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ORIGINAL ARTICLE

Nonhomogeneous multiparameter problems in Orlicz–Sobolev spaces

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Abstract

This paper is concerned with the existence and multiplicity of solutions for a class of problems involving the Φ-Laplacian operator with general assumptions on the nonlinearities, which include both semipositone cases and critical concave convex problems. The research is based on the subsupersolution technique combined with a truncation argument and an application of the Mountain Pass Theorem. The results in this paper improve and complement some recent contributions to this field.

KEYWORDS

Φ-Laplacian, concentration-compactness principle, Mountain Pass Theorem, semipositone problem, subsupersolution, variational method

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1 | INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary. In this paper, we consider the problem

$$\begin{cases} -\Delta_\Phi u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\lambda > 0$ and $\mu \in \mathbb{R}$ are parameters, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, and Δ_Φ denotes the Φ-Laplacian operator, which is defined by

$$\Delta_\Phi w := \operatorname{div}(\phi(|\nabla w|)\nabla w), \tag{1.1}$$

with

$$\Phi(t) := \int_0^{|t|} \phi(s)s\,ds, \tag{1.2}$$

where $\phi : [0, +\infty) \to [0, +\infty)$ belongs to $C^1([0, +\infty), \mathbb{R})$ and satisfies the following properties:

$$\phi(t), (t\phi(t))' > 0, \quad \text{for all } t > 0, \tag{ϕ_1}$$

$$\lim_{t\to 0^+} t\phi(t) = 0, \quad \lim_{t\to+\infty} t\phi(t) = +\infty, \tag{ϕ_2}$$

and

$$\ell - 1 \leq \frac{(\phi(t)t)'}{\phi(t)} \leq m - 1, \text{ for all } t > 0, \tag{ϕ_3'}$$

for some $\ell, m \in (1, N)$, such that $\ell \leq m < \ell^\star := N\ell/(N - \ell)$.

Note that $(\phi_3)'$ is a particular case of the more general condition

$$\ell \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m, \text{ for all } t > 0. \tag{ϕ_3}$$

Regarding the nonlinearities f and g, it will be considered that g maps bounded sets in bounded sets and that there are constants $C_1, C_2 \geq 0$ such that

$$0 \leq f(x, t) \leq C_1 t^{q-1} + C_2, \text{ for all } \; t \geq 0, \text{ a.e. in } \; \Omega, \tag{f_1}$$

for some $1 \leq q < \ell$ and

$$f(x, t) \geq C_3, \text{ for all } \; t \geq t_0, \text{ a.e. in } \; \Omega, \tag{f_2}$$

for constants $C_3, t_0 > 0$.

A weak solution of (P) is a strictly positive function $u \in W_0^{1,\Phi}(\Omega)$ satisfying

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla v = \int_\Omega (\lambda f(x, u) + \mu g(x, u))v, \text{ for all } v \in W_0^{1,\Phi}(\Omega).$$

According to the hypotheses (ϕ_1)–(ϕ_3), a wide class of operators can be incorporated in the problem (P), for instance:

(1) $\Phi(t) = |t|^p, p > 1$. The operator Δ_Φ is the p-Laplacian operator.
(2) $\Phi(t) = |t|^{p-2} + |t|^{q-2}, 1 < p < q$. The operator Δ_Φ is the (p, q)-Laplacian operator, which arises in applications in quantum physics, see, for intance, [7].
(3) $\Phi(t) = (1 + t^2)^\alpha - 1, \alpha \in (1, N/(N - 2))$. The associated operator Δ_Φ is considered in nonlinear elasticity problems as pointed in [17, 19].
(4) $\Phi(t) = |t|^p \ln(1 + |t|), 1 < \left(-1 + \sqrt{1 + 4N}\right)/2 < p < N - 1, N \geq 3$. The operator Δ_Φ is used to model plasticity problems, see [16, 18].
(5) $\Phi(t) = \int_0^{|t|} s^{1-\alpha}\left(\sinh^{-1} s\right)^\beta ds, 0 \leq \alpha \leq 1, \beta > 0$. As quoted in [18], the operator Δ_Φ arises in the study of generalized Newtonian fluids.

Related problems to (P) were considered previously in the literature. For example, in [10], existence, multiplicity, and nonexistence of solutions was obtained, by means of subsupersolutions, for the problem

$$\begin{cases} -\Delta u = \lambda f(u) + \mu g(u) & \text{in } \; \Omega, \\ \quad u > 0 & \text{in } \; \Omega, \\ \quad u = 0 & \text{on } \; \partial\Omega, \end{cases}$$

where $\lambda, \mu > 0$ are constants and f and g are functions, which behave with a sublinear and superlinear growth, respectively, $g(0) > 0$, $f(0) < 0$, with f eventually strictly positive.

Perera and Shivaji [26], by means of subsupersolutions and the Mountain Pass Theorem, obtained existence and multiplicity of solutions for the problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $p > 1$, is the p-Laplacian operator, $\lambda > 0$ and $\mu \in \mathbb{R}$ are parameters, and f and g are functions satisfying some conditions that allow the inequality $\lambda f(x,0) + \mu g(x,0) < 0$ in a set of positive measures. By using classical arguments, the authors considered a subcritical problem, and using the abstract tools of [29], they considered a case with critical behavior. An important point to quote is that the homogeneity of the p-Laplacian operator played an important role in the construction of the subsupersolutions.

On the other hand, there is by now an increasing interest in problems involving the operator (1.1), see, for instance, [2, 4, 8, 12, 14, 15, 18, 19, 24, 25, 27, 28, 31] and the references therein. In [2], which was motivated by Castro, de Figueiredo, and Lopera [11], the existence results for the semipositone problem were obtained, given by

$$\begin{cases} -\Delta_\Phi u = f(u) - a & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : [0,+\infty) \to [0,+\infty)$ is a continuous function with subcritical growth and $a > 0$ is a parameter. By using variational methods, the existence of a solution for the above problem for a small enough was obtained. In the best of our knowledge, the paper [2] was the first one to consider semipositone problems in Orlicz–Sobolev spaces.

Motivated by the classical paper by Ambrosetti, Brezis, and Cerami [5], the authors of [12] used the Nehari method to consider a concave-convex problem with a critical superlinear term of the form

$$\begin{cases} -\Delta_\Phi u = \lambda a(x) f(u) + b(x) g(u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\overline{P}$}$$

where Φ is an N-function satisfying certain conditions, $\lambda > 0$ is a parameter, $f, g : [0,+\infty) \to [0,+\infty)$ are continuous functions, and $a, b : \Omega \to \mathbb{R}$ are functions that can change sign. We also quote [15, Theorem 2] where, by means of subsupersolution arguments, a class of problems that include a critical concave-convex problem related to $\left(\overline{P}\right)$ was considered.

In [23], the problem with critical growth is considered:

$$\begin{cases} -\Delta_\Phi u = \lambda |u|^{\ell^\star - 2} u + f(x,u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f satisfy a symmetry condition. By an application of the Symmetric Mountain Pass Theorem and a concentration-compactness principle, it is proved that there is $\lambda_i > 0$ such that the problem admits i pairs of nontrivial weak solutions for $\lambda \in (0, \lambda_i)$.

Motivated by the papers [2, 12, 26, 27], we propose to obtain, by using sub-supersolutions and the Mountain Pass Theorem, existence and multiplicity results for (P). In what follows, we describe the results obtained.

In the first result, we combine subsupersolutions with a minimization argument in convex sets to prove the existence of a weak solution for (P). We quote that no growth restrictions are imposed on g.

Theorem 1.1. *Consider (f_1)–(f_2) and that g maps bounded sets in bounded sets. There exists $\lambda_0 > 0$ such that for each $\lambda \geq \lambda_0$, there is $\mu_0(\lambda) > 0$ for which (P) has a $C^{1,\alpha}\left(\overline{\Omega}\right)$ solution for $\mu \in \mathbb{R}$ with $|\mu| \leq \mu_0(\lambda)$.*

Consider $\Phi_\star(t) = \int_0^{|t|} \phi_\star(s)s$ the Sobolev conjugate N-function of (1.2) (see Section 2).

By using a truncation of the nonlinearities, the subsolution obtained in the proof of Theorem 1.1, and the Mountain Pass Theorem [6, Theorem 2.1], we obtain the multiplicity result below.

Theorem 1.2. *Consider the conditions of Theorem 1.1. Let λ_0 be as in Theorem 1.1. Then, for each $\lambda \geq \lambda_0$, there is $\overline{\mu} \in (0, \mu_0(\lambda))$ for which (P) has two solutions whenever $0 < \mu < \overline{\mu}(\lambda)$ under one the following assumptions:*

(g_1) *(Subcritical case) There are $1 \leq r < \ell^\star$, $\theta > m$ and $t_1 > 0$ such that $|g(x,t)| \leq C_4 t^{r-1} + C_5$, for all $t \geq 0$, a.e. in Ω and $0 < \theta G(x,t) \leq t g(x,t)$, $t \geq t_1$, a.e. in Ω, where $G(x,t) := \int_0^t g(x,s)$;*
(g_2) *(Critical case) $g(x,t) = \phi_\star(|t|)t$ and $q < (\ell^\star/m^\star)\ell$, where q is provided in (f_1).*

Note that in the case $\mu > 0$, we have that Theorem 1.1 and the first part of Theorem 1.2 contain, for example, the semipositone case $f(x,t) = |t|^{q-2}t, 1 < q < l$, and $g(x,t) = |t|^{r-2}t - 1, m < r < \ell^\star$. Observe also that the second part of Theorem 1.2 allows to consider a problem with critical growth.

Remark 1.3. We point out that our arguments can be adapted to prove the result of the second part of Theorem 1.2 in the case $g(x,t) = |t|^{\ell^\star - 2}t$.

Regarding the above results, we highlight the following points.

(1) The lack of homogeneity of (1.1) implies additional difficulties when one intends to consider a subsupersolution approach. Thus, the arguments of [26] are not applicable to (P). There are few papers that consider subsupersolution arguments for problems involving Orlicz–Sobolev spaces, see, for instance, [15].
(2) The content of Theorems 1.1 and 1.2 completes the existence and multiplicity results contained in [26, Theorem 1.1] and [26, Theorem 1.2], respectively. The multiplicity result pointed in Remark 1.3 completes the study of [23] due to the fact that no symmetry condition is required in nonlinearity f in (P).
(3) The results obtained allow to obtain existence and multiplicity of solutions for (P) for a class of semipositone problems that was not considered in [2].
(4) The proof of the second part of Theorem 1.2 is based on the ideas of the proof of [26, Theorem 1.2] that depends on several results of [29], which are not available in the case considered. In order to overcome such difficulties, we used some ideas of [27, 29].
(5) In the best of our knowledge, only the papers [12, 15] consider critical concave-convex problems in the Orlicz–Sobolev spaces setting. In the mentioned papers, it was needed to consider a small positive parameter in the sublinear term to obtain the existence of solutions, which does not occur in Theorem 1 and Remark 1. Consequently, we complete the classical results by Ambrosetti, Brezis, and Cerammi [5].

The remainder of this paper is organized as follows: In Section 2, we present the needed properties in Orlicz and Orlicz–Sobolev spaces. Section 3 contains the proofs of Theorems 1.1 and 1.2. We also quote that $C_1, C_2, \ldots$ will denote (possibly different) strictly positive constants.

2 | PRELIMINARIES

In this section, we present some basic facts regarding Orlicz and Orlicz–Sobolev spaces and results that will be used in this work.

We say that $\Phi : \mathbb{R} \to [0, +\infty)$ is an N-function if it is continuous, convex, even, $\Phi(t) = 0$, if and only if $t = 0$, $\lim_{t\to 0} \Phi(t)/t = 0$ and $\lim_{t\to+\infty} \Phi(t)/t = +\infty$.

An N-function Φ verifies the Δ_2-condition, if $\Phi(2t) \leq K\Phi(t)$ for all $t \geq \overline{t}$, for some constants $K, \overline{t} > 0$, which can be rewritten as: for each $s > 0$, there are numbers M_s and $\overline{t} > 0$ such that $\Phi(st) \leq M_s \Phi(t)$, for all $t \geq \overline{t}$.

Consider an open set $\Omega \subset \mathbb{R}^N$ and an N-function Φ. Unless otherwise stated, it will be considered that $\partial\Omega$ is smooth. The Orlicz space $L^\Phi(\Omega)$ is defined as

$$L^\Phi(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable; } \int_\Omega \Phi\left(\frac{u}{\theta}\right) < +\infty \text{ for some } \theta > 0 \right\}.$$

The space $L^{\Phi}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{\Phi} := \inf\left\{\theta > 0; \int_{\Omega} \Phi\left(\frac{u}{\theta}\right) \le 1\right\}$$

is a Banach space. In addition, if Φ satisfies the Δ_2-condition, then

$$L^{\Phi}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable}; \int_{\Omega} \Phi(u) < +\infty\right\}.$$

The complement function of Φ, denoted by $\widetilde{\Phi}$, is given by the Legendre transformation, that is,

$$\widetilde{\Phi}(s) := \sup_{t\ge 0}\{st - \Phi(t)\}.$$

We have the Young inequality given by

$$st \le \Phi(s) + \widetilde{\Phi}(t),\ s, t \ge 0.$$

By using the previous inequality, it is possible to prove a Hölder-type inequality

$$\left|\int_{\Omega} uv\right| \le 2\|u\|_{L^{\Phi}(\Omega)}\|u\|_{L^{\widetilde{\Phi}}(\Omega)},\ u \in L^{\Phi}(\Omega) \text{ and } v \in L^{\widetilde{\Phi}}(\Omega). \tag{2.1}$$

If Φ is an N-function of the form (1.2) where ϕ satisfies (ϕ_1)–(ϕ_3), then Φ and $\tilde{\Phi}$ verify the Δ_2-condition, see [17].
If Φ is an N-function satisfying the Δ_2 condition, it holds that

$$u_n \to u \text{ in } L^{\Phi}(\Omega) \Longleftrightarrow \int_{\Omega} \Phi(u_n - u) \to 0.$$

For an N-function Φ, the corresponding Orlicz–Sobolev space is defined as the Banach space

$$W^{1,\Phi}(\Omega) = \left\{u \in L^{\Phi}(\Omega) \ :\ \frac{\partial u}{\partial x_i} \in L^{\Phi}(\Omega),\ i = 1, \dots, N\right\},$$

endowed with the norm

$$\|u\|_{1,\Phi} = \|\nabla u\|_{L^{\Phi}} + \|u\|_{L^{\Phi}}.$$

We denote by $W_0^{1,\Phi}(\Omega)$ the completion of $C_0^{\infty}(\Omega)$ with respect to the norm defined above, and so, it is a Banach with this norm. It is important to point out that if Φ and $\tilde{\Phi}$ satisfy the Δ_2-condition, then $L^{\Phi}(\Omega), W^{1,\Phi}(\Omega), W_0^{1,\Phi}(\Omega)$ are reflexive. If Φ is an N-function that satisfies the Δ_2 condition, then it holds the Poincaré-type inequality given by

$$\int_{\Omega} \Phi(u) \le \Lambda \int_{\Omega} \Phi(|\nabla u|),\ u \in W_0^{1,\Phi}(\Omega), \tag{2.2}$$

for some $\Lambda > 0$, see [21]. Thus, the norms $\|u\| := \||\nabla u|\|_{L^{\Phi}(\Omega)}$ and $\|u\|_{1,\Phi}$ on $W_0^{1,\Phi}(\Omega)$ are equivalent.
As quoted in [1], if Φ is an N-function such that

$$\int_0^1 \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} < +\infty \text{ and } \int_1^{+\infty} \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} = +\infty, \tag{2.3}$$

then the Sobolev conjugate N-function $\Phi_\star$ of Φ is defined by

$$\Phi_\star^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} \quad \text{for} \quad t > 0 \tag{2.4}$$

and $\Phi_\star(-t) = \Phi_\star(t)$. Note that if an N-function of the form (1.2) satisfies (ϕ_1)–(ϕ_3), then (2.3) holds.

The following inequality holds:

$$\|u\|_{L^{\Phi_\star}} \leq S_N \||\nabla u|\|_{L^\Phi},\ u \in W_0^{1,\Phi}(\Omega), \tag{2.5}$$

see [13]. In [13], it is also proved that if $\Omega \subset \mathbb{R}^N$ is an open set and admissible, that is, it holds the continuous embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ and Ψ is an N-function such that

$$\lim_{t\to+\infty} \frac{\Psi(kt)}{\Phi_\star(t)} = 0, \quad \text{for all } k > 0,$$

then the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\Psi(\Omega)$ is compact.

Below we point out some results that will be often used in this work and which can be found in [1, 2, 17, 27, 31]

Lemma 2.1. *Let Φ be an N-function satisfying $(\phi_1), (\phi_2)$, and (ϕ_3). Then, $\Phi_\star$ given by (2.4) is a well-defined N-function and there exists a right continuous function $\phi_\star : [0,+\infty) \to [0,+\infty)$ such that $\Phi_\star(t) = \int_0^{|t|} \phi_\star(s)s$ and*

$$\ell^\star \leq \frac{\phi_\star(t)t^2}{\Phi_\star(t)} \leq m^\star, \text{ for all } t > 0,$$

where $\ell^\star := Nl/(N-l)$ and $m^\star := Nm/(N-m)$.

Lemma 2.2. *Let Φ be an N-function satisfying $(\phi_1), (\phi_2)$, and (ϕ_3) and $\Phi_\star$ given by Lemma 2.1. The N-functions $\Phi, \tilde{\Phi}$, and $\Phi_\star$ satisfy*

$$\tilde{\Phi}(\phi(t)t) \leq \Phi(2t) \text{ and } \widetilde{\Phi_\star}(\phi_\star(t)t) \leq \Phi_\star(2t),\ t \geq 0,$$

where $\widetilde{\Phi}$ and $\widetilde{\Phi_\star}$ are the Legendre transforms of Φ and $\Phi_\star$ respectively.

Lemma 2.3. *Let Φ be an N-function satisfying $(\phi_1), (\phi_2)$, and (ϕ_3). Define*

$$\zeta_0(t) = \min\{t^\ell, t^m\} \text{ and } \zeta_1(t) = \max\{t^\ell, t^m\},\ \ t \geq 0.$$

Then, Φ satisfies

$$\zeta_0(t)\Phi(\rho) \leq \Phi(\rho t) \leq \zeta_1(t)\Phi(\rho),\ \ \rho, t > 0,$$

and

$$\zeta_0(\|u\|_\Phi) \leq \int_\Omega \Phi(u) \leq \zeta_1(\|u\|_\Phi),\ u \in L^\Phi(\Omega).$$

Lemma 2.4. *Let $\Phi_\star$ be the N-function given in Lemma 2.1. Define*

$$\zeta_2(t) = \min\{t^{\ell^\star}, t^{m^\star}\} \text{ and } \zeta_3(t) = \max\{t^{\ell^\star}, t^{m^\star}\},\ \ t \geq 0,$$

where $\ell^\star := N\ell/(N-\ell)$ and $m^\star := Nm/(N-m)$. Then $\Phi_\star$ satisfies

$$\zeta_2(t)\Phi_\star(\rho) \leq \Phi_\star(\rho t) \leq \zeta_3(t)\Phi_\star(\rho),\ \ \rho, t \geq 0,$$

and

$$\zeta_2(\|u\|_{\Phi_\star}) \leq \int_\Omega \Phi_\star(u(x)) \leq \zeta_3(\|u\|_{\Phi_\star}),\ u \in L^{\Phi_\star}(\Omega).$$

The following Simon-type inequality, which can be found in [3], will be needed.

Lemma 2.5. *Let Φ be an N-function satisfying (ϕ_1)–(ϕ_3). Then there is a constant $\Gamma > 0$ such that*

$$\left\langle \phi(|\eta|)\eta - \phi(|\eta'|)\eta', \eta - \eta' \right\rangle \geq \Gamma \frac{|\eta - \eta'|}{1 + |\eta| + |\eta'|} \Phi\left(\frac{|\eta - \eta'|}{4}\right),$$

for all $\eta, \eta' \in \mathbb{R}^N$, where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product.

Consider $u, v \in W_0^{1,\Phi}(\Omega)$. We say that $-\Delta_\Phi u \leq -\Delta_\Phi v$ in Ω if

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla \varphi \leq \int_\Omega \phi(|\nabla v|)\nabla v \nabla \varphi,$$

for all $\varphi \in W_0^{1,\Phi}(\Omega)$ with $\varphi \geq 0$.

The results below can be found in [31].

Lemma 2.6. *Let $u, v \in W_0^{1,\Phi}(\Omega)$ with $-\Delta_\Phi u \leq -\Delta_\Phi v$ in Ω and $u \leq v$ in $\partial\Omega$ $\left(i.e., (u - v)^+ \in W_0^{1,\Phi}(\Omega)\right)$, then $u(x) \leq v(x)$ a.e. in Ω.*

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and admissible. Let $\lambda > 0$ be a constant. Then, the unique solution u of the problem*

$$\begin{cases} -\Delta_\Phi u = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

belongs to $L^\infty(\Omega)$ with

$$\|u\|_{L^\infty(\Omega)} \leq C \max\left\{\lambda^{1/(\ell-1)}, \lambda^{1/(m-1)}\right\},$$

where C is a constant that does not depend on u and λ.

3 | PROOF OF THEOREM 1.1

In order to prove Theorem 1.1, we will combine subsupersolutions and a minimization argument.

We say that $\underline{u}, \overline{u} \in W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ are a subsolution and a supersolution, respectively, for (P) if

(1) $0 < \underline{u}(x) \leq \overline{u}(x)$ a.e. in Ω,
(2) for each $v \in W_0^{1,\Phi}(\Omega)$ with $v(x) \geq 0$ a.e. in Ω, the following inequalities hold:

$$\int_\Omega \phi(|\nabla \underline{u}|)\nabla \underline{u} \nabla v \leq \int_\Omega (\lambda f(x, \underline{u}) + \mu g(x, \underline{u}))v$$

and

$$\int_\Omega \phi(|\nabla \overline{u}|)\nabla \overline{u} \nabla v \geq \int_\Omega (\lambda f(x, \overline{u}) + \mu g(x, \overline{u}))v.$$

The next result will be needed.

Lemma 3.1. *Suppose that (f_1)–(f_2) hold and that g is bounded in bounded sets. Then, there exists $\lambda_0 > 0$ with the following property: There exists $\mu_0 > 0$ such that for each $\lambda \geq \lambda_0$ and $|\mu| \leq \mu_0$, the problem (P) has a subsolution and a supersolution $(\underline{u}, \overline{u}) \in \left(W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)\right) \times \left(W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)\right)$.*

Proof. The proof will begin by considering a subsolution $\underline{u}$. Since $\partial\Omega$ is C^2, there is a constant $\delta > 0$ such that $d \in C^2\left(\overline{\Omega_{3\delta}}\right)$ with $|\nabla d| \equiv 1$ in Ω, where $d(x) := dist(x, \partial\Omega)$ and $\overline{\Omega_{3\delta}} := \left\{x \in \overline{\Omega}; d(x) \leq 3\delta\right\}$ (see [20, Lemma 14.16] and its proof). Let

$\sigma \in (0, \delta)$. As pointed in [15, p. 4156], the function defined by

$$\eta(x) = \begin{cases} e^{kd(x)} - 1, & \text{if } d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_{\sigma}^{d(x)} ke^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{m/(l-1)} dt, & \text{if } \sigma \le d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_{\sigma}^{2\delta} ke^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{m/(l-1)} dt, & \text{if } 2\delta \le d(x), \end{cases}$$

belongs to $C_0^1(\overline{\Omega})$, where $k > 0$ is an arbitrary number and satisfies

$$-\Delta_\Phi \eta = \begin{cases} -k^2 e^{kd(x)} \frac{d}{dt}(\phi(t)t)\Big|_{t=ke^{kd(x)}} - \phi\big(ke^{kd(x)}\big)ke^{kd(x)}\Delta d \quad \text{if} \quad d(x) < \sigma, \\ ke^{k\sigma}\left(\frac{m}{l-1}\right)\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{m/(l-1)-1}\left(\frac{1}{2\delta-\sigma}\right)\frac{d}{dt}(\phi(t)t)\Big|_{t=ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta-\sigma}\right)} \\ -\phi\left(ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{m/(l-1)}\right)ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{m/(l-1)}\Delta d \quad \text{if} \quad \sigma < d(x) < 2\delta, \\ 0 \quad \text{if} \quad 2\delta < d(x). \end{cases} \tag{3.1}$$

Consider $n \in \mathbb{N}$ such that $e + n - 1 \ge t_0$, where t_0 is given in (f_2) and define $\sigma := [\ln(e+n)]/k, k > 0$.

We will estimate $-\Delta_\Phi \eta$ in the case $d(x) < \sigma$ for $x \in \Omega$. There exist $C_5 > 0$ and $k_0 \ge 1$ such that $k(l-1) + \Delta d(x) \ge C_5$ if $d(x) < \delta$ and for all $k \ge k_0$. Note that by $(\phi_3), (\phi_3)'$ and Lemma 2.3, there exists $C_6 > 0$ such that

$$\begin{aligned} -\Delta_\Phi \eta &= -k^2 e^{kd(x)} \frac{d}{dt}(\phi(t)t)\Big|_{t=ke^{kd(x)}} - \phi\big(ke^{kd(x)}\big)ke^{kd(x)}\Delta d \\ &\le -k^2 e^{kd(x)}(l-1)\phi\big(ke^{kd(x)}\big) - \phi\big(ke^{kd(x)}\big)ke^{kd(x)}\Delta d \\ &= ke^{kd(x)}\phi\big(ke^{kd(x)}\big)(-k(\ell-1) - \Delta d) \\ &\le \ell\frac{\Phi\big(ke^{kd(x)}\big)}{ke^{kd(x)}}(-C_6) \\ &\le -\ell C_6 \Phi(1)\frac{\zeta_0\big(ke^{kd(x)}\big)}{ke^{kd(x)}} \\ &= -C_7 k^{l-1} \end{aligned} \tag{3.2}$$

for a larger k_0 and $k \ge k_0$, where $C_7 > 0$ is a constant that does not depend on k. Consider $\lambda > 0$. From (f_1) and (3.2), we obtain in the case $d(x) < \sigma$ that

$$-\Delta_\Phi \eta \le -C_7 k^{l-1} \le -1 \le \lambda f(x, \eta) - 1, \tag{3.3}$$

for all $\lambda > 0$ and for $k \ge k_0$ with k_0 large enough, which does not depend on $\lambda > 0$.

Now suppose that $\sigma \le d(x) < 2\delta$. A similar argument with respect to [15, p. 4157] implies

$$-\Delta_\Phi \eta \le C_8 k^{m-1}, \tag{3.4}$$

for $\sigma \le d(x) < 2\delta$, where $C_8 > 0$ is a constant that does not depend on $k \ge k_0 \ge 1$.

Consider $\lambda_0 > 0$, depending only on the fixed $k > 0$, such that $C_8 k^{m-1} \le \lambda C_3 - 1$ for all $\lambda \ge \lambda_0$, where C_3 is the constant given in (f_2). If $\sigma < d(x) < 2\delta$, we have $\eta(x) \ge e^{k\sigma} - 1 = e + n - 1 \ge t_0$. Thus, it follows from (f_2) that

$$-\Delta_\Phi \eta \le C_8 k^{m-1} \le \lambda C_3 - 1 \le \lambda f(x, \eta) - 1, \tag{3.5}$$

for all $\lambda \ge \lambda_0$ and $\sigma < d(x) < 2\delta$.

Consider $2\delta < d(x)$. In this case, we have $\eta(x) \geq e^{k\sigma} - 1 \geq t_0$. Then, by using (3.1), we have

$$-\Delta_\Phi \eta = 0 \leq \lambda C_3 - 1 \leq \lambda f(x, \eta) - 1, \tag{3.6}$$

for $2\delta < d(x)$. Then, it follows from (3.3), (3.5), and (3.6) that

$$-\Delta_\Phi \eta \leq \lambda f(x, \eta) - 1 \text{ in } \Omega \tag{3.7}$$

for all $\lambda \geq \lambda_0$.

By using the fact that g is bounded in bounded sets, we have that there exists $\mu_0 > 0$ small enough such that $\mu_0 \|g(\cdot, \eta)\|_{L^\infty} \leq 1/2$. Thus, by (3.7) we obtain in Ω that

$$-\Delta_\Phi \eta \leq \lambda f(x, \eta) - 1 \leq \lambda f(x, \eta) + \mu g(x, \eta), \tag{3.8}$$

for all $\mu \in \mathbb{R}$ with $|\mu| \leq \mu_0$.

Now, the supersolution will be considered. Fix $\lambda \geq \lambda_0$ satisfying (3.8). Let $\theta > 0$ be a constant to be chosen before and consider $z_\theta \in W_0^{1,\Phi}(\Omega)$ the solution of the problem

$$\begin{cases} -\Delta_\Phi z_\theta = \theta & \text{in } \Omega, \\ z_\theta = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $q < l$, it is possible to choose $\theta > 0$ large enough such that

$$\max\left\{C_1, C_2, C^{q-1}\right\}\lambda\left(1 + \theta^{(q-1)/(l-1)}\right) < \theta, \tag{3.9}$$

where $C_1, C_2 > 0$ and $C > 0$ are the constants given in (f_1) and Lemma 2.7, respectively.

Consider a smaller $\mu_0 > 0$, which depends only on λ, such that

$$\max\{C_1, C_2, C^{q-1}\}\lambda\left(1 + \theta^{(q-1)/(l-1)}\right) + \mu_0 \|g(x, z_\theta)\|_{L^\infty} < \theta. \tag{3.10}$$

Then by (3.9), (3.10), and Lemma 2.7, we have

$$\lambda f(x, z_\theta) + \mu g(x, z_\theta) \leq -\Delta_\Phi z_\theta \text{ in } \Omega \tag{3.11}$$

for all $\mu \in \mathbb{R}$ with $|\mu| \leq \mu_0$. Since $-\Delta_\Phi \eta$ is bounded, it is possible to choose $\theta > 0$ such that (3.11) occurs and $-\Delta_\Phi \eta \leq -\Delta_\Phi z_\theta$ in Ω. From Lemma 2.6, we have $\eta(x) \leq z_\theta(x)$ a.e. in Ω. $\square$

Proof of Theorem 1.1. Consider the function

$$w(x,t) = \begin{cases} \lambda f(x, \overline{u}(x)) + \mu g(x, \overline{u}(x)), & t > \overline{u}(x), \\ \lambda f(x,t) + \mu g(x,t), & \underline{u}(x) \leq t \leq \overline{u}(x), \\ \lambda f(x, \underline{u}(x)) + \mu g(x, \underline{u}(x)), & t < \underline{u}(x), \end{cases} \tag{3.12}$$

for $(x,t) \in \Omega \times \mathbb{R}$, where $\underline{u}, \overline{u} \in W_0^{1,\Phi}(\Omega)$ are the functions given in Lemma 3.1. Define the energy functional

$$J(u) := \int_\Omega \Phi(|\nabla u|) - \int_\Omega W(x,u), \ u \in W_0^{1,\Phi}(\Omega),$$

where $W(x,t) := \int_0^t w(x,s)$. From the hypothesis on f and g, we have $J \in C^1\left(W_0^{1,\Phi}(\Omega), \mathbb{R}\right)$ with

$$J'(u)v = \int_\Omega \phi(|\nabla u|)\nabla u \nabla v - \int_\Omega w(x,u)v, \ u, v \in W_0^{1,\Phi}(\Omega).$$

We claim that J is coercive. In fact, we have from Lemma 2.3 and the continuous embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\Phi(\Omega)$ and $L^\Phi(\Omega) \hookrightarrow L^1(\Omega)$ that

$$\begin{aligned} J(u) &\geq \int_\Omega \Phi(|\nabla u|) - C_9 \int_\Omega |u| \\ &\geq \zeta_0\left(\|\nabla u\|_{L^\Phi}\right) - C_{10}\|\nabla u\|_{L^\Phi}. \end{aligned}$$

Thus, $J(u) \to +\infty$ as $\|u\| \to +\infty$. Since w is bounded, we have that J is a weak lower semicontinuous functional. The set

$$\mathcal{K} := \left\{v \in W_0^{1,\Phi}(\Omega); \underline{u}(x) \leq v(x) \leq \overline{u}(x) \text{ a.e. in } \Omega\right\}$$

is convex and closed in $W_0^{1,\Phi}(\Omega)$, thus by the reflexivity of $W_0^{1,\Phi}(\Omega)$ and [30, Theorem 1.2], we obtain that $J\big|_{\mathcal{K}}$ attains its infimum at a point u in $\mathcal{K}$. Repeating the arguments of [30, Theorem 2.4], we see that u weakly solves the problem

$$\begin{cases} -\Delta_\Phi v = w(x, v) & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, since $u \in \mathcal{K}$, from the definition of w given in (3.12), we have that u solves (P). Since the function w is bounded, it follows from the $C^{1,\alpha}$ estimates up to the boundary (see [24]) that $u \in C^{1,\alpha}\left(\overline{\Omega}\right)$. □

4 | PROOF OF THEOREM 1.2

Before proving Theorem 1.2, some facts will be needed.

Let $\underline{u}$ be as in Lemma 3.1. Consider the functions

$$\widetilde{f}(x,t) = \begin{cases} f(x,t), & t \geq \underline{u}(x), \\ f(x,\underline{u}(x)), & t < \underline{u}(x), \end{cases} \quad \text{and} \quad \widetilde{g}(x,t) = \begin{cases} g(x,t), & t \geq \underline{u}(x), \\ g(x,\underline{u}(x)), & t < \underline{u}(x), \end{cases}$$

where g satisfies (g_1) or (g_2). Consider the problem

$$\begin{cases} -\Delta_\Phi u = \lambda \widetilde{f}(x,u) + \mu \widetilde{g}(x,u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

whose solutions coincide with the critical points of the C^1 functional

$$L(u) = \int_\Omega \Phi(|\nabla u|) - \lambda \int_\Omega \widetilde{F}(x,u) - \mu \int_\Omega \widetilde{G}(x,u), \; u \in W_0^{1,\Phi}(\Omega),$$

where $\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)$ and $\widetilde{G}(x,t) = \int_0^t \widetilde{g}(x,s)$.

4.1 | Subcritical case

In order to prove the first part of Theorem 1.2, the result below will be needed.

Lemma 4.1. *Suppose that (g_1) holds. Then, the functional L satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$.*

Proof. Consider $(x,t) \in \Omega \times \mathbb{R}$ with $t < \underline{u}(x)$. Note that

$$\begin{aligned} \widetilde{g}(x,t)t - \theta\widetilde{G}(x,t) &= g(x,\underline{u}(x))t - \theta\widetilde{G}(x,t) \\ &\geq -C_{11}|t| - C_{12}. \end{aligned} \tag{4.2}$$

On the other hand, if $t \geq \underline{u}(x)$ with $t \geq t_1$ or $t < t_1$, we have from (g_1) and the fact that g is bounded on bounded sets that

$$\begin{aligned}\widetilde{g}(x,t)t - \theta\widetilde{G}(x,t) &= g(x,t)t - \theta\left(\int_0^{\underline{u}(x)} \widetilde{g}(x,s)\,ds + \int_{\underline{u}(x)}^t \widetilde{g}(x,s)\,ds\right)\\ &= g(x,t)t - \theta G(x,t) - \theta\underline{u}(x)g(x,\underline{u}(x)) + \theta G(x,\underline{u}(x))\\ &\geq -C_{13},\end{aligned} \tag{4.3}$$

where $C_{13} := \sup_{(x,t)\in\Omega\times[0,t_1]} |g(x,t)t - \theta G(x,t)| + \sup_{x\in\Omega} |-\theta\underline{u}(x)g(x,\underline{u}(x)) + \theta G(x,\underline{u}(x))|$.

Consider $c \in \mathbb{R}$ and let (u_n) be a sequence in $W_0^{1,\Phi}(\Omega)$ such that $L(u_n) \to c$ and $L'(u_n) \to 0$. By using (f_1), (4.2), (4.3), and (ϕ_3), we have

$$L(u_n) - \frac{1}{\theta}L'(u_n)u_n \geq \left(1 - \frac{m}{\theta}\right)\int_\Omega \Phi(|\nabla u_n|) - C_{14}\left(\int_\Omega |u_n|^q + |u_n|\right) - C_{15}. \tag{4.4}$$

Let $\varepsilon > 0$ be an arbitrary number. Note that

$$t + t^q \leq C_{16} + \varepsilon\Phi(t) \tag{4.5}$$

for all $t \geq 0$, where $C_{16} > 0$ is a constant that depends on $\varepsilon > 0$. In fact, we have from Lemma 2.3 that

$$0 \leq \frac{t + t^q}{\Phi(t)} \leq \frac{t + t^q}{\Phi(1)t^\ell},$$

for all $t \geq 1$. Since $1 < q < \ell$, we have $\lim_{t\to+\infty} \frac{t+t^q}{\Phi(t)} = 0$. Then, we conclude that given $\varepsilon > 0$, there is $\bar{t} > 1$ such that

$$t + t^q \leq \varepsilon\Phi(t),$$

for all $t \geq \bar{t}$. By continuity, the function $\Lambda(t) := t + t^q$, $t \in [0, \bar{t}]$ is bounded. Therefore, we have the estimate (4.5).

Thus, by using (2.2), (4.4), and (4.5), for a suitable choice of $\varepsilon > 0$ we obtain

$$c + 1 + \|u_n\| \geq L(u_n) - \frac{1}{\theta}L'(u_n)u_n \geq C_{17}\int_\Omega \Phi(|\nabla u_n|) - C_{18}. \tag{4.6}$$

By using (4.6) and Lemma 2.3, we obtain that (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$.

From the fact that $W_0^{1,\Phi}(\Omega)$ is reflexive and the compact embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\xi(\Omega), 1 \leq \xi < l^\star$, we have up to a subsequence, still denoted by (u_n), that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,\Phi}(\Omega),\\ u_n \to u & \text{in } L^\xi(\Omega), 1 \leq \xi < l^\star,\\ u_n(x) \to u(x) \text{ a.e.} & \text{in } \Omega,\end{cases} \tag{4.7}$$

for some $u \in W_0^{1,\Phi}(\Omega)$. Note that

$$\begin{aligned}&\int_\Omega \langle \phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u|)\nabla u, \nabla(u_n - u)\rangle\\ &\quad= J'(u_n)(u_n - u) + \int_\Omega \lambda\widetilde{f}(x,u_n)(u_n - u) + \int_\Omega \mu\widetilde{g}(x,u_n)(u_n - u) - \int_\Omega \phi(|\nabla u|)\nabla u\nabla(u_n - u).\end{aligned}$$

The weak convergence in (4.7) implies that

$$\int_\Omega \phi(|\nabla u|)\nabla u\nabla(u_n - u) \to 0. \tag{4.8}$$

By using the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_\Omega \lambda \tilde{f}(x,u_n)(u_n-u) \to 0 \text{ and } \int_\Omega \mu \tilde{g}(x,u_n)(u_n-u) \to 0. \tag{4.9}$$

From (4.8), (4.9), the boundness of (u_n) in $W_0^{1,\Phi}(\Omega)$, and the fact that $L'(u_n) \to 0$, we obtain

$$\int_\Omega \langle \phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u|)\nabla u, \nabla(u_n-u)\rangle \to 0.$$

Then, it follows from [2, Lemma 2.2] that $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$. □

Proof of Theorem 1.2 (first part). Note that (g_1) imply that L satisfy the Palais–Smale condition at any level. From (f_1), (g_1), Lemma 2.3 and the embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^q(\Omega)$ and $W_0^{1,\Phi}(\Omega) \hookrightarrow L^r(\Omega)$, we have

$$L(u) \geq \zeta_0(\|u\|) - C_{19}(\|u\|^q + 1) - C_{20}\mu(\|u\|^r + 1),$$

where $C_{19}, C_{20} > 0$ are constants with C_{20} not depending on $\mu > 0$. Then, it follows that

$$\inf_{\partial B_R} L > 0, \tag{4.10}$$

where $B_R = \{u \in W_0^{1,\Phi}(\Omega); \|u\| < R\}$, for $R > 0$ large enough and $\mu > 0$ small. Since $L(0) = 0$, it follows from the Ekeland variational principle that L attains its minimum on $\overline{B_R}$. Then, by (4.10), we conclude that the minimum is attained at a critical point $u \in B_R$. We claim that $u(x) \geq \underline{u}(x)$ a.e. in Ω. In fact, by considering the test function function $(\underline{u} - u)^+ \in W_0^{1,\Phi}(\Omega)$, we obtain

$$\begin{aligned}
\int_\Omega \phi(|\nabla u|)\nabla u \nabla(\underline{u}-u)^+ &= \int_{\{u<\underline{u}\}} (\lambda \tilde{f}(x,u) + \mu \tilde{g}(x,u))(\underline{u}-u)^+ \\
&= \int_{\{u<\underline{u}\}} (\lambda \tilde{f}(x,\underline{u}) + \mu \tilde{g}(x,\underline{u}))(\underline{u}-u)^+ \\
&\geq \int_\Omega \phi(|\nabla \underline{u}|)\nabla \underline{u} \nabla(\underline{u}-u)^+.
\end{aligned}$$

Thus, by using Lemma 2.5, we have

$$\begin{aligned}
0 &\geq \int_{\{\underline{u}>u\}} \langle \phi(|\nabla \underline{u}|)\nabla \underline{u} - \phi(|\nabla u|)\nabla u, \nabla \underline{u} - \nabla u\rangle \\
&\geq \Gamma \int_\Omega \frac{|\nabla(\underline{u}-u)^+|}{1 + |\nabla \underline{u}| + |\nabla u|} \Phi\left(\frac{|\nabla(\underline{u}-u)^+|}{4}\right),
\end{aligned}$$

which imply that $\nabla(\underline{u}-u)^+(x) = 0$ a.e. in Ω. From (2.5), we obtain $\underline{u}(x) \leq u(x)$ a.e. in Ω. From the definition of $\tilde{f}$ and $\tilde{g}$, it follows that u is a solution for (P).

Now, the existence of a second solution for (P) will be proved. From (g_1), we have

$$\tilde{G}(x,t) \geq C_{21}t^\theta - C_{22}, t > 0, \text{ a.e. in } \Omega. \tag{4.11}$$

Let $\varphi \in C_0^\infty(\Omega) \setminus \{0\}$ be a nonnegative function. Then, by (4.11) we have $L(\bar{t}\varphi) < 0$ for $\bar{t} > 0$ large enough. Consider also that $\bar{t} > R/\|\varphi\|$. The Mountain Pass Theorem provides a critical point $\tilde{u} \in W_0^{1,\Phi}(\Omega)$ for L at the level

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\max\gamma([0,1])} L(u) \geq \inf_{\partial B_R} L > 0,$$

where $\Gamma = \{\gamma \in C([0,1], W_0^{1,\Phi}(\Omega)); \gamma(0) = 0, \gamma(1) = \bar{t}\varphi\}$. As before, we have $\tilde{u}(x) \geq \underline{u}(x)$ a.e. in Ω. Since $L(u) \leq 0 < L(\tilde{u})$, the result is proved. □

4.2 | Critical case

In the next results, the behavior of the Palais–Smale sequences for the functional L it will be considered. By adapting the ideas of Lemma 4.1, we have the result below.

Lemma 4.2. *Suppose that* (g_2) *holds and consider* $\lambda \geq \lambda_0$ *as in Lemma 3.1. If* $(u_n) \subset W_0^{1,\Phi}(\Omega)$ *is a Palais–Smale sequence at the level c, then* (u_n) *is bounded in* $W_0^{1,\Phi}(\Omega)$*. Fix* $M \in \mathbb{R}$ *and* $\widetilde{\mu} > 0$*. If* $c < M$*, then there exists a constant* $C_M > 0$*, depending only on* $M, \widetilde{\mu}$*, and* ℓ*, such that* $\|u_n\| \leq C_M$ *for all* $0 < \mu \leq \widetilde{\mu}$*.*

The next result will play an important role for proving that the functional L satisfies the Palais–Smale condition at certain levels.

Lemma 4.3. *Consider the conditions of Lemma 4.2. Fix* $M \in \mathbb{R}$ *and* $\widetilde{\mu} > 0$ *and let* $(u_n) \subset W_0^{1,\Phi}(\Omega)$ *be a Palais–Smale sequence at the level c with* $c < M$*. Then*

$$M + C_M \geq L(u_n) - \frac{1}{m}L'(u_n)u_n \geq \mu C_{36}\int_\Omega \Phi_\star(u_n) - C_{37} - C_{38}\left(\int_\Omega \Phi_\star(u_n)\right)^{q/\ell^\star} - C_{39}\widetilde{\mu}, \tag{4.12}$$

for all $n \in \mathbb{N}$ *and* $0 < \mu \leq \widetilde{\mu}$*, where* $C_{36}, C_{39} > 0$ *are constants depending on M, the constants* $C_{36}, C_{37}, C_{38}, C_{39}$ *do not depend on* μ*, and* C_M *is given in Lemma 4.2.*

Proof. From (f_1), we have

$$\frac{\widetilde{f}(x,t)t}{m} - \widetilde{F}(x,t) \geq -C_{29} - C_{30}|t|^q, \text{ for all } t \in \mathbb{R}, \text{ a.e. in } \Omega. \tag{4.13}$$

From (ϕ_3) we have

$$\begin{aligned} L(u_n) - \frac{1}{m}L'(u_n)u_n &= \int_\Omega \Phi(u_n) - \frac{1}{m}\phi(|\nabla u_n|)|\nabla u_n|^2 + \lambda\int_\Omega\left[\frac{1}{m}\widetilde{f}(x,u_n)u_n - \widetilde{F}(x,u_n)\right] + \mu\int_\Omega\left[\frac{1}{m}\widetilde{g}(x,u_n)u_n - \widetilde{G}(x,u_n)\right] \\ &\geq \lambda\int_\Omega\left[\frac{1}{m}\widetilde{f}(x,u_n)u_n - \widetilde{F}(x,u_n)\right] + \mu\int_\Omega\left[\frac{1}{m}\widetilde{g}(x,u_n)u_n - \widetilde{G}(x,u_n)\right]. \end{aligned} \tag{4.14}$$

If $u_n(x) < \underline{u}(x)$, then

$$\begin{aligned} \frac{1}{m}\widetilde{g}(x,u_n)u_n - \widetilde{G}(x,u_n) &= \left(\frac{1}{m}\phi_\star(\underline{u})\underline{u}u_n - \int_0^{u_n(x)} \widetilde{g}(x,s)\,ds\right) \\ &= \frac{1}{m}\phi_\star(\underline{u})\underline{u}u_n - \int_0^{u_n(x)} \phi_\star(\underline{u})\underline{u}\,ds \geq -2|u_n|\sup_{x\in\Omega}(\phi_\star(\underline{u}(x))\underline{u}(x)). \end{aligned} \tag{4.15}$$

Suppose that $u_n(x) \geq \underline{u}(x)$. From Lemma 2.4, we have

$$\begin{aligned} \frac{1}{m}\widetilde{g}(x,u_n)u_n - \widetilde{G}(x,u_n) &= \frac{1}{m}\phi_\star(u_n)u_n^2 - \int_0^{u_n(x)} \widetilde{g}(x,s)\,ds \\ &= \frac{1}{m}\phi_\star(u_n)u_n^2 - \left(\int_0^{\underline{u}(x)} \widetilde{g}(x,s)\,ds + \int_{\underline{u}(x)}^{u_n(x)} \widetilde{g}(x,s)\,ds\right) \\ &= \frac{1}{m}\phi_\star(u_n)u_n^2 - \left(\int_0^{\underline{u}(x)} \phi_\star(\underline{u})\underline{u}\,ds + \int_{\underline{u}(x)}^{u_n(x)} \phi_\star(s)s\,ds\right) \\ &= \frac{1}{m}\phi_\star(u_n)u_n^2 - (\phi_\star(\underline{u})\underline{u}^2 + \Phi_\star(u_n) - \Phi_\star(\underline{u})) \\ &\geq \left(\frac{\ell^\star}{m} - 1\right)\Phi_\star(u_n) + (1 - m^\star)\Phi_\star(\underline{u}). \end{aligned} \tag{4.16}$$

By using (4.13), (4.14), (4.15), (4.16), Lemma 2.4, the continuous embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^1(\Omega), W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\Phi_\star}(\Omega)$, and the fact that $\|u_n\| \le C_M, n \in \mathbb{N}$, where C_M is given in Lemma 4.2, we have

$$\begin{aligned}
L(u_n) - \frac{1}{m}L'(u_n)u_n &\ge \lambda \int_\Omega \left[\frac{1}{m}\widetilde{f}(x,u_n)u_n - \widetilde{F}(x,u_n)\right] + \mu \int_\Omega \left[\frac{1}{m}\widetilde{g}(x,u_n)u_n - \widetilde{G}(x,u_n)\right] \\
&\ge \mu\left(\frac{\ell^\star}{m} - 1\right)\int_{\{u_n \ge \underline{u}\}} \Phi_\star(u_n) + \lambda \int_\Omega \left[\frac{1}{m}\widetilde{f}(x,u_n)u_n - \widetilde{F}(x,u_n)\right] + \mu \int_{\{u_n < \underline{u}\}} \left[\frac{1}{m}\widetilde{g}(x,u_n)u_n - \widetilde{G}(x,u_n)\right] \\
&\quad + \mu(1-m^\star)\int_\Omega \Phi_\star(\underline{u}) \\
&\ge \mu\left(\frac{\ell^\star}{m} - 1\right)\left(\int_\Omega \Phi_\star(u_n) - \int_{\{u_n<\underline{u}\}} \Phi_\star(u_n)\right) + \lambda\left(-C_{29}|\Omega| - C_{30}\int_\Omega |u_n|^q\right) \\
&\quad -\mu C_{31}\int_\Omega |u_n| + \mu(1-m^\star)\int_\Omega \Phi_\star(\underline{u}) \\
&\ge \mu\left(\frac{\ell^\star}{m} - 1\right)\left(\int_\Omega \Phi_\star(u_n) - \zeta_3(\|u_n\|_{\Phi_\star})\right) + \lambda\left(-C_{29}|\Omega| - C_{30}\int_\Omega |u_n|^q\right) \\
&\quad -\mu C_{31}\int_\Omega |u_n| + \mu(1-m^\star)\int_\Omega \Phi_\star(\underline{u}) \\
&\ge \mu\left(\frac{\ell^\star}{m} - 1\right)\int_\Omega \Phi_\star(u_n) + \lambda\left(-C_{29}|\Omega| - C_{30}\int_\Omega |u_n|^q\right) - \widetilde{\mu}C_{32},
\end{aligned} \tag{4.17}$$

where $C_{32} > 0$ is a constant not depending on $n \in \mathbb{N}$ and $\widetilde{\mu}$. Note that

$$|t|^{\ell^\star} \le C_{33}(1 + \Phi_\star(t)),\ t \in \mathbb{R}. \tag{4.18}$$

In fact, we have from Lemma 2.4 that $\Phi_\star(s) \ge s^{\ell^\star}\Phi_\star(1)$ for all $s \ge 1$. By continuity, the function $\Lambda_\star := \Phi_\star(s) - s^{\ell^\star}$, $s \in [0,1]$ is bounded. Thus, there is a constant $C_{33} > 0$ such that

$$s^{\ell^\star} \le C_{33}(1 + \Phi_\star(s))$$

for all $s \ge 0$. Considering $s = |t|, t \in \mathbb{R}$, in the last inequality and since $\Phi_\star(t) = \Phi_\star(|t|)$ for all $t \in \mathbb{R}$, (4.18) follows.

By using the continuous embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\Phi_*}(\Omega)$ and the inequality (4.18), it follows that $u_n \in L^{\ell^\star}(\Omega)$ for all $n \in \mathbb{N}$. By using the Hölder inequality, (4.18), and the fact that $(a+b)^{q/\ell^\star} \le a^{q/\ell^\star} + b^{q/\ell^\star}$ for all $a, b \ge 0$ (see, for instance, [22, Lemma 2.5]), we obtain

$$\begin{aligned}
-C_{29}|\Omega| - C_{30}\int_\Omega |u_n|^q &\ge -C_{29}|\Omega| - C_{34}\left(\int_\Omega |u_n|^{\ell^\star}\right)^{q/\ell^\star} \\
&\ge -C_{29}|\Omega| - C_{35}\left(\int_\Omega \Phi_\star(u_n)\right)^{q/\ell^\star} - C_{34}(C_{33}|\Omega|)^{q/\ell^\star}.
\end{aligned} \tag{4.19}$$

By using (4.17) and (4.19), we have

$$L(u_n) - \frac{1}{m}L'(u_n)u_n \ge \mu C_{36}\int_\Omega \Phi_\star(u_n) - C_{37} - C_{38}\left(\int_\Omega \Phi_\star(u_n)\right)^{q/\ell^\star} - C_{39}\widetilde{\mu},$$

where $C_{36}, C_{39} > 0$ are constants depending on M and the constants $C_{36}, C_{37}, C_{38}, C_{39}$ do not depend on μ. □

As a consequence of Lemma 4.2, if (u_n) is a Palais–Smale sequence at some level, then there is a subsequence, still denoted by (u_n), and $u \in W_0^{1,\Phi}(\Omega)$, such that

(1) $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$;
(2) $u_n \rightharpoonup u$ in $L^{\Phi_\star}(\Omega)$;
(3) $u_n \to u$ in $L^{\Phi}(\Omega)$;
(4) $u_n(x) \to u(x)$ a.e. in Ω.

By using the Concentration Compactness Lemma by Lions for Orlicz–Sobolev spaces found in [17], it follows that there are two nonnegative measures $\iota, \nu \in \mathcal{M}(\mathbb{R}^N)$ (the space of the Radon measures in $\mathbb{R}^N$), a countable set $\mathcal{J}$, points $(x_j)_{j\in\mathcal{J}}$ in $\overline{\Omega}$, and sequences $(\iota_j)_{j\in\mathcal{J}}, (\nu_j)_{j\in\mathcal{J}} \subset (0,+\infty)$, such that

$$\Phi(|\nabla u_n|) \rightharpoonup \iota \geq \Phi(|\nabla u|) + \sum_{j\in\mathcal{J}} \iota_j \delta_{x_j} \text{ weakly in } \mathcal{M}(\mathbb{R}^N), \tag{4.20}$$

$$\Phi_\star(u_n) \rightharpoonup \nu = \Phi_\star(u) + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} \text{ weakly in } \mathcal{M}(\mathbb{R}^N), \tag{4.21}$$

$$\nu_j \leq \max\left\{ S_N^{\ell^\star} \iota_j^{\ell^\star/\ell}, S_N^{m^\star} \iota_j^{m^\star/\ell}, S_N^{\ell^\star} \iota_j^{\ell^\star/m}, S_N^{m^\star} \iota_j^{m^\star/m} \right\}, \tag{4.22}$$

where S_N is the constant provided in (2.5) and δ_{x_j} is the Dirac mass at x_j.

In what follows, an important estimate for $(\nu_j)_{j\in\mathcal{J}}$ is proved.

Lemma 4.4. *Suppose that* (g_2) *holds. If* (u_n) *is a Palais–Smale sequence for* L *with* $\mu > 0$ *and* $(\upsilon_j)_{j\in\mathcal{J}}$ *is given as above, then for each* $j \in \mathcal{J}$, *it holds that*

$$\nu_j \geq \left(\frac{\ell}{\mu m^\star}\right)^{\beta/(\beta-1)} S_N^{-\alpha/(\beta-1)} \text{ or } \nu_j = 0,$$

for some $\alpha \in \{\ell^\star, m^\star\}$ *and* $\beta \in \{\ell^\star/\ell, m^\star/\ell, \ell^\star/m, m^\star/m\}$.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be a function satisfying

$$\psi \equiv 1 \text{ in } B_{1/2}, \ \operatorname{supp} \psi \subset B_1 \text{ and } 0 \leq \psi(x) \leq 1, x \in \mathbb{R}^N.$$

For each $j \in \mathcal{J}$ and $\varepsilon > 0$, define

$$\psi_\varepsilon(x) = \psi\left(\frac{x - x_j}{\varepsilon}\right), \ x \in \mathbb{R}^N.$$

The sequence $(\psi_\varepsilon u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\Phi}(\Omega)$. Since $L'(u_n) \to 0$, we obtain

$$L'(u_n)(\psi_\varepsilon u_n) = o_n(1).$$

From the fact that $\widetilde{g}(x,t) \geq 0$ for all $t \in \mathbb{R}$, a.e. in Ω and the definition of $\widetilde{g}$, we obtain

$$\begin{aligned}\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n\psi_\varepsilon) &= o_n(1) + \lambda\int_\Omega \widetilde{f}(x,u_n)u_n\psi_\varepsilon + \mu\int_{\{u_n<\underline{u}\}} g(x,\underline{u})u_n\psi_\varepsilon + \mu\int_{\{u_n\geq\underline{u}\}} \widetilde{g}(x,u_n)u_n\psi_\varepsilon \\ &\leq o_n(1) + \lambda\int_\Omega \widetilde{f}(x,u_n)u_n\psi_\varepsilon + \mu C_{40}\int_\Omega |u_n|\psi_\varepsilon + \mu\int_{\{u_n\geq\underline{u}\}} \phi_\star(u_n)u_n^2\psi_\varepsilon \\ &\leq o_n(1) + \lambda\int_\Omega \widetilde{f}(x,u_n)u_n\psi_\varepsilon + \mu C_{40}\int_\Omega |u_n|\psi_\varepsilon + \mu m^\star\int_\Omega \Phi_\star(u_n)\psi_\varepsilon.\end{aligned} \tag{4.23}$$

Since the embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\ell(\Omega)$ and $L^\ell(\Omega) \hookrightarrow L^q(\Omega)$ are compact and continuous, respectively, it follows from the Lebesgue Dominated Convergence Theorem that

$$\lim_{n\to+\infty} \int_\Omega \widetilde{f}(x,u_n)u_n\psi_\varepsilon = \int_\Omega \widetilde{f}(x,u)u\psi_\varepsilon.$$

On the other hand, we have from Lemma 2.3 that

$$\begin{aligned}\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n\psi_\varepsilon) &= \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2\psi_\varepsilon + \int_\Omega \phi(|\nabla u_n|)(\nabla u_n\nabla\psi_\varepsilon)u_n \\ &\geq \ell\int_\Omega \Phi(|\nabla u_n|)\psi_\varepsilon + \int_\Omega \phi(|\nabla u_n|)(\nabla u_n\nabla\psi_\varepsilon)u_n.\end{aligned} \tag{4.24}$$

We claim that the sequence $\phi(|\nabla u_n|)\frac{\partial u_n}{\partial x_i}$, $i = 1,\dots,N$, is bounded in $L^{\widetilde{\Phi}}(\Omega)$. In fact, from Lemma 2.2, we have that

$$\widetilde{\Phi}\left(\phi(|\nabla u_n|)\frac{\partial u_n}{\partial x_i}\right) \leq \widetilde{\Phi}(\phi(|\nabla u_n|)|\nabla u_n|) \leq \Phi(2|\nabla u_n|) \leq C_{41}\Phi(|\nabla u_n|),$$

where $C_{41} > 0$ is a constant not depending on $n \in \mathbb{N}$, verifies the claim. Thus, for each $i = 1,\dots,N$, there exists $w_i \in L^{\widetilde{\Phi}}(\Omega)$ such that

$$\phi(|\nabla u_n|)\frac{\partial u_n}{\partial x_i} \rightharpoonup w_i \text{ in } L^{\widetilde{\Phi}}(\Omega), \tag{4.25}$$

which implies that

$$\int_\Omega \phi(|\nabla u_n|)(\nabla u_n\nabla\psi_\varepsilon)u_n \to \int_\Omega (w\nabla\psi_\varepsilon)u, \tag{4.26}$$

where $w = (w_1,\dots,w_N) \in \big(L^{\widetilde{\Phi}}(\Omega)\big)^N$.

From the boundness of $\phi(|\nabla u_n|)\frac{\partial u_n}{\partial x_i}$, $i = 1,\dots,N$, in $L^{\widetilde{\Phi}}(\Omega)$, (4.26), and the fact that $u_n \to u$ in $L^\Phi(\Omega)$, we get

$$\int_\Omega \phi(|\nabla u_n|)(\nabla u_n\nabla\psi_\varepsilon)u_n - (w\nabla\psi_\varepsilon)u = \int_\Omega \phi(|\nabla u_n|)\nabla u_n\nabla\psi_\varepsilon(u_n-u) + o_n(1). \tag{4.27}$$

From (4.20), (4.21), (4.23), (4.24), (4.27), and that $u_n \to u$ in $L^\Phi(\Omega)$, we obtain

$$\ell\int_\Omega \psi_\varepsilon\, d\iota + \int_\Omega (w\nabla\psi_\varepsilon)u \leq \mu m^\star\int_\Omega \psi_\varepsilon\, d\nu + \int_\Omega \widetilde{f}(x,u)u\psi_\varepsilon + \mu\|u\|_{L^\Phi}\|\psi_\varepsilon\|_{L^{\widetilde{\Phi}}}. \tag{4.28}$$

From Lebesgue's Dominated Convergence Theorem, we have $\|\psi_\varepsilon\|_{L^{\widetilde{\Phi}}} \to 0$ as $\varepsilon \to 0^+$.

Now we will prove that the second term on the left-hand side of (4.28) converges to 0 as $\varepsilon \to 0^+$.

We claim that $\big(\widetilde{f}(x,u_n)\big)$ is bounded in $L^{\widetilde{\Phi_\star}}(\Omega)$. In fact, from (f_1) and Lemma 2.4, we have $\lim_{|t|\to+\infty}\left|\frac{\widetilde{f}(x,t)}{\phi_\star(|t|)t}\right| = 0$ uniformly a.e. in Ω. Therefore, $|\widetilde{f}(x,t)| \leq C_{41} + C_{42}\phi_\star(|t|)|t|$, for all $t \in \mathbb{R}$, a.e. in Ω. Then, it follows from the convexity of $\widetilde{\Phi_\star}$ that

$$\widetilde{\Phi_\star}\big(|\widetilde{f}(x,t)|\big) \leq \widetilde{\Phi_\star}(C_{41} + C_{42}\phi_\star(|t|)|t|) \leq C_{43}\big(1 + \widetilde{\Phi_\star}(\phi_\star(|t|)|t|)\big).$$

Therefore, by using Lemma 2.2

$$\int_\Omega \widetilde{\Phi_\star}\big(|\widetilde{f}(x,u_n)|\big) \leq C_{44}\left(1 + \int_\Omega \Phi_\star(u_n)\right),$$

for some constant $C_{44} > 0$ not depending on $n \in \mathbb{N}$. From the continuous embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\Phi_\star}(\Omega)$, the boundness of (u_n) in $W_0^{1,\Phi}(\Omega)$ and Lemma 2.4, the claim follows. Note that $\widetilde{g}(x,u_n)$ is bounded in $L^{\widetilde{\Phi_\star}}(\Omega)$, thus it follows that there exists $\overline{w} \in L^{\widetilde{\Phi_\star}}(\Omega)$ such that

$$\lambda \widetilde{f}(x, u_n) + \mu \widetilde{g}(x, u_n) \rightharpoonup \overline{w} \text{ in } L^{\widetilde{\Phi_\star}}(\Omega). \tag{4.29}$$

Since $L'(u_n) \to 0$, we have from (4.25) and (4.29) that

$$\int_\Omega w \nabla v - \overline{w} v = 0,$$

for all $v \in W_0^{1,\Phi}(\Omega)$. By considering $v = u\psi_\varepsilon$, we have

$$\int_\Omega w \nabla (u\psi_\varepsilon) - \overline{w} u \psi_\varepsilon = 0,$$

which implies that

$$\int_\Omega (w \nabla \psi_\varepsilon) u = -\int_\Omega (w \nabla u - \overline{w} u)\psi_\varepsilon.$$

By Lebesgue's Dominated Convergence Theorem, we have $\int_\Omega (w\nabla u - \overline{w}u)\psi_\varepsilon \to 0$ as $\varepsilon \to 0^+$, therefore $\int_\Omega (w\nabla\psi_\varepsilon)u \to 0$ as $\varepsilon \to 0^+$.

Letting $\varepsilon \to 0^+$ in (4.28) we have $\ell \iota_j \leq \mu m^\star \nu_j$. Thus, it follows from (4.22) that

$$S_N^{-\alpha} \nu_j \leq \iota_j^\beta \leq \left(\frac{\mu m^\star}{\ell}\right)^\beta \nu_j^\beta,$$

for some $\alpha \in \{\ell^\star, m^\star\}$ and $\beta \in \left\{\ell^\star/l, m^\star/l, \ell^\star/m, m^\star/m\right\}$, which implies the result. □

The next result will be needed.

Lemma 4.5. *Consider the conditions of Lemma 4.4. Then, given $M, \widetilde{\mu} > 0$ there exists $\overline{\mu} \in (0, \widetilde{\mu})$, depending on M and $\widetilde{\mu}$, such that J satisfies the Palais–Smale condition at the level c for $c < M$ and $0 < \mu < \overline{\mu}$.*

Proof. Let $c < M$ and let (u_n) be a sequence in $W_0^{1,\Phi}(\Omega)$ such that $L(u_n) \to c$ and $L'(u_n) \to 0$. Consider $\overline{\mu} \in (0, \widetilde{\mu})$ such that

$$1 \leq \left(\frac{\ell}{m^\star \overline{\mu}}\right)^{\beta/(\beta-1)} S_N^{-\alpha/(\beta-1)}$$

and

$$\overline{\mu}^{\beta/(\beta-1)-1/(1-q/\ell^\star)} < \frac{\left(\frac{\ell}{m^\star}\right)^{\beta/(\beta-1)} S_N^{-\alpha/(\beta-1)}}{[C_{36}(M + C_M + C_{37} + C_{38} + C_{39}\widetilde{\mu})]^{1/(1-q/\ell^\star)}}, \tag{4.30}$$

where $C_{36}, C_{37}, C_{38}, C_{39} > 0$ are the constants given in (4.12). Note that it is possible to choose $\overline{\mu} > 0$ satisfying (4.30) because $1/(1 - q/\ell^\star) = \ell^\star/(\ell^\star - q), q < (\ell^\star/m^\star)\ell$, and

$$0 < \frac{1}{1 - \frac{q}{\ell^\star}} < \frac{\beta}{\beta - 1},$$

for all $\beta \in \left\{\ell^\star/\ell, m^\star/\ell, \ell^\star/m, m^\star/m\right\}$.

We claim that $\int_\Omega 1\, d\nu < \left(\frac{\ell}{m^\star \mu}\right)^{\beta/(\beta-1)} S_N^{-\alpha/(\beta-1)}$ for all $0 < \mu < \overline{\mu}$.

If $\int_\Omega 1\, d\nu \leq 1$, then

$$\int_\Omega 1\, d\nu \leq 1 \leq \left(\frac{\ell}{m^\star \overline{\mu}}\right)^{\beta/(\beta-1)} S_N^{-\alpha/(\beta-1)} \leq \left(\frac{\ell}{m^\star \mu}\right)^{\beta/(\beta-1)} S_N^{-\alpha/(\beta-1)},$$

for all $\mu \in (0, \overline{\mu})$.

If $\int_\Omega 1\,d\nu > 1$, then it follows from (4.12) that

$$\mu C_{36}\int_\Omega 1\,d\nu \le M + C_M + C_{37} + C_{38}\left(\int_\Omega 1\,d\nu\right)^{q/\ell^\star} + C_{39}\widetilde{\mu}$$
$$\le (M + C_M + C_{37} + C_{38} + C_{39}\widetilde{\mu})\left(\int_\Omega 1\,d\nu\right)^{q/\ell^\star},$$

which implies that

$$\int_\Omega 1\,d\nu \le \left[\frac{M + C_M + C_{37} + C_{38} + C_{39}\widetilde{\mu}}{\mu C_{36}}\right]^{1/(1-q/\ell^\star)}.$$

For $\mu \in (0, \overline{\mu})$, from (4.30) we have

$$\mu^{\beta/(\beta-1)-1/(1-q/\ell^\star)} < \overline{\mu}^{\beta/(\beta-1)-1/(1-q/\ell^\star)} < \frac{\left(\frac{\ell}{m^\star}\right)^{\beta/(\beta-1)} S_N^{-\alpha/(\beta-1)}}{[C_{36}(M + C_M + C_{37} + C_{38} + C_{39}\widetilde{\mu})]^{1/(1-q/\ell^\star)}}.$$

Therefore,

$$\nu_j \le \int_\Omega 1\,d\nu < \left(\frac{\ell}{m^\star \mu}\right)^{\beta/(\beta-1)} S_N^{-\alpha/(\beta-1)},$$

for all $j \in \mathcal{J}$ and $\mu \in (0, \overline{\mu})$. Thus, by using Lemma 4.4, we obtain $\nu_j = 0$, for all $j \in \mathcal{J}$, which leads to

$$\int_\Omega \Phi_\star(u_n) \to \int_\Omega \Phi_\star(u). \tag{4.31}$$

Combining (4.31) with the Brézis–Lieb lemma [9], we obtain

$$\int_\Omega \Phi_\star(u_n - u) \to 0.$$

Therefore, $u_n \to u$ in $L^{\Phi_\star}(\Omega)$. Since $L'(u_n)u_n = o_n(1)$, we have

$$\int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 = o_n(1) + \mu\int_\Omega \left(\phi_\star(u_n)u_n^2 + \lambda \widetilde{f}(x, u_n)u_n\right).$$

Define

$$P_n(x) = \langle \phi(|\nabla u_n(x)|)\nabla u_n(x) - \phi(|\nabla u(x)|)\nabla u(x), \nabla u_n(x) - \nabla u(x)\rangle.$$

Since $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$, we have

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla(u_n - u) \to 0,$$

which leads to

$$\int_\Omega P_n = o_n(1) + \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 - \int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla u. \tag{4.32}$$

Combining (4.32) with the fact that $o_n(1) = L'(u_n)u_n - L'(u_n)u$, we obtain

$$o_n(1) = \int_\Omega P_n + \mu\int_\Omega \phi_\star(u_n)(u - u_n) + \lambda\int_\Omega \widetilde{f}(x, u_n)(u - u_n).$$

From Lemma 2.2 and the boundness of (u_n) in $L^{\Phi_\star}(\Omega)$, we have that the sequence $\phi_\star(u_n)u_n$ is bounded in $L^{\widetilde{\Phi_\star}}(\Omega)$. By using the Hölder inequality (2.1), we have

$$\left|\int_\Omega \phi_\star(u_n)(u-u_n)\right| \leq 2\|\phi_\star(u_n)u_n\|_{L^{\widetilde{\Phi_\star}}}\|u-u_n\|_{L^{\Phi_\star}} \to 0.$$

From Lebesgue's Dominated Convergence Theorem, we obtain

$$\int_\Omega \widetilde{f}(x,u_n)(u-u_n) \to 0.$$

Therefore, $\int_\Omega P_n \to 0$. Then, it follows from [2, Lemma 2.2] that $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$. $\square$

Proof of Theorem 1.2 (second part). Consider the notations of the proof of the first part of Theorem 1.2. Let γ_0 be the line segment joining 0 and $\bar{t}\varphi$ and let L_0 be the functional obtained by setting $\mu = 0$ in L. Since $\mu > 0$ and $\widetilde{G}(x,t) \geq 0$ for $t \geq 0$, we have

$$c \leq \max_{u\in\gamma_0([0,1])} L(u) \leq \max_{u\in\gamma_0([0,1])} L_0(u) := c_0.$$

By Lemma 4.5, the functional L satisfies the Palais–Smale condition at all levels $\leq c_0$ for $\mu > 0$ small enough, so the conclusion follows as in the proof of the first part of the result. $\square$

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REFERENCES

[1] R. A. Adams and J. F. Fournier, *Sobolev spaces*, Academic Press, New York, 2003.
[2] C. O. Alves, A. R. de Holanda, and J. A. Santos, *Existence of positive solutions for a class of semipositone quasilinear problems through Orlicz–Sobolev space*, Proc. Amer. Math. Soc. **147** (2019), no. 1, 285–299.
[3] C. O. Alves, J. V. A. Gonçalves, and K. O. Silva, *Multiple sign-changing radially symmetric solutions in a general class of quasilinear elliptic equations*, Z. Angew. Math. Phys. **66** (2015), no. 5, 2601–2623.
[4] C. O. Alves, V. D. Rădulescu, and L. S. Tavares, *Generalized choquard equations driven by nonhomogeneous operators*, Mediterr. J. Math. **16** (2019), no. 1, 24 pp.
[5] A. Ambrosetti, H. Brezis, and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), no. 2, 519–543.
[6] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.
[7] V. Benci, D. Fortunato, and L. Pisani, *Solitons like solutions of a Lorentz invariant equation in dimension 3*, Rev. Math. Phys. **10** (1998), no. 3, 315–344.
[8] G. Bonanno, G. Molica Bisci, and V. D. Rădulescu, *Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz–Sobolev spaces*, Monatsh. Math. **165** (2012), no. 3-4, 305–318.
[9] H. Brezis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490.
[10] S. Caldwell, A. Castro, R. Shivaji, and S. Unsurangsie, *Positive solutions for classes of multiparameter elliptic semipositone problems*, Electron. J. Differential Equations **2007** (2007), 10.
[11] A. Castro, D. G. de Figueiredo, and E. Lopera, *Existence of positive solutions for a semipositone p-Laplacian problem*, Proc. R. Soc. Edinb. A **146** (2016), no. 3, 475–482.
[12] E. D. da Silva, M. L. M. Carvalho, J. V. Gonçalves, and L. S. Tavares, *Critical quasilinear elliptic problems using concave-convex nonlinearities*, Ann. Mat. Pura Appl. **198** (2019), no. 3, 693–726.
[13] T. K. Donaldson and N. S. Trudinger, *Orlicz–Sobolev spaces and embedding theorems*, J. Funct. Anal. **8** (1971), no. 8, 52–75.

[14] G. M. Figueiredo, G. C. G. dos Santos, and L. S. Tavares, *Sub-supersolution method for a singular problem involving the ϕ-Laplacian and Orlicz–Sobolev spaces*, Complex Var. Elliptic. Equ. **65** (2020), no. 3, 409–422.
[15] G. M. Figueiredo, A. Moussaoui, G. C. G. dos Santos, and L. S. Tavares, *A sub-supersolution approach for some classes of nonlocal problems involving Orlicz spaces*, J. Differential Equations **267** (2019), no. 7, 4148–4169.
[16] M. Fuchs and V. Osmolovski, *Variational integrals on Orlicz–Sobolev spaces*, Z. Anal. Anwend. **17** (1998), no. 2, 393–415.
[17] N. Fukagai, M. Ito, and K. Narukawa, *Positive solutions of quasilinear elliptic equations with critical Orlicz–Sobolev nonlinearity on $\mathbb{R}^N$*, Funkcial. Ekvac. **49** (2006), no. 2, 235–267.
[18] N. Fukagai, M. Ito, and K. Narukawa, *Quasilinear elliptic equations with slowly growing principal part and critical Orlicz–Sobolev nonlinear term*, Proc. Roy. Soc. Edinburgh Sect. A **139** (2009), no. 1, 73–106.
[19] N. Fukagai and K. Narukawa, *On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems*, Ann. Mat. Pura Appl. **186** (2007), no. 3, 539–564.
[20] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 2001.
[21] J.-P. Gossez, *Orlicz–Sobolev spaces and nonlinear elliptic boundary value problems*, Nonlinear Analysis, Function Spaces and Applications, Proceedings of a Spring School held in Horni Bradlo, 1978, Vol. 1, (S. Fučk and A. Kufner, eds.), BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1979, pp. 59–94.
[22] R. Leitao, O. de Queiroz, and E. Teixeira, *Regularity for degenerate two-phase free boundary problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), no. 4, 741–762.
[23] X. Li and G. Jia, *Multiplicity of solutions for quasilinear elliptic problems involving ϕ-Laplacian operator and critical growth*, Electron. J. Qual. Theory Differ. Equ. **2019** (2019), 15.
[24] G. M. Liebermann, *The natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), no. 2–3, 311–361.
[25] M. Mihăilescu, V. D. Rădulescu, and D. Repovš, *On a non-homogeneous eigenvalue problem involving a potential: an Orlicz–Sobolev space setting*, J. Math. Pures Appl. **93** (2010), no. 2, 132–148.
[26] K. Perera and R. Shivaji, *Positive solutions of multiparameter semipositone p-Laplacian problems*, J. Math. Anal. Appl. **338** (2008), no. 2, 1397–1400.
[27] J. A. Santos, *Multiplicity of solutions for quasilinear equations involving critical Orlicz–Sobolev nonlinear terms*, Electron. J. Differential Equations **2013** (2013), no. 249, 13.
[28] J. A. Santos and S. H. Monari Soares, *A limiting free boundary problem for a degenerate operator in Orlicz–Sobolev spaces*, Rev. Mat. Iberoam. **36** (2020), no. 6, 1687–1720.
[29] E. Silva and M. Xavier, *Multiplicity of solutions for quasilinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. H. Poincaré Anal. Non Linéaire **36** (2020), no. 6, 1687–1720.
[30] M. Struwe, *Variational methods, applications to nonlinear partial differential equations and Hamiltonian systems*, Springer, Berlin, 1996.
[31] Z. Tan and F. Fang, *Orlicz–Sobolev versus Hölder local minimizer and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **402** (2013), no. 1, 348–370.

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