–(1.3), we usually consider three different levels of the initial energy related the potential well depth d, namely sub-critical, critical and sup-critical initial energy levels, included in the arbitrary positive case in the present paper. As we have known for the sub-critical initial energy and critical initial energy that we can well divide the manifold of the initial data for the global existence (W defined in (2.1)) and the finite time blow-up (V defined in (2.2)) of the solution by the signs of the Nehari functional. In other words, we have well classified the initial data by above two manifolds, the stable manifold W and the unstable manifold V, for both sub-critical and critical initial energy cases, which can be regarded as the "threshold result" or so-called sharp condition described in Theorem 3.1 and Theorem 3.3 for the sub-critical initial energy case, also Theorem 4.1 and Theorem 4.3 for the critical initial energy case. Naturally, we are expected to extend above results parallelly to the sup-critical initial energy case, i.e., $J(u_0) > d$. Unfortunately, for this case ($J(u_0) > d$) we can only obtain some sufficient conditions of the finite time blow-up of solution as shown in Theorem 5.2 for $J(u_0) > 0$. Obviously, there is an overlap between $0 < J(u_0) \leq d$ and $J(u_0) > 0$. In the proof of Theorem 5.2 for the sup-critical initial energy case, the condition (5.1) is valid for both $J(u_0) > d$ and $0 < J(u_0) \leq d$. The "sharp condition" for the well-posedness of solution obtained for the sub-critical and critical initial energy cases implies that the condition (5.1) for the high energy blow-up is stronger than the condition required in Theorem 3.3 and Theorem 4.3 for the sub-critical and critical initial energy case respectively, i.e., $I(u_0) < 0$. Although the condition (5.1) is the best we can find up to now, how to find the better conditions for the finite time blow-up of the solution at high initial energy level is still an interesting problem. Another unsolved problem is how to prove the global existence of the solution at the sup-critical initial energy level. Indeed, for the standard parabolic equation [14, 41], it has been proved that for the arbitrary positive initial energy, there exist initial data leading to global solution, and also the initial data leading to the finite time blow-up solution. However, for the m-Laplacian parabolic equations, we cannot establish the similar conclusion because of the absence of the maximum principle and the comparison principle, which leads to the method for treating the classical parabolic equations invalid for the corresponding problem at the sup-critical initial energy level.

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