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Multiplicity of solutions for nonlinear coercive problems

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A B S T R A C T

We are concerned in this paper with problems that involve nonlinear potential mappings satisfying condition (S) and whose potentials are coercive. We first provide mild sufficient conditions for the minimizing sequence in the Weierstrass-Tonelli theorem in order to have strongly convergent subsequences. Next, we establish a three critical point theorem which is based on the Pucci-Serrin type mountain pass lemma and which is an infinite dimensional counterpart of the Courant theorem. Ricceri-type three critical point results then follow. Some applications to Dirichlet boundary value problems driven by the perturbed Laplacian are given in the final part of this paper.

1. Introduction

Let E be a separable reflexive real Banach space and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between E^* and E. In this paper we consider nonlinear potential equations of the following type

$$A(u) = 0 \text{ for } u \in E, \tag{1}$$

where $A : E \to E^*$ is coercive bounded (that is, A is bounded on bounded sets) and potential operator which satisfies condition (S) and where $\mathcal{A} : E \to \mathbb{R}$ stands for the potential of A.

We examine the solvability of problem (1) by introducing a version of the Weierstrass-Tonelli theorem in which we obtain that the minimizer is the limit of a norm convergent minimizing sequence, as is the case

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in the finite dimensional case. We examine the multiple solvability for problem (1) in the above introduced setting by introducing some version of the three critical point theorem working in the coercive case. Concerning the multiplicity of solutions to (1) in the coercive case, there are also some results mostly pertaining to the usage of three critical point theorems of Ricceri-type, see for example [17], [18], [19], [20] for the theoretical background. We aim to derive a three critical point result for coercive C^1 functionals, which can be viewed as an infinite-dimensional version of the Courant theorem, see [12] for the classical version in $\mathbb{R}^N$. This classical result asserts that a functional having two distinct local minima must have another critical point which is not a minimizer (i.e., a point which realizes the minimum).

Theorem 1 *(Courant). [12] Suppose that a C^1 functional $\mathcal{J} : \mathbb{R}^N \to \mathbb{R}$ is coercive and possesses two distinct strict relative minima x_1 and x_2. Then $\mathcal{J}$ possesses a third critical point x_3 distinct from x_1 and x_2, which is not a relative minimizer, that is in every neighborhood of x_3, there exists a point x such that $\mathcal{J}(x) < \mathcal{J}(x_3)$.*

In the infinite dimensional version of Theorem 1 coercivity is replaced by the fact that the Palais-Smale condition is satisfied and then the following is known:

Theorem 2. *[16, Theorem 2] Let E be a Banach space and let $\mathcal{J} : E \to \mathbb{R}$ be a C^1 functional satisfying Palais-Smale condition with 0_E its strict local minimum. If there exists $e \neq 0_E$ such that $\mathcal{J}(e) \leqslant \mathcal{J}(0_E)$, then there is a critical point $\bar{x}$ of $\mathcal{I}$, with $\mathcal{J}(\bar{x}) > \mathcal{J}(0_E)$, which is not a local minimum.*

Our multiplicity result relies on connecting two distinct local minimizers via suitably chosen mountain pass approach cited above. Due to the necessity of obtaining the Palais-Smale compactness condition for the coercive action functional we investigate further relations between this condition and the coercivity. It is well known that a bounded from below C^1 functional satisfying the Palais-Smale condition is coercive, while the converse holds necessarily in finite dimensional spaces. We wish to obtain the converse in infinite dimensional space under some additional assumption on mapping A. Therefore we first investigate when the minimizing sequence generated by the Weierstrass-Tonelli Theorem is strongly convergent and provide a version of the Weierstrass-Tonelli Theorem which guarantees such a behavior. This is in compliance with what is known about the Weierstrass-Tonelli Theorem in the finite dimensional setting. Such convergence result is next used in providing conditions which guarantee that a coercive functional satisfies the Palais-Smale condition and as a consequence we obtain our three critical point type theorem by demonstrating the suitable mountain geometry. We comment also on the usage of our results about convergence on minimizing sequences in derivation of a Ricceri type multiplicity theorem by providing some version of an already known multiplicity theorem from [4]. In our approach we rely on using tools commonly applied in the monotonicity theory while investigating potential problems as far as the existence and multiplicity are concerned.

Applications are provided for the Dirichlet boundary value problems driven by the perturbed $p-$Laplacian and its various generalizations. Boundary value problems driven by the $p-$Laplacian attracted a lot of attention from different point of view. Let us mention for example [10] where the Authors determine the structure of the set of the solutions to the Dirichlet problem for the $p-$Laplacian on the line. Bifurcation-type results describing the set of positive solutions as the parameter varies are considered in [9], while the three critical point theorem due to Ricceri, see for example [17], is employed for the Dirichlet problem with the $p-$Laplacian in [1] among many sources which exploit its usage.

2. Auxiliary results

For the monotonicity we follow [7] and for variational tools [14]. Operator $A : E \to E^*$ is called:
i) *monotone*, if for all $u, v \in E$

$$\langle A(u) - A(v), u - v \rangle \geq 0$$

and *strictly monotone*, if the above inequality is strict for $u \neq v$;
ii) *strongly monotone*, if for some constant $\alpha > 0$ it holds for all $u, v \in E$

$$\langle A(u) - A(v), u - v \rangle \geq \alpha \|u - v\|^2;$$

iii) *strongly continuous*, if $u_n \rightharpoonup u_0$ implies $A(u_n) \to A(u_0)$;
iv) *potential*, if there exists a Gâteaux differentiable functional $\mathcal{A} : E \to \mathbb{R}$, called *the potential* of A such that $\mathcal{A}' = A$;
v) *satisfying condition (S) if*,

$$u_n \rightharpoonup u_0 \text{ in } E \text{ and } \langle A(u_n) - A(u_0), u_n - u_0 \rangle \to 0 \text{ imply } u_n \to u_0 \text{ in } E$$

vi) *coercive if*,

$$\lim_{\|v\| \to \infty} \frac{\langle A(v), v \rangle}{\|v\|} = +\infty.$$

Remark 3. There are some related conditions pertaining to condition (S) which make weakly convergent sequences strongly convergent upon some additional convergence condition on the operator involved. These are as follows:
i) Condition $(S)_+$: $u_n \rightharpoonup u_0$ and $\limsup_{n\to+\infty} \langle A(u_n) - A(u_0), u_n - u_0 \rangle \leq 0$ imply that $u_n \to u_0$ in E;
ii) Condition $(S)_0$: $u_n \rightharpoonup u_0$, $A(u_n) \rightharpoonup b$, $\langle A(u_n), u_n \rangle \to \langle b, u_0 \rangle$ imply that $u_n \to u_0$ in E.
It is known that condition $(S)_+$ implies that condition (S) is satisfied and this in turn implies condition $(S)_0$. We decided to apply condition (S) due to the fact that it is satisfied by the perturbed $p-$Laplacian operator which we consider further on and is much more intuitive than the technical condition $(S)_0$, while being less demanding than $(S)_+$. We mention that adding a strongly continuous perturbation to operator satisfying any of the above mentioned conditions does not violate this condition.

A Gâteaux differentiable functional $\mathcal{J} : E \to \mathbb{R}$ satisfies the Palais-Smale condition, the (PS) condition, if any sequence $(u_n) \subset E$ such that

i) $|\mathcal{J}(u_n)| \leq M$ for all $n \in \mathbb{N}$ and some $M > 0$,

ii) $\lim_{n\to\infty} \mathcal{J}'(u_n) = 0$ in E^*

admits a norm convergent subsequence.

Theorem 4 *(Ekeland Variational Principle - differentiable form). [14] Let $\mathcal{J} : E \to \mathbb{R}$ be a Gâteaux differentiable functional which is bounded from below. Then there exists a minimizing sequence (u_n) consisting of almost critical points, i.e. such that*

$$\mathcal{J}(u_n) \to \inf_{u\in E} \mathcal{J}(u) \text{ and } \mathcal{J}'(u_n) \to 0 \ (\text{in } E^*).$$

We will require the Lagrange Multiplier Rule in the form of Karush-Kuhn-Tucker providing necessary optimality conditions taken after [11]. Let $f : E \to \mathbb{R}$ be a given functional and let $g : E \to \mathbb{R}$ be a constraint functional. Let

$$S = \{x : g(x) \leq 0\}.$$

Theorem 5. *[11] Assume that $u_0 \in E$ is such that*

$$\inf_{u \in S} f(u) = f(u_0).$$

Let f and g be Fréchet differentiable at u_0. Assume that the Slater constraint qualification holds, i.e. if there is some x_0 that $g(x_0) < 0$. Then there is a nonnegative real number μ such that

$$f'(u_0) + \mu g'(u_0) = 0 \ (in\ E^*).$$

The following theorem about the continuity of the Niemytskij operator is rewritten after [7]:

Theorem 6 *(Generalized Krasnosel'skii Theorem). Let $p_1, p_2 \geq 1$ and $N \geq 1$ be a fixed natural number. Assume that $f : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function. If for any sequence $(u_n)_{n=1}^{\infty} \subset L^{p_1}(0,1)$ convergent to $u \in L^{p_1}(0,1)$ there exists a function $h \in L^{p_2}(0,1)$ such that*

$$|f(t, u_n)| \leq h(t), \ \text{for } n \in \mathbb{N} \text{ and a.e. } t \in [0,1],$$

then the Niemytskij operator induced by f

$$N_f : L^{p_1}(0,1) \ni u(\cdot) \longmapsto f(\cdot, u(\cdot)) \in L^{p_2}(0,1),$$

is well defined and continuous.

3. On the Weierstrass-Tonelli Theorem

In this section we undertake the question about the type of convergence of the minimizing sequence in the Weierstrass-Tonelli Theorem. It is well known that if one minimizes a lower semicontinuous functional $\mathcal{J}$ on a closed bounded set S in a finite dimensional space or else if the set is unbounded but $\mathcal{J}$ is coercive, then the minimizer is approximated by a convergent minimizing sequence (consisting of points from S). In case we work in an infinite dimensional reflexive Banach space we must require the set S to be sequentially weakly compact and the functional $\mathcal{J}$ to be sequentially weakly lower semicontinuous (for the latter to hold it suffice to assume continuity and convexity). Then a minimizer is obtained as a limit a *weakly* convergent minimizing sequence. When a functional is defined on the whole space, we must again assume that $\mathcal{J}$ is additionally coercive. However, in the Weierstrass-Tonelli Theorem applied in the infinite dimensional setting the minimizing sequence is strongly convergent as well under the additional condition (S) on the derivative. There is a result by the second author, see [8], answering the question about a convergence of minimizing sequences for a coercive functional with some monotonicity imposed on the derivative. In the proof of this result the Minty Lemma, see Lemma 3.6 in [7], is used. Here we not only drop the assumption about the monotonicity, do not impose a special structure on the action functional but also we simplify the proof methodology.

Theorem 7. *Assume that $A : E \to E^*$ satisfies condition (S) and it is potential with a sequentially weakly l.s.c. and coercive potential $\mathcal{A}$. Let $h \in E^*$ be fixed. Then there is a solution u_0 to*

$$A(u) = h \tag{2}$$

which minimizes action functional $\mathcal{J} : E \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \mathcal{A}(u) - \langle h, u \rangle$$

over E and moreover there is a sequence $(u_n) \subset E$, $u_n \to u_0$ such that

$$\mathcal{J}(u_n) \to \inf_{u \in E} \mathcal{J}(u) \text{ and } \mathcal{J}'(u_n) \to 0 \ (\text{in } E^*). \tag{3}$$

If in addition A is strictly monotone, then the solution is unique. Moreover, functional $\mathcal{J}$ satisfies the (PS) condition.

Proof. From the assumptions it follows that functional $\mathcal{J}$ is Gâteaux differentiable, coercive and sequentially weakly l.s.c. Hence it is bounded from below. Moreover, it has at least one minimizer u_0 which is a critical point, i.e. a solution to (2). By Theorem 4 there is a minimizing sequence $(u_n) \subset E$, $u_n \rightharpoonup u_0$ (the weak convergence follows by coercivity) and which is such that (3) holds. We see from $\mathcal{J}'(u_n) \to 0$ by writing the derivative explicitly that

$$\langle A(u_n) - A(u_0), u_n - u_0 \rangle \to 0. \tag{4}$$

Since A satisfies condition (S), we obtain that $u_n \to u_0$.

Let us take a Palais-Smale sequence, i.e. such a sequence $(u_n) \subset E$ that $\mathcal{J}(u_n)$ is bounded and $\mathcal{J}'(u_n) \to 0$. Due to the assumptions we can assume that $u_n \rightharpoonup u_0$, possibly up to a subsequence which we chose and do not renumber. Hence (4) holds. Since $\mathcal{J}' = A - h$ satisfies condition (S) and since $u_n \rightharpoonup u_0$, we see that $u_n \to u_0$, so the remaining assertion follows. ∎

We mention also here that checking condition (S) is technically similar to checking the strong convergence of bounded (PS) sequences which is also why we decided to apply this condition in our reasoning. From the proof of the above result we immediately obtain:

Proposition 8. *Assume that a Gâteaux differentiable functional $\mathcal{J} : E \to \mathbb{R}$ has a derivative $\mathcal{J}' : E \to E^*$ which satisfies condition (S). Then any bounded (PS) sequence for functional $\mathcal{J}$ admits a strongly convergent subsequence.*

Now we are in position to formulate the result about the existence of minimizers for coercive functionals following directly from the above:

Corollary 9. *Assume that functional $\mathcal{J} : E \to \mathbb{R}$ is bounded from below, coercive, Gâteaux differentiable and that its derivative $\mathcal{J}' : E \to E^*$ satisfies condition (S). Then there is some $u_0 \in E$ such that*

$$\mathcal{J}(u_0) = \inf_{u \in E} \mathcal{J}(u).$$

Proof. From Theorem 7 it follows that functional $\mathcal{J}$ satisfies the (PS) condition. Since it is bounded from below and satisfies the (PS) condition, it necessarily has at least one minimizer. The result now readily follows. ∎

Remark 10. We may replace condition (S) with conditions (S^+) or $(S)_0$ with retaining the same conclusion in the above, see also Remark 3 and remarks in [6].

Now we give some direct applications of our version of the Weierstrass-Tonelli Theorem that are related the known results. We provide a version of Theorem 2.1 from [15]. We recall that a functional whose derivative is monotone and coercive is necessarily bounded from below. In the result that follows we show that for a potential problem which can be tackled by the Banach fixed point theorem the sequence obtained by the method of successive approximations which converges to the unique solution stands also for a minimizing sequence.

Proposition 11. *Let E be a Hilbert space with a scalar product $(\cdot,\cdot)$. Let $N : E \to E$ be a contraction with the unique fixed point $u^* \in E$ (guaranteed by the Banach contraction theorem). If there exists a C^1 functional $\mathcal{J}$ such that*

$$\mathcal{J}^{'}(u) = u - N(u) \text{ for all } u \in E,$$

then u^* minimizes the functional E, i.e.

$$\mathcal{J}(u^*) = \inf_{u\in E} \mathcal{J}(u).$$

Moreover, a sequence $(u_n) \subset E$, defined by $u_{n+1} = N(u_n)$ for any $u_0 \in E$ is such that $u_n \to u^*$ and

$$\mathcal{J}(u_n) \to \inf_{u\in E} \mathcal{J}(u) \text{ and } \mathcal{J}^{'}(u_n) \to 0. \tag{5}$$

Proof. Since N is a contraction we see by a direct calculation that $\mathcal{J}^{'}$ is strongly monotone. We see that $\mathcal{J}$ as a potential of a strongly monotone mapping is coercive and sequentially weakly l.s.c. It also holds that $\mathcal{J}$ is strictly convex. Moreover, since $\mathcal{J}^{'}$ is strongly monotone, it satisfies condition (S). Then by Theorem 7 there is minimizer, which is unique by the strict convexity and therefore equal to u^*. Assertion (5) follows by the continuity of N. ■

4. On a three critical point theorem for a coercive functional

In this section we are going to derive the infinite dimensional multiplicity result corresponding to finite dimensional mountain pass theorem due to Courant. For $r > 0$ we put

$$B_r := \{x : \|x\| \le r\},\ S_r = \{x : \|x\| = r\}.$$

Theorem 12. *Assume that $\mathcal{I} \in C^1(E)$ has a strongly continuous derivative $\mathcal{I}^{'} : E \to E^*$ and that operator $A : E \to E^*$ is continuous, bounded, monotone, coercive and satisfies condition (S) and it is potential with the potential $\mathcal{A}$. Denote $\mathcal{J} := \mathcal{A}+\mathcal{I}$. Let $\widetilde{x} \in E$ and $r > 0$ be fixed. Assume that the following conditions are satisfied:*

(A.1) $\liminf_{\|x\|\to\infty} \frac{\mathcal{I}(x)}{\mathcal{A}(x)} \ge 0$;

(A.2) $\inf_{x\in E} \mathcal{J}(x) < \inf_{x\in B_r} \mathcal{J}(x)$;

(A.3) $\|\widetilde{x}\| < r$ *and* $\mathcal{J}(\widetilde{x}) < \inf_{x\in S_r} \mathcal{J}(x)$.

Then functional $\mathcal{J}$ has at least three distinct critical points in E, i.e. equation

$$A(u) + \mathcal{I}^{'}(u) = 0$$

has at least three distinct solutions two of which are necessarily nontrivial.

Proof. By the coercivity and the boundedness of operator A it follows that its potential is also coercive, see Lemma 5.5 from [7]. From (**A.1**) it holds for all $x \in E$ with sufficiently large norms that

$$\mathcal{I}(x) > -\frac{1}{2}\mathcal{A}(x).$$

Indeed, according to [13] $\liminf_{\|x\|\to\infty}\frac{\mathcal{I}(x)}{\mathcal{A}(x)} \geq 0$ means that for any $\alpha < 0$ there is some $R > 0$ such that for all $\|x\| > R$ we have $\frac{\mathcal{I}(x)}{\mathcal{A}(x)} > \alpha$. This implies, for all $x \in E$ with $\|x\| > R$, that

$$\mathcal{J}(x) = \mathcal{A}(x) + \mathcal{I}(x) > \frac{1}{2}\mathcal{A}(x)$$

and therefore functional J is coercive as well. Since $\mathcal{I}'$ is strongly continuous, we see that its $\mathcal{I}$ potential is sequentially weakly continuous. Since A is monotone, it follows that $\mathcal{A}$ is sequentially weakly l.s.c. and so is functional $\mathcal{J}$. Now from Theorem 7 we see that $\mathcal{J}$ satisfies the (PS) condition. Since $\mathcal{J}$ is sequentially weakly l.s.c. and coercive it has a global minimizer (and a critical point) which due to **(A.2)** lies outside B_r. Moreover, in B_r functional $\mathcal{J}$ has a local minimizer which by (**A.2**) lies inside a ball and therefore it is a critical point as well. By the application of Theorem 2 we get the existence of a third critical point which is distinct from these two mentioned. ■

Remark 13. We can replace **(A.1)** with the assumption that functional $\mathcal{I}$ is bounded from below. As it can be expected from Theorem 7 we see that Theorem 12 has also a more general formulation which does not involve monotonicity. Hence the special structure of $\mathcal{J}$ need not be assumed. However, due to the fact that we minimize a functional over a closed ball, we have to assume the sequential weak lower semicontinuity of the functional.

In order to conclude this section we will work on exploiting the monotonicity theory in derivation of the Ricceri type of a three critical point theorem for coercive functionals following [4]. In the proof the following technical lemma will be utilized and which obtained as a special case of results from [17, Proposition 2.2] and [2, Theorem 1]:

Lemma 14. *Let $D \subseteq \mathbb{R}_+$ be an interval. Assume that $\Phi \in C^1(E)$ is such that its derivative $\Phi : E \to E^*$ is strictly monotone, coercive and satisfies condition (S). Assume that $\mathcal{I} \in C^1(E)$ is such that $\mathcal{I}' : E \to E^*$ is strongly continuous. Moreover, assume that there exist $x_1, x_2 \in E$ and $\sigma \in \mathbb{R}$ such that*

(B.1) $\Phi(x_1) < \sigma < \Phi(x_2)$;

(B.2) $\inf_{\Phi(x)\leq\sigma} \mathcal{I}(x) > \dfrac{(\Phi(x_2) - \sigma)\mathcal{I}(x_1) + (\sigma - \Phi(x_1))\mathcal{I}(x_2)}{\Phi(x_2) - \Phi(x_1)}$;

(B.3) $\lim_{\|x\|\to\infty} [\Phi(x) + \lambda\mathcal{I}(x)] = +\infty$ *for all* $\lambda \in D$.

Then there exists a nonempty open set $C \subseteq D$ such that for all $\lambda \in C$ the functional $\Phi + \lambda\mathcal{I}$ has at least three critical points in E.

Lemma 15. *Assume that $\mathcal{I} \in C^1(E)$ is sequentially weakly l.s.c. and that $A : E \to E^*$ is continuous, strictly monotone, coercive, satisfies condition (S) and is potential with the potential $\mathcal{A}$ such that $\mathcal{A}(0) = 0$. Let $\widetilde{x} \in E$ and $r > 0$ be fixed. Assume that **(A.1)** holds and also the following conditions are satisfied:*

(C.1) $\inf_{x\in E} \mathcal{I}(x) < \inf_{\mathcal{A}(x)\leq r} \mathcal{I}(x)$;

(C.2) $\mathcal{A}(\widetilde{x}) < r$ *and* $\mathcal{I}(\widetilde{x}) < \inf_{\mathcal{A}(x)=r} \mathcal{I}(x)$.

Then there exists a nonempty open set $C \subseteq (0, +\infty)$ such that for all $\lambda \in C$ the functional $\mathcal{A} + \lambda\mathcal{I}$ has at least three critical points in E, two of which are necessarily non-trivial.

Proof. By **(A.1)** we see that condition **(B.3)** is satisfied.

Since mapping A is strictly monotone, we see that its potential is necessarily convex. Then set the Lebesgue level set $B = \{x \in E : \mathcal{A}(x) \leq r\}$ is convex and due to the continuity of $\mathcal{A}$ it is also closed. This means that B is weakly closed. Since $\mathcal{A}$ is the potential of a coercive mapping it follows that B is bounded which means that B is sequentially weakly compact. Now, since functional $\mathcal{J}$ is sequentially weakly l.s.c. it attains its infimum on B at some $x_1 \in B$. We claim that

$$\mathcal{A}(x_1) < r \text{ and } \mathcal{I}(x_1) = \inf_{x \in B} \mathcal{I}(x). \tag{6}$$

Since $\mathcal{A}\,(0) = 0 < r$ we observe that the Slater constraint qualification is satisfied. Therefore it follows by the Karush-Kuhn-Tucker Theorem (Theorem 5) that there is a Lagrange multiplier $\zeta \geq 0$ for which it holds

$$\mathcal{I}^{'}\,(x_1) + \zeta A\,(x_1) = 0 \text{ and } \zeta\,(\mathcal{A}\,(x_1) - r) = 0.$$

By (**C.2**) it follows that $\mathcal{A}\,(x_1) < r$ and therefore we must take $\zeta = 0$ which implies (6).

By **(C.1)** there exists $x_2 \in E$ such that

$$\mathcal{A}(x_2) > r \text{ and } \mathcal{I}(x_2) < \mathcal{I}(x_1). \tag{7}$$

Set $\sigma = r$ and noting that $\mathcal{A}(x_1) < \sigma < \mathcal{A}(x_2)$ we see that **(B.1)** is satisfied. Moreover by (6) and (7) we have

$$\inf_{\mathcal{A}(x) \leq r} \mathcal{I}(x) = \mathcal{I}(x_1) = \frac{\mathcal{I}(x_1)\,(\mathcal{A}(x_2) - \mathcal{A}(x_1))}{\mathcal{A}(x_2) - \mathcal{A}(x_1)} =$$

$$\frac{\mathcal{I}(x_1)\mathcal{A}(x_2) - \sigma\mathcal{I}(x_1) + \sigma\mathcal{I}(x_1) - \mathcal{I}(x_1)\mathcal{A}(x_1)}{\mathcal{A}(x_2) - \mathcal{A}(x_1)} =$$

$$\frac{(\mathcal{A}(x_2) - \sigma)\,\mathcal{I}(x_1) + (\sigma - \mathcal{A}(x_1))\,\mathcal{I}(x_1)}{\mathcal{A}(x_2) - \mathcal{A}(x_1)} >$$

$$\frac{(\mathcal{A}(x_2) - \sigma)\,\mathcal{I}(x_1) + (\sigma - \mathcal{A}(x_1))\,\mathcal{I}(x_2)}{\mathcal{A}(x_2) - \mathcal{A}(x_1)},$$

so **(B.2)** is satisfied too. The assertion now follows by Lemma 14. ■

Remark 16. While in the proof of Lemma 15 we use some ideas from [4], we include new arguments, like the usage of the Karush-Kuhn-Tucker Theorem and we also exploit the usage of the monotonicity theory. Condition (**C1**) can be replaced with the following

(C.4). $\mathcal{A}(x) + \lambda\mathcal{I}(x) \to +\infty$ as $\|x\| \to \infty$

with the assertion retained. The assertion is also retained if we assume $\mathcal{I}$ to be bounded from below, see also [5] for some research in this direction. Comparing Lemma 15 with Theorem 12 we see that their applicability coincides when operator A is strongly monotone, i.e. for the case of the (negative) Laplacian. In contrast when the (negative) $p-$Laplacian is considered finding a minimizer on a ball is much more convenient than examining the behavior of the Euler action functional.

5. Applications

5.1. Applications of the Weierstrass-Tonelli Theorem

We note that the application of our version of the Weierstrass-Tonelli Theorem is similar to checking the classical version but it requires verifying that the condition (S) or any related is satisfied instead of checking the sequential weak lower semicontinuity of the action functional. Nevertheless both approaches use similar arguments. Towards the uniqueness we need to determine that the derivative defines the strictly coercive operator.

We will need some preparation prior to introduction of the problem under consideration. Let $p \geq 2$ and set $E := W_0^{1,p}(0,1)$. Then E is a separable, uniformly convex (and thus reflexive) space, see [3]. Recall that for any $u \in E$ it holds

$$\|u\|_C := \max_{t\in[0,1]} |u(t)| \leq \|u\| := \|\dot{u}\|_{L^p} = \left(\int_0^1 |\dot{u}(t)|^p \, dt \right)^{1/p}.$$

Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function and define $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$F(t,u) = \int_0^u f(t,s)\, ds \text{ for } a.e.\ \ t \in [0,1] \text{ and all } u \in \mathbb{R}. \tag{8}$$

We assume that

(φ**1**) $\varphi : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ is a Carathéodory function for which there is a constant $M > 0$ such that

$$|\varphi(t,x)| \leq M \text{ for a.e. } t \in [0,1] \text{ and all } x \in \mathbb{R}_+;$$

(φ**2**) there exists a constant $\gamma > 0$ *such that*

$$\varphi(t,x)\,x - \varphi(t,y)\,y \geq \gamma(x-y)$$

for all $x \geq y \geq 0$ and a.e. $t \in [0,1]$;

(F.1) $f(t,0) = 0$ for a.e. $t \in [0,1]$, $g \in L^{p'}(0,1)$, $g \neq 0$;

(F.2) for a.e. $t \in [0,1]$ function $x \mapsto f(t,x)$ is nondecreasing.

Now we can consider the existence and the uniqueness for the following Dirichlet problem

$$\begin{cases} -\frac{d}{dt}\left(\varphi\left(t, \left|\frac{d}{dt}u\right|^{p-1}\right) \left|\frac{d}{dt}u\right|^{p-2} \frac{d}{dt}u\right) + f(t,u(t)) = g(t), \text{ a.e. on } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{9}$$

The solutions are understood in the weak sense. We say that a function $u \in E$ is a weak solution of (9) if for all $v \in E$ it holds

$$\int_0^1 \varphi\left(t, |\dot{u}(t)|^{p-1}\right) |\dot{u}(t)|^{p-2} \dot{u}(t)\, \dot{v}(t)\, dt + \int_0^1 f(t,u(t))\, v(t)\, dt = \int_0^1 g(t)\, v(t)\, dt.$$

We see that with assumption **(F.1)** any weak solution is non-zero which we prove by a direct calculation assuming to the contrary.

Let us define $\mathcal{I}: E \to \mathbb{R}$ by

$$\mathcal{I}(u) = \int_0^1 F(u(t))\,dt + \int_0^1 g(t)\,u(t)\,dt.$$

By direct arguments we see that the Euler Lagrange functional to (9) reads

$$\mathcal{J} := \mathcal{A} + \mathcal{I} \in C^1(E).$$

In order to apply Corollary 9 we define operators A, $A_1 : E \to E^*$ by

$$\langle A(u), v\rangle = \int_0^1 \varphi\left(|\dot{u}(t)|^{p-1}\right) |\dot{u}(t)|^{p-2}\,\dot{u}(t)\,\dot{v}(t)\,dt \text{ for } u, v \in E,$$

$$\langle A_1(u), v\rangle = \int_0^1 f(t, u(t))v(t)\,dt.$$

From [7], Theorem 3.3, we get the following:

Lemma 17. *Assume that conditions* $(\varphi\mathbf{1})$, $(\varphi\mathbf{2})$ *are satisfied. Then operator* A *is continuous, bounded, strictly monotone, coercive and satisfies condition (S). Moreover, it is potential with the potential* $\mathcal{A} : E \to \mathbb{R}$ *defined by*

$$\mathcal{A}(u) = \int_0^1 \int_0^{|\dot{u}(t)|} \varphi\left(s^{p-1}\right) s^{p-1} ds dt \text{ for } u \in E. \tag{10}$$

Additionally, operator A *is invertible with a continuous inverse.*

Proposition 18. *Assume that conditions* $(\varphi\mathbf{1})$, $(\varphi\mathbf{2})$ *and* ***(F.1)-(F.2)*** *are satisfied. Then problem (9) has exactly one nontrivial solution.*

Proof. Under conditions (**F.1**), (**F.2**) we see that operator A_1 is strongly continuous, bounded and monotone. Moreover, it is potential with the potential $\mathcal{I}$. Monotonicity of A follows from (**F.2**). As for the strong continuity we see that since a weakly convergent sequence $(u_n) \subset W_0^{1,p}(0,1)$ is bounded by some $d > 0$, then since f is an L^1−Carathéodory function there is some function $f_d \in L^1(0,1)$ such that

$$|f(t,x)| \leq f_d(t) \text{ for a.e. } t \in [0,1] \text{ and } x \in [-d,d].$$

It suffices to apply the Krasnosel'skii Theorem and the compact embedding of $W_0^{1,p}(0,1)$ into $L^p(0,1)$ to get the strong continuity. Since strongly continuous perturbations do not violate the condition (S) we see that operator $A + A_1$ satisfies condition (S). By adding a monotone operator to a coercive operator we obtain the coercive operator and this is why we see that operator $A + A_1$ is coercive since by Lemma 17 operator A is coercive. Moreover, operator $A + A_1$ is bounded and continuous. Then we reach the conclusion by Corollary 9. ∎

5.2. Applications of the three critical point theorem to Dirichlet problems

We retain the assumptions about the nonlinear differential operator from the previous section, i.e. $(\varphi\mathbf{1})$, $(\varphi\mathbf{2})$, we let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ and define $F:[0,1]\times\mathbb{R}\to\mathbb{R}$ by (8). We assume that

(F.3) $f,F:[0,1]\times\mathbb{R}\to\mathbb{R}$ are L^1-Carathéodory functions;
(F.4) there is a constant $c_1>0$ such that

$$\lim_{|x|\to\infty}\frac{F(t,x)}{|x|^{p-1}}=-c_1$$

uniformly for a.e. $t\in[0,1]$;
(F.5) $\lim_{x\to 0}\frac{f(t,x)}{|x|^{p-1}}=0$ uniformly for a.e. $t\in[0,1]$.

For a numerical parameter $\lambda>0$ we now consider the following Dirichlet problem

$$\begin{aligned}&-\tfrac{d}{dt}\left(\varphi\left(\left|\tfrac{d}{dt}u\right|^{p-1}\right)\left|\tfrac{d}{dt}u\right|^{p-2}\tfrac{d}{dt}u\right)+\lambda f(t,u)=0,\ \text{a.e. on }(0,1),\\&u(0)=u(1)=0.\end{aligned}\tag{11}$$

The solutions are again understood in the weak sense which is as follows:

$$\int_0^1\varphi\left(t,|\dot u(t)|^{p-1}\right)|\dot u(t)|^{p-2}\dot u(t)\dot v(t)\,dt+\int_0^1 f(t,u(t))v(t)\,dt=0\text{ for all }v\in E.$$

Let us define a C^1 functional $\mathcal{I}:E\to\mathbb{R}$ by

$$\mathcal{I}(u)=\int_0^1F(t,u(t))\,dt.$$

For any fixed $\lambda>0$, critical points to

$$\mathcal{J}_\lambda=\mathcal{A}+\lambda\mathcal{I},$$

where $\mathcal{A}$ is defined by (10), correspond to solutions to (11), as well as solutions to (11) are critical points to $\mathcal{J}_\lambda$. Now we can proceed with the application of Theorem 12 to problem (11).

Proposition 19. *Assume that conditions* $(\varphi\mathbf{1})$, $(\varphi\mathbf{2})$ *and* ***(F.3)***, ***(F.5)*** *hold. Then there is a constant* $\bar\lambda$ *such that for* $\lambda>\bar\lambda$ *problem (11) has at least two nontrivial solutions.*

Proof. Under conditions (**F.3**), (**F.4**) for any fixed $\lambda>0$ functional $\mathcal{J}_\lambda$ is sequentially weakly l.s.c. and coercive. Indeed, from condition (**F.4**) by using the Hölder and the Poincaré inequality we obtain for some constant $c_2>0$ (which depends on p, on the Poincaré constant and also on c_1)

$$\int_0^1F(t,u(t))\,dt\le -c_2\|u\|^{p-1}\ \text{ for }u\in E.\tag{12}$$

From the coercivity of $\mathcal{A}$ and from (12) it obviously follows that $\mathcal{J}_\lambda$ is coercive for every positive λ.

We see that functional $\mathcal{I}$ is sequentially weakly continuous on E. Indeed, let us take $(u_n) \subset E$ such that $u_n \rightharpoonup u_0$. Then $u_n \to u_0$ in $L^2(0,1)$ and by the Lebesgue Dominated Convergence Theorem we see that

$$\lim_{n\to\infty} \int_0^1 F(t, u_n(t))\, dt = \int_0^1 F(t, u_0(t))\, dt.$$

Since $\mathcal{A}$ is a continuous potential of a monotone mapping, it is sequentially weakly l.s.c. Hence $\mathcal{J}_\lambda$ is sequentially weakly l.s.c. as well. From (12) it directly follows that there is a function $\bar{u} \in E$ such that

$$\int_0^1 F(t, \bar{u}(t))\, dt < 0. \tag{13}$$

By $(\varphi\mathbf{1})$, $(\varphi\mathbf{2})$ we reach at the following estimation for all $u \in E$

$$\frac{M}{p}\|u\|^p \geq \mathcal{A}(u) \geq \frac{\gamma}{p}\|u\|^p. \tag{14}$$

We take a function $\bar{u}$ that $\mathcal{I}(\bar{u}) < 0$. Since $\mathcal{A}(\bar{u}) > 0$ we define

$$\bar{\lambda} := -\frac{\mathcal{A}(\bar{u})}{\mathcal{I}(\bar{u})} > 0.$$

Then for $\lambda > \bar{\lambda}$ it holds that $\mathcal{J}_\lambda(\bar{u}) < 0$. Assume now that some $\lambda > \bar{\lambda}$ is fixed. Following the known technique applied in checking the mountain geometry we see from (**F.4**) that for any $\varepsilon > 0$ there is $\delta > 0$ such that for $|x| \leq \delta$ it holds

$$|F(t,x)| \leq \varepsilon \frac{|x|^p}{p} \text{ for a.e. } t \in [0,1].$$

Using the above we have that for $\varepsilon \in \left(0, \frac{\gamma}{\lambda}\right)$ there is a constant $\delta \leq \varepsilon$ such that for $\|u\| \leq \delta$ it holds from (14)

$$\mathcal{J}_\lambda(u) \geq \frac{1}{p}(\gamma - \varepsilon\lambda)\|u\|^p.$$

This means that **(A.2)** is satisfied since $\mathcal{J}_\lambda(u) \geq 0$ for $\|u\| \leq \delta$ and since

$$\inf_E \mathcal{J}_\lambda \leq \mathcal{J}_\lambda(\bar{u}) < 0.$$

Condition **(A.3)** is fulfilled since

$$\mathcal{J}_\lambda(0) = 0 < \frac{1}{p}(\gamma - \varepsilon\lambda)\delta^p.$$

Now the assertion follows from Theorem 12. ■

Remark 20. When $f : \mathbb{R} \to \mathbb{R}$ is continuous, i.e. if we drop the dependence on t, we see that for the above results to hold we need not assume **(F.3)**. The continuity of F suffices in this case.

Example 21. Let $p \geq 2$. We put $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} -|x|^{p-1} x, & |x| \leq 1, \\ -|x|^{p-3} x, & |x| \geq 1 \end{cases}$$

and hence

$$F(x) = \begin{cases} -(p+1)|x|^{p+1}, & |x| \leq 1, \\ -(p+1) - (p-1)|x|^{p-1} + (p-1), & |x| \geq 1. \end{cases}$$

We see that assumptions **(F.4)**, **(F.5)** hold in this case.

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References

[1] G.A. Afrouzi, S. Heidarkhani, Three solutions for a Dirichlet boundary value problem involving the $p-$Laplacian, Nonlinear Anal., Theory Methods Appl. 66 (2007) 2281–2288.
[2] G. Bonanno, A minimax inequality and its applications to ordinary differential equations, J. Math. Anal. Appl. 270 (2002) 210–229.
[3] H. Brézis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2010.
[4] A. Cabada, A. Iannizzotto, A note on a question of Ricceri, Appl. Math. Lett. 25 (2012) 215–219.
[5] A. Cabada, S. Tersian, Multiplicity of solutions of a two point boundary value problem for a fourth-order equation, Appl. Math. Comput. 219 (2013) 5261–5267.
[6] R. Chiappinelli, D.E. Edmunds, Remarks on surjectivity of gradient operators, Mathematics 2020 (1538) 8, https://doi.org/10.3390/math8091538.
[7] M. Galewski, Basic Monotonicity Methods with Some Applications, Compact Textbooks in Mathematics, Birkhäuser/SpringerNature, Basel, Switzerland/Basingstoke, UK, ISBN 978-3-030-75308-5, 2021.
[8] M. Galewski, On variational nonlinear equations with monotone operators, Adv. Nonlinear Anal. 10 (2021) 289–300, https://doi.org/10.1515/anona-2020-0102.
[9] L. Gasiński, N.S. Papageorgiou, Positive solutions for the Robin $p-$Laplacian problem with competing nonlinearities, Adv. Calc. Var. 12 (2019) 31–56.
[10] M. Guedda, L. Véron, Bifurcation phenomena associated to the Laplace $p-$operator, Trans. Am. Math. Soc. 310 (1988) 419–431.
[11] A.D. Ioffe, V.M. Tikhomirov, Theory of Extremal Problems (in Russian), Series in Nonlinear Analysis and Its Applications, Izdat. Nauka, Moscow, 1974.
[12] Y. Jabri, The Mountain Pass Theorem. Variants, Generalizations and Some Applications, Encyclopedia of Mathematics and Its Applications, vol. 95, Cambridge University Press, Cambridge, 2003.
[13] S. Łojasiewicz, An Introduction to the Theory of Real Functions, Wiley-Interscience Publication, Wiley, Chichester (UK) etc., 1988.
[14] N.S. Papageorgiou, V.D. Radulescu, D.D. Repovš, Nonlinear Analysis: Theory and Methods, Springer, Cham, 2019.
[15] R. Precup, Nash-type equilibria and periodic solutions to nonvariational systems, Adv. Nonlinear Anal. 3 (4) (2014) 197–207.
[16] P. Pucci, J. Serrin, Extensions of the mountain pass theorem, J. Funct. Anal. 59 (1984) 185–210.
[17] B. Ricceri, A three critical points theorem revisited, Nonlinear Anal. 70 (2009) 3084–3089.
[18] B. Ricceri, On a minimax theorem: an improvement, a new proof and an overview of its applications, Minimax Theory Appl. 2 (2017) 99–152.
[19] B. Ricceri, A class of functionals possessing multiple global minima, Stud. Univ. Babeş–Bolyai, Math. 66 (1) (2021) 75–84.
[20] B. Ricceri, A more complete version of a minimax theorem, Appl. Anal. Optim. 5 (2) (2021) 251–261.