Discrete and Continuous Dynamical Systems - Series S

doi:10.3934/dcdss.2022114

SEQUENCES OF HIGH AND LOW ENERGY SOLUTIONS FOR WEIGHTED (p,q)-EQUATIONS

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Abstract. We consider a Dirichlet elliptic equation driven by a weighted (p,q)-Laplace differential operator. The weights are in general different. When the reaction is "superlinear", using the fountain theorem, we show the existence of a sequence of distinct smooth solutions with energies diverging to $+\infty$. When the reaction is "sublinear" (possibly resonant), we establish the existence of a sequence of nodal solutions converging to zero in $C_0^1(\bar{\Omega})$ (in particular, the energies converge to zero).

1. **Introduction.** Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following Dirichlet problem driven by the weighted (p,q)-Laplacian

$$\left\{ \begin{array}{l} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, 1 < q < p. \end{array} \right. \tag{1}$$

Given $a \in C^{0,1}(\bar{\Omega})$ with $0 < \widehat{c} \le a(z)$ for all $z \in \bar{\Omega}$ and $r \in (1,\infty)$, by Δ_r^a we denote the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \operatorname{div}(a(z)|Du|^{r-2}Du) \text{ for all } u \in W_0^{1,r}(\Omega).$$

In problem (1) we have the sum of two such operators with different exponents $1 < q < p$ and also different weight functions $a_1(\cdot)$ and $a_2(\cdot)$. So, in problem (1), the differential operator is not homogeneous and this of course leads to difficulties in the analysis of (1). Moreover, the fact that the weights $a_1(\cdot)$ and $a_2(\cdot)$ are in general

2020 *Mathematics Subject Classification.* Primary: 35J20; Secondary: 35J60, 58E05.
Key words and phrases. Nonlinear regularity, nonlinear maximum principle, low and high energy solutions, fountain theorem, extremal constant sign solutions, nodal solutions.
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different, does not permit the use of the nonlinear maximum principle of Pucci-Serrin [22, pp.111, 120]. Instead we employ a strengthened version of a result due to Papageorgiou-Vetro-Vetro [20, Proposition 2.4], exploiting the stronger regularity theory available for our problem.

Our aim is to prove the existence of a whole sequence of distinct solutions of (1) with energy levels which tend to $+\infty$ and to zero. Such multiplicity results were obtained by Kajikiya [9], Pan-Tang [14], Papageorgiou-Rădulescu [15] (semilinear equations), Zhao-Zhao [28] (equations driven by the p-Laplacian), Gasinski-Papageorgiou [7], Leonardi-Papageorgiou [11] (parametric Robin problems driven by a nonhomgeneous differential operator) and Papageorgiou-Rădulescu-Repovš [17] (parametric double phase equations). They impose more restrictive conditions on the reaction and with the exception of Zhao-Zhao [28], produce only sequences of low energy solutions. For related existence and properties of ground state solutions for the case $p = q = 2$, we also refer the readers to the recent paper [26, 27].

2. **Mathematical background and auxiliary results.** The main spaces in the analysis of problem (1) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space

$$C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}.$$

On account of the Poincaré inequality, on $W_0^{1,p}(\Omega)$ we can use the equivalent norm

$$\|u\| = \|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\bar{\Omega})$ is ordered with positive cone

$$C_+ = \{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\},$$

where $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

By $C^{0,1}(\bar{\Omega})$ we denote the space of all Lipschitz continuous functions on $\bar{\Omega}$. Let $a \in C^{0,1}(\bar{\Omega})$ and assume that $0 < \widehat{c} \leq a(z)$ for all $z \in \bar{\Omega}$. For $r \in (1, \infty)$, let

$$A_r^a : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^* \ (\frac{1}{r} + \frac{1}{r'} = 1)$$

be the nonlinear operator defined by

$$\langle A_r^a(u), h\rangle = \int_\Omega a(z)|Du|^{r-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z.$$

This operator has the following properties (see Gasinski-Papageorgiou [6, Problem 2.192]).

Proposition 2.1. *The operator $A_r^a(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus, maximal monotone too) and of type $(S)_+$, that is,*

$$\text{“}u_n \xrightarrow{w} u \text{ in } W_0^{1,r}(\Omega), \limsup_{n\to\infty}\langle A_r^a(u_n), u_n - u\rangle \leq 0 \Rightarrow u_n \to u \text{ in } W_0^{1,r}(\Omega).\text{”}$$

Consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_r^a u(z) = \hat{\lambda} a(z)|u(z)|^{r-2}u(z) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of (2), if the problem admits a nontrivial solution $\hat{u} \in W_0^{1,r}(\Omega)$ known as an eigenfunction corresponding to $\hat{\lambda}$. Problem (2) has a smallest eigenvalue $\hat{\lambda}_1^a(r) > 0$ which has the following variational characterization

$$\hat{\lambda}_1^a(r) = \inf \left\{ \frac{\int_\Omega a(z)|Du|^r \mathrm{d}z}{\int_\Omega a(z)|u|^r \mathrm{d}z} : u \in W_0^{1,r}(\Omega), u \neq 0 \right\}. \tag{3}$$

This eigenvalue is isolated and simple (that is, if $\hat{u}$, $\hat{v}$ are two eigenfunctions corresponding to $\hat{\lambda}_1^a(r)$, then $\hat{u} = \vartheta \hat{v}$ for some $\vartheta \in \mathbb{R} \setminus \{0\}$). The infimum in (3) is realized on the corresponding one dimensional eigenspace. It is easy to see from (3) that the eigenfunctions corresponding to $\hat{\lambda}_1^a(r)$ have constant sign. The nonlinear regularity theory (see Lieberman [12]) implies that all eigenfunctions of (2) belong in $C_0^1(\bar{\Omega})$. By $\hat{u}_1(r)$ we denote the positive eigenfunction corresponding to $\hat{\lambda}_1^a(r) > 0$ such that $\int_\Omega a(z)|\hat{u}_1(r)|^r \mathrm{d}z = 1$. The nonlinear maximum principle implies that $\hat{u}_1(r) \in \operatorname{int} C_+$. We mention that in addition to $\hat{\lambda}_1^a(r) > 0$ the minimax scheme of Ljusternik-Schnirelmann (see Gasinski-Papageorgiou [5]) gives a whole strictly increasing unbounded sequence of eigenvalues $\{\hat{\lambda}_n^a(r)\}_{n \in \mathbb{N}}$. We do not know if this sequence exhausts the spectrum of (2).

From the aforementioned properties of $\hat{\lambda}_1^a$, we infer the following simple lemma (see Mugnai-Papageorgiou [13, Lemma 4.11]).

Proposition 2.2. *If $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leq \hat{\lambda}_1^a(r) a(z)$ for a.a. $z \in \Omega$ and $\vartheta \not\equiv \hat{\lambda}_1^a(r) a$, then there exists $c_0 > 0$ such that*

$$c_0 \|Du\|_r^r \leq \int_\Omega a(z)|Du|^r \mathrm{d}z - \int_\Omega \vartheta(z)|u|^r \mathrm{d}z$$

for all $u \in W_0^{1,r}(\Omega)$.

For our problem there is a strong regularity theory (see Lieberman [12]) and so we can have a stronger version of the maximum principle of Papageorgiou-Vetro-Vetro [20, Proposition 2.4].

So, let $a_1, a_2 \in C^{0,1}(\bar{\Omega})$ with $0 < \widehat{c} \leq a_1(z), a_2(z)$ for all $z \in \bar{\Omega}$ and $\xi, h \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$. We consider the following Dirichlet problem

$$\begin{cases} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) + \xi(z)|u(z)|^{p-2}u(z) = h(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, 1 < q < p < \infty. \end{cases} \tag{4}$$

Proposition 2.3. *If $u \in C_0^1(\bar{\Omega})$ is a solution of (4), $u(z) \geq 0$ for all $z \in \bar{\Omega}$, $u \neq 0$, then $u \in \operatorname{int} C_+$.*

Proof. First we show that

$$u(z) > 0 \text{ for all } z \in \Omega.$$

We argue by contradiction. So, suppose that the strict positivity of $u(\cdot)$ on Ω is not true. Then we can find $z_1, z_2 \in \Omega$ and $\rho > 0$ such that

$$\bar{B}_{2\rho}(z_2) \subseteq \Omega, \ z_1 \in \partial B_{2\rho}(z_2), \ u(z_1) = 0, \ u|_{B_{2\rho}(z_2)} > 0.$$

Here, $B_{2\rho}(z_2) = \{z \in \mathbb{R}^N : |z - z_2| < 2\rho\}$. Clearly, by fixing z_1 and varying z_2, we can always have $\rho > 0$ small. Let $m = \min_{\partial B_\rho(z_2)} u > 0$. We have

$$Du(z_1) = 0, m \to 0^+ \text{ and } \frac{m}{\rho} \to 0^+ \text{ as } \rho \to 0^+ \text{ (L'Hospital's rule)}. \tag{5}$$

Consider the annulus

$$A = \{z \in \Omega : \rho < |z - z_2| < 2\rho\}$$

and let

$$\eta = \max\left\{\sup_{\Omega} |Da_1|, \sup_{\Omega} |Da_2|\right\} > 0.$$

Since a_1, a_2 are by hypothesis Lipschitz continuous, by Rademacher's theorem (see Papageorgiou and Winkert [21, p.476]) they are almost everywhere differentiable. We define

$$\mu = -\ln\frac{m}{\rho} + \frac{N-1}{\rho} + 2\eta$$

and consider the function

$$y(t) = \frac{m[e^{\frac{\mu t}{q-1}} - 1]}{e^{\frac{\mu t}{q-1}} - 1},\ 0 \le t \le \rho.$$

For $\rho > 0$ small we have

$$0 < y(t), y'(t) < 1 \text{ for all } t \in [0, \rho] \text{ (see (5))}, \tag{6}$$

$$y''(t) = \frac{\mu}{q-1} y'(t) \text{ for all } t \in [0, \rho]. \tag{7}$$

To simplify the presentation, without any loss of generality we assume that $z_2 = 0$. Let $r = |z|(= |z - z_2|)$, $t = 2\rho - r$. For $t \in [0, \rho]$, $r \in [\rho, 2\rho]$ we define

$$v(r) = y(2\rho - r) = y(t) \Rightarrow v'(t) = -y'(t), v''(t) = y''(t).$$

We set $\hat{v}(z) = v(r)$ for $z \in \Omega$, $|z| = r$. We have $\hat{v} \in C^2(A)$. Then

$$\begin{aligned}
&\operatorname{div}\left[a_1(z)|D\hat{v}|^{p-2}D\hat{v} + a_2(z)|D\hat{v}|^{q-2}D\hat{v}\right] - \xi(z)|\hat{v}|^{p-2}\hat{v} + h(z)\\
=&(p-1)a_1(z)y'(t)^{p-2}y''(t) - a_1(z)\frac{N-1}{r}y'(t)^{p-1} - y'(t)^{p-1}\sum_{k=1}^{N}\frac{\partial a_1}{\partial z_k}\frac{z_k}{r}\\
&+ (q-1)a_2(z)y'(t)^{q-2}y''(t) - a_2(z)\frac{N-1}{r}y'(t)^{q-1} - y'(t)^{p-1}\sum_{k=1}^{N}\frac{\partial a_2}{\partial z_k}\frac{z_k}{r}\\
&- \xi(z)y(t)^{p-1} + h(z)\\
\ge&\hat{c}\left[\mu - \frac{N-1}{r} - 2\eta\right]y'(t)^{q-1} - c_1\ (c_1 = \|\xi\|_\infty + \|h\|_\infty \ge 0)\\
\ge&\hat{c}(-\ln\frac{m}{\rho})y'(t)^{p-1} - c_1 \text{ (see (6) and recall } q < p).
\end{aligned}$$

So, for $\rho > 0$ small we have

$$-\Delta_p^{a_1}\hat{v} - \Delta_q^{a_2}\hat{v} + \xi(z)\hat{v}^{p-1} \le h(z) \text{ in } \Omega.$$

Then the weak comparison principle (see [22, p.61]) implies that $v(z) \le u(z)$ for all $z \in \bar{A}$. Hence we have

$$\lim_{s\to 0^+}\frac{u(z_1 + s(z_2 - z_1))}{s} \ge \lim_{s\to 0^+}\frac{\hat{v}(z_1 + s(z_2 - z_1)) - \hat{v}(z_1)}{s} = v'(0) > 0.$$

Hence $Du(z_1) \ne 0$, a contradiction. So, $u(z) > 0$ for all $z \in \Omega$.

Now let $z_1 \in \partial\Omega$ and for $\rho > 0$ small let $z_2 = z_1 - 2\rho n(z_1)$. Let $0 < d < \inf\{u(z) : z \in \partial B_\rho(z_2)\}$. From the first part of the proof, we know that there exists $\hat{v} \in C^1(\bar{A}) \cap C^2(A)$ such that

$$\hat{v}(z) \leq u(z) \text{ for all } z \in \bar{A}, \hat{v}(z_1) = 0, \frac{\partial \hat{v}}{\partial n}(z_1) < 0,$$
$$\Rightarrow u \in \operatorname{int} C_+.$$

The proof is now complete. □

Let X be a Banach space and $\varphi \in C^1(X)$. We say that $\varphi(\cdot)$ satisfies that "C-condition", if the following property holds:

If $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ is a sequence such that
$\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded,
and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
then it has a strongly convergent subsequence.

This is a compactness-type condition on the functional $\varphi(\cdot)$ which compensates for the fact that the ambient space X is not, in general, locally compact (being infinite dimensional). It leads to a deformation theorem from which one deduces the minimax theorems characterizing the critical points of $\varphi(\cdot)$ (see [5]). We also refer to Tang and Cheng [24] who proposed a new approach to restore the compactness of Palais-Smale sequences and to Tang and Chen [23] who introduced an original method to recover the compactness of minimizing sequences. A related approach has been developed by Chen and Tang [3] in the framework of Cerami sequences.

If $u : \Omega \to \mathbb{R}$ is a measurable function, then we define

$$u^\pm(z) = \max\{\pm u(z), 0\} \text{ for all } z \in \Omega.$$

We know that $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,p}(\Omega)$, then $u^\pm \in W_0^{1,p}(\Omega)$. If $u, v : \Omega \to \mathbb{R}$ are measurable functions and $u(z) \leq v(z)$ for all $z \in \Omega$, then

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\}.$$

Finally, for $\varphi \in C^1(X)$, we set

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \text{ (the critical set of } \varphi).$$

3. **High energy solutions.** In this section we produce a sequence of smooth solutions with energy levels diverging to $+\infty$. The hypotheses on the data of problem (1) are the following:

H_0 : $a_1, a_2 \in C^{0,1}(\bar{\Omega})$ and $0 < \widehat{c} \leq a_1(z), a_2(z)$ for all $z \in \bar{\Omega}$.

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot)$ is odd and

(i) $|f(z, x)| \leq \hat{a}(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $\hat{a} \in L^\infty(\Omega)$ and $p < r < p^*$, where $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ if $N \leq p$;

(ii) if $F(z, x) = \int_0^x f(z, s)\mathrm{d}s$, then $\lim_{x\to\pm\infty} \frac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\mu \in ((r - p) \max\{\frac{N}{p}, 1\}, p^*)$ such that

$$0 < \widehat{c}_0 \leq \liminf_{x\to\pm\infty} \frac{f(z, x)x - pF(z, x)}{|x|^\mu} \text{ uniformly for a.a. } z \in \Omega.$$

Remark 1. We mention that no restriction on the behavior of $f(z,\cdot)$ near zero is imposed. Hypotheses H_1-(ii) and H_1-(iii) imply that for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-superlinear as $x \to \pm\infty$. However, this superlinearity of $f(z,\cdot)$ is not expressed via the usual for superlinear problems Ambrosetti-Rabinowitz condition (the AR-condition for short, see Willem [25, p.46]). The condition in hypothesis H_1-(iii) is less restrictive and incorporates superlinear nonlinearities with "slower" growth. For example, the function $|x|^{p-2}x\ln|x|$ satisfies hypotheses H_1 but fails to satisfy the AR-condition.

We introduce the energy functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p \mathrm{d}z + \frac{1}{q}\int_\Omega a_2(z)|Du|^q \mathrm{d}z - \int_\Omega F(z,u)\mathrm{d}z$$

for all $u \in W_0^{1,p}(\Omega)$. Evidently, $\varphi \in C^1(W_0^{1,p}(\Omega))$.

Proposition 3.1. *If hypotheses H_0 and H_1 hold, then the functional $\varphi(\cdot)$ satisfies the C-condition.*

Proof. Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\varphi(u_n)| \le c_1 \text{ for some } c_1 > 0, \text{ all } n \in \mathbb{N}, \tag{8}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{9}$$

From (9) we have

$$\left|\langle A_p^{a_1}(u_n), h\rangle + \langle A_q^{a_2}(u_n), h\rangle - \int_\Omega f(z,u_n)h\mathrm{d}z\right| \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{10}$$

for all $h \in W_0^{1,p}(\Omega)$, with $\varepsilon_n \to 0^+$.

In (10) we use the test function $h = u_n \in W_0^{1,p}(\Omega)$ and obtain

$$-\int_\Omega a_1(z)|Du_n|^p\mathrm{d}z - \int_\Omega a_2(z)|Du_n|^q\mathrm{d}z + \int_\Omega f(z,u_n)u_n\mathrm{d}z \le \varepsilon_n \tag{11}$$

for all $n \in \mathbb{N}$. From (8) we have

$$\int_\Omega a_1(z)|Du_n|^p\mathrm{d}z + \frac{p}{q}\int_\Omega a_2(z)|Du_n|^q\mathrm{d}z - \int_\Omega pF(z,u_n)\mathrm{d}z \le pc_1. \tag{12}$$

We add (11) and (12). Recalling that $q < p$, we obtain

$$\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]\mathrm{d}z \le c_2 \text{ for some } c_2 > 0, \text{ all } n \in \mathbb{N}. \tag{13}$$

From hypotheses H_1-(i) and H_1-(ii), we see that we can find $\widehat{c}_1 \in (0, \widehat{c}_0)$ and $c_3 > 0$ such that

$$\widehat{c}_1|x|^\mu - c_3 \le f(z,x)x - pF(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{14}$$

We use (14) in (13) and infer that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq L^\mu(\Omega) \text{ is bounded.} \tag{15}$$

From hypothesis H_1-(iii), it is clear that we may assume that $\mu < r < p^*$. First we assume that $p \neq N$ and choose $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{p^*}. \tag{16}$$

Invoking the interpolation inequality (see Papageorgiou-Winkert [21, p.116]), we have

$$\begin{aligned} & \|u_n\|_r \leq \|u_n\|_\mu^{1-t} \|u_n\|_{p^*}^t \\ \Rightarrow\ & \|u_n\|_r^r \leq c_4 \|u_n\|^{tr} \text{ for some } c_4 > 0, \text{ all } n \in \mathbb{N}. \\ & \text{(see (15) and use the Sobolev embedding theorem)} \end{aligned} \tag{17}$$

From (10) with $h = u_n \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \|u_n\|^p \leq & c_5 [1 + \|u_n\|_r^r] \\ & \text{for some } c_5 > 0, \text{ all } n \in \mathbb{N} \text{ (see hypothesis } H_1\text{-(i))} \\ \leq & c_6 [1 + \|u_n\|^{tr}] \\ & \text{for some } c_6 > 0, \text{ all } n \in \mathbb{N} \text{ (see (13))}. \end{aligned} \tag{18}$$

If $p < N$, the from (12) and since $p^* = \frac{Np}{N-p}$ we have

$$\begin{aligned} & t\left(\frac{p^* - \mu}{p^*}\right) = \frac{r - \mu}{r}, \\ \Rightarrow\ & tr = \frac{p^*(r-\mu)}{p^* - \mu} = \frac{(r-\mu)Np}{Np - N\mu + p\mu} < p, \\ & \text{(see hypothesis } H_1\text{-(iii))}. \end{aligned}$$

If $p > N$, then $p^* = +\infty$ and so (16) becomes

$$\begin{aligned} & \frac{1}{r} = \frac{1-t}{\mu}, \\ \Rightarrow\ & r(t) = r - \mu < p, \text{ (see hypothesis } H_1\text{-(iii))}. \end{aligned}$$

So, when $p \neq N$, we have that $tr < p$ and then from (18), it follows that

$$\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{19}$$

If $N = p$, then by definition $p^* = +\infty$, but the Sobolev embedding theorem says that $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ continuously (in fact, compactly) for all $s \in [1, \infty)$. So, in the previous argument we need to replace p^* with $s > r$ big so that $tr = \frac{s(r-\mu)}{s-\mu} < p$ (see hypothesis H_1-(iii)). Then again we infer that (15) holds.

On account of (19), we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega). \tag{20}$$

In (10) we choose $h = u - u_n \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (20), we obtain

$$\begin{aligned} & \lim_{n \to \infty} \left[\langle A_p^{a_1}(u_n), u_n - u \rangle + \langle A_q^{a_2}(u_n), u_n - u \rangle \right] = 0, \\ \Rightarrow\ & \limsup_{n \to \infty} \left[\langle A_p^{a_1}(u_n), u_n - u \rangle + \langle A_q^{a_2}(u), u_n - u \rangle \right] \leq 0 \\ & \text{(since } A_q^{a_2}(\cdot) \text{ is monotone)}, \\ \Rightarrow\ & \limsup_{n \to \infty} \langle A_p^{a_1}(u_n), u_n - u \rangle \leq 0 \text{ (see (20))}, \\ \Rightarrow\ & u_u \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 1)}. \end{aligned}$$

This proves that $\varphi(\cdot)$ satisfies the C-condition. □

The Sobolev space $W_0^{1,p}(\Omega)$ is a separable and reflexive Banach space. So, we can find two sequences

$$\{e_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ and } \{e_n^*\}_{n\in\mathbb{N}} \subseteq W^{-1,p'}(\Omega)$$

such that

$$\left\{ \begin{array}{l} W_0^{1,p}(\Omega) = \overline{\text{span}}\{e_n\}_{n\in\mathbb{N}}, \ W^{-1,p'}(\Omega) = \overline{\text{span}}\{e_n^*\}_{n\in\mathbb{N}}, \\ \langle e_m^*, e_n\rangle = \delta_{mn} \text{ for all } m, n \in \mathbb{N}. \end{array} \right. \tag{21}$$

(see Bogachev-Smolyanov [2, p.245]). Here, δ_{mn} denotes the Kronecker symbol defined by

$$\delta_{mn} = \left\{ \begin{array}{ll} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{array} \right.$$

We set

$$E_k = \mathbb{R}e_k, k \in \mathbb{N}, \ Y_n = \oplus_{k=1}^n E_k \text{ and } V_n = \overline{\oplus_{k\geq n+1} E_k}, n \in \mathbb{N}.$$

Let

$$\vartheta_n = \sup\left\{\|u\|_r : u \in V_n, \|u\| = 1\right\}. \tag{22}$$

Lemma 3.2. *$\vartheta_n \to 0$ as $n \to \infty$.*

Proof. Clearly, the sequence $\{\vartheta_n\}_{n\in\mathbb{N}} \subseteq (0, \infty)$ is decreasing. So

$$\vartheta_n \to \vartheta \geq 0 \text{ as } n \to \infty.$$

Choose $u_n \in V_n$ such that

$$\vartheta_n - \frac{1}{n} \leq \|u_n\|_r, \|u_n\| = 1 \text{ for all } n \in \mathbb{N}. \tag{23}$$

From (23) we see that we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{24}$$

We have

$$\begin{aligned} &\langle e_k^*, u_n\rangle \to \langle e_k^*, u\rangle \text{ as } n \to \infty, \text{ for all } k \in \mathbb{N}, \\ \Rightarrow\ &\langle e_k^*, u_n\rangle \to 0 \text{ as } n \to \infty, \text{ for all } k \in \mathbb{N} \text{ (see (21))}. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\langle e_k^*, u\rangle = 0 \text{ for all } k \in \mathbb{N}, \\ \Rightarrow\ &u = 0 \text{ (see (21))}, \\ \Rightarrow\ &\vartheta = 0 \text{ (see (23) and (24))}. \end{aligned}$$

The proof is now complete. □

We set

$$\begin{aligned} a_n^* &= \max\{\varphi(u) : u \in Y_n, \|u\| = \rho_n\}, \\ b_n^* &= \inf\{\varphi(u) : u \in V_n, \|u\| = l_n\}, n \in \mathbb{N}. \end{aligned}$$

Proposition 3.3. *If hypotheses H_0 and H_1 hold, then there exist $\rho_n \geq l_n > 0$ for all $n \in \mathbb{N}$ such that $a_n^* \leq 0$ for all $n \in \mathbb{N}$, $b_n^* \to +\infty$ as $n \to \infty$.*

Proof. Hypotheses H_1-(i) and H_1-(ii) imply that given $\eta > 0$, we can find $c_7 > 0$ such that

$$F(z,x) \geq \frac{\eta}{p}|x|^p - c_7 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{25}$$

Let $u \in Y_n$ with $\|u\| \geq 1$. We have

$$\varphi(u) \leq \frac{1}{p}\int_\Omega a_1(z)|Du|^p \mathrm{d}z + \frac{1}{q}\int_\Omega a_2(z)|Du|^q \mathrm{d}z + c_8 - \frac{\eta}{p}\|u\|_p^p$$

for some $c_8 > 0$ (see (25)).

Since Y_n is finite dimensional, all norms are equivalent (see Papageorgiou-Winkert [21, p.183]). We have

$$\varphi(u) \leq (c_9 - \eta c_{10})\|u\|^p \text{ for some } c_9, c_{10} > 0 \text{ (recall } q < p). \tag{26}$$

Since $\eta > 0$ is arbitrary, from (26) we infer that

$$\varphi(u) \to -\infty \text{ as } \|u\| \to \infty.$$

Therefore, we can find $\rho_n > 0$, $n \in \mathbb{N}$ with $\rho_n \to +\infty$ such that

$$a_n^* \leq 0 \text{ for all } n \in \mathbb{N}.$$

Hypothesis H_1-(i) implies that

$$|F(z,x)| \leq c_{11}(|x| + |x|^r) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{11} > 0.$$

Let $u \in V_n$ with $\|u\| \geq 1$. We know that

$$\|u\|_r \leq \vartheta_n \|u\| \text{ (see (22))}. \tag{27}$$

So, we have

$$\varphi(u) \geq \frac{\widehat{c}}{p}\|u\|^p - c_{12}[\|u\| + \vartheta_n^r \|u\|^r]$$
$$\text{for some } c_{12} > 0, \text{ all } n \in \mathbb{N} \text{ (see hypotheses } H_0 \text{ and (27))}.$$

Let $l_n = 1/\vartheta_n^{r-p}$, $n \in \mathbb{N}$. Then $l_n \to +\infty$ as $n \to \infty$ (see Lemma 3.2 and recall that $p < r$). Clearly we can always choose $\rho_n > 0$ such that $\rho_n > l_n$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned}
&\varphi(u) \geq \frac{\widehat{c}}{p}l_n^p - c_{12}l_n - c_{12}\vartheta_n^p, \\
\Rightarrow\ & b_n^* \geq \frac{\widehat{c}}{p}l_n^p - c_{12}l_n - c_{12}\vartheta_n^p, \\
\Rightarrow\ & b_n^n \to +\infty \text{ (recall } p > 1 \text{ and see Lemma 3.2)}.
\end{aligned}$$

□

The proof is now complete.

Now we can produce a sequence of high energy solutions with the energies diverging to $+\infty$.

Theorem 3.4. *If hypotheses H_0 and H_1 hold, then problem (1) has a sequence of distinct solutions $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^1(\bar{\Omega})$ such that $\varphi(u_n) \to +\infty$ as $n \to \infty$.*

Proof. Since $\varphi(\cdot)$ is even, on account of Propositions 3.1 and 3.3, we can apply the Fountain Theorem (see Willem [25, p.58]) and generate a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$u_n \in K_\varphi \text{ for all } n \in \mathbb{N} \text{ and } \varphi(u_n) \to +\infty.$$

Then each u_n is a weak solution of problem (1). From [10, Theorem 7.1, p.286] of Ladyzhenskaya and Uraltseva, we have $u_n \in L^\infty(\Omega)$ for all $n \in \mathbb{N}$ and then the regularity theory of Lieberman [12], implies that $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^1(\bar{\Omega})$. □

4. **Low energy solutions.** In this section, we have a $(p-1)$-sublinear reaction and we generate a whole sequence of distinct smooth nodal (sign-changing) solutions with low energies which converge to zero.

In this case the hypotheses on the reaction $f(z,x)$ are the following:

H_2 : $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z,0) = 0$ and

(i) $|f(z,x)| \le \hat{a}(z)[1 + |x|^{p-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $\hat{a} \in L^\infty(\Omega)$;

(ii) $\limsup\limits_{x\to\pm\infty} \frac{f(z,x)}{a_1(z)|x|^{p-2}x} \le \hat{\lambda}_1^{a_1}(p)$ uniformly for a.a. $z \in \Omega$;

(iii) if $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$, then

$$\lim_{x\to\pm\infty} [f(z,x)x - pF(z,x)] = +\infty \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exists a function $\eta \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_1^{a_2}(q)a_2(z) \le \eta(z) \text{ for a.a. } z \in \Omega, \eta \not\equiv \hat{\lambda}_1^{a_2}(q)a_2,$$

$\eta(z) \le \liminf\limits_{x\to 0} \frac{qF(z,x)}{a_2(z)|x|^q}$ uniformly for a.a. $z \in \Omega$;

(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function

$$x \mapsto f(z,x) + \hat{\xi}_\rho|x|^{p-2}x$$

is nondecreasing on $[-\rho,\rho]$.

Remark 2. Hypothesis H_2-(ii) implies that we can have resonance with respect to the principal eigenvalue of $(-\Delta_p^{a_1}, W_0^{1,p}(\Omega))$. Hypothesis H_2-(iii) implies that the resonance occurs from the left of $\hat{\lambda}_1^{a_1}(p)$ in the sense that

$$\hat{\lambda}_1^{a_1}(p)a_1(z)|x|^p - pF(z,x) \to +\infty$$

uniformly for a.a. $z \in \Omega$, as $x \to \pm\infty$. This makes the energy functional $\varphi(\cdot)$ and its positive and negative truncations coercive (see Proposition 4.1 below).

The positive and negative truncations of the energy functional $\varphi(\cdot)$, are the functionals $\varphi_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\pm(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\mathrm{d}z + \frac{1}{q}\int_\Omega a_2(z)|Du|^q\mathrm{d}z - \int_\Omega F(z,\pm u^\pm)\mathrm{d}z$$

for all $u \in W_0^{1,p}(\Omega)$. We have that $\varphi_\pm \in C^1(W_0^{1,p}(\Omega))$.

Proposition 4.1. *If hypotheses H_0 and H_2 hold, then the functionals $\varphi_\pm(\cdot)$ and $\varphi(\cdot)$ are coercive.*

Proof. We have

$$\begin{aligned}\frac{d}{dx}\left[\frac{F(z,x)}{|x|^p}\right] =&\frac{f(z,x)|x|^p - p|x|^{p-2}xF(z,x)}{|x|^{2p}} \\ =&\frac{|x|^{p-2}x[f(z,x)x - pF(z,x)]}{|x|^{2p}} \\ =&\frac{f(z,x)x - pF(z,x)}{|x|^p x}.\end{aligned} \tag{28}$$

On account of hypothesis H_2-(iii) given $\gamma > 0$, we can find $M_\gamma > 0$ such that

$$f(z,x)x - pF(z,x) \geq \gamma \text{ for a.a. } z \in \Omega, \text{ all } |x| \geq M_\gamma. \tag{29}$$

We use (29) in (28) and obtain

$$\begin{aligned}&\frac{d}{dx}\left[\frac{F(z,x)}{|x|^p}\right] = \begin{cases} \geq \frac{\gamma}{x^{p+1}}, & \text{if } x \geq M_\gamma, \\ \leq \frac{\gamma}{|x|^p x}, & \text{if } x < -M_\gamma, \end{cases} \\ \Rightarrow\ &\frac{F(z,x)}{|x|^p} - \frac{F(z,y)}{|y|^p} \geq \frac{\gamma}{p}\left[\frac{1}{|y|^p} - \frac{1}{|x|^p}\right] \text{ for a.a. } z \in \Omega, \text{ all } |x| \geq |y| \geq M_\gamma. \end{aligned} \tag{30}$$

In (30) we let $|x| \to \infty$. Using hypothesis H_2-(ii), we obtain

$$\begin{aligned}&\frac{\hat{\lambda}_1^{a_1}(p)a_1(z)}{p} - \frac{F(z,y)}{|y|^p} \geq \frac{\gamma}{p}\frac{1}{|y|^p} \\ \Rightarrow\ &\hat{\lambda}_1^{a_1}(p)a_1(z)|y|^p - pF(z,y) \geq \gamma \text{ for a.a. } z \in \Omega, \text{ all } |y| \geq M_\gamma.\end{aligned}$$

Since $\gamma > 0$ is arbitrary, we conclude that

$$\hat{\lambda}_1^{a_1}(p)a_1(z)|y|^p - pF(z,y) \to +\infty \text{ uniformly for a.a. } z \in \Omega, \text{ as } |y| \to \infty. \tag{31}$$

We will show that (31) implies the coercivity of three functionals. We will do the proof for $\varphi_+(\cdot)$, the proofs for $\varphi_-(\cdot)$ and $\varphi(\cdot)$ being similar.

Arguing by contradiction, suppose that $\varphi_+(\cdot)$ is not coercive. Then we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$\begin{cases} \varphi_+(u_n) \leq c_{13} \text{ for some } c_{13} > 0, \text{ all } n \in \mathbb{N}, \\ \|u_n\| \to \infty \text{ as } n \to \infty. \end{cases} \tag{32}$$

From the inequality in (32), we see that if $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded, then so is $\{u_n^-\}_{n\in\mathbb{N}}$ and we infer that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded, a contradiction (see (32)). Therefore, we must have

$$\|u_n^+\| \to \infty. \tag{33}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$ for all $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega), y \geq 0. \tag{34}$$

From the inequality in (32), we have

$$\begin{aligned}&\frac{1}{p}\int_\Omega a_1(z)|Dy_n|^p \mathrm{d}z + \frac{1}{q\|u_n^+\|^{p-q}}\int_\Omega a_2(z)|Dy_n|^q \mathrm{d}z \\ &\leq \frac{c_{13}}{\|u_n^+\|^p} + \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\mathrm{d}z \text{ for all } n \in \mathbb{N}.\end{aligned} \tag{35}$$

Hypothesis H_2-(i) implies that

$$\left\{\frac{F(\cdot,u_n^+)}{\|u_n^+\|^p}\right\}_{n\in\mathbb{N}}\subseteq L^{p'}(\Omega) \text{ is bounded.}$$

Hence, by passing to a subsequence if necessary and using hypothesis H_2-(ii), we obtain

$$\frac{F(\cdot,u_n^+)}{\|u_n^+\|^p}\xrightarrow{w}\frac{1}{p}\vartheta(\cdot)y^p \text{ in } L^{p'}(\Omega), \tag{36}$$

with $\vartheta\in L^\infty(\Omega)$, $\vartheta(z)\le\hat{\lambda}_1^{a_1}(p)a_1(z)$ for a.a. $z\in\Omega$ (see Aizicovici-Papageorgiou-Staicu [1] (proof of Proposition 16)). Passing to the limit as $n\to\infty$ in (35) and using (33), (34) and (36) we obtain

$$\begin{aligned}&\int_\Omega a_1(z)|Dy|^p\mathrm{d}z\le\int_\Omega\vartheta(z)y^p\mathrm{d}z\le\hat{\lambda}_1^{a_1}(p)\int_\Omega a_1(z)y^p\mathrm{d}z, \qquad (37)\\ \Rightarrow&\int_\Omega a_1(z)|Dy|^p\mathrm{d}z=\hat{\lambda}_1^{a_1}(p)\int_\Omega a_1(z)y^p\mathrm{d}z \text{ (see (3))},\\ \Rightarrow& \; y=0 \text{ or } y=\hat{u}_1(p)\in\operatorname{int}C_+.\end{aligned}$$

If $y=0$, then from (35) we see that

$$y_n\to 0 \text{ in } W_0^{1,p}(\Omega),$$

which contradicts the fact that $\|y_n\|=1$ for all $n\in\mathbb{N}$. If $y=\hat{u}_1(p)\in\operatorname{int}C_+$ and $\vartheta\not\equiv\hat{\lambda}_1^{a_1}(p)a_1$, then from (37) and Proposition 2.2 we have

$$\begin{aligned}&c_0\int_\Omega a_1(z)|Dy|^p\mathrm{d}z\le 0,\\ \Rightarrow&\; y=0,\end{aligned}$$

which as above leads to a contradiction.

Finally we consider the case $y=\hat{u}_1(p)\in\operatorname{int}C_+$ and $\vartheta\equiv\hat{\lambda}_1^{a_1}(p)a_1$. From (31) we have

$$\begin{aligned}&\hat{\lambda}_1^{a_1}(p)a_1(z)u_n^+(z)-pF(z,u_n^+(z))\to+\infty \text{ for a.a. } z\in\Omega,\\ \Rightarrow&\int_\Omega\left[\hat{\lambda}_1^{a_1}(p)a_1(z)u_n^+-pF(z,u_n^+)\right]\mathrm{d}z\to+\infty \text{ (by Fatou's lemma, see (31)).} \qquad (38)\end{aligned}$$

From (35) and (3), we have

$$\begin{aligned}\int_\Omega\left[\hat{\lambda}_1^{a_1}(p)a_1(z)u_n^+-pF(z,u_n^+)\right]\mathrm{d}z+\frac{p}{q\|u_n^+\|^{p-q}}\int_\Omega a_2(z)|Dy_n|^q\mathrm{d}z&\\ \le\frac{pc_{13}}{\|u_n^+\|^p} \text{ for all } n\in\mathbb{N}.&\end{aligned} \tag{39}$$

Comparing (38) and (39), we have a contradiction. Therefore we infer that

$$\begin{aligned}&\{u_n^+\}_{n\in\mathbb{N}}\subseteq W_0^{1,p}(\Omega) \text{ is bounded},\\ \Rightarrow&\{u_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,p}(\Omega) \text{ is bounded}\end{aligned}$$

and this contradicts (32). This proves that $\varphi_+(\cdot)$ is coercive. Similarly, we show that $\varphi_-(\cdot)$ and $\varphi(\cdot)$ are coercive. ☐

Remark 3. In the process of the above proof we saw that the resonance occurs from the left of $\hat{\lambda}_1^{a_1}(p)$ (see (31)).

The coercivity of $\varphi_{\pm}(\cdot)$ permits the use of the direct method of calculus of variations in order to generate constant sign solutions for problem (1).

Proposition 4.2. *If hypotheses H_0 and H_2 hold, then problem (1) has at least two constant sign solutions $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$, both with negative energy.*

Proof. From Proposition 4.1 we know $\varphi_+(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\varphi_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_+(u_0) = \inf\{\varphi_+(u) : u \in W_0^{1,p}(\Omega)\}. \tag{40}$$

On account of hypothesis H_2-(iv), we see that given $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{1}{q}[\eta(z) - \varepsilon] \leq F_+(z, x) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta. \tag{41}$$

Consider the eigenfunction $\hat{u}_1(q) \in \operatorname{int} C_+$. We choose $t \in (0, 1)$ small such that $0 \leq t\hat{u}_1(q)(z) \leq \delta$ for all $z \in \bar{\Omega}$. We have

$$\begin{aligned}
\varphi_+(t\hat{u}_1(q)) \leq & \frac{t^p}{p}\int_\Omega a_1(z)|D\hat{u}_1(q)|^p \mathrm{d}z + \frac{t^q}{q}\int_\Omega a_2(z)|D\hat{u}_1(q)|^q \mathrm{d}z \\
& - \frac{t^q}{q}\int_\Omega \eta(z)|\hat{u}_1(q)|^q \mathrm{d}z + \frac{\varepsilon}{q}t^q \\
& \quad \text{(see (41) and recall that } \|\hat{u}_1(q)\|_q = 1) \\
\leq & c_{14}t^p + \frac{t^q}{q}\left[\int_\Omega (\hat{\lambda}_1^{a_2}(q) - \eta(z))a_2(z)|\hat{u}_1(q)|^q \mathrm{d}z + \varepsilon\right] \\
& \quad \text{(for some } c_{14} > 0) \\
\leq & c_{14}t^p - c_{15}t^q \text{ for some } c_{15} > 0 \\
& \quad \text{(choosing } \varepsilon > 0 \text{ small; see hypothesis } H_2\text{-(iv))}.
\end{aligned}$$

Since $q < p$, choosing $t \in (0, 1)$ small, we have

$$\begin{aligned}
& \varphi_+(t\hat{u}_1(q)) < 0, \\
\Rightarrow\ & \varphi_+(u_0) < 0 = \varphi_+(0) \text{ see (40)} \\
\Rightarrow\ & u_0 \neq 0.
\end{aligned}$$

From (40) we have

$$\begin{aligned}
& \varphi'_+(u_0) = 0, \\
\Rightarrow\ & \langle A_p^{a_1}(u_0), h\rangle + \langle A_q^{a_2}(u_0), h\rangle = \int_\Omega f(z, u_0^+)h\mathrm{d}z
\end{aligned} \tag{42}$$

for all $h \in W_0^{1,p}(\Omega)$. In (42) we use the test function $h = -u_0^- \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned}
& \widehat{c}\left[\|Du_0^-\|_p^p + \|Du_0^-\|_q^q\right] \leq 0, \text{ (see hypotheses } H_0), \\
\Rightarrow\ & u_0 \geq 0, u_0 \neq 0.
\end{aligned}$$

From (42), it follows that u_0 is a positive solution of (1). [10, Theorem 7.1, p.286] of Ladyzhenskaya-Uraltseva implies that $u_0 \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [12] implies that $u_0 \in C_+ \setminus \{0\}$. Using Proposition 2.3 (see also hypothesis H_2-(v)), we conclude that $u_0 \in \operatorname{int} C_+$.

Similarly working this time with the functional $\varphi_-(\cdot)$, we produce a negative solution $v_0 \in$ -int C_+ with $\varphi(v_0) < 0$. □

On account of hypotheses H_2-(i) and H_2-(iv), given $\varepsilon > 0$ and $r \in (p, p^*)$, we can find $c_{16} = c_{16}(\varepsilon, r) > 0$ such that

$$f(z,x)x \geq [\eta(z) - \varepsilon]a_2(z)|x|^q - c_{16}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{43}$$

This unilateral growth condition on $f(z,\cdot)$, leads to the following auxiliary Dirichlet problem

$$\begin{cases} -\Delta_p^{a_1} u - \Delta_q^{a_2} u = [\eta(z) - \varepsilon]|u|^{q-2}u - c_{16}|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{44}$$

Proposition 4.3. *If hypotheses H_0 and H_2 hold, then for all $\varepsilon > 0$ small problem (44) has a unique positive solution $\bar{u} \in$ intC_+, and since problem (44) is odd $\bar{v} = -\bar{u} \in$ -intC_+ is the unique negative solution of (44).*

Proof. We consider the C^1-functional $\psi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p \mathrm{d}z + \frac{1}{q}\int_\Omega a_2(z)|Du|^q \mathrm{d}z + \frac{c_{15}}{r}\|u^+\|_r^r - \frac{1}{q}\int_\Omega [\eta(z) - \varepsilon]a_2(z)(u^+)^q \mathrm{d}z$$

for all $u \in W_0^{1,p}(\Omega)$. Evidently, $\psi_+(\cdot)$ is coercive and sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W_0^{1,p}(\Omega)$ such that

$$\psi_+(\bar{u}) = \inf\left\{\psi_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{45}$$

As in the proof of Proposition 4.2, we show that for $\varepsilon > 0$ small, we have

$$\psi_+(\bar{u}) < 0 = \psi_+(0),$$
$$\Rightarrow \bar{u} \neq 0.$$

From (45), we have

$$\psi_+'(\bar{u}) = 0,$$
$$\Rightarrow \langle \psi_+'(\bar{u}), h \rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega).$$

Choosing $h = -\bar{u}^- \in W_0^{1,p}(\Omega)$, we infer that

$$\bar{u} \geq 0, \bar{u} \neq 0.$$

The nonlinear regularity theory and Proposition 2.3 imply that

$$\bar{u} \in \operatorname{int} C_+.$$

Note that for a.a. $z \in \Omega$, the function

$$x \mapsto [\eta(z) - \varepsilon]\frac{1}{x^{p-q}} - c_{15}x^{r-p}$$

is strictly decreasing on $(0, +\infty)$. So, [4, Theorem 3.5] of Fragnelli-Mugnai -Papageorgiou, implies that $\bar{u} \in$ int C_+ is the unique positive solution of (44). Since the problem is odd, $\bar{v} = -\bar{u} \in$ -int C_+ is the unique negative solution of problem (44). □

Let S_+ (resp. S_-) be the set of positive (resp. negative) solutions of problem (1). From Proposition 4.2, we know that

$$\emptyset \neq S_+ \subseteq \operatorname{int} C_+ \text{ and } \emptyset \neq S_- \subseteq \text{-int}\, C_+.$$

Proposition 4.4. *If hypotheses H_0 and H_2 hold, then $\bar{u} \leq u$ for all $u \in S_+$ and $v \leq \bar{v}$ for all $v \in S_-$.*

Proof. Let $u \in S_+ \subseteq \operatorname{int} C_+$ and let $\varepsilon > 0$ be small as postulated by Proposition 4.3. We introduce the Carathéodory function $k_+(z,x)$ defined by

$$k_+(z,x) = \begin{cases} [\eta(z)-\varepsilon]a_2(z)(x^+)^{q-1} - c_{16}(x^+)^{r-1} & \text{if } x \leq u(z), \\ [\eta(z)-\varepsilon]a_2(z)u(z)^{q-1} - c_{16}u(z)^{r-1} & \text{if } u(z) < x. \end{cases} \tag{46}$$

We set $K_+(z,x) = \int_0^x k_+(z,s)\mathrm{d}s$ and consider the C^1-functional $\delta_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\delta_+(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p \mathrm{d}z + \frac{1}{q}\int_\Omega a_2(z)|Du|^q\mathrm{d}z - \int_\Omega K_+(z,u)\mathrm{d}z$$

for all $u \in W_0^{1,p}(\Omega)$. It is clear from (46) that $\delta_+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\delta_+(\tilde{u}) = \inf\left\{\delta_+(\tilde{u}) : u \in W_0^{1,p}(\Omega)\right\} < 0 = \delta_+(0), \tag{47}$$

(see the proof of Proposition 4.2)

$$\Rightarrow \tilde{u} \neq 0.$$

From (47), we have

$$\delta_+'(\tilde{u}) = 0,$$

$$\Rightarrow \langle \delta_+'(\tilde{u}), h\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega). \tag{48}$$

In (48) first we use the test function $h = -\tilde{u}^- \in W_0^{1,p}(\Omega)$ and obtain that $\tilde{u} \geq 0$.

Next in (48) we choose $h = [\tilde{u}-u]^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} &\langle A_p^{a_1}(\tilde{u}), (\tilde{u}-u)^+\rangle + \langle A_q^{a_2}(\tilde{u}), (\tilde{u}-u)^+\rangle \\ &= \int_\Omega \left([\eta(z)-\varepsilon]a_2(z)u^{q-1} - c_{16}u^{r-1}\right)(\tilde{u}-u)^+\mathrm{d}z \\ &\leq \int_\Omega f(z,u)(\tilde{u}-u)^+\mathrm{d}z \quad (\text{see } (43)) \\ &= \langle A_p^{a_1}(u), (\tilde{u}-u)^+\rangle + \langle A_q^{a_2}(u), (\tilde{u}-u)^+\rangle \quad (\text{since } u \in S_+), \\ \Rightarrow\ &\tilde{u} \leq u \quad (\text{see Proposition 2.1}). \end{aligned}$$

So, we have proved that

$$\tilde{u} \in [0,u],\ \tilde{u} \neq 0. \tag{49}$$

Then (46), (48), (49) and Proposition 4.3, implies that

$$\tilde{u} = u,$$

$$\Rightarrow \bar{u} \leq u \text{ for all } u \in S_+ \text{ (see (49))}.$$

Similarly we show that

$$v \leq \bar{v} \text{ for all } v \in S_-.$$

The proof is now complete. □

Using these bounds, we can show the existence of external constant sign solutions, that is, we show the existence of a smallest positive solution and of a biggest negative solution.

Proposition 4.5. *If hypotheses H_0 and H_2 hold, then there exist $u^* \in S_+ \subseteq \operatorname{int} C_+$ and $v^* \in S_- \subseteq -\operatorname{int} C_+$ such that*

$$u^* \leq u \text{ for all } u \in S_+, \; v \leq v^* \text{ for all } v \in S_-.$$

Proof. From Proposition 7 of Papageorgiou-Rădulescu-Repovš [18] we know that S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leq u_1$, $u \leq u_2$). Hence, invoking Lemma 3.10 of Hu-Papageorgiou [8], we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\langle A_p^{a_1}(u_n), h\rangle + \langle A_q^{a_2}(u_n), h\rangle = \int_\Omega f(z, u_n) h \mathrm{d}z \tag{50}$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$,

$$\bar{u} \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N} \quad \text{(see Proposition 4.4)}. \tag{51}$$

In (50) we use the test function $h = u_n \in W_0^{1,p}(\Omega)$. Using (51) and hypothesis H_2-(i), we infer that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u^* \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u^* \text{ in } L^p(\Omega).$$

In (50) we choose $h = u_n - u^* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use Proposition 2.1 (as in the proof of Proposition 3.1). We obtain that

$$u_n \to u^* \text{ in } W_0^{1,p}(\Omega). \tag{52}$$

Passing to the limit as $n \to \infty$ in (50) and using (52) we obtain

$$\langle A_p^{a_1}(u^*), h\rangle + \langle A_q^{a_2}(u^*), h\rangle = \int_\Omega f(z, u^*) h \mathrm{d}z \tag{53}$$

for all $h \in W_0^{1,p}(\Omega)$. From (51) we have

$$\bar{u} \leq u^*. \tag{54}$$

Then (53) and (54) imply that

$$u^* \in S_+, u^* \leq u \text{ for all } u \in S_+.$$

Similarly, we produce

$$v^* \in S_-, v \leq v^* \text{ for all } v \in S_-.$$

We mention that in this case, the set S_- is upward directed (that is, if $v_1, v_2 \in S_-$, then there exists $v \in S_-$ such that $v_1 \leq v$, $v_2 \leq v$). ☐

Since our aim is to produce a whole sequence of distinct nodal solutions with vanishing energy levels, we need to strengthen the conditions on the reaction. So, we introduce a symmetry condition on $f(z, \cdot)$ and also strengthen the condition on $f(z, \cdot)$ near zero. The new conditions on the reaction $f(z, x)$ are the following:

H_2': $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot)$ is odd, hypotheses H_2'-(i),(ii),(iii), (v) are the same as the corresponding hypotheses H_2-(i),(ii),(iii),(v) and

(iv) $\lim\limits_{x\to 0} \frac{F(z,x)}{|x|^q} = +\infty$ uniformly for a.a. $z \in \Omega$.

Remark 4. The new condition at zero, reflects the presence of a "concave" term near zero.

Let $V \subseteq W_0^{1,p}(\Omega)$ be a finite dimensional subspace.

Proposition 4.6. *If hypotheses H_0 and H_2' hold, then there exists $\rho_V > 0$ such that*

$$\sup\{\varphi(u) : u \in V, \|u\| = \rho_V\} < 0.$$

Proof. On account of hypothesis H_2'-(iv), given $\eta > 0$, we can find $\delta = \delta(\eta) > 0$ such that

$$F(z,x) \geq \eta|x|^q \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{55}$$

Let $u \in V$. Since V is finite dimensional, all norms are equivalent (see Papageorgiou-Winkert [21, p.183]). Therefore, we can find $\rho_V \in (0,1)$ such that

$$\|u\| \leq \rho_V \Rightarrow |u(z)| \leq \delta \text{ for a.a. } z \in \Omega. \tag{56}$$

Hence if $u \in V$ with $\|u\| \leq \rho_V$, then

$$\varphi(u) \leq \frac{1}{p}\|u\|^p + [c_{17} - \eta c_{18}]\,\|u\|^q$$

for some c_{17}, $c_{18} > 0$ (see (55) and (56)). Choosing $\eta > \frac{c_{17}}{c_{18}}$, we obtain

$$\varphi(u) \leq \frac{1}{p}\|u\|^p - c_{19}\|u\|^q \text{ for some } c_{19} > 0.$$

Since $q < p$, choosing $\rho_V \in (0,1)$ even smaller if necessary, we have

$$\varphi(u) \leq -c^* < 0 \text{ for all } u \in V \text{ with } \|u\|_{\rho_V}.$$

This completes the proof. ☐

Now we can generate a sequence of low energy nodal solutions. In fact our conclusion is stronger since we have that the nodal solutions themselves converge to zero in $C_0^1(\bar{\Omega})$.

Theorem 4.7. *If hypotheses H_0 and H_2' hold, then problem (1) has a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^1(\bar{\Omega})$ of nodal solutions such that $u_n \to 0$ in $C_0^1(\bar{\Omega})$ (hence $\varphi(u_n) \to 0$).*

Proof. Clearly, $\varphi(\cdot)$ is even. Also from Proposition 4.1 we know that $\varphi(\cdot)$ is coercive. Therefore, $\varphi(\cdot)$ is bounded below and satisfies the C-condition (see Proposition 5.1, p.369, of Papageorgiou-Rădulescu-Repovš [19]). Then these facts and Proposition 4.6, permit the use of Theorem 1 of Kajikiya [9]. So, we can find $u_n \in W_0^{1,p}(\Omega)$, $n \in \mathbb{N}$ such that

$$u_n \in K_\varphi, n \in \mathbb{N} \;\text{ and }\; \|u_n\| \to 0 \text{ as } n \to \infty. \tag{57}$$

From Ladyzhenskaya-Uraltseva [10, p.286] (see also Papageorgiou-Rădulescu [16, Proposition 2.10]), we have that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq L^\infty(\Omega) \text{ is bounded.}$$

Then the nonlinear regularity theory of Lieberman [12] implies that there exist $\alpha \in (0,1)$ and $c_{20} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\bar{\Omega}), \|u_n\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq c_{20} \text{ for all } n \in \mathbb{N}. \tag{58}$$

The compact embedding of $C_0^{1,\alpha}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$ and (57), imply that

$$u_n \to 0 \text{ in } C_0^1(\bar{\Omega}) \text{ as } n \to \infty. \tag{59}$$

Since $u^* \in \operatorname{int} C_+$, $v^* \in$ -$\operatorname{int} C_+$, we see that

$$\operatorname{int}_{C_0^1(\bar{\Omega})}[v^*, u^*] \neq \emptyset.$$

So, we can find $n_0 \in \mathbb{N}$ such that

$$u_n \in \operatorname{int}_{C_0^1(\bar{\Omega})}[v^*, u^*] \text{ for all } n \geq n_0 \text{ (see (59))}.$$

The extremity of u^* and of v^* imply that

$$\{u_n\}_{n \geq n_0} \text{ are nodal solutions of (1)}$$

and we have $u_n \to 0$ in $C_0^1(\bar{\Omega})$. □

Acknowledgments. The research of V. D. Rădulescu was supported by a grant of the Romanian Ministry of Research, Innovation and Digitization, CNCS/CCCDI-UEFISCDI, project number PCE 137/2021, within PNCDI III. The research of J. Zhang was supported by Key project of Scientific Research Project of Department of Education of Hunan Province (21A0387), Natural Science Foundation of Hunan Province (2021JJ30189), Funding scheme for Young Backbone Teachers of universities in Hunan Province (Hunan Education Notification (2018) No. 574), and the China Scholarship Council (201908430218) for visiting the University of Craiova (Romania). J. Zhang would like to thank the China Scholarship Council and the Embassy of the People's Republic of China in Romania.

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Received for publication February 2022; early access May 2022.

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