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Existence and concentration properties for the 1-biharmonic equation with lack of compactness

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ABSTRACT

In this work, we are interested in the following 1-biharmonic problem with potentials

$$\begin{cases} \varepsilon^2\Delta_1^2 u - \varepsilon\Delta_1 u + V(x)\frac{u}{|u|} = K(x)f(u) & \text{in } \mathbb{R}^N, \\ u \in BL(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$, $\varepsilon > 0$ is a positive parameter and V, K, f satisfy some proper conditions. Under the Nehari manifold technique, the Concentration-Compactness Principle and some analysis techniques, we establish the existence and concentration properties of ground state solutions to the 1-biharmonic equation.

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1. Introduction

In this paper, we consider the existence and concentration of solutions to the quasilinear elliptic problems with potentials

$$\begin{cases} \varepsilon^2\Delta_1^2 u - \varepsilon\Delta_1 u + V(x)\frac{u}{|u|} = K(x)f(u) \quad \text{in } \mathbb{R}^N, \\ u \in BL(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $\varepsilon > 0$ is a positive parameter, V, K, f satisfy some proper conditions and $BL(\mathbb{R}^N)$ is a space of functions of bounded variation, which will be defined in Section 2. The 1-Laplacian operator is formally defined as

$$\Delta_1 u = \mathrm{div}(\frac{Du}{|Du|}),$$

and the 1-biharmonic operator is given by

$$\Delta_1^2 u = \Delta(\frac{\Delta u}{|\Delta u|}).$$

In the last years, the interest in elliptic problems involving the 1-Laplacian operator has increased a lot. The 1-Laplacian operator comes out from an optimal design problem in the theory of torsion and from the level set formulation of the Inverse Mean Curvature Flow, and also appears in the variational approach to image denoising restoration. On the other hand, from a purely mathematical point of view, there are a lot of papers on this highly singular operator. For example, in [2], F. Andreu, C. Ballesteler, V. Caselles and J.M. Mazón made a pioneering study of problems involving this operator, and produced the monograph [3]; in [1,8,11,12], the authors analyzed related questions based on the energy functional of the space BV, and in [17–20], the authors used a method based on approximation through p-Laplacian problems. The 1-Laplacian operator can be seen as the p-Laplacian ones, as the parameter $p \to 1^+$. As pointed out in [10], this fact has great mathematical significance because diffusion processes involving this operator do not have diffusion on different levels.

One can consider its higher-order counterparts, including problems involving the 1-biharmonic operator $\Delta_1^2 u = \Delta(\frac{\Delta u}{|\Delta u|})$, which can be seen as the limit of the p-biharmonic ones, as the parameter p goes to 1^+. The difference is that few articles discuss the problem of designing this operator. Indeed, in [21], E. Parini, B. Ruf and C. Tarsi first studied the problem of such operator and dealt with the related eigenvalue problem; they proved that

$$\Lambda_{1,1}(\Omega) = \inf_{u \in BL_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|}{\|u\|_1}$$

is attained by a non-negative and superharmonic function v that belongs to the space

$$BL_0(\Omega) = \{u \in W_0^{1,1}(\Omega); \Delta u \in \mathcal{M}(\Omega)\}$$

where $\mathcal{M}(\Omega)$ is the space of the Radon measures defined on Ω. In fact, their result is more complete, as it also provides information about the shape of the domain Ω that maximizes $\Lambda_{1,1}(\Omega)$. In [23], the same authors also dealt with the 1-biharmonic operator; they studied the following minimization problem

$$\Lambda_{1,1}^c(\Omega) = \inf_{u \in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|}{\|u\|_1}.$$

Similarly, in [23] the authors also studied the shape of the subset that maximizes the quantity $\Lambda_{1,1}^c(\Omega)$. In [22], these authors studied some optimal constants of Sobolev embeddings in some spaces of functions related to 1-biharmonic operator. In [6], S. Barile and M.T. Pimenta studied some existence results of bounded variation solutions to the following quasilinear fourth-order problem

$$\begin{cases} \Delta_1^2 u = f(x,u) & \text{in } \Omega, \\ u = \frac{\Delta u}{|\Delta u|} = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is superlinear and subcritical at infinity which satisfies the Ambrosetti-Rabinowitz condition and a monotonicity one or f is sublinear. In [13], E.J. Hurtado, M.T. Pimenta and O.H. Miyagaki proved some compactness results of $BL_{rad}(\mathbb{R}^N)$, the space $BL(\mathbb{R}^N)$ of radially symmetric functions and the existence of the ground state solution for the quasilinear elliptic problem

$$\begin{cases} \Delta_1^2 u - \Delta_1 u + \frac{u}{|u|} = f(u) & \text{in } \mathbb{R}^N, \\ u \in BL(\mathbb{R}^N). \end{cases}$$

Moreover, Alves and Pimenta [1], Che [9], Liu and Guo [15] considered existence and concentration of solutions for quasilinear elliptic problems with potentials; see also the references there. The above applications make one consider if these relevant results can be generalized to problems involving the 1-biharmonic operator.

Motivated by the aforementioned works, we will deal with some generalizations of the above results to problem (1.1) under the following assumptions:

(f_1) f is continuous in $\mathbb{R}$;
(f_2) $\lim_{|s|\to 0} f(s) = 0$;

(f_3) There exist constants $c_1, c_2 > 0$ and $p \in (1, 1^*)$ such that

$$|f(s)| \leq c_1 + c_2|s|^{p-1}, \quad \forall s \in \mathbb{R};$$

(f_4) There exists $\kappa > 1$ such that

$$0 < \kappa F(s) \leq f(s)s, \quad \text{for } s \neq 0,$$

where $F(s) = \int_0^s f(t)dt$;

(f_5) f is increasing.

Moreover, the potentials are assumed to satisfy some of the following conditions:

(V_1) $V(x) \in C(\mathbb{R}^N)$ and $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0$;
(V_2) $V(x) \in C(\mathbb{R}^N), V_\infty = \lim_{|x|\to\infty} V(x) \geq V(x) > 0$ for $x \in \mathbb{R}^N$;
(K_1) $K(x) \in C(\mathbb{R}^N)$, $K_0 = \max_{x\in\mathbb{R}^N} K(x) \geq K(x) \geq K_\infty = \lim_{|x|\to\infty} K(x) > 0$ for $x \in \mathbb{R}^N$;
(K_2) $K(x) \in C(\mathbb{R}^N)$, $\lim_{|x|\to\infty} K(x) = K_\infty \geq K(x) > 0$ for $x \in \mathbb{R}^N$;
(VK) $\Lambda = \{x \in \mathbb{R}^N : V(x) = V_0\}$ and $\Lambda_1 = \{x \in \mathbb{R}^N : K(x) = K_0\}$, and $\Lambda \cap \Lambda_1 \neq \emptyset$.

Obviously, note that $V(x)$ satisfies the Rabinowitz's condition when (V_1) hold, and $K(x)$ is a bounded continuous function.

The main results in this work are the following.

Theorem 1.1. *Suppose that assumptions $(f_1) - (f_5)$, (V_1), (K_1) and (VK) hold. Then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, problem (1.1) has a nontrivial ground state solution u_ε. Moreover, u_ε has a global maximum point $x_\varepsilon \in \mathbb{R}^N$ such that $\lim_{\varepsilon\to 0} V(x_\varepsilon) = V_0$ and $\lim_{\varepsilon\to 0} K(x_\varepsilon) = K_0$. More specifically, there exists $C > 0$ such that for all $\delta > 0$, there exist $\overline{R} > 0$ and $n_0 \in \mathbb{N}$ such that,*

$$\int\limits_{B^c_{\varepsilon_n \overline{R}}(x_0)} f(v_n)v_n dx < \varepsilon_n^N \delta \quad \text{and} \quad \int\limits_{B_{\varepsilon_n \overline{R}}(x_0)} f(v_n)v_n dx \geq C\varepsilon_n^N,$$

for all $n \geq n_0$.

Our second result shows the existence of solution for all $\varepsilon > 0$ when V is asymptotically constant and it has the following statement.

Theorem 1.2. *Suppose that assumptions $(f_1) - (f_5)$, (V_2) and (K_2) hold. Then for each $\varepsilon > 0$, problem (1.1) has a nontrivial ground state solution u_ε.*

Our third result shows the existence of a ground state solution to the autonomous problem

$$\begin{cases} \Delta_1^2 u - \Delta_1 u + V_\infty \frac{u}{|u|} = K_\infty f(u) & \text{in } \mathbb{R}^N, \\ u \in BL(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where V_∞ and K_∞ are constants.

Theorem 1.3. *Suppose that assumptions $(f_1) - (f_5)$, (V_1) and (K_1) hold. Then problem (1.2) has a ground state solution w_∞.*

The proofs of Theorem 1.1- 1.3 are based on an abstract version of the Mountain-Pass Theorem which can be applied to the space of functions (see Theorem 4.1). The difficulties arise mainly from the following facts:

- The energy functional associated with the problem (1.1) lacks smoothness;
- The space $BL(\mathbb{R}^N)$ lacks reflexivity.

Therefore, we need to use the critical point theory of nonsmooth functional.

Since a version of the Lions' Lemma to $BL(\mathbb{R}^N)$ seems not available in the literature, we will give its proof by drawing on the literature [14]. We consider it is interesting in its own way because it is a classical and largely used tool in the analysis of quasilinear elliptic problems.

Theorem 1.4. *(Lions' Lemma in $BL(\mathbb{R}^N)$). Suppose there exist $R > 0$, $1 \le q < 1^*$, and a bounded sequence (u_n) in $BL(\mathbb{R}^N)$ such that*

$$\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^q \, dx \to 0, \qquad \text{as } n \to \infty.$$

Then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in (1, 1^*)$.

This paper is arranged as follows. In Section 2 we give a detailed description of the variational framework and the properties of the working space defined by the energy functional. In Section 3 we prove the Lions' Lemma in $BL(\mathbb{R}^N)$. In Section 4 we consider the autononous case and give the proof of Theorem 1.3. In Section 5 we present the proof of Theorem 1.1, studying separately the arguments on existence and concentration of the solutions. Finally, we give the proof of Theorem 1.2 in Section 6.

2. Preliminaries

2.1. The space and the energy functional

First of all, making the change of variable $x = \varepsilon z$, we note that the problem (1.1) is equivalent to the problem

$$\begin{cases} \Delta_1^2 - \Delta_1 u + V(\varepsilon x)\frac{u}{|u|} = K(\varepsilon x)f(u) & \text{in } \mathbb{R}^N, \\ u \in BL(\mathbb{R}^N). \end{cases} \tag{2.1}$$

Let us introduce the space we are going to deal with, which is defined by

$$BL(\mathbb{R}^N) := \left\{u \in W^{1,1}(\mathbb{R}^N) : \Delta u \in \mathcal{M}(\mathbb{R}^N)\right\},$$

where we recall $\mathcal{M}(\mathbb{R}^N)$ is the set of Radon measures in $\mathbb{R}^N$. In [21], it is proved that $u \in W^{1,1}(\mathbb{R}^N)$ belongs to $BL(\mathbb{R}^N)$ if and only if

$$\int_{\mathbb{R}^N} |\Delta u| < +\infty,$$

where

$$\int_{\mathbb{R}^N} |\Delta u| := \sup\left\{\int_{\mathbb{R}^N} u\Delta\varphi dx : \varphi \in C_0^\infty(\mathbb{R}^N), \|\varphi\|_\infty \leqslant 1\right\}.$$

The space $BL(\mathbb{R}^N)$ is a Banach space when endowed with the following norm

$$\|u\|_{BL(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\Delta u| + \|\nabla u\|_1 + \|u\|_1,$$

and it is continuously embedded into $L^r(\mathbb{R}^N)$ for all $r \in [1, 1^*]$ (see [13]).

Nevertheless, as one can see, the space of smooth functions is not dense in $BL(\mathbb{R}^N)$ with respect to the topology of the norm, and it is dense with respect to another, weaker, notion of convergence. We say that a sequence $(u_n) \subset BL(\mathbb{R}^N)$ converges to $u \in BL(\mathbb{R}^N)$ in the sense of the "strict convergence" if both of the following conditions are satisfied

$$u_n \to u \quad \text{in } W^{1,1}(\mathbb{R}^N),$$

and

$$\int_{\mathbb{R}^N} |\Delta u_n| \to \int_{\mathbb{R}^N} |\Delta u|,$$

as $n \to +\infty$. In fact, with respect to the "strict convergence", $C^\infty(\mathbb{R}^N) \cap BL(\mathbb{R}^N)$ is dense in $BL(\mathbb{R}^N)$ and $C_0^\infty(\mathbb{R}^N)$ is dense in $BL(\mathbb{R}^N)$.

For a vector Radon measure $\mu \in \mathcal{M}(\mathbb{R}^N, \mathbb{R}^N)$, we denote by $\mu = \mu^a + \mu^s$ the usual decomposition stated in the Radon Nikodym Theorem, where μ^a and μ^s are, respectively, the absolute continuous and the singular parts with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$. Denoting with $|\mu|$ as the scalar Radon measure defined like in [5], the usual Lebesgue-Radon-Nikodym derivative of μ with respect to $|\mu|$ is given by

$$\frac{\mu}{|\mu|}(x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{|\mu|(B_r(x))}.$$

It is easy to see that $\mathcal{J} : BL(\mathbb{R}^N) \to \mathbb{R}$, given by

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| dx + \int_{\mathbb{R}^N} |u| dx \tag{2.2}$$

is a convex functional which is Lipschitz continuous in its domain and lower semicontinuous with respect to the $W^{1,r}(\mathbb{R}^N)$ topology, for $r \in [1, 1^*]$. Meanwhile, $\mathcal{J}$ is lower semicontinuous with respect to the $L^r(\mathbb{R}^N)$ topology, for $r \in [1, 1^*)$ (see [13]). Although nonsmooth, the functional $\mathcal{J}$ admits some directional derivatives. More precisely, as is shown in [4], given $u \in BL(\mathbb{R}^N)$, for all $v \in BL(\mathbb{R}^N)$ such that $(\Delta v)^s$ is absolutely continuous with respect to $(\Delta u)^s$, $(\Delta v)^a$ vanishes $\mathcal{L}^N$-a.e. in $\{x \in \mathbb{R}^N; (\Delta u)^a(x) = 0\}$, ∇v vanishes a.e. in the set where ∇u vanishes and $v \equiv 0$, a.e. in the set where u vanishes, it follows that

$$\begin{aligned}\mathcal{J}'(u)v = &\int_{\mathbb{R}^N} \frac{(\Delta u)^a (\Delta v)^a}{|(\Delta u)^a|} dx + \int_{\mathbb{R}^N} \frac{\Delta u}{|\Delta u|}(x) \frac{\Delta v}{|\Delta v|}(x)\, |(\Delta v)^s| + \int_{\mathbb{R}^N} \frac{\nabla u \cdot \nabla v}{|\nabla u|} dx \\ &+ \int_{\mathbb{R}^N} \operatorname{sgn}(u) v dx,\end{aligned} \tag{2.3}$$

where $\operatorname{sgn}(u(x)) = 0$ if $u(x) = 0$ and $\operatorname{sgn}(u(x)) = u(x)/|u(x)|$ if $u(x) \neq 0$. In particular, taking (2.3) into account, for all $u \in BL(\mathbb{R}^N)$, we have

$$\mathcal{J}'(u)u = \mathcal{J}(u). \tag{2.4}$$

Now let us define in the space $BL(\mathbb{R}^N)$ the following norm

$$\|u\|_\varepsilon = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| dx + \int_{\mathbb{R}^N} V(\varepsilon x)|u| dx. \tag{2.5}$$

Then we present the energy functional associated to (2.1). Let $\Phi_\varepsilon : BL(\mathbb{R}^N) \to \mathbb{R}$ be given by

$$\Phi_\varepsilon(u) = \mathcal{J}_\varepsilon(u) - \mathcal{F}(u),$$

where $\mathcal{J}_\varepsilon = \|u\|_\varepsilon$ and $\mathcal{F} : BL(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} K(\varepsilon x) F(u) dx.$$

It can be a plain matter to prove that $\mathcal{J}_\varepsilon$ is a convex functional which is Lipschitz continuous in its domain and $\mathcal{F} \in C^1(BL(\mathbb{R}^N), \mathbb{R})$. Similar to (2.4), we have

$$\mathcal{J}_\varepsilon'(u)v = \int\limits_{\mathbb{R}^N} \frac{(\Delta u)^a (\Delta v)^a}{|(\Delta u)^a|} dx + \int\limits_{\mathbb{R}^N} \frac{\Delta u}{|\Delta u|}(x) \frac{\Delta v}{|\Delta v|}(x) \, |(\Delta v)^s| + \int\limits_{\mathbb{R}^N} \frac{\nabla u \cdot \nabla v}{|\nabla u|} dx \\ + \int\limits_{\mathbb{R}^N} V(\varepsilon x) \operatorname{sgn}(u) v dx. \tag{2.6}$$

In particular, note that, for all $u \in BL(\mathbb{R}^N)$, $\mathcal{J}_\varepsilon'(u)u = \mathcal{J}_\varepsilon(u)$. Moreover, taking $v = u$ in (2.6), it shows that

$$\Phi_\varepsilon'(u)u = \mathcal{J}_\varepsilon'(u)u - \int\limits_{\mathbb{R}^N} K(\varepsilon x) f(u) u dx = \|u\|_\varepsilon - \int\limits_{\mathbb{R}^N} K(\varepsilon x) f(u) u dx. \tag{2.7}$$

Let us give a precise definition of the solution we are considering. Since Φ_ε can be written as the difference between the Lipschitz functional $\mathcal{J}_\varepsilon$ and a smooth functional $\mathcal{F}$, we say that $u_\varepsilon \in BL(\mathbb{R}^N)$ is a solution of (1.1) if $0 \in \partial\Phi_\varepsilon(u_\varepsilon)$, where $\partial\Phi_\varepsilon(u_\varepsilon)$ denotes the subdifferential of Φ_ε in u_ε, as defined in [7]. This, in turn, is equivalent to $\mathcal{F}'(u_\varepsilon) \in \partial\mathcal{J}_\varepsilon(u_\varepsilon)$. However, since the convexity of $\mathcal{J}_\varepsilon$, it implies that $\mathcal{F}'(u_\varepsilon) \in \partial\mathcal{J}_\varepsilon(u_\varepsilon)$ if and only if

$$\mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) \geq \mathcal{F}'(u_\varepsilon)(v - u_\varepsilon), \quad \forall v \in BL(\mathbb{R}^N),$$

or equivalently

$$\|v\|_\varepsilon - \|u_\varepsilon\|_\varepsilon \geq \int\limits_{\mathbb{R}} K(\varepsilon x) f(u_\varepsilon)(v - u_\varepsilon) dx, \quad \forall v \in BL(\mathbb{R}^N). \tag{2.8}$$

Hence, every $u_0 \in X_\varepsilon$ such that (2.8) holds is going to be called a solution of (1.1).

2.2. The Euler-Lagrange equation

In this section, we give the precise version of the problem satisfied by critical points of Φ_ε, whose formal version is given by (2.1). More specifically, we prove that if $u \in BL(\mathbb{R}^N)$ is such that $0 \in \partial\Phi_\varepsilon(u)$, then there exists $z \in W^{1,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, which plays the role of $\frac{\Delta u}{|\Delta u|}$ in (2.1) and is well defined even where $\Delta u = 0$, $\nabla u = 0$ or $u = 0$. In order to do so, we start by extending the functionals $\mathcal{J}_\varepsilon$, $\mathcal{F}$ and Φ_ε to the space $E = L^1(\mathbb{R}^N) \cap L^{1^*}(\mathbb{R}^N)$ equipped with the norm

$$\|u\|_E = \|u\|_1 + \|u\|_{1^*}, \quad u \in E.$$

More precisely, define the functionals $\overline{\mathcal{J}_\varepsilon}, \overline{\mathcal{F}}, \overline{\Phi_\varepsilon} : E \to \mathbb{R} \cup \{+\infty\}$, by

$$\overline{\mathcal{J}_\varepsilon}(u) = \begin{cases} \mathcal{J}_\varepsilon, & \text{if } u \in BL(\mathbb{R}^N), \\ +\infty, & \text{if } u \in E \backslash BL(\mathbb{R}^N), \end{cases}$$
$$\overline{\mathcal{F}}(u) = \mathcal{F}(u), \quad \text{and} \quad \overline{\Phi_\varepsilon}(u) = \overline{\mathcal{J}_\varepsilon}(u) - \overline{\mathcal{F}}(u), \quad \forall u \in E.$$

By standard arguments, one can easily see that the function $\overline{\mathcal{F}}$ belongs to $C^1(E, \mathbb{R})$ and $\overline{\mathcal{J}_\varepsilon}$ is convex and lower semicontinuous, then the subdifferential ([24]) of $\overline{\mathcal{J}_\varepsilon}$, introduced by $\partial\overline{\mathcal{J}_\varepsilon}$, is well defined. The following result is immediate.

Lemma 2.1. *If $u_\varepsilon \in BL(\mathbb{R}^N)$ is such that $0 \in \partial\Phi_\varepsilon(u_\varepsilon)$, then $0 \in \partial\overline{\Phi_\varepsilon}(u_\varepsilon)$.*

Proof. Suppose that $u_\varepsilon \in BL(\mathbb{R}^N)$ is such that $0 \in \partial\Phi_\varepsilon(u_\varepsilon)$, then u_ε satisfies (2.8). Consider $v \in E$ and note that:

- if $v \in BL(\mathbb{R}^N) \cap E$, then

$$\begin{aligned}\overline{\mathcal{J}_\varepsilon}(v) - \overline{\mathcal{J}_\varepsilon}(u_\varepsilon) &= \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon)\\ &\geqslant \mathcal{F}'(u_\varepsilon)(v - u_\varepsilon)\\ &= \int_{\mathbb{R}^N} f(u_\varepsilon)(v - u_\varepsilon)dx\\ &= \overline{\mathcal{F}}'(u_\varepsilon)(v - u_\varepsilon);\end{aligned}$$

- if $v \in E \setminus BL(\mathbb{R}^N)$, since $\overline{\mathcal{J}_\varepsilon}(v) = +\infty$ and $\overline{\mathcal{J}_\varepsilon}(u_\varepsilon) < +\infty$, it follows that

$$\begin{aligned}\overline{\mathcal{J}_\varepsilon}(v) - \overline{\mathcal{J}_\varepsilon}(u_\varepsilon) &= +\infty\\ &\geqslant \overline{\mathcal{F}}'(u_\varepsilon)(v - u_\varepsilon).\end{aligned}$$

Then, in any case $0 \in \partial\overline{\Phi_\varepsilon}(u_\varepsilon)$. □

Let us assume that $u_\varepsilon \in BL(\mathbb{R}^N)$ is a bounded variation solution of (2.1). Since $0 \in \partial\Phi_\varepsilon(u_\varepsilon)$, from the last result it follows that $0 \in \partial\overline{\Phi_\varepsilon}(u_\varepsilon)$. Since $\overline{\mathcal{J}_\varepsilon}$ is convex and $\overline{\mathcal{F}}$ is smooth, it follows that $\overline{\mathcal{F}}'(u_\varepsilon) \in \partial\overline{\mathcal{J}_\varepsilon}(u_\varepsilon)$. Then, we set the functionals $\overline{\mathcal{J}_\varepsilon^1}, \overline{\mathcal{J}_\varepsilon^2} : E \to \mathbb{R} \cup \{+\infty\}$, by

$$\overline{\mathcal{J}_\varepsilon^1}(u) = \begin{cases} \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u|dx, & \text{if } u \in BL(\mathbb{R}^N),\\ +\infty, & \text{if } u \in E\backslash BL(\mathbb{R}^N),\end{cases}$$

and

$$\overline{\mathcal{J}_\varepsilon^2}(u) = \int_{\mathbb{R}^N} V(\varepsilon u)|u|dx.$$

Note that $\overline{\mathcal{J}_\varepsilon^2} \in C(E, \mathbb{R}), \overline{\mathcal{J}_\varepsilon^2} \in C\left(BL(\mathbb{R}^N), \mathbb{R}\right)$ and

$$\overline{\mathcal{J}_\varepsilon}(u) = \overline{\mathcal{J}_\varepsilon^1}(v) + \overline{\mathcal{J}_\varepsilon^2}(u), \quad \forall u \in E.$$

Since $\overline{\mathcal{J}_\varepsilon^1}$ and $\overline{\mathcal{J}_\varepsilon^2}$ are convex, and $\overline{\mathcal{J}_\varepsilon^2}$ is finite and continuous in every point of $BL(\mathbb{R}^N)$, it follows that

$$\overline{\mathcal{F}}'(u) \in \partial\overline{\mathcal{J}_\varepsilon}(u) = \partial\overline{\mathcal{J}_\varepsilon^1}(u) + \partial\overline{\mathcal{J}_\varepsilon^2}(u).$$

By E. Parini, B. Ruf and C. Tarsi [21], there exist a function $\gamma \in L_{\infty,N}(\mathbb{R}^N)$ and a vector field $\mathbf{z} \in W^{1,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty \leq 1$ and

$$\begin{cases} \operatorname{div}\mathbf{z} \in L_{\infty,N}(\mathbb{R}^N), \Delta\mathbf{z} \in L_{\infty,N}(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} u_\varepsilon \Delta\mathbf{z} - \int_{\mathbb{R}^N} u_\varepsilon \operatorname{div}\mathbf{z} dx = \int_{\mathbb{R}^N} |\Delta u_\varepsilon| + \int_{\mathbb{R}^N} |\nabla u_\varepsilon|\, dx, \\ \gamma\,|u_\varepsilon| = V(\varepsilon x)u_\varepsilon \quad \text{ a.e. in } \mathbb{R}^N, \\ \Delta\mathbf{z} - \operatorname{div}\mathbf{z} + \gamma = K(\varepsilon x)f(u_\varepsilon), \quad \text{ a.e. in } \mathbb{R}^N, \end{cases} \tag{2.9}$$

where

$$L_{\infty,N}\left(\mathbb{R}^N\right) = \left\{g : \mathbb{R}^N \to \mathbb{R} \quad \text{ measurable };\|g\|_{\infty,N} < \infty\right\}$$

and

$$\|g\|_{\infty,N} = \sup_{\|\phi\|_1+\|\phi\|_{1^*}\leq 1} \left| \int_{\mathbb{R}^N} g\phi dx \right|.$$

Hence, (2.9) is the precise version of (1.1).

Analogously, we can define the norms

$$\|u\|_{V_\infty} = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| dx + \int_{\mathbb{R}^N} V_\infty |u| dx,$$

$$\|u\|_{V_0} = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| dx + \int_{\mathbb{R}^N} V_0 |u| dx,$$

and the functionals

$$\Phi_\infty(u) = \|u\|_{V_\infty} - \int_{\mathbb{R}^N} K_\infty F(u) dx,$$

$$\Phi_0(u) = \|u\|_{V_0} - \int_{\mathbb{R}^N} K_0 F(u) dx.$$

Similarly we define critical points of the functionals $\Phi_\infty(u)$ and $\Phi_0(u)$, since they have the same properties that $\Phi_\varepsilon(u)$.

3. Proof of Theorem 1.4

In this section, let us present the proof of the Lions' type result.

Proof of Theorem 1.4. Let $q < s < 1^*$ and $u \in BL(\mathbb{R}^N)$. Since $BL(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for all $r \in [1, 1^*]$, then $u \in L^q(\mathbb{R}^N)$ and $u \in L^{1^*}(\mathbb{R}^N)$.

For $R > 0$, by interpolation inequality with $\theta = \frac{s-q}{1^*-q}\frac{1^*}{s}$ and embedding inequality, we have

$$\begin{aligned}\|u\|_{L^s(B_R(y))} &\le \|u\|^{1-\theta}_{L^q(B_R(y))}\|u\|^{\theta}_{L^{1^*}(B_R(y))}\\ &\le c\|u\|^{1-\theta}_{L^q(B_R(y))}\|u\|^{\theta}_{BL(B_R(y))}.\end{aligned}$$

Let us cover $\mathbb{R}^N$ by balls of radius R and center in (y_n) in such a way that each point in $\mathbb{R}^N$ belongs at most $N+1$ balls, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^s dx &\le \sum_{n=1}^{+\infty} \int_{B_R(y_n)} |u|^s dx\\ &\le c\sum_{n=1}^{+\infty} \|u\|^{(1-\theta)s}_{L^q(B_R(y_n))}\|u\|^{\theta s}_{BL(B_R(y_n))}\\ &\le c\left(\sup_{y\in\mathbb{R}^N}\int_{B_R(y)} |u|^q dx\right)^{\frac{(1-\theta)s}{q}} \lim_{k\to+\infty}\sum_{n=1}^{k}\Bigg(\int_{B_R(y_n)} |\Delta u|\\ &\quad + \int_{B_R(y_n)} |\nabla u| dx + \int_{B_R(y_n)} |u| dx\Bigg)^{\theta s}\\ &= c\left(\sup_{y\in\mathbb{R}^N}\int_{B_R(y)} |u|^q dx\right)^{\frac{(1-\theta)s}{q}} \lim_{k\to+\infty}\sum_{n=1}^{k}\Bigg(\int_{\mathbb{R}^N} \chi_{B_R(y_n)}|\Delta u|\\ &\quad + \int_{\mathbb{R}^N} \chi_{B_R(y_n)}(|\nabla u| + |u|) dx\Bigg)^{\theta s}\\ &= c\left(\sup_{y\in\mathbb{R}^N}\int_{B_R(y)} |u|^q dx\right)^{\frac{(1-\theta)s}{q}} \lim_{k\to+\infty}\Bigg(\int_{\mathbb{R}^N}\sum_{n=1}^{k} \chi_{B_R(y_n)}|\Delta u|\\ &\quad + \int_{\mathbb{R}^N}\sum_{n=1}^{k} \chi_{B_R(y_n)}(|\nabla u| + |u|) dx\Bigg)^{\theta s}\\ &\le c\left(\sup_{y\in\mathbb{R}^N}\int_{B_R(y)} |u|^q dx\right)^{\frac{(1-\theta)s}{q}} (N+1)\|u\|^{\theta s}.\end{aligned}$$

Suppose that (u_n) is bounded in $BL(\mathbb{R}^N)$, by the last inequality and the hypothesis, we have

$$u_n \to 0 \quad \text{in } L^s(\mathbb{R}^N), \tag{3.1}$$

for all $q < s < 1^*$.

Then, if $q = 1$ we are done. Otherwise, if $1 < q < 1^*$, let us consider $1 < s \leq q$ and take $s_0 \in (q, 1^*)$ in such a way that (3.1) holds. Note that $u \in L^1(\mathbb{R}^N) \cap L^{s_0}(\mathbb{R}^N)$ and, since $s \in (1, s_0)$, by doing

$$\theta = \frac{s_0 - s}{s(s_0 - 1)},$$

we have that

$$\frac{1}{s} = \frac{\theta}{1} + \frac{1-\theta}{s_0} \quad \text{and} \quad 0 < \theta < 1.$$

Then, again, the interpolation inequality, the embedding of $BL(\mathbb{R}^N)$ and (3.1), imply that

$$\|u_n\|_s \leq \|u_n\|_1 \|u_n\|_{s_0} \leq \|u_n\| \|u_n\|_{s_0} \to 0,$$

as $n \to \infty$, since (u_n) is bounded in $BL(\mathbb{R}^N)$. $\square$

4. The autonomous case

Let us first recall the Mountain-Pass Theorem in its version from [13].

Theorem 4.1 *(Mountain-Pass Theorem). Let E be a Banach space, $\Psi = I_0 - I$, where $I \in C^1(E, \mathbb{R})$ and I_0 is a locally Lipschitz convex functional defined in E. Suppose that the functional Ψ satisfies the following conditions:*

(g_1) *There exist $\rho > 0$ and $\alpha > \Psi(0)$ such that $\Psi|_{\partial B_\rho(0)} \geq \alpha$.*
(g_2) *$\Psi(e) < \Psi(0)$, for some $e \in E \setminus \overline{B_\rho(0)}$.*

Then for all $\tau > 0$, there exists $x_\tau \in E$ such that

$$c - \tau < \Psi(x_\tau) < c + \tau,$$

and

$$I_0(y) - I_0(x_\tau) \geq I'(x_\tau)(y - x_\tau) - \tau \|y - x_\tau\| \quad \textit{for all } y \in E,$$

where $c \geq \alpha$ is characterized by

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Psi(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0$ *and* $\gamma(1) = e\}$.

Now let us verify that the functional $\Phi_\varepsilon : BL(\mathbb{R}^N) \to \mathbb{R}$ satisfies the geometrical conditions of the Mountain-Pass Theorem.

Lemma 4.2. *There exist* $\varrho, \rho > 0$ *such that,*

(g_1) $\Phi_\varepsilon(u) \geq \varrho$ *for each* $u \in BL(\mathbb{R}^N)$ *such that* $\|u\|_\varepsilon = \rho$*;*
(g_2) *There exists* $e \in BL(\mathbb{R}^N)$ *such that* $\|u\|_\varepsilon > \rho$ *and* $\Phi_\varepsilon(e) < 0$.

Proof. We start to verify the first condition. Note that, from $(f_2) - (f_3)$, for all $\eta > 0$, there exists $A_\eta > 0$ such that

$$F(s) \leq \eta|s| + A_\eta|s|^p, \quad \forall s \in \mathbb{R}, \tag{4.1}$$

where p is as in (f_3). Then, by (K_1), (4.1) and the continuous embeddings of $BL(\mathbb{R}^N)$ we have that

$$\begin{aligned}
\Phi_\varepsilon(u) &= \int\limits_{\mathbb{R}^N} |\Delta u| + \int\limits_{\mathbb{R}^N} |\nabla u|dx + \int\limits_{\mathbb{R}^N} V(\varepsilon x)|u|dx - \int\limits_{\mathbb{R}^N} K(\varepsilon x)F(u)dx \\
&= \|u\|_\varepsilon - \int\limits_{\mathbb{R}^N} K(\varepsilon x)F(u)dx \\
&\geq \|u\|_\varepsilon - K_0\eta \int\limits_{\mathbb{R}^N} |u|dx - K_0 A_\eta \int\limits_{\mathbb{R}^N} |u|^p dx \\
&\geq \|u\|_\varepsilon - K_0\eta C\|u\|_\varepsilon - K_0 A_\eta C\|u\|_\varepsilon^p \\
&= (1 - K_0\eta C)\|u\|_\varepsilon - K_0 A_\eta C\|u\|_\varepsilon^p.
\end{aligned} \tag{4.2}$$

Let us consider $\eta > 0$ such that $(1 - K_0\eta C) > 0$ and ρ such that

$$0 < \rho < (\frac{1 - K_0\eta C}{K_0 A_\eta C})^{\frac{1}{p-1}}.$$

Hence, from (4.2), it implies that

$$\Phi_\varepsilon(u) \geq \varrho > 0, \tag{4.3}$$

for all $u \in BL(\mathbb{R}^N)$ such that $\|u\|_\varepsilon = \rho$, where $\varrho = (1 - K_0\eta C)\rho + K_0 A_\eta C\rho^p > 0$. Hence we have verified the condition (g_1) in Lemma 4.2.

Now let us prove that Φ satisfies (g_2). Note that condition (f_4) implies that there exist constants $d_1, d_2 > 0$ such that

$$F(s) \geq d_1|s|^\kappa - d_2, \quad \forall s \in \mathbb{R}. \tag{4.4}$$

Let $u \in BL(\mathbb{R}^N)$, with compact support, $u \neq 0$ and let $t > 0$. Then

$$\Phi_\varepsilon(tu) \leq t\|u\|_\varepsilon - K_\infty d_1 t^\kappa |u|_\kappa^\kappa + K_0 d_2 |\operatorname{supp}(u)| \to -\infty, \tag{4.5}$$

as $t \to +\infty$. Since $\kappa > 1$ and then we can choose $e \in X$ such that $\Phi_\varepsilon(e) < 0$. $\square$

Analogously, $\Phi_\infty(u)$ and $\Phi_0(u)$ also satisfy the geometrical conditions of the Mountain-Pass Theorem. Then the following minimax levels are well defined

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \sup_{t \in [0,1]} \Phi_\varepsilon(\gamma(t)),$$
$$c_\infty = \inf_{\gamma \in \Gamma_\infty} \sup_{t \in [0,1]} \Phi_\infty(\gamma(t)),$$

and

$$c_0 = \inf_{\gamma \in \Gamma_0} \sup_{t \in [0,1]} \Phi_0(\gamma(t)),$$

where $\Gamma_\varepsilon = \left\{\gamma \in C^0([0,1], BL(\mathbb{R}^N)); \gamma(0) = 0, \Phi_\varepsilon(\gamma(1)) < 0\right\}$ and Γ_∞, Γ_0 are defined in an analogous way.

Now let us define the Nehari manifolds associated to Φ_ε, which are well defined by

$$\mathcal{N}_\varepsilon = \left\{u \in BL(\mathbb{R}^N)\backslash\{0\}; \Phi'_\varepsilon(u)u = 0\right\}.$$

This set is going to give us a better characterization of the minimax level c_ε. Arguing as in [11] it is possible to show that $\mathcal{N}_\varepsilon$ contains all nontrivial solutions of (1.1). More specifically, Figueiredo and Pimenta performed a study of the *fibering maps* $\gamma_u(t) := \Phi_\varepsilon(tu)$, by using $(f_1) - (f_5)$ to show that $\mathcal{N}_\varepsilon$ is radially homeomorphic to the unit sphere in X_ε. In fact, for each $u \in X_\varepsilon\backslash\{0\}$, by (f_2) and (f_3), it can be seen that there exists $t_0 > 0$ such that $\gamma_u(t_0) > 0$. On the other hand, (f_4) implies that $\gamma_u(t) \to -\infty$ as $t \to +\infty$. Then there exists $t_u > 0$ such that $\gamma_u(t_u) = \max_{t>0} \gamma_u(t)$ and then that $\gamma'_u(t_u) = 0$. But (f_5) implies that such t_u is unique. Then for each $u \in BL(\mathbb{R}^N)\backslash\{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$. This establishes such a radial homeomorphism. Still with arguments presented in Rabinowitz [16], one can prove that the minimax level c_ε satisfies

$$c_\varepsilon = \inf_{u \in BL(\mathbb{R}^N)\backslash\{0\}} \max_{t \geq 0} \Phi_\varepsilon(tu) = \inf_{u \in \mathcal{N}_\varepsilon} \Phi_\varepsilon(u).$$

Similarly, setting

$$\mathcal{N}_\infty = \left\{u \in BL(\mathbb{R}^N)\backslash\{0\}; \Phi'_\infty(u)u = 0\right\},$$

and

$$\mathcal{N}_0 = \left\{u \in BL(\mathbb{R}^N)\backslash\{0\}; \Phi_0'(u)u = 0\right\},$$

then one can easily prove that $c_\infty = \inf_{\mathcal{N}_\infty} \Phi_\infty$ and $c_0 = \inf_{\mathcal{N}_0} \Phi_0$.

4.1. Proof of Theorem 1.3

In this section, let us consider that existence of ground-state solutions to the autonomous problem

$$\begin{cases} \Delta_1^2 - \Delta_1 u + V_\infty \frac{u}{|u|} = K_\infty f(u) & \text{in } \mathbb{R}^N, \\ u \in BL(\mathbb{R}^N). \end{cases}$$

Since $\Phi_\infty(u)$ satisfies the geometrical conditions of the Mountain-Pass Theorem, then by the Mountain-Pass Theorem, given a sequence $\tau_n \to 0$, there exists $(w_n) \subset BL(\mathbb{R}^N)$ from [13] such that $\Phi_\infty(w_n) \to c_\infty$ and moreover,

$$\|v\|_{V_\infty} - \|w_n\|_{V_\infty} \geq \int_{\mathbb{R}^N} K_\infty f(w_n)(v - w_n)dx - \tau_n \|v - w_n\|_{V_\infty}, \quad \forall v \in BL(\mathbb{R}^N), \tag{4.6}$$

as $n \to \infty$.

Let us show that the sequence (w_n) is bounded in $BL(\mathbb{R}^N)$.

Lemma 4.3. *The sequence (w_n) is a bounded sequence in $BL(\mathbb{R}^N)$.*

Proof. Taking as test function $v = 2w_n$ in (4.6), we get

$$\|w_n\|_{V_\infty} \geq \int_{\mathbb{R}^N} K_\infty f(w_n) w_n dx - \tau_n \|w_n\|_{V_\infty},$$

which implies that

$$(1 + \tau_n)\|w_n\|_{V_\infty} \geq \int_{\mathbb{R}^N} K_\infty f(w_n) w_n dx. \tag{4.7}$$

Then, by (f_4) and (4.7), it yields

$$\begin{aligned} c_\infty + o_n(1) &\geq \Phi_\infty(w_n) \\ &= \|w_n\|_{V_\infty} + \int_{\mathbb{R}^N} K_\infty(\frac{1}{\kappa} f(w_n)w_n - F(w_n))dx - \int_{\mathbb{R}^N} \frac{1}{\kappa} K_\infty f(w_n) w_n dx \\ &\geq \|w_n\|_{V_\infty}(1 - \frac{1}{\kappa} - \frac{\tau_n}{\kappa}) \\ &\geq C\|w_n\|_{V_\infty}, \end{aligned} \tag{4.8}$$

for some $C > 0$ uniform in $n \in \mathbb{N}$. Then we conclude that (w_n) is bounded in $BL(\mathbb{R}^N)$. $\square$

Since the sequence (w_n) is bounded in $BL(\mathbb{R}^N)$ and the compactness of the embeddings of $BL(\mathbb{R}^N)$ in $L^q_{loc}(\mathbb{R}^N)$ for $1 \le q < 1^*$, there exists $w_\infty \in BL_{loc}(\mathbb{R}^N)$ such that

$$w_n \to w_\infty \text{ in } L^q_{loc}(\mathbb{R}^N), \quad \text{for } 1 \le q < 1^*, \tag{4.9}$$

and

$$w_n \to w_\infty \quad \text{a.e. in } \mathbb{R}^N, \tag{4.10}$$

as $n \to +\infty$. Note that $w_\infty \in BL(\mathbb{R}^N)$. In fact, by Fatou's Lemma, it follows that $w_\infty \in L^1(\mathbb{R}^N)$. For a given $R > 0$, from the semicontinuity of the norm in $BL\,(B_R(0))$ with respect to the $L^q(B_R(0))$ convergence, we have that

$$\int_{B_R(0)} |\Delta w_\infty| \le \liminf_{n\to+\infty} \int_{B_R(0)} |\Delta w_n| \le \liminf_{n\to+\infty} \|w_n\|_{BL(\mathbb{R}^N)} \le C, \tag{4.11}$$

where C does not depend on n or R. Since the last inequality (4.11) holds for every $R > 0$, then $\Delta u \in \mathcal{M}(\mathbb{R}^N)$. Hence, by Proposition 2.1 of [13], it follows that $w_\infty \in BL(\mathbb{R}^N)$. Moreover, there exist $R, \beta > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} |w_n|\, dx \ge \beta.$$

Indeed, otherwise, from [13], $w_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $1 < q < 1^*$ and then, by (f_2) and (f_3), $\Phi_\infty(w_n) \to 0$, which lead to a contradiction with the fact that $c_\infty > 0$. Since Φ_∞ is invariant by translation, we can assume that $w_\infty \neq 0$ without lack of generality.

Now let us consider the following Lemma.

Lemma 4.4. $\Phi'_\infty(w_\infty)w_\infty \le 0$.

Proof. Note that, if $\varphi \in C_0^\infty(\mathbb{R}^N)$, $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_R(0)$, $\varphi \equiv 0$ in $B_{2R}(0)^c$ and there exists a fixed positive constant $C > 0$ such that $|\nabla\varphi| \le C$ and $|\Delta\varphi| \le C$, for $\varphi_R := \varphi(\cdot/R)$, it shows that for all $u \in BL(\mathbb{R}^N)$,

$$(\Delta(\varphi_R u))^s \quad \text{is absolutely continuous w.r.t.} \quad (\Delta u)^s. \tag{4.12}$$

Indeed, note that

$$\begin{aligned}\Delta(\varphi_R u) =& \Delta\varphi_R u + 2\nabla\varphi_R \cdot \nabla u + \varphi_R \Delta u\\ =& \Delta\varphi_R u + 2\nabla\varphi_R \cdot \nabla u + \varphi_R(\Delta u)^a + \varphi_R(\Delta u)^s, \quad \text{in } \mathcal{D}'(\mathbb{R}^N).\end{aligned}$$

Then it follows that

$$(\Delta(\varphi_R u))^s = (\varphi_R(\Delta u)^s)^s = \varphi_R(\Delta u)^s.$$

Taking (4.12) into account and the fact that $\varphi_R w_n$ is equal to 0 a.e. in the set where u_n vanishes, note that $\varphi_R w_n$ and w_n fulfill two of the three requirements that would allow us to calculate $\Phi'_\infty(w_n)(\varphi_R w_n)$. However, we have no ensure that $(\Delta(\varphi_R w_n))^a = \Delta\varphi_R w_n + 2\nabla\varphi_R \nabla w_n + \varphi_R(\Delta w_n)^a$ vanishes a.e. in the set

$$\left\{x \in \mathbb{R}^N; (\Delta w_n)^a(x) = 0\right\}.$$

Hence, it might not be possible to calculate the Gateaux derivative $\Phi'_\infty(w_n)(\varphi_R w_n)$ and then we have to work in a slightly different way. In fact, it will be enough to work with the left Gateaux derivative

$$\lim_{t\to 0^-} \frac{\Phi_\infty(w_n + t\varphi_R w_n) - \Phi_\infty(w_n)}{t},$$

which, by (4.6), satisfy

$$\lim_{t\to 0^-} \frac{\Phi_\infty(w_n + t\varphi_R w_n) - \Phi_\infty(w_n)}{t} \le o_n(1). \tag{4.13}$$

In order to calculate the limit above, let us first calculate separately a part of it. Let us define for all $u \in BL(\mathbb{R}^N)$,

$$\mathcal{J}_a(u) = \int\limits_{\mathbb{R}^N} |(\Delta u)^a(x)|\, dx.$$

Then, for all $u, v \in BL(\mathbb{R}^N)$, we have that

$$\begin{aligned} \lim_{t\to 0^-} \frac{\mathcal{J}_a(u+tv) - \mathcal{J}_a(u)}{t} &= \lim_{t\to 0^-} \frac{1}{t} \int\limits_{\mathbb{R}^N} (|(\Delta u)^a + t(\Delta v)^a| - |(\Delta u)^a|)dx \\ &= -\int\limits_{T_u} |(\Delta v)^a|\, dx + \int\limits_{\mathbb{R}^N\setminus T_u} \frac{(\Delta u)^a(\Delta v)^a}{|(\Delta u)^a|} dx, \end{aligned} \tag{4.14}$$

where $T_u = \left\{x \in \mathbb{R}^N; (\Delta u)^a(x) = 0\right\}$.

Taking into account (4.13) and (4.14), it follows that

$$
\begin{aligned}
o_n(1) \geq & \int\limits_{\mathbb{R}^N \setminus T_{w_n}} \frac{(\Delta w_n)^a[\Delta\varphi_R w_n + 2\nabla\varphi_R \cdot \nabla w_n + \varphi_R(\Delta w_n)^a]}{|(\Delta w_n)^a|} dx \\
& - \int\limits_{T_{w_n}} |(\Delta\varphi_R w_n + 2\nabla\varphi_R \cdot \nabla w_n)| dx \\
& + \int\limits_{\mathbb{R}^N} \frac{\Delta w_n}{|\Delta w_n|}\varphi_R(\Delta w_n)^s + \int\limits_{\mathbb{R}^N} \frac{\nabla w_n \cdot (\nabla\varphi_R w_n + \varphi_R \nabla w_n)}{|\nabla w_n|} dx \\
& + \int\limits_{\mathbb{R}^N} V_\infty \operatorname{sgn}(w_n)(\varphi_R w_n) dx - \int\limits_{\mathbb{R}^N} K_\infty f(w_n)\varphi_R w_n dx \\
= & \int\limits_{\mathbb{R}^N \setminus T_{w_n}} \varphi_R |(\Delta w_n)^a| dx + \int\limits_{\mathbb{R}^N \setminus T_{w_n}} \frac{(\Delta w_n)^a(\Delta\varphi_R w_n + 2\nabla\varphi_R \cdot \nabla w_n)}{|(\Delta w_n)^a|} dx \\
& - \int\limits_{T_{w_n}} |(\Delta\varphi_R w_n + 2\nabla\varphi_R \cdot \nabla w_n)| dx \\
& + \int\limits_{\mathbb{R}^N} \frac{\Delta w_n}{|\Delta w_n|}\varphi_R(\Delta w_n)^s + \int\limits_{\mathbb{R}^N} \frac{\nabla w_n \cdot (\nabla\varphi_R w_n + \varphi_R \nabla w_n)}{|\nabla w_n|} dx \\
& + \int\limits_{\mathbb{R}^N} V_\infty |w_n| \varphi_R dx - \int\limits_{\mathbb{R}^N} K_\infty f(w_n)\varphi_R w_n dx.
\end{aligned}
$$

Noting that $\int_{\mathbb{R}^N \setminus T_{w_n}} \varphi_R |(\Delta w_n)^a| dx = \int_{\mathbb{R}^N} \varphi_R |(\Delta w_n)^a| dx$ and calculating the lim inf in the inequality above as $n \to +\infty$, we have that

$$
\begin{aligned}
0 \geq & \liminf_{n\to+\infty} \left(\int\limits_{\mathbb{R}^N} \varphi_R |(\Delta w_n)^a| dx + \int\limits_{\mathbb{R}^N} \frac{(\Delta w_n)^s}{|(\Delta w_n)^s|} \varphi_R (\Delta w_n)^s \right) \\
& + \liminf_{n\to+\infty} \int\limits_{\mathbb{R}^N \setminus T_{w_n}} \frac{(\Delta w_n)^a(\Delta\varphi_R w_n + 2\nabla\varphi_R \cdot \nabla w_n)}{|(\Delta w_n)^a|} dx \\
& - \limsup_{n\to+\infty} \int\limits_{T_{w_n}} |(\Delta\varphi_R w_n + 2\nabla\varphi_R \cdot \nabla w_n)| dx \\
& + \liminf_{n\to+\infty} \int\limits_{\mathbb{R}^N} \frac{\nabla w_n \cdot (\nabla\varphi_R w_n + \varphi_R \nabla w_n)}{|\nabla w_n|} dx \\
& + \int\limits_{\mathbb{R}^N} V_\infty |w_\infty| \varphi_R dx - \int\limits_{\mathbb{R}^N} K_\infty f(w_\infty)\varphi_R w_\infty dx.
\end{aligned}
\tag{4.15}
$$

Now, by the lower semicontinuity of the norm in $BL(B_R(0))$ w.r.t. the $L^1(B_R(0))$ convergence and also the fact that $\frac{\varphi_R \mu}{|\varphi_R \mu|} = \frac{\mu}{|\mu|}$ a.e. in $B_R(0)$ with (4.15), we have that

$$\begin{aligned}\int_{B_R(0)} |\Delta w_\infty| \leq &- \liminf_{n\to+\infty} \int_{\mathbb{R}^N\setminus T_{w_n}} \frac{(\Delta w_n)^a(\Delta\varphi_R w_n + 2\nabla\varphi_R\cdot\nabla w_n)}{|(\Delta w_n)^a|}dx \\ &+ \limsup_{n\to+\infty}\int_{T_{w_n}} |(\Delta\varphi_R w_n + 2\nabla\varphi_R\cdot\nabla w_n)|\,dx \\ &- \liminf_{n\to+\infty}\int_{\mathbb{R}^N} \frac{\nabla w_n\cdot(\nabla\varphi_R w_n + \varphi_R\nabla w_n)}{|\nabla w_n|}dx \\ &- \int_{\mathbb{R}^N} V_\infty\,|w_\infty|\,\varphi_R dx + \int_{\mathbb{R}^N} K_\infty f(w_\infty)\varphi_R w_\infty dx.\end{aligned} \tag{4.16}$$

And since (w_n) is a bounded sequence in $L^1(\mathbb{R}^N)$, it shows that

$$\begin{aligned}&\lim_{R\to+\infty}\left|\lim\ \inf_{n\to\infty}\int_{\mathbb{R}^N\setminus T_{w_n}} \frac{w_n(\Delta w_n)^a\cdot\Delta\varphi_R}{|(\Delta w_n)^a|}dx\right| \\ \leq &\lim_{R\to+\infty}(\lim\ \inf_{n\to\infty}\int_{\mathbb{R}^N\setminus T_{w_n}} |w_n|\,|\Delta\varphi_R|\,dx) \\ \leq &\lim_{R\to+\infty}\frac{C}{R^2}(\lim\ \inf_{n\to\infty}\int_{\mathbb{R}^N\setminus T_{w_n}} |w_n|\,dx) = 0.\end{aligned} \tag{4.17}$$

Similarly, we can also get that

$$\begin{aligned}&\lim_{R\to+\infty}\left|\lim\ \inf_{n\to\infty}\int_{\mathbb{R}^N\setminus T_{w_n}} \frac{(\Delta w_n)^a(2\nabla\varphi_R\cdot\nabla w_n)}{|(\Delta w_n)^a|}dx\right| = 0, \\ &\lim_{R\to+\infty}\left|\liminf_{n\to+\infty}\int_{T_{w_n}} |(w_n\Delta\varphi_R + 2\nabla\varphi_R\cdot\nabla w_n)|\,dx\right| = 0,\end{aligned} \tag{4.18}$$

and

$$\lim_{R\to+\infty}\left|\liminf_{n\to+\infty}\int_{\mathbb{R}^N} \frac{w_n\nabla w_n\cdot\nabla\varphi_R}{|\nabla w_n|}dx\right| = 0. \tag{4.19}$$

By doing $R\to+\infty$ in both sides of (4.16) and taking into account (4.17), (4.18) and (4.19), we get that

$$\int_{\mathbb{R}^N} |\Delta w_\infty| + \int_{\mathbb{R}^N} |\nabla w_\infty| dx + \int_{\mathbb{R}^N} V_\infty\,|w_\infty|\,dx \leq \int_{\mathbb{R}^N} K_\infty f(w_\infty)w_\infty dx, \tag{4.20}$$

and the Lemma is proved. □

Proof of Theorem 1.3. By the last results, then there exists $t_\infty \in (0,1]$ such that $t_\infty w_\infty \in \mathcal{N}_\infty$. Note also that

$$c_\infty + o_n(1) = \Phi_\infty(w_n) \geq \int\limits_{\mathbb{R}^N} K_\infty(f(w_n)w_n - F(w_n))dx + o_n(1), \tag{4.21}$$

and under (f_5), it is easy to see that $t \mapsto f(t)t - F(t)$ is increasing for $t \in (0, +\infty)$ and decreasing for $t \in (-\infty, 0)$, then by Fatou Lemma in the last inequality, we derive that

$$\begin{aligned} c_\infty &\geq \int\limits_{\mathbb{R}^N} K_\infty(f(w_\infty)w_\infty - F(w_\infty))dx \\ &\geq \int\limits_{\mathbb{R}^N} K_\infty(f(t_\infty w_\infty)t_\infty w_\infty - F(t_\infty w_\infty))dx \\ &= \Phi_\infty(t_\infty w_\infty) - \Phi'_\infty(t_\infty w_\infty)t_\infty w_\infty \\ &= \Phi_\infty(t_\infty w_\infty) \\ &\geq c_\infty, \end{aligned}$$

which implies that $t_\infty = 1$, $\Phi_\infty(w_\infty) = c_\infty$. And it follows by [11] that w_∞ is a ground-state solution of problem (1.2). □

Remark 4.5. Note that, by the same reason, there exists a critical point of Φ_0, $w_0 \in BL(\mathbb{R}^N)$, such that $\Phi_0(w_0) = c_0$.

5. Proof of Theorem 1.1

5.1. Existence of solution

First of all, we study the behavior of the minimax levels c_ε, when $\varepsilon \to 0^+$. Without lack of generality, let us suppose that $V(0) = V_0$ and $K(0) = K_0$.

Lemma 5.1. $\lim_{\varepsilon\to 0^+} c_\varepsilon = c_0$.

Proof. Let $\varepsilon_n \to 0$ as $n \to +\infty$. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \leq \psi \leq 1, \psi \equiv 0$ in $B_2(0)^c, \psi \equiv 1$ in $B_1(0)$, $|\nabla\psi| \leq C$ and $|\Delta\psi| \leq C$ in $\mathbb{R}^N$. Let us define

$$w_{\varepsilon_n}(x) = \psi(\varepsilon_n x)w_0(x),$$

where w_0 is a ground-state critical point of Φ_0. And note that $w_{\varepsilon_n} \to w_0$ in $BL(\mathbb{R}^N)$ and $\Phi_0(w_{\varepsilon_n}) \to \Phi_0(w_0)$, as $n \to +\infty$. Let $t_{\varepsilon_n} > 0$ be such that $t_{\varepsilon_n} w_{\varepsilon_n} \in \mathcal{N}_{\varepsilon_n}$ and let us suppose just for a while that $t_{\varepsilon_n} \to 1$ as $n \to +\infty$. Then we have

$$\begin{aligned}c_{\varepsilon_n} &\le \Phi_{\varepsilon_n}(t_{\varepsilon_n}w_{\varepsilon_n})\\ &= \int\limits_{\mathbb{R}^N}|\Delta(t_{\varepsilon_n}w_{\varepsilon_n})| + \int\limits_{\mathbb{R}^N}|\nabla(t_{\varepsilon_n}w_{\varepsilon_n})|dx + \int\limits_{\mathbb{R}^N}V(\varepsilon_n x)|t_{\varepsilon_n}w_{\varepsilon_n}|dx\\ &\quad - \int\limits_{\mathbb{R}^N}K(\varepsilon_n x)F(t_{\varepsilon_n}w_{\varepsilon_n})dx\\ &= \Phi_0\left(t_{\varepsilon_n}w_{\varepsilon_n}\right) + \int\limits_{\mathbb{R}^N}\left(V\left(\varepsilon_n x\right) - V_0\right)\left|t_{\varepsilon_n}w_{\varepsilon_n}\right|dx - \int\limits_{\mathbb{R}^N}(K(\varepsilon_n x) - K_0)F(t_{\varepsilon_n}w_{\varepsilon_n})dx.\end{aligned}$$

Using the Lebesgue Dominated Theorem, it follows that

$$\limsup_{n\to+\infty} c_{\varepsilon_n} \le \Phi_0\left(w_0\right) = c_0.$$

On the other hand, since $\Phi_0(u) \le \Phi_{\varepsilon_n}(u)$ for all $u \in BL(\mathbb{R}^N)$, it follows that $c_0 \le c_{\varepsilon_n}$. Then

$$\lim_{n\to+\infty} c_{\varepsilon_n} = c_0.$$

Now let us prove that in fact $t_{\varepsilon_n} \to 1$, as $n \to +\infty$. Since $\Phi'_{\varepsilon_n}\left(t_{\varepsilon_n}w_{\varepsilon_n}\right)w_{\varepsilon_n} = 0$, it shows that

$$\int\limits_{\mathbb{R}^N}|\Delta w_{\varepsilon_n}| + \int\limits_{\mathbb{R}^N}|\nabla w_{\varepsilon_n}|dx + \int\limits_{\mathbb{R}^N}V(\varepsilon_n x)|w_{\varepsilon_n}|dx = \int\limits_{\mathbb{R}^N}K(\varepsilon_n x)f(t_{\varepsilon_n}w_{\varepsilon_n})w_{\varepsilon_n}dx.$$

We claim that $(t_{\varepsilon_n})_{\varepsilon_n>0}$ is bounded. Indeed, on the contary, up to a subsequence, $t_{\varepsilon_n} \to +\infty$. Let $\Sigma \subset \mathbb{R}^N$ be such that $|\Sigma| > 0$ and $w_0(x) \neq 0$ for all $x \in \Sigma$. Hence it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned}\|w_{\varepsilon_n}\|_{\varepsilon_n} &= \int\limits_{\mathbb{R}^N}\frac{K(\varepsilon_n x)f\left(t_{\varepsilon_n}w_{\varepsilon_n}\right)t_{\varepsilon_n}w_{\varepsilon_n}}{t_{\varepsilon_n}}dx\\ &\ge \int\limits_{\Sigma}\frac{\kappa K(\varepsilon_n x)F\left(t_{\varepsilon_n}w_{\varepsilon_n}\right)}{t_{\varepsilon_n}}dx.\end{aligned}$$

Then by (f_4) and Fatou's Lemma, it follows that

$$\left\|w_{\varepsilon_n}\right\|_{\varepsilon_n} \to +\infty, \qquad \text{as } n \to \infty,$$

which contradicts the fact that $w_{\varepsilon_n} \to w_0$ in $BL(\mathbb{R}^N)$ as $n \to \infty$. Now we have to verify that $t_{\varepsilon_n} \nrightarrow 0$ as $n \to +\infty$. Indeed, on the contrary, from (f_2) and the fact that $t_{\varepsilon_n}w_{\varepsilon_n} \in \mathcal{N}_{\varepsilon_n}$, we would have that

$$\left\|w_{\varepsilon_n}\right\|_{\varepsilon_n} = \int\limits_{\mathbb{R}^N}K(\varepsilon_n x)f\left(t_{\varepsilon_n}w_{\varepsilon_n}\right)w_{\varepsilon_n}dx = o_n(1),$$

which lead to a clear contradiction. Then there exist $\alpha, \beta > 0$ such that

$$\alpha \leq t_{\varepsilon_n} \leq \beta \text{ for all } n \in \mathbb{N}$$

and then, up to a subsequence, $t_{\varepsilon_n} \to \bar{t} > 0$, as $n \to +\infty$. Since

$$\int_{\mathbb{R}^N} |\Delta w_{\varepsilon_n}| + \int_{\mathbb{R}^N} |\nabla w_{\varepsilon_n}| dx + \int_{\mathbb{R}^N} V(\varepsilon_n x)|w_{\varepsilon_n}| dx \to \int_{\mathbb{R}^N} |\Delta w_0| + \int_{\mathbb{R}^N} |\nabla w_0| dx + \int_{\mathbb{R}^N} V(0)\, |w_0|\, dx,$$

and

$$\int_{\mathbb{R}^N} K\left(\varepsilon_n x\right) f(w_{\varepsilon_n}) dx \to \int_{\mathbb{R}^N} K(0) f(w_0) dx,$$

from the definition of w_0, it follows by (f_5) that $\bar{t} = 1$. Hence, $\lim_{\varepsilon \to 0^+} c_\varepsilon = c_0$. The proof is complete. □

Since by (V_1) and (K_1), we know $V_0 < V_\infty$ and $K_0 \geq K_\infty$, it shows from the monotonicity of the energy functional w.r.t. the potentials that

$$c_0 < c_\infty. \tag{5.1}$$

Thus, from Lemma 5.1 and (5.1), we can easily get the following Corollary.

Corollary 5.2. *There exists $\varepsilon_0 > 0$ such that $c_\varepsilon < c_\infty$ for all $\varepsilon \in (0, \varepsilon_0)$.*

As in the proof of Lemma 4.3, it is possible to prove that (u_n) is a bounded sequence in $BL(\mathbb{R}^N)$. By Theorem 4.1, for each $\varepsilon > 0$, there exists a Palais–Smale sequence $(u_n) \subset BL(\mathbb{R}^N)$ to Φ_ε in the level c_ε, i.e.

$$\lim_{n \to \infty} \Phi_\varepsilon(u_n) = c_\varepsilon, \tag{5.2}$$

and

$$\|v\|_\varepsilon - \|u_n\|_\varepsilon \geq \int_{\mathbb{R}} K(\varepsilon x) f(u_n)(v - u_n) dx - \tau_n \|v - u_n\|_\varepsilon, \quad \forall v \in BL(\mathbb{R}^N), \tag{5.3}$$

where $\tau_n \to 0$ as $n \to \infty$. As in the proof of Lemma 4.3, it is possible to prove that the sequence (u_n) is a bounded sequence in $BL(\mathbb{R}^N)$ and by the compactness of the embeddings of $BL(\mathbb{R}^N)$ in $L^q_{loc}(\mathbb{R}^N)$ for $1 \leq q < 1^*$, there exists $u_\varepsilon \in BL_{loc}(\mathbb{R}^N)$ such that

$$u_n \to u_\varepsilon \text{ in } L^q_{loc}(\mathbb{R}^N), \quad \text{for } 1 \le q < 1^*, \tag{5.4}$$

and

$$u_n \to u_\varepsilon \quad \text{a.e. in } \mathbb{R}^N, \tag{5.5}$$

as $n \to +\infty$. Note that as in the last section, it is possible to prove that $u_\varepsilon \in BL(\mathbb{R}^N)$.

The next result is a key tool in our work. We will use Concentration of Compactness Principle due to Lions [14] to get its proof.

Lemma 5.3. *If $(u_n) \subset BL(\mathbb{R}^N)$ is a sequence satisfying (5.2) and (5.3). Then there exists $u_\varepsilon \in BL(\mathbb{R}^N)$ such that for all $\varepsilon \in (0, \varepsilon_0)$, where ε_0 like in Corollary 5.2, and $u_n \to u_\varepsilon$ in $L^q(\mathbb{R}^N)$ for all $1 \le q < 1^*$.*

Proof. Let us use the Concentration of Compactness Principle of Lions to the following bounded sequence in $L^1(\mathbb{R}^N)$, and set the function $\rho_n(x) = \frac{|u_n(x)|}{\|u_n\|_1}$. Note that

$$\|u_n\|_1 \nrightarrow 0, \quad \text{as } n \to +\infty. \tag{5.6}$$

Indeed, otherwise, by interpolation inequality (u_n) would converge to 0 in $L^q(\mathbb{R}^N)$ for all $1 \le q < 1^*$. By taking $w = u_n + tu_n$ in (5.3) and doing $t \to 0$, it is easy to see that

$$\|u_n\|_\varepsilon = \int_{\mathbb{R}^N} K(\varepsilon x) f(u_n) u_n dx + o_n(1). \tag{5.7}$$

Then, by the last equality, $(f_2), (f_3)$, the fact that $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $1 \le q < 1^*$ and the Lebesgue Convergence Theorem, we have that $u_n \to 0$ in $BL(\mathbb{R}^N)$, implying that $c_\varepsilon = 0$, which is a contradiction.

It is easy to see that (ρ_n) is bounded in $L^1(\mathbb{R}^N)$, then the Concentration of Compactness Principle implies that one and only one of the following statements holds:

(i) (Vanishing) For all $R > 0$, there holds $\lim_{n\to+\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} \rho_n dx = 0$;

(ii) (Compactness) There exist $(y_n) \subset \mathbb{R}^N$ such that for all $\eta > 0$, there exists $R > 0$ such that

$$\int_{B_R(y_n)} \rho_n dx \ge 1 - \eta, \quad \forall n \in \mathbb{N}; \tag{5.8}$$

(iii) (Dichotomy) There exist $(y_n) \subset \mathbb{R}^N, \alpha \in (0,1), R_1 > 0, R_n \to +\infty$ such that the functions $\rho_{n,1}(x) := \chi_{B_{R_1}(y_n)}(x)\rho_n(x)$ and $\rho_{n,2}(x) := \chi_{B^c_{R_n}(y_n)}(x)\rho_n(x)$ satisfy

$$\int_{\mathbb{R}^N} \rho_{n,1} dx \to \alpha \quad \text{and} \quad \int_{\mathbb{R}^N} \rho_{n,2} dx \to 1 - \alpha. \tag{5.9}$$

Our target is to show that (ρ_n) verifies the Compactness condition and in order to do so we act by ruling out the others two possibilities. Note that the Vanishing case does not occur. Indeed, otherwise, it will hold that $\rho_n \to 0$ in $L^q(\mathbb{R}^N)$, for all $1 \leq q < 1^*$. Taking (5.6) into account, this implies that $u_n \to 0$ in $L^q(\mathbb{R}^N)$, for all $1 \leq q < 1^*$ and then, this will lead us to $c_\varepsilon = 0$, which is a clear contradiction.

Now we show the Dichotomy case also does not hold. Firstly, it follows from (5.3) that $\Phi'_\varepsilon(u_n)u_n = o_n(1)$, as $n \to \infty$. As far as the sequence (y_n) is concerned, let us consider the following two possible situations.

- (y_n) is bounded.
  In this case, the function u_ε is nontrivial, since $\int_{B_R(y_n)} \frac{|u_n|}{\|u_n\|_1} dx \to \alpha$, it implies that

$$\int\limits_{B_R(y_n)} |u_n|\, dx \geq \delta, \quad \text{for } n \text{ sufficiently large.}$$

  Thus, by taking $R_0 > 0$ such that $B_R(y_n) \subset B_{R_0}(0)$ for all $n \in \mathbb{N}$, then $\int_{B_{R_0}(0)} |u_n|\, dx \geq \delta$ for n large enough. It follows from the Sobolev embedding inequality that

$$\int\limits_{B_{R_0}(0)} |u_\varepsilon|\, dx \geq \delta, \quad \text{for } n \text{ sufficiently large.} \tag{5.10}$$

Similarly to the proof of Lemma 4.4, we obtain that

$$\Phi'_\varepsilon(u_\varepsilon)u_\varepsilon \leq 0.$$

Then, there exists $t_\varepsilon \in (0,1]$ such that $t_\varepsilon u_\varepsilon \in \mathcal{N}_\varepsilon$. Note also that

$$c_\varepsilon + o_n(1) = \Phi_\varepsilon(u_n) \geq \int\limits_{\mathbb{R}^N} K(\varepsilon x)(f(u_n)u_n - F(u_n))dx + o_n(1), \tag{5.11}$$

and by (f_5) and Fatou Lemma in the last inequality, we have that

$$\begin{aligned}
c_\varepsilon &\geq \int\limits_{\mathbb{R}^N} K(\varepsilon x)(f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon))dx \\
&\geq \int\limits_{\mathbb{R}^N} K(\varepsilon x)(f(t_\varepsilon u_\varepsilon)t_\varepsilon u_\varepsilon - F(t_\varepsilon u_\varepsilon))dx \\
&= \Phi_\varepsilon(t_\varepsilon u_\varepsilon) - \Phi'_\varepsilon(t_\varepsilon u_\varepsilon)t_\varepsilon u_\varepsilon \\
&= \Phi_\varepsilon(t_\varepsilon u_\varepsilon) \\
&\geq c_\varepsilon.
\end{aligned}$$

Hence, $t_\varepsilon = 1$, $\Phi_\varepsilon(u_\varepsilon) = c_\varepsilon$, and by (5.11),

$$K(\varepsilon x)(f(u_n)u_n - F(u_n)) \to K(\varepsilon x)(f(u_\varepsilon)u_\epsilon - F(u_\varepsilon)) \text{ in } L^1(\mathbb{R}^N). \tag{5.12}$$

Moreover, by (f_4), we have

$$0 \le (1 - \frac{1}{\kappa})f(u_n)u_n \le f(u_n)u_n - F(u_n),$$

and

$$0 \le (\kappa - 1)F(u_n) \le f(u_n)u_n - F(u_n).$$

Then, by (5.12), we can apply the Lebesgue Dominated Convergence Theorem to get

$$K(\varepsilon x)f(u_n)u_n \to K(\varepsilon x)f(u_\varepsilon)u_\varepsilon \quad \text{in } L^1(\mathbb{R}^N), \tag{5.13}$$

and

$$K(\varepsilon x)F(u_n) \to K(\varepsilon x)F(u_\varepsilon) \quad \text{in } L^1(\mathbb{R}^N). \tag{5.14}$$

Since that $\|u_\varepsilon\|_\varepsilon = \int_{\mathbb{R}^N} K(\varepsilon x)f(u_\varepsilon)u_\varepsilon dx$, by the limit (5.13) combines with (5.7), we obtain

$$\|u_n\|_\varepsilon \to \|u_\varepsilon\|_\varepsilon , \tag{5.15}$$

from where it follows that

$$\|u_n\|_1 \to \|u_\varepsilon\|_1, \tag{5.16}$$

as $n \to \infty$. As a consequence, since (y_n) is bounded and $R_n \to +\infty$, the $L^1(\mathbb{R}^N)$ convergence of (u_n) leads to

$$\int\limits_{B^c_{R_n}(y_n)} |u_n|\, dx \to 0, \qquad \text{as } n \to +\infty. \tag{5.17}$$

On the other hand, since $u_n \to u_\varepsilon \neq 0$ in $L^1(\mathbb{R}^N)$ and by (5.9), it follows that

$$\int\limits_{B^c_{R_n}(y_n)} |u_n|\, dx \to (1-\alpha)\|u_\varepsilon\|_1 > 0, \qquad \text{as } n \to +\infty,$$

which is a contradiction with (5.17).

- (y_n) is unbounded.
  In this case, we should define the sequence $\{\widetilde{u}_n\}$, where $\widetilde{u}_n = u_n(\cdot + y_n)$. Indeed, note that $\{\widetilde{u}_n\}$ is bounded in $BL(\mathbb{R}^N)$ and then converges, up to a subsequence, to some function $\widetilde{u} \in BL(\mathbb{R}^N)$ in $L^1_{loc}(\mathbb{R}^N)$, where $\widetilde{u} \neq 0$ by (5.9).

Now let us show a claim.

Claim 5.4. $\Phi'_\infty(\widetilde{u})\widetilde{u} \le 0$.

Let us define

$$\|u\|_{\varepsilon,y_n} = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| dx + \int_{\mathbb{R}^N} V(\varepsilon x + \varepsilon y_n)|u| dx$$

and

$$\Phi_{\varepsilon,y_n}(u) = \|u\|_{\varepsilon,y_n} - \int_{\mathbb{R}^N} K(\varepsilon x + \varepsilon y_n)F(u)dx,$$

for all $u \in BL(\mathbb{R}^N)$. Note that, just like before, $\Phi'_{\varepsilon,y_n}(u)v$ is well denoted for all $u, v \in BL(\mathbb{R}^N)$ such that $(\Delta v)^s$ is absolutely continuous with respect to $(\Delta u)^s$, $(\Delta v)^a$ vanishes $\mathcal{L}^N$-a.e. in $\{x \in \mathbb{R}^N; (\Delta u)^a(x) = 0\}$, ∇v vanishes a.e. in the set where ∇u vanishes and $v \equiv 0$, a.e. in the set where u vanishes, it follows that

$$\begin{aligned}\Phi'_{\varepsilon,y_n}(u)v = &\int_{\mathbb{R}^N} \frac{(\Delta u)^a(\Delta v)^a}{|(\Delta u)^a|}dx + \int_{\mathbb{R}^N} \frac{\Delta u}{|\Delta u|}(x)\frac{\Delta v}{|\Delta v|}(x)\,|(\Delta v)^s| + \int_{\mathbb{R}^N} \frac{\nabla u \cdot \nabla v}{|\nabla u|}dx \\ &+ \int_{\mathbb{R}^N} V(\varepsilon x + \varepsilon y_n)\,\mathrm{sgn}(u)v dx - \int_{\mathbb{R}^N} K(\varepsilon x + \varepsilon y_n)f(u)v dx.\end{aligned} \tag{5.18}$$

Similar to the Lemma 4.4, taking into account that $|y_n| \to +\infty$ and the invariance of translation, we can get

$$\Phi'_{\varepsilon,y_n}(\tilde{u}_n)(\varphi_R\tilde{u}_n) = o_n(1), \tag{5.19}$$

and

$$\int_{\mathbb{R}^N} |\Delta\widetilde{u}| + \int_{\mathbb{R}^N} |\nabla\widetilde{u}| dx + \int_{\mathbb{R}^N} V_\infty|\widetilde{u}| dx \le \int_{\mathbb{R}^N} K_\infty f(\widetilde{u})\widetilde{u} dx, \tag{5.20}$$

and the Claim 5.4 is proved.

Then from Claim 5.4 and $\widetilde{u} \neq 0$, it follows that there exists $\widetilde{t} \in (0,1]$ such that $\widetilde{t}\widetilde{u} \in \mathcal{N}_\infty$. Note that

$$\begin{aligned} c_\varepsilon + o_n(1) &= \Phi(u_n) + o_n(1) \\ &= \Phi(u_n) - \Phi'(u_n)u_n \\ &= \int\limits_{\mathbb{R}^N} K(\varepsilon x)(f(u_n)u_n - F(u_n))dx \\ &= \int\limits_{\mathbb{R}^N} K(\varepsilon x)(f(\widetilde{u})\widetilde{u} - F(\widetilde{u}))dx. \end{aligned} \tag{5.21}$$

Then from (5.21), (K_1) and Fatou's Lemma, we have that

$$c_\varepsilon \geq \int\limits_{\mathbb{R}^N} K(\varepsilon x)(f(\widetilde{u})\widetilde{u} - F(\widetilde{u}))dx \geq \int\limits_{\mathbb{R}^N} K(\varepsilon x)(f(\widetilde{t\widetilde{u}})\widetilde{t\widetilde{u}} - F(\widetilde{t\widetilde{u}}))dx = \Phi_\infty(\widetilde{t\widetilde{u}}) \geq c_\infty,$$

which is a contradiction with Corollary 5.2.

Therefore, we can inference that Dichotomy case does not hold and then, it follows that Compactness must happen.

Claim 5.5. *The sequence (y_n) in (5.8) is bounded in $\mathbb{R}^N$.*

Now we can prove this claim by contradiction that, up to a subsequence, $|y_n| \to +\infty$ and then proceed as in the case of Dichotomy, where (y_n) is unbounded, getting that $c_\varepsilon \geq c_\infty$, which is also a contradiction with Corollary 5.2.

By Claim 5.5, for $\eta > 0$, it follows from (5.8) that there exists $R > 0$ such that

$$\int\limits_{B_R^c(0)} \rho_n dx < \eta, \quad \forall n \in \mathbb{N},$$

that is,

$$\int\limits_{B_R^c(0)} |u_n| \, \mathrm{d}x < \eta \|u_n\|_1 \leq C\eta, \quad \forall n \in \mathbb{N}. \tag{5.22}$$

Since $u_\varepsilon \in L^1(\mathbb{R}^N)$, there exists $R_0 > 0$ such that

$$\int\limits_{B_{R_0}^c(0)} |u_\varepsilon| \, \mathrm{d}x \leq \eta. \tag{5.23}$$

Therefore, for $R_1 \geq \max\{R, R_0\}$, since $u_n \to u_\varepsilon$ in $L^1(B_{R_1}(0))$, there exists $n_0 \in \mathbb{N}$ such that

$$\int\limits_{B_{R_1}(0)} |u_n - u_\varepsilon| \, \mathrm{d}x \leq \eta, \quad \forall n \geq n_0. \tag{5.24}$$

Then from (5.22), (5.23) and (5.24), it follows that if $n \geq n_0$,

$$\int_{\mathbb{R}^N} |u_n - u_\varepsilon| \, \mathrm{d}x \leq \int_{B^c_{R_1}(0)} |u_n - u_\varepsilon| \, \mathrm{d}x + \eta \leq \int_{B^c_{R_1}(0)} |u_n| \, \mathrm{d}x + \int_{B^c_{R_1}(0)} |u_\varepsilon| \, \mathrm{d}x + \eta \leq C_1 \eta.$$

Thus, $u_n \to u_\varepsilon$ in $L^1(\mathbb{R}^N)$, and since (u_n) is bounded in $L^{1^*}(\mathbb{R}^N)$, we have from the interpolation inequality that

$$u_n \to u_\varepsilon \quad \text{in } L^q(\mathbb{R}^N), \quad 1 \leq q < 1^*.$$

The proof of Lemma 5.3 is complete. □

Proposition 5.6. *Under the assumptions of Theorem 1.1, problem (1.1) has at least a nontrivial ground state solution.*

Proof. Firstly, from (f_2), (f_3) and Lemma 5.3, it follows that

$$\int_{\mathbb{R}^N} K(\varepsilon x) f(u_n) u_n dx \to \int_{\mathbb{R}^N} K(\varepsilon x) f(u_\varepsilon) u_\varepsilon dx, \quad \text{as } n \to +\infty. \tag{5.25}$$

From (5.3), (5.25) and the lower semicontinuity of the norm w.r.t. the $L^1(\mathbb{R}^N)$ convergence, we have that

$$\|v\|_\varepsilon - \|u_\varepsilon\|_\varepsilon \geq \int_{\mathbb{R}^N} K(\varepsilon x) f(u_\varepsilon)(v - u_\varepsilon) dx, \quad \forall v \in BL(\mathbb{R}^N). \tag{5.26}$$

Hence, u_ε is a nontrivial solution of problem (2.1). Moreover, by (5.2), we have that

$$\begin{aligned} c_\varepsilon &\leq \Phi_\varepsilon(u_\varepsilon) \\ &= \Phi_\varepsilon(u_\varepsilon) - \Phi'_\varepsilon(u_\varepsilon) u_\varepsilon \\ &= \int_{\mathbb{R}^N} K(\varepsilon x)(f(u_\varepsilon) u_\varepsilon - F(u_\varepsilon)) dx \\ &\leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} K(\varepsilon x)(f(u_n) u_n - F(u_n)) dx \\ &= \Phi_\varepsilon(u_n) + o_n(1) \\ &= c_\varepsilon. \end{aligned} \tag{5.27}$$

Thus, u_ε is a ground state solution of problem (2.1) and then $v_\varepsilon = u_\varepsilon(\cdot/\varepsilon)$ is a ground state solution of problem (1.1). □

5.2. Concentration behavior

In the last section, we have proved that for each $\varepsilon \in (0, \varepsilon_0)$, there exists a solution $u_\varepsilon \in BL(\mathbb{R}^N)$ of problem (2.1) such that $\Phi_\varepsilon(u_\varepsilon) = c_\varepsilon$. Now let us show that the sequence of solutions concentrate on the intersection set of global minimum points of $V(x)$ and maximum points of $K(x)$. And before I do that, let us prove the following preliminaries lemmas.

Lemma 5.7. *There exist $\{y_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}^N$ and $R, \delta > 0$ such that*

$$\liminf_{\varepsilon\to 0} \int_{B_R(y_\varepsilon)} |u_\varepsilon|\, dx \geq \delta > 0. \tag{5.28}$$

Moreover, the family $\{\varepsilon y_\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathbb{R}^N$.

Proof. Suppose by contradiction that (5.28) does not hold. In fact, on the contrary, thanks to Lions' Lemma in $BL(\mathbb{R}^N)$, it follows that $u_{\varepsilon_n} \to 0$ in $L^q(\mathbb{R}^N)$ for all $1 < q < 1^*$, where $\varepsilon_n \to 0$, as $n \to \infty$. Then, by $(f_2), (f_3)$ and the Lebesgue Convergence Theorem, it shows that

$$\int_{\mathbb{R}^N} K(\varepsilon_n x) f(u_{\varepsilon_n}) u_{\varepsilon_n} dx = o_n(1).$$

Taking $w = u_{\varepsilon_n} \pm t u_{\varepsilon_n}$ in (2.8) and passing to the limit as $t \to 0^+$, we obtain

$$\|u_{\varepsilon_n}\|_{\varepsilon_n} = \int_{\mathbb{R}^N} K(\varepsilon_n x) f(u_{\varepsilon_n}) u_{\varepsilon_n} dx = o_n(1),$$

which implies that $c_{\varepsilon_n} = \Phi_{\varepsilon_n}(u_{\varepsilon_n}) = o_n(1)$. This leads to a contradiction with Lemma 5.1 and then (5.28) holds.

Set $y_n := y_{\varepsilon_n}$ and $u_n := u_{\varepsilon_n}$. Suppose by contradiction that there exist $\varepsilon_n \to 0$, such that $|\varepsilon_n y_n| \to \infty$, as $n \to \infty$. Next we proceed as in the proof of Claim 5.4 of Lemma 5.3. Note that, if φ_R is as in the proof of such claim, it shows that

$$\Phi'_{\varepsilon_n, y_n}(\widetilde{u}_n)(\varphi_R \widetilde{u}_n) = 0,$$

where $\widetilde{u}_n := u_n(\cdot + y_n)$. As $(u_n), (\widetilde{u}_n)$ is bounded in $BL(\mathbb{R}^N)$ and then $\widetilde{u}_n \to \widetilde{u}$ in $L^1_{loc}(\mathbb{R}^N)$, where $\widetilde{u} \neq 0$. Then, as before, we also get that

$$\Phi'_\infty(\widetilde{u})\widetilde{u} \leq 0,$$

and then there exists $\widetilde{t} \in (0, 1]$ such that $\widetilde{t}\widetilde{u} \in \mathcal{N}_\infty$. Therefore, in the same way of Claim 5.4, we will get the contradiction that $c_0 = \lim_{n\to\infty} c_{\varepsilon_n} \geq c_\infty$. Hence, $\{\varepsilon y_n\}$ is bounded. □

Corollary 5.8. *If $\varepsilon_n \to 0$, then up to a subsequence, there exists a point $x_0 \in \mathbb{R}^N$ such that $\varepsilon_n y_n \to x_0$ where*

$$V(x_0) = V_0 = \inf_{x\in\mathbb{R}^N} V, \quad \textit{and} \quad K(x_0) = K_0 = \max_{x\in\mathbb{R}^N} K.$$

Proof. If $\varepsilon_n \to 0$, according to Lemma 5.7, we know that $\{\varepsilon_n y_n\}$ is bounded, then there exists a point $x_0 \in \mathbb{R}^N$ such that $\varepsilon_n y_n \to x_0$. Denote a functional $\Phi_{VK(x_0)}$ as follows:

$$\Phi_{VK(x_0)}(u) = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| dx + \int_{\mathbb{R}^N} V(x_0)|u| \mathrm{d}x - \int_{\mathbb{R}^N} K(x_0)F(u)dx.$$

Since $V(x_0) \geq V_0$ and $K(x_0) \leq K_0$, similarly to the arguments that used in the proof of Claim 5.4 and of Lemma 5.7, it is possible to prove that

$$c_0 = \lim_{n\to\infty} c_{\varepsilon_n} \geq c_{VK(x_0)} \geq c_0,$$

where $c_{VK(x_0)}$ is the mountain pass minimax level of problem (2.1) with $V(x_0)$ playing the role of $V(\varepsilon x)$ and $K(x_0)$ playing the role of $K(\varepsilon x)$. Then it shows that $V(x_0) = V_0 = \inf_{x\in\mathbb{R}^N} V$ and $K(x_0) = K_0 = \max_{x\in\mathbb{R}^N} K$. Thus, $x_0 \in \Lambda \cap \Lambda_1$, i.e., $V(x_0) = V_0$ and $K(x_0) = K_0$. The proof is complete. $\square$

Lemma 5.9. *If $\varepsilon_n \to 0$, then there exists $\widetilde{u} \in BL(\mathbb{R}^N)$ such that*

$$\widetilde{u}_n := u_n(\cdot - y_n) \to \widetilde{u} \quad \textit{in } L^1_{loc}(\mathbb{R}^N),$$

and

$$f(\widetilde{u}_n)\widetilde{u}_n \to f(\widetilde{u})\widetilde{u} \quad \textit{in } L^1(\mathbb{R}^N). \tag{5.29}$$

Proof. Firstly, note that as in Lemma 4.3, it is possible to prove that $(\widetilde{u}_n)$ is bounded in $BL(\mathbb{R}^N)$ and then that $\widetilde{u}_n \to \widetilde{u}$ in $L^q_{loc}(\mathbb{R}^N)$ for all $1 \leq q < 1^*$, where $\widetilde{u} \in BL(\mathbb{R}^N)$. As in the proof of Lemma 5.7, we can prove that $\widetilde{t} \in (0, 1]$ such that $\widetilde{t}\widetilde{u} \in \mathcal{N}_0$ and verify $\widetilde{t} = 1$. Hence, $\widetilde{u} \in \mathcal{N}_0$ and note that $\Phi_0(\widetilde{u}) = c_0$. Indeed,

$$
\begin{aligned}
c_0 &\le \Phi_0(\widetilde{u}) \\
&= \Phi_0(\widetilde{u}) - \Phi_0'(\widetilde{u})\widetilde{u} \\
&= \int_{\mathbb{R}^N} K_0(f(\widetilde{u})\widetilde{u} - F(\widetilde{u}))dx \\
&\le \liminf_{n\to\infty} \int_{\mathbb{R}^N} K_0(f(\widetilde{u}_n)\widetilde{u}_n - F(\widetilde{u}_n))dx \\
&= \lim_{n\to\infty} (\Phi_{\varepsilon_n}(u_n) - \Phi_{\varepsilon_n}'(u_n)u_n) \\
&= \lim_{n\to\infty} c_{\varepsilon_n} \\
&= c_0.
\end{aligned}
$$

Then

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K_0(f(\widetilde{u}_n)\widetilde{u}_n - F(\widetilde{u}_n))dx = \int_{\mathbb{R}^N} K_0(f(\widetilde{u})\widetilde{u} - F(\widetilde{u}))dx,$$

and hence $K_0(f(\widetilde{u}_n)\widetilde{u}_n - F(\widetilde{u}_n)) \to K_0(f(\widetilde{u})\widetilde{u} - F(\widetilde{u}))$ in $L^1(\mathbb{R}^N)$. Therefore, by (f_4), we have

$$f(u_n)u_n \to f(\widetilde{u})\widetilde{u} \quad \text{in } L^1(\mathbb{R}^N). \quad \square$$

Proof of the last part to Theorem 1.1. Based on the previous results, we can finish the proof of Theorem 1.1. Indeed, if $\varepsilon_n \to 0$, as $n \to \infty$, denoting $L = \int_{\mathbb{R}^N} f(\widetilde{u})\widetilde{u}dx > 0$, for a given $\delta > 0$, by (5.29), there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \ge n_0$,

$$\int_{B_R^c(0)} f(\widetilde{u}_n)\widetilde{u}_n dx < \delta,$$

which implies that

$$\int_{B_R(0)} f(\widetilde{u}_n)\widetilde{u}_n dx \ge L - \delta + o_n(1).$$

By the change of variable $\widetilde{u}_n(x) = v_n(\varepsilon_n x + \varepsilon_n y_n)$ and the above inequalities indicate that

$$\int_{B^c_{\varepsilon_n R}(\varepsilon_n y_n)} f(v_n)v_n dx < \varepsilon_n^N \delta, \tag{5.30}$$

and

$$\int_{B_{\varepsilon_n R}(\varepsilon_n y_n)} f(v_n)v_n dx \ge C\varepsilon_n^N, \tag{5.31}$$

where $n \geq n_0$ and $C > 0$. Taking into account the fact that $\varepsilon_n y_n \to x_0$ where $V(x_0) = V_0$ and $K(x_0) = K_0$, we can consider $\overline{R} > 0$ such that, for $n \geq n_0$, $B_R(\varepsilon_n y_n) \subset B_{\overline{R}}(x_0)$, and then from (5.30) and (5.31), for all $n \geq n_0$

$$\int\limits_{B^c_{\varepsilon_n \overline{R}}(x_0)} f(v_n)v_n dx < \varepsilon_n^N \delta,$$

and

$$\int\limits_{B_{\varepsilon_n \overline{R}}(x_0)} f(v_n)v_n dx \geq C\varepsilon_n^N.$$

The proof of Theorem 1.1 is complete. □

6. The asymptotic constant case

In the section, we can prove Theorem 1.2 under $(f_1) - (f_5)$, (V_2) and (K_2). It can be seen from the statement of Theorem 1.2 that our existence result is related to $\varepsilon > 0$, and then we can assume $\varepsilon = 1$ without lack of generality.

Proof of Theorem 1.2. Let us denote $\|u\|_Y = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| dx + \int_{\mathbb{R}^N} V(x)|u|dx$, $\Phi(u)_Y = \|u\|_Y - \int_{\mathbb{R}^N} K(x)F(u)dx$ and consider Φ_∞ as in Sect. 2.

As in proof of Lemma 4.2, it is easy to see that Φ_Y and Φ_∞ satisfy the geometric conditions of the Mountain Pass Theorem and then the minimax levels are well defined

$$c_Y = \inf_{\gamma \in \Gamma_Y} \sup_{t \in [0,1]} \Phi_Y(\gamma(t)),$$
$$c_\infty = \inf_{\gamma \in \Gamma_\infty} \sup_{t \in [0,1]} \Phi_\infty(\gamma(t)),$$

where $\Gamma_Y = \left\{\gamma \in C^0([0,1], BL(\mathbb{R}^N)); \gamma(0) = 0, \Phi_Y(\gamma(1)) < 0\right\}$ and Γ_∞ is defined in a similar way.

By the results in Sect. 4, it follows that there exists a critical point of $\Phi_\infty, w_\infty \in BL(\mathbb{R}^N)$, such that $\Phi_\infty(w_\infty) = c_\infty$. Now let us define the Nehari manifolds associated to Φ_Y and Φ_∞, which are well defined by

$$\mathcal{N}_Y = \left\{u \in BL(\mathbb{R}^N)\backslash\{0\}; \Phi_Y'(u)u = 0\right\},$$
$$\mathcal{N}_\infty = \left\{u \in BL(\mathbb{R}^N)\backslash\{0\}; \Phi_\infty'(u)u = 0\right\}.$$

From [11], we know that $c_Y = \inf_{\mathcal{N}_Y} \Phi_Y$ and $c_\infty = \inf_{\mathcal{N}_\infty} \Phi_\infty$. Moreover, it has been proved there that if there exists $u_0 \in BL(\mathbb{R}^N)$ such that $\Phi_Y(u_0) = \inf_{\mathcal{N}_Y} \Phi_Y$, then u_0 is a solution of (1.1).

By (V_2) and (K_2), it is easy to see that $\Phi_Y(u) \leq \Phi_\infty(u)$ for all $u \in BL(\mathbb{R}^N)$ and as a consequence,

$$c_Y \leq c_\infty. \tag{6.1}$$

To prove Theorem 1.2, let us consider two possible cases of c_Y and c_∞.

- *Case* A: $c_Y = c_\infty$.

If this case holds, problem (1.1) has a ground state solution. Indeed, since $w_\infty \in \mathcal{N}_\infty$, then

$$\begin{aligned}
\|w_\infty\|_Y &= \int_{\mathbb{R}^N} |\Delta w_\infty| + \int_{\mathbb{R}^N} |\nabla w_\infty| dx + \int_{\mathbb{R}^N} V(x)|w_\infty| dx \\
&\leq \int_{\mathbb{R}^N} |\Delta w_\infty| + \int_{\mathbb{R}^N} |\nabla w_\infty| dx + \int_{\mathbb{R}^N} V_\infty |w_\infty| dx \\
&= \int_{\mathbb{R}^N} K_\infty f(w_\infty) w_\infty dx,
\end{aligned}$$

is that,

$$\Phi'_Y(w_\infty)w_\infty \leq 0.$$

Then there exists $t \in (0,1]$ such that $tw_\infty \in \mathcal{N}_Y$. Hence, from (f_5), we have that

$$\begin{aligned}
c_Y &\leq \Phi_Y(tw_\infty) \\
&= \Phi_Y(tw_\infty) - \Phi'_Y(tw_\infty)tw_\infty \\
&= \int_{\mathbb{R}^N} K(x)(f(tw_\infty)tw_\infty - F(tw_\infty))dx \\
&\leq \int_{\mathbb{R}^N} K_\infty(f(w_\infty)w_\infty - F(w_\infty))dx \\
&= \Phi_\infty(w_\infty) \\
&= c_\infty \\
&= c_Y.
\end{aligned}$$

It indicates that $t = 1$ and w_∞ is also a minimizer of Φ_Y on $\mathcal{N}_Y$ and then is a ground-state solution of (1.1).

- *Case* B: $c_Y < c_\infty$.

From [13], there exists $(u_n) \subset BL(\mathbb{R}^N)$ such that

$$\lim_{n\to\infty} \Phi_Y(u_n) = c_Y, \tag{6.2}$$

and

$$\|v\|_Y - \|u_n\|_Y \geq \int\limits_{\mathbb{R}^N} K(x)f(u_n)(v-u_n)dx - \tau_n \|v-u_n\|_Y, \quad \forall v \in BL(\mathbb{R}^N), \tag{6.3}$$

where $\tau_n \to 0$, as $n \to \infty$. Note that as in Lemma 4.3, it is possible to prove that (u_n) is bounded in $BL(\mathbb{R}^N)$. From the compactness of the embeddings of $BL(\mathbb{R}^N)$, it implies that there exists $u_0 \in BL_{loc}(\mathbb{R}^N)$ such that

$$u_n \to u_0 \quad \text{in } L^q_{loc}(\mathbb{R}^N) \quad \text{for all } 1 \leq q < 1^*,$$

and

$$u_n \to u_0 \quad \text{a.e. in } \mathbb{R}^N,$$

as $n \to +\infty$. In addition, note that as in Section 4, it is possible to prove that $u_0 \in BL(\mathbb{R}^N)$. And as in the proof of Lemma 5.3, let us use the Concentration of Compactness Principle of Lions to the following bounded sequence in $L^1(\mathbb{R}^N)$, and set the function $\rho_n(x) = \frac{|u_n(x)|}{\|u_n\|_1}$. Note that

$$\|u_n\|_1 \nrightarrow 0, \quad \text{as } n \to +\infty. \tag{6.4}$$

Similarly, the Concentration of Compactness Principle implies that one and only one of the following statements holds:

(i) (Vanishing) For all $R > 0$, there holds $\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n dx = 0$;
(ii) (Compactness) There exist $(y_n) \subset \mathbb{R}^N$ such that for all $\eta > 0$, there exists $R > 0$ such that

$$\int\limits_{B_R(y_n)} \rho_n dx \geq 1 - \eta, \quad \forall n \in \mathbb{N}; \tag{6.5}$$

(iii) (Dichotomy) There exist $(y_n) \subset \mathbb{R}^N, \alpha \in (0,1), R_1 > 0, R_n \to +\infty$ such that the functions $\rho_{n,1}(x) := \chi_{B_{R_1}(y_n)}(x)\rho_n(x)$ and $\rho_{n,2}(x) := \chi_{B^c_{R_n}(y_n)}(x)\rho_n(x)$ satisfy

$$\int\limits_{\mathbb{R}^N} \rho_{n,1} dx \to \alpha \quad \text{and} \quad \int\limits_{\mathbb{R}^N} \rho_{n,2} dx \to 1 - \alpha.$$

As can be seen in Lemma 5.3, we also have that the Vanishing case and the Dichotomy case do not hold. It follows that Compactness must occur and then, the sequence (y_n) is bounded. Then, for $\eta > 0$, it follows from (6.5) that there exists $R > 0$ such that

$$\int_{B_R^c(0)} \rho_n dx < \eta, \quad \forall n \in \mathbb{N},$$

that is,

$$\int_{B_R^c(0)} |u_n| \mathrm{d}x < \eta \|u_n\|_1 \leq C\eta, \quad \forall n \in \mathbb{N}. \tag{6.6}$$

Since $u_0 \in L^1(\mathbb{R}^N)$, there exists $R_0 > 0$ such that

$$\int_{B_{R_0}^c(0)} |u_0| \,\mathrm{d}x \leq \eta. \tag{6.7}$$

Therefore, for $R_1 \geq \max\{R, R_0\}$, since $u_n \to u_0$ in $L^1(B_{R_1}(0))$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_{R_1}(0)} |u_n - u_0| \, dx \leq \eta, \quad \forall n \geq n_0. \tag{6.8}$$

Then from (6.6), (6.7) and (6.8), it follows that when $n \geq n_0$,

$$\int_{\mathbb{R}^N} |u_n - u_0| \,\mathrm{d}x \leq \int_{B_{R_1}^c(0)} |u_n - u_0| \,\mathrm{d}x + \eta \leq \int_{B_{R_1}^c(0)} |u_n| \,\mathrm{d}x + \int_{B_{R_1}^c(0)} |u_0| \,\mathrm{d}x + \eta \leq C_1\eta.$$

Thus, $u_n \to u_0$ in $L^1(\mathbb{R}^N)$, and since (u_n) is bounded in $L^{1^*}(\mathbb{R}^N)$, we have from the interpolation inequality that

$$u_n \to u_0 \quad \text{in } L^q(\mathbb{R}^N), \quad \forall 1 \leq q < 1^*. \tag{6.9}$$

From (f_2), (f_3) and (6.9), it follows that

$$\int_{\mathbb{R}^N} K(x) f(u_n) u_n dx \to \int_{\mathbb{R}^N} K(x) f(u_0) u_0 dx, \tag{6.10}$$

as $n \to +\infty$. From (6.3), (6.10) and the lower semicontinuity of the norm $\|\cdot\|_Y$ w.r.t. the $L^1(\mathbb{R}^N)$ convergence, we have that

$$\|v\|_Y - \|u_0\|_Y \geq \int_{\mathbb{R}^N} K(x) f(u_0)(v - u_0) \mathrm{d}x, \quad \forall v \in BL(\mathbb{R}^N).$$

Hence, u_0 is a nontrivial solution of problem (1.1). Moreover, by (6.2), we have that

$$\begin{aligned}
c_Y &\leq \Phi_Y(u_0) \\
&= \Phi_Y(u_0) - \Phi'_Y(u_0)u_0 \\
&= \int_{\mathbb{R}^N} K(x)(f(u_0)u_0 - F(u_0))dx \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} K(x)(f(u_n)u_n - F(u_n))dx \\
&= \Phi_Y(u_n) + o_n(1) \\
&= c_Y.
\end{aligned}$$

Thus, u_0 is a ground-state solution of problem (1.1). The proof is complete. □

Declaration of competing interest

There is no conflict of interest.

Data availability

No data was used for the research described in the article.

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