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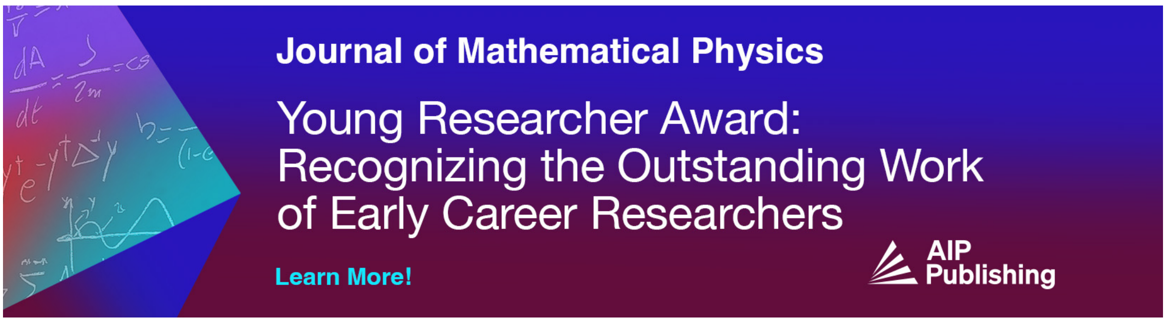

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# Qualitative properties of solution to a viscoelastic Kirchhoff-like plate equation

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## ABSTRACT

This paper is concerned with the initial boundary value problem for viscoelastic Kirchhoff-like plate equations with rotational inertia, memory, $p$-Laplacian restoring force, weak damping, strong damping, and nonlinear source terms. We establish the local existence and uniqueness of the solution by linearization and the contraction mapping principle. Then, we obtain the global existence of solutions with subcritical and critical initial energy by applying potential well theory. Then, we prove the asymptotic behavior of the global solution with positive initial energy strictly below the depth of the potential well. Finally, we conduct a comprehensive study on the finite time blow-up of solutions with negative initial energy, null initial energy, and positive initial energy strictly below the depth of the potential well and arbitrary positive initial energy, respectively.



## I. INTRODUCTION

In this paper, we study the following viscoelastic Kirchhoff-like plate equation with rotational inertia, memory, $p$-Laplacian restoring force, weak damping, strong damping, and nonlinear source terms:

$$\begin{aligned} &u_{tt} - \Delta u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,d\tau \\ &\quad - \Delta_p u + u_t - \Delta u_t = |u|^{q-2}u, \quad x \in \Omega, \ t > 0, \end{aligned} \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

and simply supported boundary conditions

$$u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega, \ t > 0, \tag{1.3}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Moreover, the memory kernel $g$ and the growth exponents $p$, $q$ satisfy the following assumptions:

$$(\mathrm{A}_1)\quad g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+),\ g(t) \ge 0,\ g'(t) \le 0,\ t \in [0,\infty),$$

and

$$\kappa := 1 - \int_0^\infty g(t)\,\mathrm{d}t > 0, \tag{1.4}$$

$$(\mathrm{A}_2)\quad 2 \le p < \infty \ \text{ if } N \le 2,\quad 2 \le p \le \frac{2N-2}{N-2} \ \text{ if } N > 2,$$

$$(\mathrm{A}_3)\quad p < q < \infty \ \text{ if } N \le p,\quad p < q \le \frac{Np+2N-2p}{2N-2p} \ \text{ if } N > p.$$

Viscoelastic materials possess the properties between those of elastic solids and viscous fluids that can be modeled by partial differential equations. In recent years, different kinds of second-order viscoelastic equations have received considerable attention; see, e.g., Refs. 1–8 and the references therein. Equation (1.1) is a fourth-order nonlinear hyperbolic equation and can be used to describe small-deflection vibrations of viscoelastic thin homogeneous and isotropic plates.[9,10] The unknown function $u(x,t)$ represents the deflection (namely, the normal component of the displacement vector) at time $t$ of a filament having position $x$ in a given reference configuration.[9,10] In order to better understand the physical background of Eq. (1.1), we sketch the derivation of Eq. (1.1), which can be derived from the Mindlin–Timoshenko plate model,[9,10]

$$\begin{cases} \rho\hbar u_{tt} - K\operatorname{div}(\nabla u + \Psi) = f, \\ \dfrac{\rho\hbar^3}{12}\Psi_{tt} - DS + K(\nabla u + \Psi) = H, \end{cases}$$

where $\rho$ is the density, $\hbar$ is the thickness, $K = \frac{kY\hbar}{2(1+r)}$ is the shear modulus, $D = \frac{Y\hbar^3}{12(1-r^2)}$ is the flexural rigidity, $0 < r < \frac{1}{2}$ is Poisson's ratio, $Y$ is Young's modulus, and $k$ is the shear correction coefficient. Unlike the Kirchhoff plate model, the Mindlin–Timoshenko plate model takes into account transverse shear effects. In the two-dimensional case, $\Psi = (\psi,\varphi)$, where $\psi = \psi(x_1,x_2,t)$ and $\varphi = \varphi(x_1,x_2,t)$ correspond to rotation angles of the filament. According to the theory of elasticity, the stress tensor $S = A\Psi$, where

$$A = \begin{bmatrix} \partial_{x_1x_1} + \dfrac{1-r}{2}\partial_{x_2x_2} & \dfrac{1+r}{2}\partial_{x_1x_2} \\ \dfrac{1+r}{2}\partial_{x_1x_2} & \dfrac{1-r}{2}\partial_{x_1x_1} + \partial_{x_2x_2} \end{bmatrix}.$$

For viscoelastic thin plates, $S$ can be expressed in the form[11]

$$S = A\Psi - \int_0^t g(s)A\Psi(t-s)\,\mathrm{d}s.$$

Thus, in nonconservative systems, we can arrive at the following viscoelastic Mindlin–Timoshenko plate model:

$$\begin{cases} \rho\hbar u_{tt} - K\operatorname{div}(\nabla u + \Psi) = f(u) - \mu_2 u_t, \\ \dfrac{\rho\hbar^3}{12}\Psi_{tt} - D\left(A\Psi - \displaystyle\int_0^t g(s)A\Psi(t-s)\,\mathrm{d}s\right) \\ \quad + K(\nabla u + \Psi) = H(-\Psi) - \mu_3\Psi_t, \end{cases} \tag{1.5}$$

where $\mu_2$, $\mu_3 \ge 0$ are the damping coefficients. Substitution of $(1.5)_2$ into $(1.5)_1$ gives

$$\begin{aligned} &\rho\hbar u_{tt} + \frac{\rho\hbar^3}{12}\operatorname{div}\Psi_{tt} - D\operatorname{div}\left(A\Psi - \int_0^t g(s)A\Psi(t-s)\,\mathrm{d}s\right) \\ &\quad - \operatorname{div} H(-\Psi) + \mu_2 u_t + \mu_3\operatorname{div}\Psi_t = f(u). \end{aligned} \tag{1.6}$$

Taking the Kirchhoff limit $k \to \infty$, we have $\Psi = -\nabla u$. Under a normalization of coefficients, (1.6) becomes

$$\begin{aligned} &u_{tt} - \mu_1\Delta u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau \\ &\quad - \operatorname{div}H(\nabla u) + \mu_2 u_t - \mu_3\Delta u_t = f(u), \end{aligned}$$

where $\mu_1 = \frac{\hbar^2}{12}$. Since the qualitative properties of solutions in this paper are independent of the coefficients in the equation, we take $\mu_1 = \mu_2 = \mu_3 = 1$ for the sake of convenience. Thus, by $H(\nabla u) = |\nabla u|^{p-2}\nabla u$ and $f(u) = |u|^{q-2}u$, Eq. (1.1) is derived. The above derivation is still valid for the higher-dimensional case. As for the one-dimensional case, Eq. (1.1) can be obtained from the Timoshenko beam model.[12]

From above discussions, the terms $-\Delta u_{tt}$, $-\int_0^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau$, $-\Delta_p u$, and $|u|^{q-2}u$ in Eq. (1.1) stand for the rotational inertia, viscoelasticity, restoring force, and external force, respectively. In the absence of the restoring force $-\Delta_p u$, (1.1) reduces to a viscoelastic Kirchhoff plate equation, and various versions of such a model equation have been extensively investigated (see, e.g., Refs. 13–20 and the references therein, and some of them will be introduced later).

Muñoz Rivera *et al.*[19] estimated the decay rates for the viscoelastic Kirchhoff plate equation,

$$u_{tt} - \mu_1 \Delta u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau = 0, \quad x \in \Omega,\ t > 0,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open set with a smooth boundary. Under certain assumptions on the memory kernel $g$, they proved that the total energy function of the solution decays to zero with the same rate of decay of $g$, which means that the memory effect produces strong dissipation capable of making uniform rate of decay for the energy. In the case $\mu_1 = 0$ (namely, the rotational inertia is neglected), Muñoz Rivera and Fatori[18] considered the viscoelastic Kirchhoff plate equation with a strong damping term,

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \mu_3 \Delta u_t = 0.$$

Under some assumptions on $g$, they established the global existence, uniqueness, and exponential decay of solutions with sufficiently small initial data. Cavalcanti *et al.*[16] studied the viscoelastic Kirchhoff plate equation with a weak damping term,

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - M\left(\int_\Omega |\nabla u|^2\,\mathrm{d}x\right) u_t = 0,$$

where $M \in C^1([0,\infty))$ and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is an open set. On the basis of certain assumptions on $g$, they obtained the global existence and uniqueness of solutions and the exponential decay of the energy by density arguments.

Alabau-Boussouira *et al.*[14] investigated the following abstract equation, including the viscoelastic Kirchhoff plate equation as a concrete model:

$$u_{tt} + Au - \int_0^t g(t-\tau)Au(\tau)\,\mathrm{d}\tau = \nabla F(u), \quad t > 0, \tag{1.7}$$

under appropriate assumptions on the linear operator $A$, the memory kernel $g$, and the Gâteaux differentiable functional $F$. Their main results showed that the energy of any mild solution to (1.7), with sufficiently small initial data, decays at infinity with the same exponential or polynomial rate as the memory kernel function $g$ does. Subsequently, Alabau-Boussouira and Cannarsa[13] considered (1.7) with $F = 0$ and presented a general method, which gives energy decay rates in terms of the asymptotic behavior of $g$ at infinity. Lasiecka and Wang[17] also studied (1.7) under the assumptions different from those in Ref. 14 and derived a general result on the decay of non-negative energy. Moreover, they provided the estimates on the decay rates with a general nonlinearity. Prüss[20] studied the following model equation:

$$u_{tt} + Au - \int_0^t g(t-\tau)Au(\tau)\,\mathrm{d}\tau = f,$$

where $g$ can be singular at $t = 0$ in contrast to Ref. 14 to obtain the exponential or polynomial decay of mild solutions with the same rate of $g$ by using the frequency domain method. Furthermore, this work also extended these results to the model

$$u_{tt} + Au - \int_0^t g(t-\tau)Au(\tau)\,\mathrm{d}\tau = f(u, u_t),$$

where the initial data are required to be sufficiently small. Combining the models in Refs. 14 and 20, Cannarsa and Sforza[15] investigated the model equation

$$u_{tt} + Au - \int_0^t g(t-\tau)Au(\tau)\,\mathrm{d}\tau = \nabla F(u) + f,$$

where $g$ is weaker than that in Ref. 14. They obtained the existence and uniqueness of mild and strong solutions and derived the exponential decay of the energy with sufficient small initial data and linear external force by using the multiplier method.

Concerning viscoelastic Kirchhoff-like plate equations, Jorge Silva and Ma[21] investigated

$$u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \Delta_p u - \Delta u_t + f(u) = h(x).$$

For non-negative energy, they obtained the global well-posedness and regularity of solutions and proved the exponential decay of the energy.

The works mentioned above well established the corresponding theories of global well-posedness, regularity, and asymptotic behavior of solutions to such a viscoelastic model equation. In order to achieve these, some restrictions on the initial energy reflected by nonlinear source terms and the initial data are natural and necessary. The above restrictions on the energy positive definitely and the initial data small enough motivate the present paper to consider the cases without positive definitely energy and focus on describing the effects of the initial data on the dynamical behavior of solutions. The main purpose of the present paper is to discuss the relationship between the initial data and the qualitative dynamical properties of solutions in order to establish some sufficient conditions for the local and global existence, uniqueness, asymptotic behavior, and finite time blow-up of solutions to problems (1.1)–(1.3). Generally speaking, the source term and the initial data have a strong influence on the dynamical properties of solutions to the nonlinear evolution equations. As mentioned above, the results of Refs. 17 and 21 require the source term to satisfy certain conditions to ensure that the total energy function is always non-negative. Although the results of Refs. 14, 15, and 20 do not need the total energy function to be non-negative, the initial data are strictly restricted to the sufficiently small ones. This paper aims to provide a rigorous mathematical description for the initial data. To this end, the potential well theory (see, e.g., Refs. 22–31) is employed, which plays an essential role in the proofs of the main results. Since the total energy function associated with problems (1.1)–(1.3) is not always non-negative, we have to overcome the additional technical difficulties in energy estimates.

The rest of this paper is organized as follows. In Sec. II, we state some notations and definitions related to problems (1.1)–(1.3) and present the main results of this paper. In Sec. III, we are engaged in the Proof of Theorem 2.3, namely, the local existence and uniqueness of solutions. In Sec. IV, we finish the Proof of Theorem 2.4, namely, the global existence of solutions with subcritical and critical initial energy. In Sec. V, we complete the Proof of Theorem 2.5, namely, the asymptotic behavior of solutions with positive initial energy strictly below the depth of the potential well. In Sec. VI, we perform the Proofs of Theorems 2.6–2.8, namely, the finite time blow-up of solutions with negative initial energy, null initial energy, and positive initial energy strictly below the depth of the potential well and arbitrary positive initial energy, respectively.

## II. NOTATIONS AND MAIN RESULTS

Throughout this paper, in order to simplify the notations, we denote

$$\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}, \quad \|\cdot\| := \|\cdot\|_2, \quad (u,w) := \int_\Omega uw\,\mathrm{d}x,$$

$$(u,w)_* := (u,w) + (\nabla u, \nabla w), \quad \|u\|_*^2 := \|u\|^2 + \|\nabla u\|^2,$$

and

$$(g\circ\Delta u)(t) := \int_0^t g(t-\tau)\|\Delta u(t) - \Delta u(\tau)\|^2\,\mathrm{d}\tau.$$

Moreover, $C$ denotes a generic constant that may vary even in the same formula, and $\mathfrak{C}_i$ $(i = 1,2,3,4)$ represent the best Sobolev constants of the embeddings $H^2(\Omega)\cap H_0^1(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$, $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $H^2(\Omega)\cap H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, and $H^2(\Omega)\cap H_0^1(\Omega) \hookrightarrow H_0^1(\Omega)$, respectively.

*Definition 2.1 (weak solution). A function $u \in C([0,T]; H^2(\Omega)\cap H_0^1(\Omega))$ with $u_t \in C([0,T]; H_0^1(\Omega))$ is called a weak solution to problems (1.1)–(1.3), provided that for $u(0) = u_0$ in $H^2(\Omega)\cap H_0^1(\Omega)$ and $u_t(0) = u_1$ in $H_0^1(\Omega)$, there holds*

$$\begin{aligned}(u_t(t),w)_* &+ \int_0^t (\Delta u(\tau),\Delta w)\,d\tau - \int_0^t\int_0^s g(s-\tau)(\Delta u(\tau),\Delta w)\,d\tau ds\\ &+ \int_0^t \left(|\nabla u(\tau)|^{p-2}\nabla u(\tau),\nabla w\right)d\tau + (u(t),w)_*\\ &= (u_1,w)_* + (u_0,w)_* + \int_0^t \left(|u(\tau)|^{q-2}u(\tau),w\right)d\tau\end{aligned}$$

*for any $w \in H^2(\Omega)\cap H_0^1(\Omega)$ and $t \in (0,T)$. Here, $T$ is the maximum existence time of solutions.*

*Remark 2.2. Definition 2.1 implies that*

$$\begin{aligned}\langle u_{tt}(t),w\rangle_* = &\left(|u(t)|^{q-2}u(t),w\right) - (\Delta u(t),\Delta w) + \int_0^t g(t-\tau)(\Delta u(\tau),\Delta w)\,d\tau\\ &- \left(|\nabla u(t)|^{p-2}\nabla u(t),\nabla w\right) - (u_t(t),w)_*\end{aligned}$$

*for a.e. $t \in (0,T)$, where $\langle\cdot,\cdot\rangle_*$ denotes the duality pairing between $H^2(\Omega)\cap H_0^1(\Omega)$ and its dual space.*

We define the total energy function associated with problems (1.1)–(1.3),

$$\begin{aligned}E(t) := &\frac{1}{2}\|u_t(t)\|_*^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u(t)\|^2 + \frac{1}{2}(g\circ\Delta u)(t)\\ &+ \frac{1}{p}\|\nabla u(t)\|_p^p - \frac{1}{q}\|u(t)\|_q^q,\end{aligned}$$

the potential energy functional

$$J(u(t)) := \frac{1}{2}\left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u(t)\|^2 + \frac{1}{2}(g \circ \Delta u)(t) + \frac{1}{p}\|\nabla u(t)\|_p^p - \frac{1}{q}\|u(t)\|_q^q,$$

and the Nehari functional

$$I(u(t)) := \left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u(t)\|^2 + (g \circ \Delta u)(t) + \|\nabla u(t)\|_p^p - \|u(t)\|_q^q.$$

Thus, all nontrivial stationary solutions belong to the Nehari manifold, defined by

$$\mathcal{N} := \left\{u \in H^2(\Omega) \cap H_0^1(\Omega)\backslash\{0\} | I(u) = 0\right\}.$$

We introduce the potential well

$$\mathcal{W} := \left\{u \in H^2(\Omega) \cap H_0^1(\Omega) | J(u) < d, I(u) > 0\right\} \cup \{0\},$$

its outside set

$$\mathcal{V} := \left\{u \in H^2(\Omega) \cap H_0^1(\Omega) | J(u) < d, I(u) < 0\right\},$$

and the depth of the potential well

$$d := \inf_{u \in \mathcal{N}} J(u).$$

We also define

$$\bar{d} := \max\{d_1, d_2, d_3\}$$

and

$$\tilde{d} := \max\{d_4, d_5\},$$

where

$$d_1 := \frac{q-2}{2q}\kappa^{\frac{q}{q-2}}\mathfrak{C}_1^{-\frac{2q}{q-2}}\mathfrak{C}_2^{-\frac{2q}{q-2}}, \tag{2.1}$$

$$d_2 := \frac{q-2}{2q}\kappa\mathfrak{C}_1^{-2}\mathfrak{C}_2^{-\frac{2q}{q-p}}, \tag{2.2}$$

$$d_3 := \frac{q-p}{pq}\mathfrak{C}_2^{-\frac{pq}{q-p}}, \tag{2.3}$$

$$d_4 := d_1 + \frac{q-p}{pq}\kappa^{\frac{p}{q-2}}\mathfrak{C}_1^{-\frac{2p}{q-2}}\mathfrak{C}_2^{-\frac{pq}{q-2}},$$

and

$$d_5 := d_2 + d_3.$$

The relationships between $d$ and $\bar{d}$ and between $d$ and $\tilde{d}$ will be discussed later. Moreover, we introduce

$$H_3(\Omega) := \left\{u \in H^3(\Omega) \cap H_0^1(\Omega) | \Delta u \in H_0^1(\Omega)\right\},$$

which is a Hilbert space equipped with an inner product and norm (see Ref. 21),

$$(u, w)_{H_3(\Omega)} := (\nabla\Delta u, \nabla\Delta w), \quad \|u\|_{H_3(\Omega)} := \|\nabla\Delta u\|.$$

The main results of this paper are stated as follows.

**Theorem 2.3** (**local existence and uniqueness**). *Let* $(A_1)$–$(A_3)$ *be fulfilled. Assume that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $u_1 \in H_0^1(\Omega)$. *Then, there exists a time* $T > 0$ *such that problems (1.1)–(1.3) admit a unique weak solution u, which satisfies*

$$E(t) + \int_0^t \left(\|u_\tau(\tau)\|_*^2 - \frac{1}{2}(g' \circ \Delta u)(\tau) + \frac{1}{2}g(\tau)\|\Delta u(\tau)\|^2\right)\mathrm{d}\tau = E(0) \tag{2.4}$$

*for all* $t \in [0, T)$. *Moreover, if* $T = \infty$, *the solution exists globally in time. If* $T < \infty$, *the solution blows up in finite time, i.e.,*

$$\lim_{t \to T}\|u(t)\|_r = \infty \tag{2.5}$$

*for all* $r \geq 1$ *such that* $r > \frac{N(q-p)}{p}$.

**Theorem 2.4** (**global existence for subcritical and critical initial energy**). *Let* $(A_1)$–$(A_3)$ *be fulfilled. Assume that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$. *Then, we have the following:*

*(i)* *If* $E(0) < d$ *and* $I(u_0) > 0$ *or* $\|\Delta u_0\| = 0$, *then the solution* $u$ *to problems (1.1)–(1.3) is global, and* $u(t) \in \mathcal{W}$ *for all* $t \in [0, \infty)$.
*(ii)* *If* $E(0) = d$ *and* $I(u_0) \ge 0$, *then problems (1.1)–(1.3) admit a unique global solution* $u(t) \in \overline{\mathcal{W}}$ *for all* $t \in [0, \infty)$.

The following theorem shows the asymptotic behavior of the solution $u$ to problems (1.1)–(1.3) with positive initial energy strictly below the depth of the potential well, i.e., $E(0) < \overline{d}$. Later, we shall illustrate $\overline{d} < d$ in Proposition 4.2.

**Theorem 2.5** (**asymptotic behavior for the positive initial energy strictly below the depth of the potential well**). *In addition to the assumptions of Theorem 2.4, suppose that there exists a constant* $\rho > 0$ *such that* $g'(t) \le -\rho g(t)$ *for all* $t \in [0, \infty)$. *If* $E(0) < \overline{d}$ *and* $I(u_0) > 0$ *or* $\|\Delta u_0\| = 0$, *then the solution* $u$ *to problems (1.1)–(1.3) possesses the following property:*

$$\|\Delta u(t)\|^2 + \|u_t(t)\|_*^2 \le \alpha e^{-\beta t} \tag{2.6}$$

*for all* $t \in [0, \infty)$ *and some constants* $\alpha, \beta > 0$.

**Theorem 2.6** (**blow-up for non-positive initial energy**). *Let* $(A_1)$–$(A_3)$ *be fulfilled, and*

$$\int_0^\infty g(\tau)\,\mathrm{d}\tau \le \frac{q(q-2)}{q(q-2)+1}. \tag{2.7}$$

*Assume that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, *and either one of the following cases occurs:*

(i) $E(0) < 0$;
(ii) $E(0) = 0$ *and* $(u_0, u_1)_* > 0$.
*Then, the solution to problems (1.1)–(1.3) blows up in finite time.*

The following theorem shows the finite time blow-up of the solution $u$ to problems (1.1)–(1.3) with positive initial energy strictly below the depth of the potential well, i.e., $0 < E(0) < \theta\tilde{d}$. Here, $\tilde{d} < d$ will be discussed in Proposition 4.2. In addition, as $\theta < \frac{2(q-p)}{p(q-2)} (\le 1)$ will be required in the following theorem, we also need to figure out that $\theta\tilde{d} < d$.

**Theorem 2.7** (**blow-up for the positive initial energy strictly below the depth of the potential well**). *Let* $(A_1)$–$(A_3)$ *be fulfilled, and*

$$\int_0^\infty g(\tau)\,d\tau \le \frac{(q-2)(1-\theta)(q-\theta(q-2))}{(q-2)(1-\theta)(q-\theta(q-2))+1}, \tag{2.8}$$

*where* $0 < \theta < \frac{2(q-p)}{p(q-2)}$. *Assume that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, $0 < E(0) < \theta\tilde{d}$, *and* $I(u_0) < 0$. *Then, the solution to problems (1.1)–(1.3) blows up in finite time.*

**Theorem 2.8** (**blow-up for arbitrary positive initial energy**). *Let* $(A_1)$–$(A_3)$ *be fulfilled,*

$$\int_0^t \varsigma(s) \int_0^s e^{\frac{s-\tau}{2}} g(s-\tau)\varsigma(\tau)\,d\tau ds \ge 0, \quad \varsigma \in C^1([0,\infty)),\ t > 0, \tag{2.9}$$

*and*

$$\int_0^\infty g(\tau)\,d\tau < \min\left\{\frac{q-2}{q}, \frac{q(q-2)}{q(q-2)+1}\right\}. \tag{2.10}$$

*Assume that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, $E(0) > 0$, $(u_0, u_1)_* > 0$, $\psi(u_0) < 0$, *and*

$$\|u_0\|_*^2 \ge \frac{2q(\mathfrak{C}_3^2 + \mathfrak{C}_4^2)}{\min\{\kappa q - 2, \varrho\}} E(0), \tag{2.11}$$

*where*

$$\psi(u(t)) := \|\Delta u(t)\|^2 + \|\nabla u(t)\|_p^p - \|u(t)\|_q^q \tag{2.12}$$

*and*

$$\varrho := q - 2 - \left(q - 2 + \frac{1}{q}\right)(1-\kappa).$$

*Then, the solution to problems (1.1)–(1.3) blows up in finite time.*

*Remark 2.9 (remark on the memory kernel in the blow-up theorems). In fact, (2.7), (2.8), and (2.10) reveal the essential relationship between the viscoelasticity of thin plates, the restoring force, and the external force, which leads to the finite time blow-up of solution. According to Ref. 19, (1.4) is the premise that the viscoelastic term produces strong dissipation effect. Consequently, once the memory kernel satisfies either one of (2.7), (2.8), and (2.10), strong dissipation effect of the viscoelastic term brings obstacles to the finite time blow-up. Thus, by means of the growth exponents of the restoring force and the external force, (2.7), (2.8), and (2.10) show that how strong is the dissipation effect at most to guarantee the finite time blow-up.*

*Remark 2.10 (unsolved problems). Our results on the global existence and asymptotic behavior of solutions to problems (1.1)–(1.3) are restricted to the cases $E(0) \le d$ and $E(0) < \overline{d}$, respectively. For the case of higher initial energy, since the invariance of $\mathcal{W}$ is inaccessible, whether the solution still exists globally and decays is now unsolved. Moreover, because the initial data and the initial energy are not accurate enough to make the expected energy estimates difficult to obtain, the estimates on the blow-up time of solutions are also unsolved.*

## III. LOCAL EXISTENCE AND UNIQUENESS (PROOF OF THEOREM 2.3)

In order to prove the local existence and uniqueness of solutions to problems (1.1)–(1.3), we shall first handle the existence, uniqueness, and regularity of solutions to the initial boundary problem for the corresponding linear equation by Galerkin approximations and study the continuity of solutions in time by density arguments. Thus, by the contraction mapping principle, we prove the local existence and uniqueness of solutions to the original nonlinear problems (1.1)–(1.3) in the sense of Definition 2.1. To deal with the nonlinearity in the problem, we need the following lemma.

*Lemma 3.1. For any $u(x,t)$ and $v(x,t)$ with $(x,t) \in \Omega \times [0,T)$, if $u \neq v$ and $p \ge 2$, then*

$$\left||u|^{p-2}u - |v|^{p-2}v\right| \le (p-1)(|u| + |v|)^{p-2}|u-v|.$$

*Proof.* Set $f(u) := |u|^{p-2}u$ and $\tilde{u} := u - v$. Then, by the property of the Gâteaux derivative, we see that

$$f(u) - f(v) = f(v+\tilde{u}) - f(v) = \int_0^1 \mathrm{d}f(v + s\tilde{u}; \tilde{u})\,\mathrm{d}s$$

for all $s \in (0,1)$. From

$$\mathrm{d}f(v+s\tilde{u};\tilde{u}) = \lim_{\tau \to 0} \frac{f(v + s\tilde{u} + \tau\tilde{u}) - f(v + s\tilde{u})}{\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau} f(v + s\tilde{u} + \tau\tilde{u})\Big|_{\tau=0},$$

we further deduce that

$$\begin{aligned} f(u) - f(v) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\left(|v + s\tilde{u} + \tau\tilde{u}|^{p-2}(v + s\tilde{u} + \tau\tilde{u})\right)\Big|_{\tau=0}\mathrm{d}s \\ &= \int_0^1 (p-1)|v + s\tilde{u} + \tau\tilde{u}|^{p-2}\tilde{u}\Big|_{\tau=0}\mathrm{d}s \\ &= \int_0^1 (p-1)|su + (1-s)v|^{p-2}(u-v)\,\mathrm{d}s \\ &\le (p-1)(|u| + |v|)^{p-2}|u-v|. \end{aligned}$$

□

In order to handle the nonlinear model in the present paper, we first consider the initial boundary value problem for the linear version of (1.1),

$$\begin{aligned} &v_{tt} - \Delta v_{tt} + \Delta^2 v - \int_0^t g(t-\tau)\Delta^2 v(\tau)\,\mathrm{d}\tau \\ &\quad - \Delta_p v + v_t - \Delta v_t = |u|^{q-2}u, \quad x \in \Omega,\ t > 0, \end{aligned} \tag{3.1}$$

$$v(x,0) = u_0(x),\ v_t(x,0) = u_1(x), \quad x \in \Omega, \tag{3.2}$$

$$v(x,t) = \Delta v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0. \tag{3.3}$$

Later, in Lemma 3.3, we shall show the existence of solutions to the linear problem, i.e., problems (3.1)–(3.3). In the Proof of Lemma 3.3, the estimates only support $v \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ with $v_t \in L^\infty(0,T;H_0^1(\Omega))$, which is not enough to give the solution with higher regularity demonstrated by (3.6). Hence, we need Lemma 3.2 to ensure that the expected (3.6) can be achieved, provided $v_t(t) \in H_0^1(\Omega)$.

*Lemma 3.2. Let* $(A_1)$–$(A_3)$ *be fulfilled. Suppose that* $v \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ *with* $v_t \in L^2(0,T;H_0^1(\Omega))$ *is a solution to problems (3.1)–(3.3) with* $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, *and* $u \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$. *Then, there holds*

$$\begin{aligned}&\left(v_{tt}(t) - \Delta v_{tt}(t) + \Delta^2 v(t) - \int_0^t g(t-\tau)\Delta^2 v(\tau)\,d\tau, v_t(t)\right)\\ &= \frac{1}{2}\frac{d}{dt}\left(\|v_t(t)\|_*^2 + \left(1 - \int_0^t g(\tau)\,d\tau\right)\|\Delta v(t)\|^2 + (g\circ\Delta v)(t)\right)\\ &\quad - \frac{1}{2}(g'\circ\Delta v)(t) + \frac{1}{2}g(t)\|\Delta v(t)\|^2.\end{aligned} \tag{3.4}$$

*Proof.* Extend $v$ to be zero outside $(0,T)$. Let $\tilde{v} := \zeta v$, where the truncation function

$$\zeta(t) := \begin{cases} 0 & \text{if } t \in \mathbb{R}\backslash(0,T),\\ \frac{1}{\delta}t & \text{if } t\in(0,\delta),\\ 1 & \text{if } t\in[\delta, T-\delta),\\ -\frac{1}{\delta}t + \frac{T}{\delta} & \text{if } t \in [T-\delta, T).\end{cases} \tag{3.5}$$

Set the mollification of $v$ to be

$$\tilde{v}_\varepsilon := \eta_\varepsilon * \tilde{v},$$

where the mollifier $\eta_\varepsilon(t) := \frac{1}{\varepsilon}\eta\left(\frac{t}{\varepsilon}\right)$, $\varepsilon > 0$, and $\eta(t)$ is a non-negative even $C^\infty$-function on the real line with compact support and $\int_{\mathbb{R}}\eta(t)\,\mathrm{d}t = 1$. According to the regularization theory, we have $\tilde{v}_\varepsilon \in C^\infty(\mathbb{R};H^2(\Omega)\cap H_0^1(\Omega))$ [but $\tilde{v}_\varepsilon \in C^2(\mathbb{R};H^2(\Omega)\cap H_0^1(\Omega))$ is enough here], and as $\varepsilon \to 0$,

$$\tilde{v}_\varepsilon \to \tilde{v} \text{ in } L^2(\mathbb{R};H^2(\Omega)\cap H_0^1(\Omega)),$$

$$\tilde{v}_{\varepsilon t} \to \tilde{v}_t \text{ in } L^2(\mathbb{R};H_0^1(\Omega)).$$

Since

$$\begin{aligned}&\left(\tilde{v}_{\varepsilon tt}(t) - \Delta\tilde{v}_{\varepsilon tt}(t) + \Delta^2\tilde{v}_\varepsilon(t) - \int_0^t g(t-\tau)\Delta^2\tilde{v}_\varepsilon(\tau)\,\mathrm{d}\tau, \tilde{v}_{\varepsilon t}(t)\right)\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{v}_{\varepsilon t}(t)\|_*^2 + \|\Delta\tilde{v}_\varepsilon(t)\|^2\right) - \int_0^t g(t-\tau)(\Delta\tilde{v}_\varepsilon(\tau), \Delta\tilde{v}_{\varepsilon t}(t))\,\mathrm{d}\tau\end{aligned}$$

and

$$\begin{aligned}&\int_0^t g(t-\tau)(\Delta\tilde{v}_\varepsilon(\tau),\Delta\tilde{v}_{\varepsilon t}(t))\,\mathrm{d}\tau\\ &= \int_0^t g(t-\tau)(\Delta\tilde{v}_\varepsilon(\tau) - \Delta\tilde{v}_\varepsilon(t), \Delta\tilde{v}_{\varepsilon t}(t))\,\mathrm{d}\tau + \int_0^t g(t-\tau)(\Delta\tilde{v}_\varepsilon(t),\Delta\tilde{v}_{\varepsilon t}(t))\,\mathrm{d}\tau\\ &= -\frac{1}{2}\int_0^t g(t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\tilde{v}_\varepsilon(\tau) - \Delta\tilde{v}_\varepsilon(t)\|^2\,\mathrm{d}\tau + \frac{1}{2}\int_0^t g(t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\tilde{v}_\varepsilon(t)\|^2\,\mathrm{d}\tau\\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left((g\circ\Delta\tilde{v}_\varepsilon)(t) - \int_0^t g(\tau)\,\mathrm{d}\tau\|\Delta\tilde{v}_\varepsilon(t)\|^2\right) + \frac{1}{2}(g'\circ\Delta\tilde{v}_\varepsilon)(t) - \frac{1}{2}g(t)\|\Delta\tilde{v}_\varepsilon(t)\|^2,\end{aligned}$$

we get

$$\begin{aligned}&\left(\tilde{v}_{\varepsilon tt}(t) - \Delta\tilde{v}_{\varepsilon tt}(t) + \Delta^2\tilde{v}_\varepsilon(t) - \int_0^t g(t-\tau)\Delta^2\tilde{v}_\varepsilon(\tau)\,\mathrm{d}\tau, \tilde{v}_{\varepsilon t}(t)\right)\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{v}_{\varepsilon t}(t)\|_*^2 + \left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{v}_\varepsilon(t)\|^2 + (g\circ\Delta\tilde{v}_\varepsilon)(t)\right)\\ &\quad - \frac{1}{2}(g'\circ\Delta\tilde{v}_\varepsilon)(t) + \frac{1}{2}g(t)\|\Delta\tilde{v}_\varepsilon(t)\|^2.\end{aligned}$$

Taking $\varepsilon \to 0$ first, the above formula still holds for $\tilde{v}$. Finally, by taking $\delta \to 0$ in (3.5) and restriction to $(0,T)$, we obtain (3.4). □

*Lemma 3.3 (existence and uniqueness of solutions to linear equations). Let* $(A_1)$–$(A_3)$ *be fulfilled. Assume that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, *and* $u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$. *Then, the linear problems (3.1)–(3.3) admit a unique solution* $v \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ *with* $v_t \in L^\infty(0,T; H_0^1(\Omega))$, *which satisfies*

$$\begin{aligned}
&\frac{1}{2}\|v_t(t)\|_*^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)\,d\tau\right)\|\Delta v(t)\|^2 + \frac{1}{2}(g \circ \Delta v)(t) + \frac{1}{p}\|\nabla v(t)\|_p^p \\
&\quad + \int_0^t \left(\|v_\tau(\tau)\|_*^2 - \frac{1}{2}(g' \circ \Delta v)(\tau) + \frac{1}{2}g(\tau)\|\Delta v(\tau)\|^2\right) d\tau \\
&= \frac{1}{2}\|u_1\|_*^2 + \frac{1}{2}\|\Delta u_0\|^2 + \frac{1}{p}\|\nabla u_0\|_p^p + \int_0^t \left(|u(\tau)|^{q-2}u(\tau), v_\tau(\tau)\right) d\tau
\end{aligned} \tag{3.6}$$

*for all* $t \in [0,T)$.

*Proof.* According to the spectral theory of a compact operator (see Ref. 32, Theorem 6.3) and the Hilbert–Schmidt theory (see Ref. 32, Theorem 6.6), we can choose $\{\omega_j\}_{j=1}^\infty$ given by eigenfunctions of $\Delta^2$ with a simply supported boundary condition as an orthogonal basis of $H^2(\Omega) \cap H_0^1(\Omega)$, which is also an orthonormal basis of $L^2(\Omega)$ with the corresponding eigenvalues $\{\lambda_j\}$. Denote $W_n := \{\omega_1, \omega_2, \ldots, \omega_n\}$. Set

$$u_{0n} := \sum_{j=1}^n (u_0, \omega_j)\omega_j$$

and

$$u_{1n} := \sum_{j=1}^n (u_1, \omega_j)\omega_j$$

such that

$$u_{0n} \to u_0 \quad \text{in} \quad H^2(\Omega) \cap H_0^1(\Omega) \tag{3.7}$$

and

$$u_{1n} \to u_1 \quad \text{in} \quad H_0^1(\Omega) \tag{3.8}$$

as $n \to \infty$. For all $n \geq 1$, we seek $n$ functions $\xi_{1n}, \xi_{2n}, \ldots, \xi_{nn} \in C^2[0,T)$ to construct the approximate solutions to problems (3.1)–(3.3),

$$v_n(t) := \sum_{j=1}^n \xi_{jn}(t)\omega_j, \quad n = 1, 2, \ldots, \tag{3.9}$$

which satisfy

$$\begin{aligned}
&\Big(v_{ntt}(t) - \Delta v_{ntt}(t) + \Delta^2 v_n(t) - \int_0^t g(t-\tau)\Delta^2 v_n(\tau)\,\mathrm{d}\tau - \Delta_p v_n(t) \\
&\quad + v_{nt}(t) - \Delta v_{nt}(t), w\Big) = \left(|u(t)|^{q-2}u(t), w\right), \quad t > 0,
\end{aligned} \tag{3.10}$$

$$v_n(0) = u_{0n}, \quad v_{nt}(0) = u_{1n}, \tag{3.11}$$

for any $w \in W_n$. Let $\xi_n(t) := (\xi_{1n}(t), \xi_{2n}(t), \ldots, \xi_{nn}(t))^T$. Then, by taking $w = \omega_i$ $(i = 1, 2, \ldots, n)$ in (3.10), the vector function $\xi_n$ solves

$$\Lambda_n \xi_n''(t) + \Lambda_n \xi_n'(t) + \mathcal{L}_n(t, \xi_n(t)) = \mathcal{F}_n(t), \quad t > 0, \tag{3.12}$$

$$\xi_n(0) = ((u_0, \omega_1), (u_0, \omega_2), \ldots, (u_0, \omega_n))^T, \tag{3.13}$$

$$\xi_n'(0) = ((u_1, \omega_1), (u_1, \omega_2), \ldots, (u_1, \omega_n))^T, \tag{3.14}$$

where $\Lambda_n := \mathrm{diag}\left(1 + \lambda_1^{\frac{1}{2}}, 1 + \lambda_2^{\frac{1}{2}}, \ldots, 1 + \lambda_n^{\frac{1}{2}}\right)$, $\mathcal{L}_n : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ is the map defined by

$$\mathcal{L}_n(t, \xi_n(t)) := (\mathcal{L}_{1n}(t, \xi_n(t)), \mathcal{L}_{2n}(t, \xi_n(t)), \ldots, \mathcal{L}_{nn}(t, \xi_n(t)))^T,$$

$$\begin{aligned}
\mathcal{L}_{in}(t, \xi_n(t)) :=& \left(\sum_{j=1}^n \xi_{jn}(t)\Delta^2\omega_j, \omega_i\right) - \int_0^t g(t-\tau)\left(\sum_{j=1}^n \xi_{jn}(\tau)\Delta^2\omega_j, \omega_i\right)\mathrm{d}\tau \\
&+ \left(\left|\sum_{j=1}^n \xi_{jn}(t)\nabla\omega_j\right|^{p-2} \sum_{j=1}^n \xi_{jn}(t)\nabla\omega_j, \nabla\omega_i\right), \quad i = 1, 2, \ldots, n,
\end{aligned}$$

and $\mathcal{F}_n : [0, T) \to \mathbb{R}^n$ is defined by

$$\mathcal{F}_n(t) := \Big(\big(|u(t)|^{q-2}u(t), \omega_1\big), \big(|u(t)|^{q-2}u(t), \omega_2\big), \dots, \big(|u(t)|^{q-2}u(t), \omega_n\big)\Big)^T.$$

By the standard theory for ODEs, the Cauchy problem (3.12)–(3.14) admits a solution $\xi_n \in C^2[0, T_n)$ with $T_n \le T$. In turn, this gives a solution $u_n(t)$ defined by (3.9) and satisfying (3.10) and (3.11).

Taking $w = v_{nt}(t)$ in (3.10), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v_{nt}(t)\|_*^2 + \|\Delta v_n(t)\|^2\big) - \int_0^t g(t-\tau)(\Delta v_n(\tau), \Delta v_{nt}(t))\,\mathrm{d}\tau \\ &\quad + \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v_n(t)\|_p^p + \|v_{nt}(t)\|_*^2 \\ &\quad = \big(|u(t)|^{q-2}u(t), v_{nt}\big).\end{aligned} \tag{3.15}$$

Note that

$$\begin{aligned}&\int_0^t g(t-\tau)(\Delta v_n(\tau), \Delta v_{nt}(t))\,\mathrm{d}\tau \\ &\quad = \int_0^t g(t-\tau)(\Delta v_n(\tau) - \Delta v_n(t), \Delta v_{nt}(t))\,\mathrm{d}\tau + \int_0^t g(t-\tau)(\Delta v_n(t), \Delta v_{nt}(t))\,\mathrm{d}\tau \\ &\quad = -\frac{1}{2}\int_0^t g(t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta v_n(\tau) - \Delta v_n(t)\|^2\,\mathrm{d}\tau + \frac{1}{2}\int_0^t g(t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta v_n(t)\|^2\,\mathrm{d}\tau \\ &\quad = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big((g\circ\Delta v_n)(t) - \int_0^t g(\tau)\,\mathrm{d}\tau\|\Delta v_n(t)\|^2\Big) \\ &\qquad + \frac{1}{2}\big(g'\circ\Delta v_n\big)(t) - \frac{1}{2}g(t)\|\Delta v_n(t)\|^2.\end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15) and integrating with respect to $t$, we deduce that

$$\begin{aligned}&\frac{1}{2}\|v_{nt}(t)\|_*^2 + \frac{1}{2}\Big(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\Big)\|\Delta v_n(t)\|^2 + \frac{1}{2}(g\circ\Delta v_n)(t) + \frac{1}{p}\|\nabla v_n(t)\|_p^p \\ &\quad + \int_0^t\Big(\|v_{n\tau}(\tau)\|_*^2 - \frac{1}{2}\big(g'\circ\Delta v_n\big)(\tau) + \frac{1}{2}g(\tau)\|\Delta v_n(\tau)\|^2\Big)\,\mathrm{d}\tau \\ &\quad = \frac{1}{2}\|v_{nt}(0)\|_*^2 + \frac{1}{2}\|\Delta v_n(0)\|^2 + \frac{1}{p}\|\nabla v_n(0)\|_p^p + \int_0^t\big(|u(\tau)|^{q-2}u(\tau), v_{n\tau}(\tau)\big)\,\mathrm{d}\tau\end{aligned} \tag{3.17}$$

for all $t \in [0, T)$. By (1.4) in $(\mathrm{A}_1)$, we have

$$1 - \int_0^t g(\tau)\,\mathrm{d}\tau \ge \kappa > 0. \tag{3.18}$$

By the assumption $g(t) \ge 0$ in $(\mathrm{A}_1)$, we discover

$$(g\circ\Delta u)(t) \ge 0 \tag{3.19}$$

for any $u \in L^2\big(0, T; H^2(\Omega)\cap H_0^1(\Omega)\big)$. By the assumptions $g'(t) \le 0$ and $g(t) \ge 0$ in $(\mathrm{A}_1)$, we have

$$-\big(g'\circ\Delta u\big)(t) + g(t)\|\Delta u(t)\|^2 \ge 0 \tag{3.20}$$

for any $u \in L^2\big(0, T; H^2(\Omega)\cap H_0^1(\Omega)\big)$. From $(\mathrm{A}_2)$ and $(\mathrm{A}_3)$ and the inequalities of Hölder, Sobolev, and Cauchy, it follows that

$$\begin{aligned}\int_0^t\big(|u(\tau)|^{q-2}u(\tau), v_{n\tau}(\tau)\big)\,\mathrm{d}\tau &\le \int_0^t \|u(\tau)\|_{2q-2}^{q-1}\|v_{n\tau}(\tau)\|\,\mathrm{d}\tau \\ &\le C\sup_{t\in[0,T)}\|\Delta u(t)\|^{2q-2} + \frac{1}{2}\int_0^t\|v_{n\tau}(\tau)\|^2\,\mathrm{d}\tau.\end{aligned} \tag{3.21}$$

Hence, by virtue of (3.17)–(3.21), (3.7), and (3.8), we get

$$\frac{1}{2}\|v_{nt}(t)\|_*^2 + \frac{\kappa}{2}\|\Delta v_n(t)\|^2 + \frac{1}{p}\|\nabla v_n(t)\|_p^p \le C \tag{3.22}$$

for all $t \in [0, T)$, where $C$ is independent of $n$.

Therefore, there exist a subsequence of $\{v_n\}$ (always relabeled as the same, and we shall not repeat) and a function $v$ such that as $n\to\infty$,

$$v_n \rightharpoonup v \quad \text{weakly star in } L^\infty\big(0,T;H^2(\Omega)\cap H_0^1(\Omega)\big)$$

and

$$v_{nt} \rightharpoonup v_t \quad \text{weakly star in } L^\infty\big(0,T;H_0^1(\Omega)\big).$$

According to the Aubin–Lions lemma, we have

$$v_n \to v \ \text{ in } L^2\big(0,T;H_0^1(\Omega)\big). \tag{3.23}$$

We claim that for all $t\in[0,T)$,

$$\int_0^t \big(|\nabla v_n(\tau)|^{p-2}\nabla v_n(\tau),\nabla w\big)\,\mathrm{d}\tau \to \int_0^t \big(|\nabla v(\tau)|^{p-2}\nabla v(\tau),\nabla w\big)\,\mathrm{d}\tau$$

as $n\to\infty$. Indeed, by Lemma 3.1 and the inequalities of Hölder and Minkowski, we get

$$\begin{aligned}
&\big|\big(|\nabla v_n(t)|^{p-2}\nabla v_n(t) - |\nabla v(t)|^{p-2}\nabla v(t),\nabla w\big)\big|\\
&\quad\le C\int_\Omega \big(|\nabla v_n(t)|^{p-2} + |\nabla v(t)|^{p-2}\big)|\nabla v_n(t)-\nabla v(t)||\nabla w|\,\mathrm{d}x\\
&\quad\le C\Big(\|\nabla v_n(t)\|_{2p-2}^{p-2} + \|\nabla v(t)\|_{2p-2}^{p-2}\Big)\|\nabla w\|_{2p-2}\|\nabla v_n(t)-\nabla v(t)\|.
\end{aligned}$$

We further deduce from $(\mathrm{A}_2)$ and the Sobolev inequality that

$$\begin{aligned}
&\big|\big(|\nabla v_n(t)|^{p-2}\nabla v_n(t) - |\nabla v(t)|^{p-2}\nabla v(t),\nabla w\big)\big|\\
&\quad\le C\big(\|\Delta v_n(t)\|^{p-2} + \|\Delta v(t)\|^{p-2}\big)\|\Delta w\|\|\nabla v_n(t)-\nabla v(t)\|.
\end{aligned}$$

Accordingly, the assertion follows from (3.23).

Integrating (3.10) with respect to $t$, we get

$$\begin{aligned}
&(v_{nt}(t),w)_* + \int_0^t (\Delta v_n(\tau),\Delta w)\,\mathrm{d}\tau - \int_0^t\int_0^s g(s-\tau)(\Delta v_n(\tau),\Delta w)\,\mathrm{d}\tau\mathrm{d}s\\
&\quad + \int_0^t \big(|\nabla v_n(\tau)|^{p-2}\nabla v_n(\tau),\nabla w\big)\,\mathrm{d}\tau + (v_n(t),w)_*\\
&\quad = (u_{1n},w)_* + (u_{0n},w)_* + \int_0^t \big(|u(\tau)|^{q-2}u(\tau),w\big)\,\mathrm{d}\tau.
\end{aligned}$$

Taking $n\to\infty$, we obtain

$$\begin{aligned}
&(v_t(t),w)_* + \int_0^t (\Delta v(\tau),\Delta w)\,\mathrm{d}\tau - \int_0^t\int_0^s g(s-\tau)(\Delta v(\tau),\Delta w)\,\mathrm{d}\tau\mathrm{d}s\\
&\quad + \int_0^t \big(|\nabla v(\tau)|^{p-2}\nabla v(\tau),\nabla w\big)\,\mathrm{d}\tau + (v(t),w)_*\\
&\quad = (u_1,w)_* + (u_0,w)_* + \int_0^t \big(|u(\tau)|^{q-2}u(\tau),w\big)\,\mathrm{d}\tau.
\end{aligned}$$

By virtue of (3.7) and (3.8), we have $v(0)=u_0$ in $H^2(\Omega)\cap H_0^1(\Omega)$ and $v_t(0)=u_1$ in $H_0^1(\Omega)$. Therefore, $v$ is a solution to problems (3.1)–(3.3).

In addition, thanks to Lemma 3.2 and

$$\begin{aligned}
&\Big(v_{tt}(t) - \Delta v_{tt}(t) + \Delta^2 v(t) - \int_0^t g(t-\tau)\Delta^2 v(\tau)\,\mathrm{d}\tau\\
&\quad - \Delta_p v(t) + v_t(t) - \Delta v_t(t), v_t(t)\Big) = \big(|u(t)|^{q-2}u(t),v_t(t)\big),
\end{aligned}$$

we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac12\|v_t(t)\|_*^2 + \frac12\Big(1-\int_0^t g(\tau)\,\mathrm{d}\tau\Big)\|\Delta v(t)\|^2 + \frac12(g\circ\Delta v)(t) + \frac1p\|\nabla v(t)\|_p^p\right)\\
&\quad + \|v_t(t)\|_*^2 - \frac12(g'\circ\Delta v)(t) + \frac12 g(t)\|\Delta v(t)\|^2\\
&= \big(|u(t)|^{q-2}u(t),v_t(t)\big).
\end{aligned}$$

Integrating this equality with respect to $t$, we obtain (3.6).

Next, we prove the uniqueness of solutions. Suppose that $v$ and $\bar{v}$ are two solutions to problems (3.1)–(3.3). Set $\tilde{v} := \bar{v} - v$. Then,

$$\Big(\tilde{v}_{tt}(t) - \Delta\tilde{v}_{tt}(t) + \Delta^2\tilde{v}(t) - \int_0^t g(t-\tau)\Delta^2\tilde{v}(\tau)\,\mathrm{d}\tau - \Delta_p\bar{v}(t) + \Delta_p v(t) + \tilde{v}_t(t) - \Delta\tilde{v}_t(t), w\Big) = 0.$$

By the arguments similar to Lemma 3.2, we have

$$\begin{aligned}\Big(\tilde{v}_{tt}(t) - \Delta\tilde{v}_{tt}(t) + \Delta^2\tilde{v}(t) - \int_0^t g(t-\tau)\Delta^2\tilde{v}(\tau)\,\mathrm{d}\tau, \tilde{v}_t(t)\Big)\\ = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\tilde{v}_t(t)\|_*^2 + \Big(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\Big)\|\Delta\tilde{v}(t)\|^2 + (g\circ\Delta\tilde{v})(t)\Big)\\ - \frac{1}{2}(g'\circ\Delta\tilde{v})(t) + \frac{1}{2}g(t)\|\Delta\tilde{v}(t)\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|\tilde{v}_t(t)\|_*^2 + \frac{1}{2}\Big(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\Big)\|\Delta\tilde{v}(t)\|^2 + \frac{1}{2}(g\circ\Delta\tilde{v})(t)\Big)\\ + \big(|\nabla\bar{v}(t)|^{p-2}\nabla\bar{v}(t) - |\nabla v(t)|^{p-2}\nabla v(t), \nabla\tilde{v}_t(t)\big)\\ + \|\tilde{v}_t(t)\|_*^2 - \frac{1}{2}(g'\circ\Delta\tilde{v})(t) + \frac{1}{2}g(t)\|\Delta\tilde{v}(t)\|^2 = 0.\end{aligned} \tag{3.24}$$

Taking into account (3.20), we deduce from (3.24) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|\tilde{v}_t(t)\|_*^2 + \frac{1}{2}\Big(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\Big)\|\Delta\tilde{v}(t)\|^2 + \frac{1}{2}(g\circ\Delta\tilde{v})(t)\Big)\\ \le \big|\big(|\nabla\bar{v}(t)|^{p-2}\nabla\bar{v}(t) - |\nabla v(t)|^{p-2}\nabla v(t), \nabla\tilde{v}_t(t)\big)\big|.\end{aligned}$$

From Lemma 3.1, $(\mathrm{A}_2)$, and the inequalities of Hölder, Minkowski, Sobolev, and Cauchy, it follows that

$$\begin{aligned}&\big|\big(|\nabla\bar{v}(t)|^{p-2}\nabla\bar{v}(t) - |\nabla v(t)|^{p-2}\nabla v(t), \nabla\tilde{v}_t(t)\big)\big|\\ &\le C\Big(\|\nabla v(t)\|_{2p-2}^{p-2} + \|\nabla\bar{v}(t)\|_{2p-2}^{p-2}\Big)\|\nabla\tilde{v}(t)\|_{2p-2}\|\nabla\tilde{v}_t(t)\|\\ &\le C\|\Delta\tilde{v}(t)\|^2 + \frac{1}{2}\|\nabla\tilde{v}_t(t)\|^2.\end{aligned}$$

We can deduce from (1.4) in $(\mathrm{A}_1)$ that

$$1 \ge 1 - \int_0^t g(\tau)\,\mathrm{d}\tau > 0. \tag{3.25}$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|\tilde{v}_t(t)\|_*^2 + \frac{1}{2}\Big(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\Big)\|\Delta\tilde{v}(t)\|^2 + \frac{1}{2}(g\circ\Delta\tilde{v})(t)\Big)\\ \le C\Big(\frac{1}{2}\|\tilde{v}_t(t)\|_*^2 + \frac{1}{2}\Big(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\Big)\|\Delta\tilde{v}(t)\|^2 + \frac{1}{2}(g\circ\Delta\tilde{v})(t)\Big),\end{aligned}$$

which combined with Gronwall's inequality, $\tilde{v}(0) = 0$, $\tilde{v}_t(0) = 0$, (3.18), and (3.19) tells us that

$$\frac{1}{2}\|\tilde{v}_t(t)\|_*^2 + \frac{\kappa}{2}\|\Delta\tilde{v}(t)\|^2 \le 0 \tag{3.26}$$

for all $t \in [0,T]$, which implies that $\tilde{v} \equiv 0$, i.e., $v \equiv \bar{v}$ for all $t \in [0,T)$. The Proof of Lemma 3.3 is completed. □

In order to show that under the conditions of Lemma 3.3, problems (3.1)–(3.3) can have a unique solution $v \in C([0,T); H^2(\Omega)\cap H_0^1(\Omega))$ with $v_t \in C([0,T); H_0^1(\Omega))$, we discuss the regularity of solutions to problems (3.1)–(3.3) with $u_0 \in H_3(\Omega)$ and $u_1 \in H^2(\Omega)\cap H_0^1(\Omega)$ in the following lemma.

*Lemma 3.4 (regularity of solutions to linear equations). Let $(\mathrm{A}_1)$–$(\mathrm{A}_3)$ be fulfilled. Assume that $u_0 \in H_3(\Omega)$, $u_1 \in H^2(\Omega)\cap H_0^1(\Omega)$, and $u \in C([0,T); H^2(\Omega)\cap H_0^1(\Omega))$. Then, problems (3.1)–(3.3) admit a unique solution $v \in L^\infty(0,T; H_3(\Omega))$ with $v_t \in L^\infty(0,T; H^2(\Omega)\cap H_0^1(\Omega))$ and $v_{tt} \in L^2(0,T; H_0^1(\Omega))$, which satisfies (3.6) for all $t \in [0,T)$.*

*Proof.* As in the Proof of Lemma 3.3, we construct the approximate solutions $v_n(t)$ to problems (3.1)–(3.3), where

$$u_{0n} \to u_0 \ \text{ in } \ H_3(\Omega)$$

and

$$u_{1n} \to u_1 \quad \text{in} \ \ H^2(\Omega) \cap H_0^1(\Omega).$$

We see from the Proof of Lemma 3.3 that estimate (3.22) holds. Taking $w = \omega_i$ in (3.10), multiplying by $\lambda_i^{\frac{1}{2}} \xi'_{in}(t)$, and summing for *i*, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v_{nt}(t)\|^2 + \|\Delta v_{nt}(t)\|^2 + \|\nabla\Delta v_n(t)\|^2\right) \\
&\quad - \int_0^t g(t-\tau)(\nabla\Delta v_n(\tau), \nabla\Delta v_{nt}(t))\,\mathrm{d}\tau \\
&\quad - \left(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_{nt}(t)\right) + \|\nabla v_{nt}(t)\|^2 + \|\Delta v_{nt}(t)\|^2 \\
&= -\left(|u(t)|^{q-2}u(t), \Delta v_{nt}(t)\right).
\end{aligned} \tag{3.27}$$

By the arguments similar to the proof of (3.16), we have

$$\begin{aligned}
&\int_0^t g(t-\tau)(\nabla\Delta v_n(\tau), \nabla\Delta v_{nt}(t))\,\mathrm{d}\tau \\
&\quad = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left((g\circ\nabla\Delta v_n)(t) - \int_0^t g(\tau)\,\mathrm{d}\tau\|\nabla\Delta v_n(t)\|^2\right) \\
&\qquad + \frac{1}{2}\left(g'\circ\nabla\Delta v_n\right)(t) - \frac{1}{2}g(t)\|\nabla\Delta v_n(t)\|^2.
\end{aligned}$$

Substituting this equality into (3.27), we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\|\nabla v_{nt}(t)\|^2 + \|\Delta v_{nt}(t)\|^2 + \left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\nabla\Delta v_n(t)\|^2 \\
&\quad + (g\circ\nabla\Delta v_n)(t)\bigg) - \left(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_{nt}(t)\right) + \|\nabla v_{nt}(t)\|^2 \\
&\quad + \|\Delta v_{nt}(t)\|^2 - \frac{1}{2}\left(g'\circ\nabla\Delta v_n\right)(t) + \frac{1}{2}g(t)\|\nabla\Delta v_n(t)\|^2 \\
&= -\left(|u(t)|^{q-2}u(t), \Delta v_{nt}(t)\right).
\end{aligned} \tag{3.28}$$

In (3.28), we have

$$\begin{aligned}
-\left(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_{nt}(t)\right) = &-\frac{\mathrm{d}}{\mathrm{d}t}\left(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_n(t)\right) \\
&- (p-1)\left(|\nabla v_n(t)|^{p-2}\nabla v_{nt}(t), \nabla\Delta v_n(t)\right).
\end{aligned}$$

Recalling $(\mathrm{A}_1)$, we observe

$$-\frac{1}{2}\left(g'\circ\nabla\Delta v_n\right)(t) + \frac{1}{2}g(t)\|\nabla\Delta v_n(t)\|^2 \geq 0.$$

Hence, (3.28) turns into

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{1}{2}\|\nabla v_{nt}(t)\|^2 + \frac{1}{2}\|\Delta v_{nt}(t)\|^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\nabla\Delta v_n(t)\|^2 \\
&\quad + (g\circ\nabla\Delta v_n)(t) - \left(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_n(t)\right)\bigg) \\
&\leq I_1 + I_2,
\end{aligned} \tag{3.29}$$

where

$$I_1 := -\left(|u(t)|^{q-2}u(t), \Delta v_{nt}(t)\right)$$

and

$$I_2 := (p-1)\left(|\nabla v_n(t)|^{p-2}\nabla v_{nt}(t), \nabla\Delta v_n(t)\right).$$

From $(\mathrm{A}_2)$ and $(\mathrm{A}_3)$ and the inequalities of Hölder, Sobolev, and Cauchy, it follows that

$$\begin{aligned}I_1 &\leq \|u(t)\|_{2q-2}^{q-1}\|\Delta v_{nt}(t)\| \\ &\leq C\|\Delta u(t)\|^{q-1}\|\Delta v_{nt}(t)\| \\ &\leq C\|\Delta u(t)\|^{2q-2} + \frac{1}{2}\|\Delta v_{nt}(t)\|^2. \end{aligned} \tag{3.30}$$

Moreover, from $(\mathrm{A}_2)$ and the inequalities of Hölder and Sobolev, we deduce that

$$\begin{aligned}I_2 &\leq (p-1)\|\nabla v_n(t)\|_{2p-2}^{p-2}\|\nabla v_{nt}(t)\|_{2p-2}\|\nabla\Delta v_n(t)\| \\ &\leq C\|\Delta v_n(t)\|^{p-2}\|\Delta v_{nt}(t)\|\|\nabla\Delta v_n(t)\|.\end{aligned}$$

Thus, there exists a constant $\epsilon_1 > 0$ to be determined such that

$$I_2 \leq \left(\frac{C}{(2\epsilon_1)^{\frac{1}{2}}}\|\Delta v_n(t)\|^{p-2}\|\Delta v_{nt}(t)\|\right)\left((2\epsilon_1)^{\frac{1}{2}}\|\nabla\Delta v_n(t)\|\right).$$

We further deduce from Cauchy's inequality that

$$I_2 \leq \frac{C^2}{4\epsilon_1}\|\Delta v_n(t)\|^{2p-4}\|\Delta v_{nt}(t)\|^2 + \epsilon_1\|\nabla\Delta v_n(t)\|^2.$$

Taking $\epsilon_1 = \frac{\kappa}{4}$, we obtain

$$I_2 \leq C\|\Delta v_n(t)\|^{2p-4}\|\Delta v_{nt}(t)\|^2 + \frac{\kappa}{4}\|\nabla\Delta v_n(t)\|^2. \tag{3.31}$$

Due to the fact that $(\mathrm{A}_1)$ implies $(g\circ\nabla\Delta v_n)(t) \geq 0$, we deduce from (3.29)–(3.31), (3.22), and (3.18) that

$$\begin{aligned}&\frac{1}{2}\|\nabla v_{nt}(t)\|^2 + \frac{1}{2}\|\Delta v_{nt}(t)\|^2 + \frac{\kappa}{2}\|\nabla\Delta v_n(t)\|^2 \\ &\quad - \left(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_n(t)\right) \\ &\leq C\int_0^t\left(\frac{1}{2}\|\nabla v_{nt}(t)\|^2 + \frac{1}{2}\|\Delta v_{nt}(t)\|^2 + \frac{\kappa}{4}\|\nabla\Delta v_n(\tau)\|^2\right)\mathrm{d}\tau + C.\end{aligned} \tag{3.32}$$

For the fourth term on the left-hand side of (3.32), we deduce from $(\mathrm{A}_2)$ and the inequalities of Hölder, Sobolev, and Cauchy that there exists a constant $\epsilon_2 > 0$ to be determined such that

$$\begin{aligned}\left|-\left(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_n(t)\right)\right| &\leq \|\nabla v_n(t)\|_{2p-2}^{p-1}\|\nabla\Delta v_n(t)\| \\ &\leq C\|\Delta v_n(t)\|^{p-1}\|\nabla\Delta v_n(t)\| \\ &\leq \left(\frac{C}{(2\epsilon_2)^{\frac{1}{2}}}\|\Delta v_n(t)\|^{p-1}\right)\left((2\epsilon_2)^{\frac{1}{2}}\|\nabla\Delta v_n(t)\right) \\ &\leq C(\epsilon_2)\|\Delta v_n(t)\|^{2p-2} + \epsilon_2\|\nabla\Delta v_n(t)\|^2.\end{aligned}$$

Taking $\epsilon_2 = \frac{\kappa}{4}$, we deduce from (3.22) that

$$C + \frac{\kappa}{4}\|\nabla\Delta v_n(t)\|^2 - \left(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_n(t)\right) \geq 0. \tag{3.33}$$

Thus, (3.32) turns into

$$\begin{aligned}&\frac{1}{2}\|\nabla v_{nt}(t)\|^2 + \frac{1}{2}\|\Delta v_{nt}(t)\|^2 + \frac{\kappa}{2}\|\nabla\Delta v_n(t)\|^2 \\ &\quad - \left(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_n(t)\right) \\ &\leq C\int_0^t\left(\frac{1}{2}\|\nabla v_{nt}(\tau)\|^2 + \frac{1}{2}\|\Delta v_{nt}(\tau)\|^2 + \frac{\kappa}{2}\|\nabla\Delta v_n(\tau)\|^2\right. \\ &\quad \left. - \left(|\nabla v_n(\tau)|^{p-2}\nabla v_n(\tau), \nabla\Delta v_n(\tau)\right)\right)\mathrm{d}\tau + C.\end{aligned}$$

Using Gronwall's inequality, we have

$$\frac{1}{2}\|\nabla v_{nt}(t)\|^2 + \frac{1}{2}\|\Delta v_{nt}(t)\|^2 + \frac{\kappa}{2}\|\nabla\Delta v_n(t)\|^2 - \left(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_n(t)\right) \leq C,$$

which together with (3.33) gives

$$\frac{1}{2}\|\nabla v_{nt}(t)\|^2 + \frac{1}{2}\|\Delta v_{nt}(t)\|^2 + \frac{\kappa}{4}\|\nabla\Delta v_n(t)\|^2 \le C, \tag{3.34}$$

where $C$ is independent of $n$.

Taking $w = v_{ntt}(t)$ in (3.10), we obtain

$$\begin{aligned}\|v_{ntt}\|_*^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{nt}(t)\|_*^2 = \big(|u(t)|^{q-2}u(t), v_{ntt}(t)\big) + (\nabla\Delta v_n, \nabla v_{ntt})\\ - \int_0^t g(t-\tau)(\nabla\Delta v_n(\tau), \nabla v_{ntt}(t))\,\mathrm{d}\tau\\ - \big(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla v_{ntt}(t)\big).\end{aligned}$$

From $(A_1)$–$(A_3)$ and the inequalities of Hölder, Sobolev, and Cauchy, it follows that there exist constants $\epsilon_i > 0$ $(i = 3,4,5,6)$ to be determined such that

$$\begin{aligned}\big(|u(t)|^{q-2}u(t), v_{ntt}(t)\big) &\le \|u(t)\|_{2q-2}^{q-1}\|v_{ntt}(t)\|\\ &\le \left(\frac{1}{(2\epsilon_3)^{\frac{1}{2}}}\|u(t)\|_{2q-2}^{q-1}\right)\left((2\epsilon_3)^{\frac{1}{2}}\|v_{ntt}(t)\|\right)\\ &\le \frac{1}{4\epsilon_3}\|\Delta u(t)\|^{2q-2} + \epsilon_3\|v_{ntt}(t)\|^2,\end{aligned}$$

$$\begin{aligned}(\nabla\Delta v_n(t), \nabla v_{ntt}(t)) &\le \|\nabla\Delta v_n(t)\|\|\nabla v_{ntt}(t)\|\\ &\le \left(\frac{1}{(2\epsilon_4)^{\frac{1}{2}}}\|\nabla\Delta v_n(t)\|\right)\left((2\epsilon_4)^{\frac{1}{2}}\|\nabla v_{ntt}(t)\|\right)\\ &\le \frac{1}{4\epsilon_4}\|\nabla\Delta v_n(t)\|^2 + \epsilon_4\|\nabla v_{ntt}(t)\|^2,\end{aligned}$$

$$\begin{aligned}&- \int_0^t g(t-\tau)(\nabla\Delta v_n(\tau), \nabla v_{ntt}(t))\,\mathrm{d}\tau\\ &\quad\le \int_0^t g(t-\tau)\|\nabla\Delta v_n(\tau)\|\,\mathrm{d}\tau\|\nabla v_{ntt}(t)\|\\ &\quad\le \left(\frac{1}{(2\epsilon_5)^{\frac{1}{2}}}\int_0^t g(t-\tau)\|\nabla\Delta v_n(\tau)\|\,\mathrm{d}\tau\right)\left((2\epsilon_5)^{\frac{1}{2}}\|\nabla v_{ntt}(t)\|\right)\\ &\quad\le \frac{1}{4\epsilon_5}\left(\int_0^t g(t-\tau)\|\nabla\Delta v_n(\tau)\|\,\mathrm{d}\tau\right)^2 + \epsilon_5\|\nabla v_{ntt}(t)\|^2\\ &\quad\le \frac{1}{4\epsilon_5}(1-\kappa)^2\left(\operatorname*{ess\,sup}_{t\in[0,T)}\|\nabla\Delta v_n(t)\|\right)^2 + \epsilon_5\|\nabla v_{ntt}(t)\|^2,\end{aligned}$$

and

$$\begin{aligned}-\big(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla v_{ntt}(t)\big) &\le \|\nabla v_n(t)\|_{2p-2}^{p-1}\|\nabla v_{ntt}(t)\|\\ &\le C\|\Delta v_n(t)\|^{p-1}\|\nabla v_{ntt}(t)\|\\ &\le \left(\frac{C}{(2\epsilon_6)^{\frac{1}{2}}}\|\Delta v_n(t)\|^{p-1}\right)\left((2\epsilon_6)^{\frac{1}{2}}\|\nabla v_{ntt}(t)\|\right)\\ &\le C(\epsilon_6)\|\Delta v_n(t)\|^{2p-2} + \epsilon_6\|\nabla v_{ntt}(t)\|^2.\end{aligned}$$

Taking $\epsilon_i = \frac{1}{5}$ $(i = 3,4,5,6)$, we further deduce from (3.22) and (3.34) that

$$\frac{1}{5}\int_0^t \|v_{n\tau\tau}(\tau)\|_*^2\,\mathrm{d}\tau + \frac{1}{2}\|v_{nt}(t)\|_*^2 \le C, \tag{3.35}$$

where $C$ is independent of $n$.

From (3.34) and (3.35), we conclude that there exist a subsequence of $\{v_n\}$ and a function $v$ such that as $n \to \infty$,

$$v_n \rightharpoonup v \quad \text{weakly star in } L^\infty(0,T;H_3(\Omega)), \tag{3.36}$$

$$v_{nt} \rightharpoonup v_t \quad \text{weakly star in } L^\infty\big(0,T;H^2(\Omega)\cap H_0^1(\Omega)\big), \tag{3.37}$$

and

$$v_{ntt} \rightharpoonup v_{tt} \quad \text{weakly in } L^2\big(0,T;H_0^1(\Omega)\big). \tag{3.38}$$

Therefore, $v$ is a solution to problems (3.1)–(3.3). In addition, by testing (3.1) with $v_t(t)$, we see that there holds (3.6) for all $t \in [0,T)$. Suppose that $v$ and $\bar{v}$ are two solutions to problems (3.1)–(3.3). Set $\tilde{v} := \bar{v} - v$. Then, (3.24) holds again for this proof. By the same arguments as in the proof of (3.26), we have $v \equiv \bar{v}$ for all $t \in [0,T)$. Thus, the Proof of Lemma 3.4 is finished. □

Lemma 3.3 tells us that if $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, then problems (3.1)–(3.3) admit a unique solution $v \in L^\infty\big(0,T;H^2(\Omega)\cap H_0^1(\Omega)\big)$ with $v_t \in L^\infty\big(0,T;H_0^1(\Omega)\big)$. Making use of Lemma 3.4, we now in Lemma 3.5 employ the density arguments to show that under the conditions of Lemma 3.3, problems (3.1)–(3.3) can have a unique solution $v \in C\big([0,T);H^2(\Omega)\cap H_0^1(\Omega)\big)$ with $v_t \in C\big([0,T);H_0^1(\Omega)\big)$. Although Lemma 3.4 itself can also achieve this conclusion, it requires higher regularity of the initial data, i.e., $u_0 \in H_3(\Omega)$ and $u_1 \in H^2(\Omega)\cap H_0^1(\Omega)$.

*Lemma 3.5 (continuity of solutions to linear equations). Under the conditions of Lemma 3.3, problems (3.1)–(3.3) admit a unique solution $v \in C\big([0,T);H^2(\Omega)\cap H_0^1(\Omega)\big)$ with $v_t \in C\big([0,T);H_0^1(\Omega)\big)$, which satisfies (3.6) for all $t \in [0,T)$.*

*Proof.* For $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, there exist $\{v_{0n}\} \subset H_3(\Omega)$ and $\{v_{1n}\} \subset H^2(\Omega)\cap H_0^1(\Omega)$ such that

$$v_{0n} \to u_0 \ \text{ in } H^2(\Omega)\cap H_0^1(\Omega) \tag{3.39}$$

and

$$v_{1n} \to u_1 \ \text{ in } H_0^1(\Omega). \tag{3.40}$$

According to Lemma 3.4, for each $n \in \mathbb{N}^+$, problems (3.1)–(3.3) admit a unique solution $v_n \in L^\infty(0,T;\ H_3(\Omega))$ with $v_{nt} \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and $v_{ntt} \in L^2(0,T;H_0^1(\Omega))$, where $v_n(x,0) = v_{0n}(x)$ and $v_{nt}(x,0) = v_{1n}(x)$. Thus,

$$\begin{aligned}\bigg(&v_{ntt}(t) - \Delta v_{ntt}(t) + \Delta^2 v_n(t) - \int_0^t g(t-\tau)\Delta^2 v_n(\tau)\,\mathrm{d}\tau - \Delta_p v_n(t)\\ &+ v_{nt}(t) - \Delta v_{nt}(t) - |u(t)|^{q-2}u(t), w\bigg) = 0.\end{aligned} \tag{3.41}$$

Taking $w = v_{nt}(t)$ in (3.41) and using the same arguments as in the Proof of Lemma 3.3, we retrieve (3.22) here. Taking $w = -\Delta v_{nt}(t)$ in (3.41), we conclude from the arguments similar to Lemma 3.2 that

$$\begin{aligned}&\bigg(v_{ntt}(t) - \Delta v_{ntt}(t) + \Delta^2 v_n(t) - \int_0^t g(t-\tau)\Delta^2 v_n(\tau)\,\mathrm{d}\tau - \Delta_p v_n(t), -\Delta v_{nt}(t)\bigg)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{1}{2}\|\nabla v_{nt}(t)\|^2 + \frac{1}{2}\|\Delta v_{nt}(t)\|^2 + \frac{1}{2}\bigg(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\bigg)\|\nabla\Delta v_n(t)\|^2\\ &\quad + (g\circ\nabla\Delta v_n)(t) - \big(|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla\Delta v_n(t)\big)\bigg)\\ &\quad - \frac{1}{2}\big(g'\circ\nabla\Delta v_n\big)(t) + \frac{1}{2}g(t)\|\nabla\Delta v_n(t)\|^2 - (p-1)\big(|\nabla v_n(t)|^{p-2}\nabla v_{nt}(t), \nabla\Delta v_n(t)\big).\end{aligned}$$

Hence, by going back to the Proof of Lemma 3.4, we retrieve (3.34). Taking $w = v_{ntt}(t)$ in (3.41) and using same arguments as in the Proof of Lemma 3.4, we have (3.35) again. Hence, (3.36)–(3.38) hold again for this proof. We infer from Ref. 33 (Theorem 4 in Sec. 5.9) that $v \in C\big([0,T);H^2(\Omega)\cap H_0^1(\Omega)\big)$ and $v_t \in C\big([0,T);H_0^1(\Omega)\big)$. Observing (3.39) and (3.40), we know that $v$ is a solution to problems (3.1)–(3.3). The remainder of the proof is same as that of the Proof of Lemma 3.3. □

Based on the above preparations, we employ the contraction mapping principle to prove the local existence and uniqueness of solutions to problems (1.1)–(1.3).

*Proof of Theorem 2.3.* The proof of this theorem is divided into two steps.
**Step 1.** The local existence and uniqueness of solutions to problems (1.1)–(1.3).
Define

$$\mathcal{S}_T := \{u \in X \mid \|u\|_X \le R, u(0) = u_0, u_t(0) = u_1\}$$

with

$$X := C\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big) \cap C^1\big([0,T]; H_0^1(\Omega)\big),$$

$$\|u\|_X^2 := \sup_{t\in[0,T)} \left( \frac{1}{2}\|u_t(t)\|_*^2 + \frac{\kappa}{2}\|\Delta u(t)\|^2 \right),$$

and

$$R^2 := \|u_1\|_*^2 + \|\Delta u_0\|^2 + \frac{2}{p}\|\nabla u_0\|_p^p.$$

By Lemma 3.5, for any $u \in \mathcal{S}_T$, we can define a map $\Phi$ such that $v := \Phi(u)$.

We claim that $\Phi$ is a contractive map from $\mathcal{S}_T$ into itself. To confirm this, we observe that a combination of (3.6) and (3.18)–(3.20) gives

$$\begin{aligned} &\frac{1}{2}\|v_t(t)\|_*^2 + \frac{\kappa}{2}\|\Delta v(t)\|^2 + \int_0^t \|v_\tau(\tau)\|_*^2 \,\mathrm{d}\tau \\ &\le \frac{1}{2}\|u_1\|_*^2 + \frac{1}{2}\|\Delta u_0\|^2 + \frac{1}{p}\|\nabla u_0\|_p^p + \int_0^t \big(|u(\tau)|^{q-2}u(\tau), v_\tau(\tau)\big)\,\mathrm{d}\tau \end{aligned} \tag{3.42}$$

for all $t \in [0,T)$. Applying the inequalities of Hölder and Cauchy, we deduce that there exists a constant $\epsilon > 0$ to be determined such that

$$\begin{aligned} \int_0^t \big(|u(\tau)|^{q-2}u(\tau), v_\tau(\tau)\big)\,\mathrm{d}\tau &\le \int_0^t \|u(\tau)\|_{2q-2}^{q-1}\|v_\tau(\tau)\|\,\mathrm{d}\tau \\ &\le \int_0^t \left(\frac{1}{(2\epsilon)^{\frac{1}{2}}}\|u(\tau)\|_{2q-2}^{q-1}\right)\left((2\epsilon)^{\frac{1}{2}}\|v_\tau(\tau)\|\right)\mathrm{d}\tau \\ &\le \frac{1}{4\epsilon}\int_0^t \|u(\tau)\|_{2q-2}^{2q-2}\,\mathrm{d}\tau + \epsilon\int_0^t \|v_\tau(\tau)\|^2\,\mathrm{d}\tau. \end{aligned}$$

Taking $\epsilon = 1$, we deduce from $(\mathrm{A}_2)$ and $(\mathrm{A}_3)$ and the Sobolev inequality that

$$\int_0^t \big(|u(\tau)|^{q-2}u(\tau), v_\tau(\tau)\big)\,\mathrm{d}\tau \le C\int_0^t \|\Delta u(\tau)\|^{2q-2}\,\mathrm{d}\tau + \int_0^t \|v_\tau(\tau)\|^2\,\mathrm{d}\tau.$$

Inserting this inequality into (3.42) and considering $u \in \mathcal{S}_T$, we can obtain

$$\begin{aligned} &\frac{1}{2}\|v_t(t)\|_*^2 + \frac{\kappa}{2}\|\Delta v(t)\|^2 \\ &\quad\le \frac{1}{2}\|u_1\|_*^2 + \frac{1}{2}\|\Delta u_0\|^2 + \frac{1}{p}\|\nabla u_0\|_p^p + C\int_0^t \|\Delta u(\tau)\|^{2q-2}\,\mathrm{d}\tau \\ &\quad\le \frac{R^2}{2} + CR^{2q-2}T. \end{aligned}$$

Hence, there exists a time $T > 0$ sufficiently small such that $\|v\|_X^2 \le R^2$. Thus, $\Phi(\mathcal{S}_T) \subseteq \mathcal{S}_T$. Next, we prove that such a map is contractive. Define $\bar{v} := \Phi(\bar{u})$, $u, \bar{u} \in \mathcal{S}_T$. Set $\tilde{v} := \bar{v} - v$. Then, by repeating analogous arguments in Lemma 3.2, we have

$$\begin{aligned} &\left(\tilde{v}_{tt}(t) - \Delta\tilde{v}_{tt}(t) + \Delta^2\tilde{v}(t) - \int_0^t g(t-\tau)\Delta^2\tilde{v}(\tau)\,\mathrm{d}\tau, \tilde{v}_t(t)\right) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{v}_t(t)\|_*^2 + \left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{v}(t)\|^2 + (g\circ\Delta\tilde{v})(t)\right) \\ &\quad - \frac{1}{2}(g'\circ\Delta\tilde{v})(t) + \frac{1}{2}g(t)\|\Delta\tilde{v}(t)\|^2. \end{aligned}$$

Substituting this equality into

$$\begin{aligned} &\left(\tilde{v}_{tt}(t) - \Delta\tilde{v}_{tt}(t) + \Delta^2\tilde{v}(t) - \int_0^t g(t-\tau)\Delta^2\tilde{v}(\tau)\,\mathrm{d}\tau - \Delta_p\bar{v}(t) + \Delta_p v(t)\right. \\ &\quad \left. + \tilde{v}_t(t) - \Delta\tilde{v}_t(t), \tilde{v}_t(t)\right) = \big(|\bar{u}(t)|^{q-2}\bar{u}(t) - |u(t)|^{q-2}u(t), \tilde{v}_t(t)\big), \end{aligned}$$

we derive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\tilde{v}_t(t)\|_*^2+\frac{1}{2}\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{v}(t)\|^2+\frac{1}{2}(g\circ\Delta\tilde{v})(t)\right)\\&\quad+\left(|\nabla\tilde{v}(t)|^{p-2}\nabla\tilde{v}(t)-|\nabla v(t)|^{p-2}\nabla v(t),\nabla\tilde{v}_t(t)\right)\\&\quad+\|\tilde{v}_t(t)\|_*^2-\frac{1}{2}(g'\circ\Delta\tilde{v})(t)+\frac{1}{2}g(t)\|\Delta\tilde{v}(t)\|^2\\&=\left(|\bar{u}(t)|^{q-2}\bar{u}(t)-|u(t)|^{q-2}u(t),\tilde{v}_t(t)\right).\end{aligned}$$

From (3.20), we further have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\tilde{v}_t(t)\|_*^2+\frac{1}{2}\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{v}(t)\|^2+\frac{1}{2}(g\circ\Delta\tilde{v})(t)\right)\le I_3+I_4, \tag{3.43}$$

where

$$I_3:=\left(|\bar{u}(t)|^{q-2}\bar{u}(t)-|u(t)|^{q-2}u(t),\tilde{v}_t(t)\right)$$

and

$$I_4:=\left|\left(|\nabla\tilde{v}(t)|^{p-2}\nabla\tilde{v}(t)-|\nabla v(t)|^{p-2}\nabla v(t),\nabla\tilde{v}_t(t)\right)\right|.$$

From Lemma 3.1, $(A_2)$ and $(A_3)$, and the inequalities of Hölder, Minkowski, Sobolev, and Cauchy, it follows that

$$\begin{aligned}I_3&\le C\int_\Omega\left(|u(t)|^{q-2}+|\bar{u}(t)|^{q-2}\right)|\bar{u}(t)-u(t)||\tilde{v}_t(t)|\,\mathrm{d}x\\&\le C\left(\|u(t)\|_{2q-2}^{q-2}+\|\bar{u}(t)\|_{2q-2}^{q-2}\right)\|\bar{u}(t)-u(t)\|_{2q-2}\|\tilde{v}_t(t)\|\\&\le C\left(\|\Delta u(t)\|^{q-2}+\|\Delta\bar{u}(t)\|^{q-2}\right)\|\Delta\bar{u}(t)-\Delta u(t)\|\|\tilde{v}_t(t)\|\\&\le C\|\Delta\bar{u}(t)-\Delta u(t)\|^2+\frac{1}{2}\|\tilde{v}_t(t)\|^2.\end{aligned}\tag{3.44}$$

Likewise, we have

$$\begin{aligned}I_4&\le C\left(\|\nabla v(t)\|_{2p-2}^{p-2}+\|\nabla\tilde{v}(t)\|_{2p-2}^{p-2}\right)\|\nabla\tilde{v}(t)\|_{2p-2}\|\nabla\tilde{v}_t(t)\|\\&\le C\|\Delta\tilde{v}(t)\|^2+\frac{1}{2}\|\nabla\tilde{v}_t(t)\|^2.\end{aligned}\tag{3.45}$$

Hence, on account of (3.25), we easily see that (3.43) becomes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\tilde{v}_t(t)\|_*^2+\frac{1}{2}\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{v}(t)\|^2+\frac{1}{2}(g\circ\Delta\tilde{v})(t)\right)\\&\le C\left(\frac{1}{2}\|\tilde{v}_t(t)\|_*^2+\frac{1}{2}\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{v}(t)\|^2+\frac{1}{2}(g\circ\Delta\tilde{v})(t)\right)\\&\quad+C\|\Delta\bar{u}(t)-\Delta u(t)\|^2.\end{aligned}$$

By Gronwall's inequality, $\tilde{v}(0)=0$, $\tilde{v}_t(0)=0$, (3.18), and (3.19), we arrive at

$$\frac{1}{2}\|\tilde{v}_t(t)\|_*^2+\frac{\kappa}{2}\|\Delta\tilde{v}(t)\|^2\le C\int_0^t\|\Delta\bar{u}(\tau)-\Delta u(\tau)\|^2\,\mathrm{d}\tau$$

for all $t\in[0,T)$. Therefore,

$$\|\tilde{v}\|_X^2\le CT\|\bar{u}-u\|_X^2,$$

which shows that there exists a constant $0<\delta<1$ such that

$$\|\Phi(\bar{u})-\Phi(u)\|_X^2\le\delta\|\bar{u}-u\|_X^2,$$

provided that $T$ is sufficiently small. This proves the assertion.

According to the contraction mapping principle, problems (1.1)–(1.3) admit a unique solution satisfying (2.4) on $[0,T)$.

**Step 2.** Finite time blow-up, i.e., the local solution blows up if the existence time is finite.

We claim that if $T = \infty$, then $u$ exists globally in time; if $T < \infty$, then

$$\lim_{t\to T}\left(\|\Delta u(t)\|^2 + \|u_t(t)\|_*^2\right) = \infty. \tag{3.46}$$

Arguing by contradiction, we suppose that $T < \infty$ and $\lim_{t\to T}\left(\|\Delta u(t)\|^2 + \|u_t(t)\|_*^2\right) < \infty$. Then, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} t_n = T$ and $\|\Delta u(t_n)\|^2 + \|u_t(t_n)\|_*^2 \le C$ for $n \in \mathbb{N}^+$. Note that, for each $n$, problems (1.1)–(1.3) with the initial data $u(t_n)$ and $u_t(t_n)$ admit a unique solution on $[t_n, t_n + \widetilde{T})$, where $\widetilde{T} > 0$ depends on $C$ only. Thus, $T < t_n + \widetilde{T}$ for large enough $n$. This contradicts the definition of the maximum existence time $T$.

Finally, we prove (2.5). From (2.4) and (3.18)–(3.20), we can get

$$\frac{1}{2}\|u_t(t)\|_*^2 + \frac{\kappa}{2}\|\Delta u(t)\|^2 + \frac{1}{p}\|\nabla u(t)\|_p^p \le E(0) + \frac{1}{q}\|u(t)\|_q^q, \quad t \in [0, T),$$

which together with (3.46) gives

$$\lim_{t\to T}\|u(t)\|_q = \infty.$$

By the Sobolev inequality and $(\mathrm{A}_2)$ and $(\mathrm{A}_3)$, we further have

$$\lim_{t\to T}\|\nabla u(t)\|_p = \infty. \tag{3.47}$$

In view of

$$\frac{1}{p}\|\nabla u(t)\|_p^p \le E(0) + \frac{1}{q}\|u(t)\|_q^q, \quad t \in [0, T),$$

we reach

$$C\|\nabla u(t)\|_p^p - C \le \|u(t)\|_q^q. \tag{3.48}$$

By the Gagliardo–Nirenberg interpolation inequality, we have

$$\|u(t)\|_q^q \le C\|\nabla u(t)\|_p^{q\vartheta}\|u(t)\|_r^{q(1-\vartheta)}, \tag{3.49}$$

where $1 \le r < q$ and

$$\vartheta := \frac{Np(q-r)}{q(Np + rp - Nr)}.$$

Owing to $r > \frac{N(q-p)}{p}$, we have $0 < \vartheta < \frac{p}{q}$. It follows from (3.48) and (3.49) that

$$C\|\nabla u(t)\|_p^{p-q\vartheta} - C\|\nabla u(t)\|_p^{-q\vartheta} \le \|u(t)\|_r^{q(1-\vartheta)},$$

which together with (3.47) implies that there holds (2.5). □

## IV. GLOBAL EXISTENCE (PROOF OF THEOREM 2.4)

First of all, in the case $0 < E(0) < d$, we show the invariance of $\mathcal{W}$ and $\mathcal{V}$, which will be used to classify the initial data for the global existence and finite time blow-up of solutions.

*Lemma 4.1. Let* $(\mathrm{A}_1)$–$(\mathrm{A}_3)$ *be fulfilled. Assume that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, *and* $0 < E(0) < d$. *Then, for the solution* $u(t)$ *to problems (1.1)–(1.3), there holds*

*(i)* $u(t) \in \mathcal{W}$ *for all* $t \in [0, T)$, *provided that* $I(u_0) > 0$ *or* $\|\Delta u_0\| = 0$;
*(ii)* $u(t) \in \mathcal{V}$ *for all* $t \in [0, T)$, *provided that* $I(u_0) < 0$.

*Proof.* Suppose that $u(t) \notin \mathcal{W}$ for some $0 < t < T$. Then, by the continuity of $u(t)$ in $t$, we see that there exists a time $0 < t_0 < T$ such that $u(t_0) \in \mathcal{N}$ or $J(u(t_0)) = d$. From (2.4) and (3.20), it is obvious that $J(u(t_0)) = d$ is impossible. On the other hand, if $u(t_0) \in \mathcal{N}$, then by recalling the definition of $d$, we get $J(u(t_0)) \ge d$, which contradicts $E(0) < d$ due to (2.4) and (3.20). Hence, $u(t) \in \mathcal{W}$ for all $t \in [0, T)$.

The proof of part (ii) is similar to that of part (i). □

We now discuss the relationship between $d$ and $\overline{d}$ and that between $d$ and $\tilde{d}$.

*Proposition 4.2. Let* $(\mathrm{A}_1)$–$(\mathrm{A}_3)$ *be fulfilled. Then,* $d > \overline{d}$ *and* $d > \overline{d}$.

*Proof.* By the definitions of $J(u)$ and $I(u)$, we get

$$J(u) = \frac{q-2}{2q}\left(\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u\|^2 + (g\circ\Delta u)\right) + \frac{q-p}{pq}\|\nabla u\|_p^p + \frac{1}{q}I(u).$$

Let $u \in \mathcal{N}$. Then, from (3.18) and (3.19) and $(\mathrm{A}_2)$, we obtain

$$\begin{aligned} J(u) &\geq \frac{(q-2)\kappa}{2q}\|\Delta u\|^2 + \frac{q-p}{pq}\|\nabla u\|_p^p \\ &\geq \frac{(q-2)\kappa}{2q\mathfrak{C}_1^2}\|\nabla u\|_p^2 + \frac{q-p}{pq}\|\nabla u\|_p^p. \end{aligned} \tag{4.1}$$

Since $u \in \mathcal{N}$ implies

$$\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u\|^2 + (g\circ\Delta u)(t) + \|\nabla u\|_p^p = \|u\|_q^q,$$

it follows from $(\mathrm{A}_2)$ and $(\mathrm{A}_3)$ that

$$\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u\|^2 + \|\nabla u\|_p^p \leq \mathfrak{C}_2^q\|\nabla u\|_p^q.$$

On the other hand, from (3.18), we get

$$\begin{aligned} \left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u\|^2 + \|\nabla u\|_p^p &\geq \kappa\|\Delta u\|^2 + \|\nabla u\|_p^p \\ &\geq \frac{\kappa}{\mathfrak{C}_1^2}\|\nabla u\|_p^2 + \|\nabla u\|_p^p. \end{aligned}$$

Hence,

$$\frac{\kappa}{\mathfrak{C}_1^2}\|\nabla u\|_p^2 + \|\nabla u\|_p^p \leq \mathfrak{C}_2^q\|\nabla u\|_p^q,$$

which gives

$$\|\nabla u\|_p > \max\left\{\kappa^{\frac{1}{q-2}}\mathfrak{C}_1^{-\frac{2}{q-2}}\mathfrak{C}_2^{-\frac{q}{q-2}}, \mathfrak{C}_2^{-\frac{q}{q-p}}\right\}. \tag{4.2}$$

From (4.1), we have

$$J(u) > \frac{(q-2)\kappa}{2q\mathfrak{C}_1^2}\|\nabla u\|_p^2 \tag{4.3}$$

and

$$J(u) > \frac{q-p}{pq}\|\nabla u\|_p^p. \tag{4.4}$$

Substituting $\|\nabla u\|_p > \kappa^{\frac{1}{q-2}}\mathfrak{C}_1^{-\frac{2}{q-2}}\mathfrak{C}_2^{-\frac{q}{q-2}}$ into (4.3) and (4.4), respectively, we deduce from the definition of $d$ that $d > d_1$ and $d > d_2$. Substituting $\|\nabla u\|_p > \mathfrak{C}_2^{-\frac{q}{q-p}}$ into (4.4), we get $d > d_3$. Consequently, $d > \bar{d}$. Moreover, substituting $\|\nabla u\|_p > \kappa^{\frac{1}{q-2}}\mathfrak{C}_1^{-\frac{2}{q-2}}\mathfrak{C}_2^{-\frac{q}{q-2}}$ and $\|\nabla u\|_p > \mathfrak{C}_2^{-\frac{q}{q-p}}$ into (4.1), respectively, we obtain $d > d_4$ and $d > d_5$, i.e., $d > \tilde{d}$. □

The following lemma shows a property of the fibering map $J(\lambda u)$, which will facilitate the proof of the global existence of solutions with critical initial energy.

*Lemma 4.3. Let* $(\mathrm{A}_1)$–$(\mathrm{A}_3)$ *be fulfilled,* $\lambda > 0$, $u \in H^2(\Omega)\cap H_0^1(\Omega)$, *and* $\|\Delta u\| \neq 0$. *Then, there exists a unique constant* $\lambda_0 > 0$, *depending on u, such that* $J(\lambda u)$ *is strictly increasing for all* $\lambda \in (0,\lambda_0)$, *strictly decreasing for all* $\lambda \in (\lambda_0,\infty)$, *and takes the maximum at* $\lambda = \lambda_0$.

*Proof.* From

$$I(\lambda u) = \left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\lambda^2\|\Delta u\|^2 + \lambda^2(g\circ\Delta u)(t) + \lambda^p\|\nabla u\|_p^p - \lambda^q\|u\|_q^q,$$

we know that there exists a unique constant $\lambda_0 > 0$, depending on $u$, such that $I(\lambda u) > 0$ for all $\lambda \in (0,\lambda_0)$, $I(\lambda u) < 0$ for all $\lambda \in (\lambda_0,\infty)$, and $I(\lambda_0 u) = 0$. Because of

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) = \frac{1}{\lambda}I(\lambda u),$$

the conclusions of this lemma follow immediately. □

Next, we prove the global existence of solutions to problems (1.1)–(1.3).

*Proof of Theorem 2.4.* (i) For the local solution $u$ to problems (1.1)–(1.3), it is easy to see from (i) in Lemma 4.1 that $u(t) \in \mathcal{W}$ for all $t \in [0, T)$. Hence, it follows from (3.18) and (3.19) that for all $t \in [0, T)$,

$$\begin{aligned} J(u(t)) &= \frac{q-2}{2q}\left(\left(1 - \int_0^t g(\tau)\,d\tau\right)\|\Delta u(t)\|^2 + (g \circ \Delta u)(t)\right) \\ &\quad + \frac{q-p}{pq}\|\nabla u(t)\|_p^p + \frac{1}{q}I(u(t)) \\ &\geq \frac{(q-2)\kappa}{2q}\|\Delta u(t)\|^2, \end{aligned}$$

which together with (2.4), $E(0) < d$, and (3.20) gives

$$\frac{1}{2}\|u_t(t)\|_*^2 + \frac{(q-2)\kappa}{2q}\|\Delta u(t)\|^2 < d.$$

Consequently, by the continuation principle, we have $T = \infty$.

(ii) We divide the proof of (ii) into two cases.

**Case 1.** $\|\Delta u_0\| \neq 0$.

Set $u_{0n} := \lambda_n u_0$, where $\lambda_n := 1 - \frac{1}{n}$, $n = 2, 3, \ldots$. We consider problems (1.1) and (1.3) with the following initial conditions:

$$u(x,0) = u_{0n}(x), \quad u_t(x,0) = u_1(x). \tag{4.5}$$

From $I(u_0) \geq 0$ and the Proof of Lemma 4.3, it is easy to see that $\lambda_0 \geq 1$. Thus, we conclude from Lemma 4.3 that $J(\lambda u)$ is strictly increasing for all $\lambda \in [\lambda_n, 1]$. Hence, $J(u_{0n}) < J(u_0)$ and $I(u_{0n}) > 0$. Moreover,

$$J(u_{0n}) = \frac{q-2}{2q}\|\Delta u_{0n}\|^2 + \frac{q-p}{pq}\|\nabla u_{0n}\|_p^p + \frac{1}{q}I(u_{0n}) > 0.$$

We further obtain

$$E_n(0) = \frac{1}{2}\|u_1\|_*^2 + J(u_{0n}) > 0$$

and

$$E_n(0) < \frac{1}{2}\|u_1\|_*^2 + J(u_0) = E(0) = d. \tag{4.6}$$

Hence, we infer from (i) in this theorem that, for each $n$, problems (1.1), (1.3), and (4.5) admit a unique solution $u_n \in C\big([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)\big)$ with $u_{nt} \in C\big([0,\infty); H_0^1(\Omega)\big)$, which satisfies $u_n(t) \in \mathcal{W}$,

$$\begin{aligned} &(u_{nt}(t), w)_* + \int_0^t (\Delta u_n(\tau), \Delta w)\,d\tau - \int_0^t \int_0^s g(s-\tau)(\Delta u_n(\tau), \Delta w)\,d\tau ds \\ &\quad + \int_0^t \left(|\nabla u_n(\tau)|^{p-2}\nabla u_n(\tau), \nabla w\right) d\tau + (u_n(t), w)_* \\ &= (u_1, w)_* + (u_{0n}, w)_* + \int_0^t \left(|u_n(\tau)|^{q-2}u_n(\tau), w\right) d\tau \end{aligned}$$

and

$$E_n(t) + \int_0^t \left(\|u_{n\tau}(\tau)\|_*^2 - \frac{1}{2}(g' \circ \Delta u_n)(\tau) + \frac{1}{2}g(\tau)\|\Delta u_n(\tau)\|^2\right) d\tau = E_n(0). \tag{4.7}$$

Next, we will conclude the existence of solutions by the energy estimates and the compactness arguments. From (i) in Lemma 4.1, it is readily seen that $u_n(t) \in \mathcal{W}$ for all $t \in [0, \infty)$. Hence, from (3.18) and (3.19), we can get

$$\begin{aligned} E_n(t) &= \frac{1}{2}\|u_{nt}(t)\|_*^2 + \frac{q-2}{2q}\left(\left(1 - \int_0^t g(\tau)\,d\tau\right)\|\Delta u_n(t)\|^2 + (g \circ \Delta u_n)(t)\right) \\ &\quad + \frac{q-p}{pq}\|\nabla u_n(t)\|_p^p + \frac{1}{q}I(u_n(t)) \\ &\geq \frac{1}{2}\|u_{nt}(t)\|_*^2 + \frac{(q-2)\kappa}{2q}\|\Delta u_n(t)\|^2 + \frac{q-p}{pq}\|\nabla u_n(t)\|_p^p, \end{aligned}$$

which together with (4.7), (4.6), and (3.20) yields

$$\frac{1}{2}\|u_{nt}(t)\|_*^2 + \frac{(q-2)\kappa}{2q}\|\Delta u_n(t)\|^2 + \frac{q-p}{pq}\|\nabla u_n(t)\|_p^p < d \tag{4.8}$$

for all $t \in [0,\infty)$. By virtue of (4.8), we get

$$\||u_n(t)|^{q-2}u_n(t)\|_\gamma^\gamma = \|u_n(t)\|_q^q \le C\|\Delta u_n(t)\|^q < C\left(\frac{2q}{(q-2)\kappa}d\right)^{\frac{q}{2}}$$

for all $t \in [0,\infty)$, where $\gamma = \frac{q}{q-1}$.

Therefore, there exist a subsequence $\{u_n\}$, $u \in \overline{\mathcal{W}}$, and $\chi$ such that as $n \to \infty$,

$$u_n \rightharpoonup u \quad \text{weakly star in } C([0,\infty); H^2(\Omega)\cap H_0^1(\Omega)),$$

$$u_{nt} \rightharpoonup u_t \quad \text{weakly star in } C([0,\infty); H_0^1(\Omega)),$$

and

$$|u_n|^{q-2}u_n \rightharpoonup \chi \quad \text{weakly star in } L^\infty(0,\infty; L^\gamma(\Omega)).$$

In light of Ref. 34 (Lemma 1.3 in Chap. 1), we get $\chi = |u|^{q-2}u$. Therefore, by the analogous arguments in the Proof of Lemma 3.3, $u$ is a global solution to problems (1.1)–(1.3).

Suppose that $u$ and $\bar{u}$ are two solutions to problems (1.1)–(1.3). Set $\tilde{u} := \bar{u} - u$. Then, by the arguments similar to Lemma 3.2, we have

$$\begin{aligned}&\left(\tilde{u}_{tt}(t) - \Delta\tilde{u}_{tt}(t) + \Delta^2\tilde{u}(t) - \int_0^t g(t-\tau)\Delta^2\tilde{u}(\tau)\,\mathrm{d}\tau, \tilde{u}_t(t)\right)\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}_t(t)\|_*^2 + \left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{u}(t)\|^2 + (g\circ\Delta\tilde{u})(t)\right)\\ &\quad - \frac{1}{2}(g'\circ\Delta\tilde{u})(t) + \frac{1}{2}g(t)\|\Delta\tilde{u}(t)\|^2.\end{aligned}$$

Substituting this equality into

$$\begin{aligned}&\left(\tilde{u}_{tt}(t) - \Delta\tilde{u}_{tt}(t) + \Delta^2\tilde{u}(t) - \int_0^t g(t-\tau)\Delta^2\tilde{u}(\tau)\,\mathrm{d}\tau - \Delta_p\bar{u}(t) + \Delta_p u(t)\right.\\ &\quad \left. + \tilde{u}_t(t) - \Delta\tilde{u}_t(t), \tilde{u}_t(t)\right) = \left(|\bar{u}(t)|^{q-2}\bar{u}(t) - |u(t)|^{q-2}u(t), \tilde{u}_t(t)\right),\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\tilde{u}_t(t)\|_*^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{u}(t)\|^2 + \frac{1}{2}(g\circ\Delta\tilde{u})(t)\right)\\ &\quad + \left(|\nabla\bar{u}(t)|^{p-2}\nabla\bar{u}(t) - |\nabla u(t)|^{p-2}\nabla u(t), \nabla\tilde{u}_t(t)\right)\\ &\quad + \|\tilde{u}_t(t)\|_*^2 - \frac{1}{2}(g'\circ\Delta\tilde{u})(t) + \frac{1}{2}g(t)\|\Delta\tilde{u}(t)\|^2\\ &= \left(|\bar{u}(t)|^{q-2}\bar{u}(t) - |u(t)|^{q-2}u(t), \tilde{u}_t(t)\right).\end{aligned}$$

By the arguments similar to the proofs of (3.43)–(3.45), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\tilde{u}_t(t)\|_*^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{u}(t)\|^2 + \frac{1}{2}(g\circ\Delta\tilde{u})(t)\right) \le I_5 + I_6,$$

$$\begin{aligned}I_5 &:= \left(|\bar{u}(t)|^{q-2}\bar{u}(t) - |u(t)|^{q-2}u(t), \tilde{u}_t(t)\right)\\ &\le C\|\Delta\tilde{u}(t)\|^2 + \frac{1}{2}\|\tilde{u}_t(t)\|^2,\end{aligned}$$

and

$$\begin{aligned}I_6 &:= \left|\left(|\nabla\bar{u}(t)|^{p-2}\nabla\bar{u}(t) - |\nabla u(t)|^{p-2}\nabla u(t), \nabla\tilde{u}_t(t)\right)\right|\\ &\le C\|\Delta\tilde{u}(t)\|^2 + \frac{1}{2}\|\nabla\tilde{u}_t(t)\|^2.\end{aligned}$$

Consequently, on account of (3.25), we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\tilde{u}_t(t)\|_*^2+\frac{1}{2}\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{u}(t)\|^2+\frac{1}{2}(g\circ\Delta\tilde{u})(t)\right)$$
$$\le C\left(\frac{1}{2}\|\tilde{u}_t(t)\|_*^2+\frac{1}{2}\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta\tilde{u}(t)\|^2+\frac{1}{2}(g\circ\Delta\tilde{u})(t)\right),$$

which combined with Gronwall's inequality, $\tilde{u}(0)=0$, $\tilde{u}_t(0)=0$, (3.18), and (3.19) tells us that

$$\frac{1}{2}\|\tilde{u}_t(t)\|_*^2+\frac{\kappa}{2}\|\Delta\tilde{u}(t)\|^2\le 0$$

for all $t\in[0,\infty)$. Accordingly, $u\equiv\tilde{u}$ for all $t\in[0,\infty)$.

**Case 2.** $\|\Delta u_0\|=0$.

In this case, $J(u_0)=0$. Thus,

$$E(0)=\frac{1}{2}\|u_1\|_*^2.$$

Set $u_{1n}:=\lambda_n u_1$, where $\lambda_n:=1-\frac{1}{n}$, $n=2,3,\ldots$, and consider problems (1.1) and (1.3) with the following initial conditions:

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_{1n}(x). \tag{4.9}$$

Note that

$$0<E_n(0)=\frac{1}{2}\|u_{1n}\|_*^2<E(0).$$

We conclude from (i) that, for each $n$, problems (1.1), (1.3), and (4.9) admit a unique global solution $u_n(t)\in\mathcal{W}$. The remainder of the proof is similar to that of the proof of case 1.

Thus, the Proof of Theorem 2.4 is completed. □

## V. ASYMPTOTIC BEHAVIOR (PROOF OF THEOREM 2.5)

*Proof of Theorem 2.5.* First of all, Theorem 2.4 and Proposition 4.2 ensure that problems (1.1)–(1.3) have a unique global solution $u(t)\in\mathcal{W}$ for all $t\in[0,\infty)$. We construct a Lyapunov function by performing a suitable modification of the total energy function as follows:

$$L(t):=E(t)+\varepsilon\Psi(t),\quad t\in[0,\infty), \tag{5.1}$$

where $\Psi(t):=(u(t),u_t(t))_*$ and $\varepsilon>0$ is a constant to be determined later.

We now claim that there exist two constants $\gamma_i>0$ $(i=1,2)$, depending on $\varepsilon$, such that

$$\gamma_1 E(t)\le L(t)\le\gamma_2 E(t) \tag{5.2}$$

for all $t\in[0,\infty)$. Indeed, by virtue of the inequalities of Schwarz and Cauchy, we know that

$$|\Psi(t)|\le\frac{1}{2}\|u(t)\|_*^2+\frac{1}{2}\|u_t(t)\|_*^2,$$

and so

$$|\Psi(t)|\le\frac{\mathfrak{C}_3^2+\mathfrak{C}_4^2}{2}\|\Delta u(t)\|^2+\frac{1}{2}\|u_t(t)\|_*^2. \tag{5.3}$$

From (3.18) and (3.19), we have

$$E(t)=\frac{1}{2}\|u_t(t)\|_*^2+\frac{q-2}{2q}\left(\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u(t)\|^2+(g\circ\Delta u)(t)\right)$$
$$+\frac{q-p}{pq}\|\nabla u(t)\|_p^p+\frac{1}{q}I(u(t))$$
$$\ge\frac{1}{2}\|u_t(t)\|_*^2+\frac{(q-2)\kappa}{2q}\|\Delta u(t)\|^2+\frac{q-p}{pq}\|\nabla u(t)\|_p^p \tag{5.4}$$
$$\ge\frac{1}{2}\|u_t(t)\|_*^2+\frac{(q-2)\kappa}{2q}\|\Delta u(t)\|^2. \tag{5.5}$$

From (5.3) and (5.5), we know that there exists a constant $Q>0$ such that $|\Psi(t)|\le QE(t)$, which together with (5.1) yields

$$(1 - \varepsilon Q)E(t) \le L(t) \le (1 + \varepsilon Q)E(t).$$

Thus, in order to guarantee $\gamma_i > 0$ $(i = 1, 2)$ in (5.2), we need to find a proper $\varepsilon$ later such that

$$1 - \varepsilon Q > 0. \tag{5.6}$$

Hence, assertion (5.2) holds.

By the arguments similar to Lemma 3.2, we have

$$\begin{aligned}&\left(u_{tt}(t) - \Delta u_{tt}(t) + \Delta^2 u(t) - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau, u_t(t)\right)\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t(t)\|_*^2 + \left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u(t)\|^2 + (g\circ\Delta u)(t)\right)\\ &\quad - \frac{1}{2}(g'\circ\Delta u)(t) + \frac{1}{2}g(t)\|\Delta u(t)\|^2.\end{aligned}$$

Inserting this equality into

$$\begin{aligned}&\left(u_{tt}(t) - \Delta u_{tt}(t) + \Delta^2 u(t) - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \Delta_p u(t)\right.\\ &\quad\left. + u_t(t) - \Delta u_t(t), u_t(t)\right) = (|u(t)|^{q-2}u(t), u_t(t)),\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u_t(t)\|_*^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u(t)\|^2 + \frac{1}{2}(g\circ\Delta u)(t) + \frac{1}{p}\|\nabla u(t)\|_p^p\right.\\ &\quad\left. - \frac{1}{q}\|u(t)\|_q^q\right) + \|u_t(t)\|_*^2 - \frac{1}{2}(g'\circ\Delta u)(t) + \frac{1}{2}g(t)\|\Delta u(t)\|^2 = 0,\end{aligned}$$

i.e.,

$$E'(t) = \frac{1}{2}(g'\circ\Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|^2 - \|u_t(t)\|_*^2.$$

Hence, from

$$\Psi'(t) = \langle u_{tt}(t), u(t)\rangle_* + \|u_t(t)\|_*^2$$

and Remark 2.2, we further obtain

$$\begin{aligned}L'(t) &= \frac{1}{2}(g'\circ\Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|^2 - \|u_t(t)\|_*^2 + \varepsilon\|u_t(t)\|_*^2\\ &\quad - \varepsilon\|\Delta u(t)\|^2 + \varepsilon\int_0^t g(t-\tau)(\Delta u(\tau), \Delta u(t))\,\mathrm{d}\tau - \varepsilon\|\nabla u(t)\|_p^p\\ &\quad - \varepsilon(u(t), u_t(t))_* + \varepsilon\|u(t)\|_q^q.\end{aligned} \tag{5.7}$$

For the sixth term on the right-hand side of (5.7), it follows from the inequalities of Schwarz and Cauchy and (3.18) that there exists a constant $\epsilon_1 > 0$ to be determined later such that

$$\begin{aligned}&\int_0^t g(t-\tau)(\Delta u(\tau), \Delta u(t))\,\mathrm{d}\tau\\ &= \int_0^t g(t-\tau)\|\Delta u(t)\|^2\,\mathrm{d}\tau + \int_0^t g(t-\tau)(\Delta u(\tau) - \Delta u(t), \Delta u(t))\,\mathrm{d}\tau\\ &\le \|\Delta u(t)\|^2\int_0^t g(\tau)\,\mathrm{d}\tau\\ &\quad + \int_0^t g(t-\tau)\left((2\epsilon_1)^{\frac{1}{2}}\|\Delta u(t)\|\right)\left(\frac{1}{(2\epsilon_1)^{\frac{1}{2}}}\|\Delta u(\tau) - \Delta u(t)\|\right)\mathrm{d}\tau\\ &\le \|\Delta u(t)\|^2\int_0^t g(\tau)\,\mathrm{d}\tau + \epsilon_1\|\Delta u(t)\|^2\int_0^t g(\tau)\,\mathrm{d}\tau + \frac{1}{4\epsilon_1}(g\circ\Delta u)(t)\\ &\le (1-\kappa)\|\Delta u(t)\|^2 + \epsilon_1(1-\kappa)\|\Delta u(t)\|^2 + \frac{1}{4\epsilon_1}(g\circ\Delta u)(t).\end{aligned}$$

For the eighth term on the right-hand side of (5.7), it follows from the inequalities of Schwarz and Cauchy that there exists a constant $\epsilon_2 > 0$ to be determined later such that

$$\begin{aligned}-(u(t), u_t(t))_* &\le \|u(t)\|_* \|u_t(t)\|_* \\ &\le \left((2\epsilon_2)^{\frac{1}{2}}\|u(t)\|_*\right)\left(\frac{1}{(2\epsilon_2)^{\frac{1}{2}}}\|u_t(t)\|_*\right) \\ &\le \epsilon_2\|u(t)\|_*^2 + \frac{1}{4\epsilon_2}\|u_t(t)\|_*^2 \\ &\le \epsilon_2\left(\mathfrak{C}_3^2 + \mathfrak{C}_4^2\right)\|\Delta u(t)\|^2 + \frac{1}{4\epsilon_2}\|u_t(t)\|_*^2.\end{aligned}$$

Hence, on account of $g'(t) \le -\rho g(t)$, we deduce from (5.7) that

$$\begin{aligned}L'(t) \le &-\left(1 - \varepsilon - \frac{\varepsilon}{4\epsilon_2}\right)\|u_t(t)\|_*^2 - \varepsilon\left(\kappa - \epsilon_1(1-\kappa) - \epsilon_2\left(\mathfrak{C}_3^2 + \mathfrak{C}_4^2\right)\right)\|\Delta u(t)\|^2 \\ &- \left(\frac{\rho}{2} - \frac{\varepsilon}{4\epsilon_1}\right)(g\circ\Delta u)(t) - \varepsilon\|\nabla u(t)\|_p^p + \varepsilon\|u(t)\|_q^q,\end{aligned}$$

and so

$$L'(t) \le -\varepsilon\eta E(t) - \sum_{i=7}^{10} I_i + I_{11}, \tag{5.8}$$

where

$$\begin{aligned}I_7 &:= \left(1 - \varepsilon - \frac{\varepsilon}{4\epsilon_2} - \frac{\varepsilon\eta}{2}\right)\|u_t(t)\|_*^2, \\ I_8 &:= \varepsilon\left(\kappa - \epsilon_1(1-\kappa) - \epsilon_2\left(\mathfrak{C}_3^2 + \mathfrak{C}_4^2\right) - \frac{\eta}{2}\right)\|\Delta u(t)\|^2, \\ I_9 &:= \left(\frac{\rho}{2} - \frac{\varepsilon}{4\epsilon_1} - \frac{\varepsilon\eta}{2}\right)(g\circ\Delta u)(t), \\ I_{10} &:= \varepsilon\left(1 - \frac{\eta}{p}\right)\|\nabla u(t)\|_p^p, \\ I_{11} &:= \varepsilon\|u(t)\|_q^q,\end{aligned}$$

and $\eta > 0$ is a constant to be determined later.

Next, we claim that

$$L'(t) \le -\varepsilon\eta E(t) \tag{5.9}$$

for sufficiently small $\eta$ and $\varepsilon$. To confirm this, we need to show that $I_{11}$ can be controlled by $I_8$ or $I_{10}$ on the right-hand side of (5.8), i.e.,

$$-I_8 - I_{10} + I_{11} \le 0. \tag{5.10}$$

To achieve this, we shall well deal with the energy identity (2.4), and thus, $E(0)$ will appear in the energy estimates. Taking into account the restriction $E(0) < \overline{d}$ and the definition of $\overline{d}$, we shall prove (5.10) by discussing the following three cases.

Case 1. $\overline{d} = d_1$.

From (5.4), (2.4), and (3.20), it follows that

$$\frac{(q-2)\kappa}{2q}\|\Delta u(t)\|^2 + \frac{q-p}{pq}\|\nabla u(t)\|_p^p \le E(0), \tag{5.11}$$

which implies

$$\|\Delta u(t)\| \le \left(\frac{2q}{(q-2)\kappa}E(0)\right)^{\frac{1}{2}}.$$

Hence,

$$\|u(t)\|_q^q \le \mathfrak{C}_1^q\mathfrak{C}_2^q\|\Delta u(t)\|^q = \mathfrak{C}_1^q\mathfrak{C}_2^q\|\Delta u(t)\|^{q-2}\|\Delta u(t)\|^2 \le \delta_1\|\Delta u(t)\|^2,$$

where

$$\delta_1 := \mathfrak{C}_1^q \mathfrak{C}_2^q \left( \frac{2q}{(q-2)\kappa} E(0) \right)^{\frac{q-2}{2}}.$$

Note that

$$0 < \delta_1 < \mathfrak{C}_1^q \mathfrak{C}_2^q \left( \frac{2q}{(q-2)\kappa} d_1 \right)^{\frac{q-2}{2}}.$$

Substituting the expression of $d_1$, i.e., (2.1) into the right-hand side of the above inequality, we get

$$\mathfrak{C}_1^q \mathfrak{C}_2^q \left( \frac{2q}{(q-2)\kappa} d_1 \right)^{\frac{q-2}{2}} = \kappa,$$

which means $\kappa > \delta_1$. Thus, we can choose sufficiently small $\epsilon_i$ $(i = 1, 2)$ and $\eta$ such that

$$\kappa - \epsilon_1(1-\kappa) - \epsilon_2(\mathfrak{C}_3^2 + \mathfrak{C}_4^2) - \frac{\eta}{2} - \delta_1 > 0$$

and

$$1 - \frac{\eta}{p} > 0.$$

These two formulas imply $-I_8 + I_{11} \le 0$ and $-I_{10} \le 0$; hence, (5.10) is proved.

Case 2. $\overline{d} = d_2$.

It is easy to see from (5.11) that

$$\|\nabla u(t)\|_p \le \mathfrak{C}_1 \|\Delta u(t)\| \le \left( \frac{2q\mathfrak{C}_1^2}{(q-2)\kappa} E(0) \right)^{\frac{1}{2}}.$$

Hence,

$$\|u(t)\|_q^q \le \mathfrak{C}_2^q \|\nabla u(t)\|_p^q = \mathfrak{C}_2^q \|\nabla u(t)\|_p^{q-p} \|\nabla u(t)\|_p^p \le \delta_2 \|\nabla u(t)\|_p^p,$$

where

$$\delta_2 := \mathfrak{C}_2^q \left( \frac{2q\mathfrak{C}_1^2}{(q-2)\kappa} E(0) \right)^{\frac{q-p}{2}}.$$

Observing

$$0 < \delta_2 < \mathfrak{C}_2^q \left( \frac{2q\mathfrak{C}_1^2}{(q-2)\kappa} d_2 \right)^{\frac{q-p}{2}},$$

we discover that the substitution of the expression of $d_2$, namely, (2.2), into the right-hand side of this inequality gives

$$\mathfrak{C}_2^q \left( \frac{2q\mathfrak{C}_1^2}{(q-2)\kappa} d_2 \right)^{\frac{q-p}{2}} = 1.$$

Thus, $\delta_2 < 1$, and we can choose sufficiently small $\epsilon_i$ $(i = 1, 2)$ and $\eta$ such that

$$\kappa - \epsilon_1(1-\kappa) - \epsilon_2(\mathfrak{C}_3^2 + \mathfrak{C}_4^2) - \frac{\eta}{2} > 0 \tag{5.12}$$

and

$$1 - \frac{\eta}{p} - \delta_2 > 0.$$

We further get $-I_8 \le 0$ and $-I_{10} + I_{11} \le 0$. Thus, (5.10) is derived here.

Case 3. $\overline{d} = d_3$.

Using (5.11) again, we have

$$\|\nabla u(t)\|_p \le \left( \frac{pq}{q-p} E(0) \right)^{\frac{1}{p}}.$$

Hence,

$$\|u(t)\|_q^q \le \mathfrak{C}_2^q \|\nabla u(t)\|_p^q = \mathfrak{C}_2^q \|\nabla u(t)\|_p^{q-p} \|\nabla u(t)\|_p^p \le \delta_3 \|\nabla u(t)\|_p^p,$$

where

$$\delta_3 := \mathfrak{C}_2^q \left( \frac{pq}{q-p} E(0) \right)^{\frac{q-p}{p}}.$$

Taking into account the expression of $d_3$, namely, (2.3), and

$$0 < \delta_3 < \mathfrak{C}_2^q \left( \frac{pq}{q-p} d_3 \right)^{\frac{q-p}{p}},$$

we get $\delta_3 < 1$. Thus, we readily choose sufficiently small $\epsilon_i$ $(i = 1, 2)$ and $\eta$ such that we have (5.12) and

$$1 - \frac{\eta}{p} - \delta_3 > 0.$$

They again show that $-I_8 \le 0$ and $-I_{10} + I_{11} \le 0$. Hence, (5.10) is demonstrated again.

Having proved (5.10), we now turn to deal with $I_7$ and $I_9$. For fixed $\epsilon_i$ $(i = 1, 2)$ and $\eta$ in the above three cases, in order to make sure the nonnegativity of $I_7$, we need

$$1 - \varepsilon - \frac{\varepsilon}{4\epsilon_2} - \frac{\varepsilon\eta}{2} > 0,$$

i.e.,

$$\varepsilon < \frac{4\epsilon_2}{4\epsilon_2 + 1 + 2\eta\epsilon_2}.$$

In addition, to ensure $I_9 \ge 0$, we require

$$\frac{\rho}{2} - \frac{\varepsilon}{4\epsilon_1} - \frac{\varepsilon\eta}{2} > 0,$$

i.e.,

$$\varepsilon < \frac{2\rho\epsilon_1}{1 + 2\eta\epsilon_1}.$$

Hence, by recalling (5.6), we choose

$$\varepsilon < \min\left\{ \frac{1}{Q}, \frac{4\epsilon_2}{4\epsilon_2 + 1 + 2\eta\epsilon_2}, \frac{2\rho\epsilon_1}{1 + 2\eta\epsilon_1} \right\}.$$

Thus, assertion (5.9) is verified.

By virtue of assertions (5.9) and (5.2), we obtain

$$L'(t) \le -\frac{\varepsilon\eta}{\gamma_2} L(t).$$

Solving this differential inequality, we get

$$L(t) \le Ce^{-\frac{\varepsilon\eta}{\gamma_2} t}, \quad t \in [0, \infty).$$

Again, by assertion (5.2), we see that

$$E(t) \le \frac{C}{\gamma_1} e^{-\frac{\varepsilon\eta}{\gamma_2} t}, \quad t \in [0, \infty),$$

which combined with (5.5) gives (2.6), where $\alpha = \frac{C}{\gamma_1}$ and $\beta = \frac{\varepsilon\eta}{\gamma_2}$. □

## VI. FINITE TIME BLOW-UP (PROOFS OF THEOREMS 2.6–2.8)

This section is devoted to the proofs of the finite time blow-up of solutions to problems (1.1)–(1.3) with negative initial energy, null initial energy, and positive initial energy strictly below the depth of the potential well and arbitrary positive initial energy. For the convenience of readers, we first give the following lemma, which plays a key role in the Proofs of Theorems 2.6–2.8.

*Lemma 6.1 (Refs. 28 and 35). If a function $\phi(t) > 0$ is twice differentiable, which satisfies $\phi'(0) > 0$ and for all $t > 0$,*

$$\phi(t)\phi''(t) - (1+\lambda)\big(\phi'(t)\big)^2 \ge 0 \tag{6.1}$$

*with some constant $\lambda > 0$, then there exists a time $0 < T_0 \le \frac{\phi(0)}{\lambda\phi'(0)}$ such that $\lim\limits_{t\to T_0}\phi(t) = \infty$.*

Now, we prove the finite time blow-up of solutions to problems (1.1)–(1.3) with non-positive initial energy, i.e., Theorem 2.6.

*Proof of Theorem 2.6.* Let $u$ be the solution to problems (1.1)–(1.3). Next, we prove $T < \infty$. Arguing by contradiction, we suppose that $T = \infty$. Then, we consider an auxiliary function $\phi : [0, T_1] \to \mathbb{R}^+$ defined by

$$\phi(t) := \|u(t)\|_*^2 + \int_0^t \|u(\tau)\|_*^2\,\mathrm{d}\tau + (T_1 - t)\|u_0\|_*^2 + \sigma(t+T_2)^2,$$

where $0 < T_1 < \infty$ due to $T = \infty$ and $\sigma$ and $T_2$ are two constants to be determined later.

**Case (i).** $E(0) < 0$.

We perform the proof along three steps.

**Step 1.** We first claim that

$$\phi(t)\phi''(t) - \frac{q+2}{4}\big(\phi'(t)\big)^2 \ge \phi(t)\varphi(t) \tag{6.2}$$

for a.e. $t \in [0, T_1]$, where

$$\varphi(t) := \phi''(t) - (q+2)\left(\|u_t(t)\|_*^2 + \int_0^t \|u_\tau(\tau)\|_*^2\,\mathrm{d}\tau + \sigma\right). \tag{6.3}$$

To see this, we first deal with the term $\big(\phi'(t)\big)^2$. Since

$$\begin{aligned}\phi'(t) &= 2(u(t), u_t(t))_* + \|u(t)\|_*^2 - \|u_0\|_*^2 + 2\sigma(t+T_2)\\ &= 2(u(t), u_t(t))_* + 2\int_0^t (u(\tau), u_\tau(\tau))_*\,\mathrm{d}\tau + 2\sigma(t+T_2),\end{aligned}$$

we get

$$\begin{aligned}\big(\phi'(t)\big)^2 = 4\Bigg(&(u(t), u_t(t))_*^2 + \left(\int_0^t (u(\tau), u_\tau(\tau))_*\,\mathrm{d}\tau\right)^2\\ &+ 2(u(t), u_t(t))_* \int_0^t (u(\tau), u_\tau(\tau))_*\,\mathrm{d}\tau + 2\sigma(t+T_2)(u(t), u_t(t))_*\\ &+ 2\sigma(t+T_2)\int_0^t (u(\tau), u_\tau(\tau))_*\,\mathrm{d}\tau + \sigma^2(t+T_2)^2\Bigg).\end{aligned}$$

By the inequalities of Schwarz and Cauchy, we see that

$$(u(t), u_t(t))_*^2 \le \|u(t)\|_*^2\|u_t(t)\|_*^2,$$

$$\left(\int_0^t (u(\tau), u_\tau(\tau))_*\,\mathrm{d}\tau\right)^2 \le \int_0^t \|u(\tau)\|_*^2\,\mathrm{d}\tau \int_0^t \|u_\tau(\tau)\|_*^2\,\mathrm{d}\tau,$$

$$\begin{aligned}&2(u(t), u_t(t))_* \int_0^t (u(\tau), u_\tau(\tau))_*\,\mathrm{d}\tau\\ &\le 2\|u(t)\|_*\|u_t(t)\|_*\left(\int_0^t \|u(\tau)\|_*^2\,\mathrm{d}\tau\right)^{\frac12}\left(\int_0^t \|u_\tau(\tau)\|_*^2\,\mathrm{d}\tau\right)^{\frac12}\\ &\le \|u(t)\|_*^2\int_0^t \|u_\tau(\tau)\|_*^2\,\mathrm{d}\tau + \|u_t(t)\|_*^2\int_0^t \|u(\tau)\|_*^2\,\mathrm{d}\tau,\end{aligned}$$

$$2\sigma(t+T_2)(u(t),u_t(t))_* \le 2\Big(\sigma^{\frac{1}{2}}\|u(t)\|_*\Big)\Big(\sigma^{\frac{1}{2}}(t+T_2)\|u_t(t)\|_*\Big)$$
$$\le \sigma\|u(t)\|_*^2 + \sigma(t+T_2)^2\|u_t(t)\|_*^2,$$

and

$$2\sigma(t+T_2)\int_0^t (u(\tau),u_\tau(\tau))_*\,\mathrm{d}\tau \le \int_0^t 2\Big(\sigma^{\frac{1}{2}}\|u(\tau)\|_*\Big)\Big(\sigma^{\frac{1}{2}}(t+T_2)\|u_\tau(\tau)\|_*\Big)\,\mathrm{d}\tau$$
$$\le \sigma\int_0^t \|u(\tau)\|_*^2\,\mathrm{d}\tau + \sigma(t+T_2)^2\int_0^t \|u_\tau(\tau)\|_*^2\,\mathrm{d}\tau.$$

Consequently,

$$\begin{aligned}(\phi'(t))^2 &\le 4\Big(\|u(t)\|_*^2 + \int_0^t \|u(\tau)\|_*^2\,\mathrm{d}\tau + \sigma(t+T_2)^2\Big)\Big(\|u_t(t)\|_*^2 + \int_0^t \|u_\tau(\tau)\|_*^2\,\mathrm{d}\tau + \sigma\Big)\\ &\le 4\phi(t)\Big(\|u_t(t)\|_*^2 + \int_0^t \|u_\tau(\tau)\|_*^2\,\mathrm{d}\tau + \sigma\Big).\end{aligned} \tag{6.4}$$

Thus, assertion (6.2) follows from (6.4).

**Step 2.** We show that $\varphi(t) \ge 0$. Indeed, a direct calculation yields

$$\phi''(t) = 2\langle u_{tt}(t),u(t)\rangle_* + 2\|u_t(t)\|_*^2 + 2(u(t),u_t(t))_* + 2\sigma.$$

Owing to Remark 2.2, we discover

$$\begin{aligned}\phi''(t) = {}& 2\|u_t(t)\|_*^2 - 2\|\Delta u(t)\|^2 + 2\int_0^t g(t-\tau)(\Delta u(\tau),\Delta u(t))\,\mathrm{d}\tau\\ &- 2\|\nabla u(t)\|_p^p + 2\|u(t)\|_q^q + 2\sigma\end{aligned} \tag{6.5}$$

for a.e. $t \in [0,T_1]$. Applying the inequalities of Schwarz and Cauchy, the third term on the right-hand side of (6.5) becomes

$$\begin{aligned}&2\int_0^t g(t-\tau)(\Delta u(\tau),\Delta u(t))\,\mathrm{d}\tau\\ &= 2\int_0^t g(t-\tau)\|\Delta u(t)\|^2\,\mathrm{d}\tau + 2\int_0^t g(t-\tau)(\Delta u(\tau)-\Delta u(t),\Delta u(t))\,\mathrm{d}\tau\\ &\ge 2\int_0^t g(\tau)\,\mathrm{d}\tau\|\Delta u(t)\|^2\\ &\quad - 2\int_0^t g(t-\tau)\Big((2\epsilon)^{\frac{1}{2}}\|\Delta u(\tau)-\Delta u(t)\|\Big)\Big(\frac{1}{(2\epsilon)^{\frac{1}{2}}}\|\Delta u(t)\|\Big)\,\mathrm{d}\tau\\ &\ge 2\int_0^t g(\tau)\,\mathrm{d}\tau\|\Delta u(t)\|^2 - 2\epsilon(g\circ\Delta u)(t) - \frac{1}{2\epsilon}\int_0^t g(\tau)\,\mathrm{d}\tau\|\Delta u(t)\|^2\end{aligned}$$

for some constant $\epsilon > 0$ to be determined later. Substituting this inequality into (6.5), we get

$$\begin{aligned}\phi''(t) &\ge 2\|u_t(t)\|_*^2 - 2\|\Delta u(t)\|^2 - 2\|\nabla u(t)\|_p^p + 2\int_0^t g(\tau)\,\mathrm{d}\tau\|\Delta u(t)\|^2\\ &\quad - 2\epsilon(g\circ\Delta u)(t) - \frac{1}{2\epsilon}\int_0^t g(\tau)\,\mathrm{d}\tau\|\Delta u(t)\|^2 + 2\|u(t)\|_q^q + 2\sigma\\ &= 2\|u_t(t)\|_*^2 - 2I(u(t)) + (2-2\epsilon)(g\circ\Delta u)(t)\\ &\quad - \frac{1}{2\epsilon}\int_0^t g(\tau)\,\mathrm{d}\tau\|\Delta u(t)\|^2 + 2\sigma.\end{aligned} \tag{6.6}$$

Combining (6.3) with (6.6), for $t \in [0,T_1]$, we obtain

$$\begin{aligned}\varphi(t) \ge {}& 2\|u_t(t)\|_*^2 - 2I(u(t)) + (2-2\epsilon)(g\circ\Delta u)(t) - \frac{1}{2\epsilon}\int_0^t g(\tau)\,\mathrm{d}\tau\|\Delta u(t)\|^2\\ &+ 2\sigma - (q+2)\Big(\|u_t(t)\|_*^2 + \int_0^t \|u_\tau(\tau)\|_*^2\,\mathrm{d}\tau + \sigma\Big).\end{aligned} \tag{6.7}$$

Next, we move on to deal with $I(u(t))$. Recalling (2.4) and (3.20), we have

$$E(0) \geq \int_0^t \|u_\tau(\tau)\|_*^2 \, d\tau + \frac{1}{2}\|u_t(t)\|_*^2 + \frac{1}{q}I(u(t)) + \frac{q-2}{2q}\left(\left(1 - \int_0^t g(\tau)\, d\tau\right)\|\Delta u(t)\|^2 + (g \circ \Delta u)(t)\right) + \frac{q-p}{pq}\|\nabla u(t)\|_p^p,$$

which leads to

$$-2I(u(t)) \geq -2qE(0) + 2q\int_0^t \|u_\tau(\tau)\|_*^2 \, d\tau + q\|u_t(t)\|_*^2 + (q-2)\left(\left(1 - \int_0^t g(\tau)\, d\tau\right)\|\Delta u(t)\|^2 + (g \circ \Delta u)(t)\right) + \frac{2(q-p)}{p}\|\nabla u(t)\|_p^p$$

for $t \in [0, T_1]$. Substituting the above inequality into (6.7), we arrive at

$$\begin{aligned}\varphi(t) &\geq (q-2)\int_0^t \|u_\tau(\tau)\|_*^2 \, d\tau + \left(q - 2 - \left(q - 2 + \frac{1}{2\epsilon}\right)\int_0^t g(\tau)\, d\tau\right)\|\Delta u(t)\|^2 \\ &\quad - 2qE(0) - q\sigma + (q - 2\epsilon)(g \circ \Delta u)(t) + \frac{2(q-p)}{p}\|\nabla u(t)\|_p^p \\ &\geq \left(q - 2 - \left(q - 2 + \frac{1}{2\epsilon}\right)\int_0^t g(\tau)\, d\tau\right)\|\Delta u(t)\|^2 - 2qE(0) - q\sigma \\ &\quad + (q - 2\epsilon)(g \circ \Delta u)(t) + \frac{2(q-p)}{p}\|\nabla u(t)\|_p^p.\end{aligned} \tag{6.8}$$

By choosing $\epsilon = \frac{q}{2}$ and $0 < \sigma \leq -2E(0)$, we see from (2.7) that

$$q - 2 - \left(q - 2 + \frac{1}{2\epsilon}\right)\int_0^t g(\tau)\, d\tau \geq 0. \tag{6.9}$$

We also see from $(A_2)$ and $(A_3)$ that

$$\frac{2(q-p)}{p} > 0. \tag{6.10}$$

To sum up, we obtain $\varphi(t) \geq 0$.

**Step 3.** We claim that there exists a finite time $T_0$ such that

$$\lim_{t \to T_0} \phi(t) = \infty. \tag{6.11}$$

Indeed, a combination of steps 1 and 2 shows that (6.1) holds for a.e. $t \in [0, T_1]$, where $\lambda = \frac{q-2}{4}$. In order to ensure $\phi'(0) = 2(u_0, u_1)_* + 2\sigma T_2 > 0$, due to the assumption $T = \infty$, we can choose

$$T_2 > \max\left\{-\frac{(u_0, u_1)_*}{\sigma}, 0\right\}.$$

Consequently, by Lemma 6.1, there exists

$$0 < T_0 \leq \frac{2\left(\|u_0\|_*^2 + T_1\|u_0\|_*^2 + \sigma T_2^2\right)}{(q-2)\left((u_0, u_1)_* + \sigma T_2\right)}$$

such that assertion (6.11) holds, which contradicts $T = \infty$. This completes the Proof of Theorem 2.6 for case (i).

**Case (ii).** $E(0) = 0$ and $(u_0, u_1)_* > 0$.

Similar to case (i), arguing by contradiction, we also suppose that $T = \infty$. For this case, (6.2) and (6.8) still hold. Taking $\epsilon = \frac{q}{2}$ and $\sigma = 0$, we conclude from (6.9) and (6.10) that $\varphi(t) \geq 0$. Hence, we verify (6.1) again. By Lemma 6.1 and $\phi'(0) = 2(u_0, u_1)_* > 0$, there exists a finite time $T_0$ such that (6.11) holds again, where

$$0 < T_0 \leq \frac{2\left(\|u_0\|_*^2 + T_1\|u_0\|_*^2\right)}{(q-2)(u_0, u_1)_*}.$$

Thus, the Proof of Theorem 2.6 for case (ii) is finished by the contradiction between (6.11) and $T = \infty$. □

Next, we prove the finite time blow-up of solutions to problems (1.1)–(1.3) with positive initial energy strictly below the depth of the potential well, i.e., Theorem 2.7.

*Proof of Theorem 2.7.* We adopt the contradictory argument, a similar structure to the Proof of Theorem 2.6 for case (i), to finish the proof of this theorem. Considering that many estimates and analyses in the Proof of Theorem 2.6 are not related to the initial condition, by quickly checking steps 1 and 2 there, we find that (6.2) and (6.8) are still available in this proof here. In addition, next, similar to step 2 in the Proof of Theorem 2.6, we shall prove $\varphi(t) > 0$.

In (6.8), choosing $\sigma = 2(\theta\tilde{d} - E(0))$, we deduce that

$$\varphi(t) \geq I_{12} + I_{13},$$

where

$$\begin{aligned} I_{12} := &\left(q-2-\left(q-2+\frac{1}{2\epsilon}\right)\int_0^t g(\tau)\,\mathrm{d}\tau - \theta(q-2)\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\right)\|\Delta u(t)\|^2 \\ &+ (q-2\epsilon-\theta(q-2))(g\circ\Delta u)(t) \\ &+ \left(\frac{2(q-p)}{p} - \theta(q-2)\right)\|\nabla u(t)\|_p^p \end{aligned}$$

and

$$I_{13} := \theta(q-2)\left(\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u(t)\|^2 + (g\circ\Delta u)(t) + \|\nabla u(t)\|_p^p\right) - 2q\theta\tilde{d}.$$

For $I_{12}$, taking $\epsilon = \frac{q-\theta(q-2)}{2}$, we deduce from $g(t) \geq 0$ in $(\mathrm{A}_1)$ that

$$\begin{aligned} q-2-&\left(q-2+\frac{1}{2\epsilon}\right)\int_0^t g(\tau)\,\mathrm{d}\tau - \theta(q-2)\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right) \\ &\geq (q-2)(1-\theta) - \frac{(q-2)(1-\theta)(q-\theta(q-2))+1}{q-\theta(q-2)}\int_0^\infty g(\tau)\,\mathrm{d}\tau, \end{aligned}$$

which together with (2.8) yields

$$q-2-\left(q-2+\frac{1}{2\epsilon}\right)\int_0^t g(\tau)\,\mathrm{d}\tau - \theta(q-2)\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right) \geq 0.$$

Recalling the condition $0 < \theta < \frac{2(q-p)}{p(q-2)}$, we see that

$$\frac{2(q-p)}{p} - \theta(q-2) > 0.$$

Hence, $I_{12} > 0$.

For $I_{13}$, we note that if

$$\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u(t)\|^2 + (g\circ\Delta u)(t) + \|\nabla u(t)\|_p^p > \frac{2q}{q-2}\tilde{d}, \tag{6.12}$$

then $I_{13} > 0$ holds for all $t \in [0, T_1]$. Therefore, it remains to prove (6.12). In fact, from $I(u_0) < 0$, $0 < E(0) < \theta\tilde{d}$, and Proposition 4.2, and (ii) in Lemma 4.1, we have $u(t) \in \mathcal{V}$ for all $t \in [0, T_1]$. This implies $I(u(t)) < 0$, which combined with the Sobolev inequality gives

$$\left(1-\int_0^t g(\tau)\,\mathrm{d}\tau\right)\|\Delta u(t)\|^2 + (g\circ\Delta u)(t) + \|\nabla u(t)\|_p^p < \mathfrak{C}_2^q\|\nabla u(t)\|_p^q.$$

Recalling (3.18) and (3.19), the above inequality becomes

$$\frac{\kappa}{\mathfrak{C}_1^2}\|\nabla u(t)\|_p^2 + \|\nabla u(t)\|_p^p < \mathfrak{C}_2^q\|\nabla u(t)\|_p^q,$$

which makes (4.2) hold again. On account of the definition of $\tilde{d}$, we shall proceed to prove (6.12) by considering the following two cases.

Case 1. $\tilde{d} = d_4$.

In view of (4.2), i.e., $\|\nabla u\|_p > \kappa^{\frac{1}{q-2}} \mathfrak{C}_1^{-\frac{2}{q-2}} \mathfrak{C}_2^{-\frac{q}{q-2}}$, we obtain

$$\begin{aligned}\kappa\|\Delta u(t)\|^2 + \|\nabla u(t)\|_p^p &\geq \frac{\kappa}{\mathfrak{C}_1^2}\|\nabla u(t)\|_p^2 + \|\nabla u(t)\|_p^p \\ &> \kappa^{\frac{q}{q-2}} \mathfrak{C}_1^{-\frac{2q}{q-2}} \mathfrak{C}_2^{-\frac{2q}{q-2}} + \kappa^{\frac{p}{q-2}} \mathfrak{C}_1^{-\frac{2p}{q-2}} \mathfrak{C}_2^{-\frac{pq}{q-2}} \\ &\geq \frac{2q}{q-2} d_4,\end{aligned} \tag{6.13}$$

which together with (3.18) and (3.19) gives (6.12).

Case 2. $\tilde{d} = d_5$.

In such a case, from (6.13) and $\|\nabla u\|_p > \mathfrak{C}_2^{-\frac{q}{q-p}}$ indicated by (4.2), we derive

$$\begin{aligned}\kappa\|\Delta u(t)\|^2 + \|\nabla u(t)\|_p^p &> \kappa \mathfrak{C}_1^{-2} \mathfrak{C}_2^{-\frac{2q}{q-p}} + \mathfrak{C}_2^{-\frac{pq}{q-p}} \\ &\geq \frac{2q}{q-2} d_5,\end{aligned}$$

which together with (3.18) and (3.19) still gives (6.12).

Therefore, $\varphi(t) > 0$.

The remainder of the Proof of Theorem 2.7 can be finished by a repetition of step 3 in the Proof of Theorem 2.6 for case (i). □

To prove Theorem 2.8, we need Lemma 6.2 to construct an increasing auxiliary function. Its proof is similar to that in Ref. 36, Lemma 2.1.

*Lemma 6.2. Let* $(A_1)$–$(A_3)$ *and (2.9) be fulfilled. If a function* $M(t)$ *is twice continuously differentiable and also satisfies* $M(0) > 0$, $M'(0) > 0$, *and*

$$M''(t) + M'(t) > \int_0^t g(t-\tau)(\Delta u(\tau), \Delta u(t))\, d\tau$$

*for all* $t \in [0, T)$, *then* $M(t)$ *is strictly increasing on* $[0, T)$, *where* $u$ *is the solution to problems (1.1)–(1.3).*

Set $M(t) := \|u(t)\|_*^2$, where $u$ is the solution to problems (1.1)–(1.3). Then, making use of Lemma 6.2, we can derive the monotonicity of $M(t)$ stated in Lemma 6.3.

*Lemma 6.3. Let* $(A_1)$–$(A_3)$ *and (2.9) be fulfilled. Assume that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, *and* $(u_0, u_1)_* > 0$. *Then,* $M(t)$ *is strictly increasing on* $[0, T)$, *provided* $\psi(u(t)) < 0$, *where* $\psi(u(t))$ *is the functional defined by (2.12).*

*Proof.* Since

$$M'(t) = 2(u(t), u_t(t))_*$$

and

$$\begin{aligned}M''(t) &= 2\|u_t(t)\|_*^2 + 2\langle u_{tt}(t), u(t)\rangle_* \\ &= 2\|u_t(t)\|_*^2 - 2\psi(u(t)) + 2\int_0^t g(t-\tau)(\Delta u(\tau), \Delta u(t))\, d\tau - 2(u(t), u_t(t))_*,\end{aligned}$$

we get

$$M''(t) + M'(t) = 2\|u_t(t)\|_*^2 - 2\psi(u(t)) + 2\int_0^t g(t-\tau)(\Delta u(\tau), \Delta u(t))\, d\tau.$$

By $M'(0) = 2(u_0, u_1)_* > 0$ and Lemma 6.2, we have $M'(t) > 0$ on $[0, T)$. □

For the functional $\psi(u(t))$ defined by (2.12), we have the following conclusion.

*Lemma 6.4. Let* $(A_1)$–$(A_3)$, *(2.9), and (2.10) be fulfilled. Assume that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, $E(0) > 0$, $(u_0, u_1)_* > 0$, $\psi(u_0) < 0$, *and there holds (2.11). Then,* $\psi(u(t)) < 0$ *for all* $t \in [0, T)$.

*Proof.* Arguing by contradiction, from $\psi(u_0) < 0$ and the continuity of $u(t)$ in $t$, we suppose that there exists the first time $0 < t_0 < T$ such that $\psi(u(t_0)) = 0$. Then, by Lemma 6.3 and (2.11), we get

$$M(t_0) > M(0) = \|u_0\|_*^2 \geq \frac{2q(\mathfrak{C}_3^2 + \mathfrak{C}_4^2)}{\min\{\kappa q - 2, \varrho\}} E(0). \tag{6.14}$$

On the other hand, by using (3.18) and (3.19) again, we have

$$E(t_0) \geq \frac{\kappa}{2}\|\Delta u(t_0)\|^2 + \frac{1}{p}\|\nabla u(t_0)\|_p^p - \frac{1}{q}\|u(t_0)\|_q^q$$
$$= \frac{\kappa q - 2}{2q}\|\Delta u(t_0)\|^2 + \frac{q-p}{pq}\|\nabla u(t_0)\|_p^p + \frac{1}{q}\psi(u(t_0)),$$

which together with $\psi(u(t_0)) = 0$, (2.4), and (3.20) allows us to derive

$$\frac{\kappa q - 2}{2q}\|\Delta u(t_0)\|^2 \leq E(0).$$

Hence,

$$M(t_0) = \|u(t_0)\|_*^2 \leq \left(\mathfrak{C}_3^2 + \mathfrak{C}_4^2\right)\|\Delta u(t_0)\|^2 \leq \frac{2q\left(\mathfrak{C}_3^2 + \mathfrak{C}_4^2\right)}{\kappa q - 2}E(0),$$

which contradicts (6.14). Thus, the Proof of Lemma 6.4 is completed. □

In the end, we prove the finite time blow-up of solutions to problems (1.1)–(1.3) with arbitrary positive initial energy, i.e., Theorem 2.8.

*Proof of Theorem 2.8.* Again, by the arguments, by contradiction similar to the Proof of Theorem 2.6 for case (i), we see that (6.2) and (6.8) hold here. Taking $\epsilon = \frac{q}{2}$ and $\sigma = 0$ in (6.8), we have

$$\varphi(t) \geq \left(q - 2 - \left(q - 2 + \frac{1}{q}\right)(1-\kappa)\right)\|\Delta u(t)\|^2 - 2qE(0),$$

which together with (2.11) gives

$$\varphi(t) \geq \varrho\|\Delta u(t)\|^2 - \frac{\min\{\kappa q - 2, \varrho\}}{\mathfrak{C}_3^2 + \mathfrak{C}_4^2}\|u_0\|_*^2.$$

According to Lemmas 6.4 and 6.3, we obtain

$$\varrho\|\Delta u(t)\|^2 \geq \frac{\varrho}{\mathfrak{C}_3^2 + \mathfrak{C}_4^2}\|u(t)\|_*^2 > \frac{\varrho}{\mathfrak{C}_3^2 + \mathfrak{C}_4^2}\|u_0\|_*^2.$$

Hence, $\varphi(t) > 0$. The remainder of the proof is same as that of the Proof of Theorem 2.6 for case (ii). □




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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

### Author Contributions

**Yang Liu**: Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Byungsoo Moon**: Investigation (equal); Methodology (equal); Writing – review & editing (equal). **Vicenţiu D. Rădulescu**: Investigation (equal); Methodology (equal); Writing – review & editing (equal). **Runzhang Xu**: Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Chao Yang**: Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).

## DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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