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Indefinite Perturbations of the Eigenvalue Problem for the Nonautonomous p-Laplacian

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Abstract. We consider an indefinite perturbation of the eigenvalue problem for the nonautonomous p-Laplacian. The main result establishes an exhaustive analysis in the nontrivial case that corresponds to noncoercive perturbations of the reaction. Using variational tools and truncation and comparison techniques, we prove an existence and multiplicity theorem which is global in the parameter. The main result of this paper establishes the existence of at least two positive solutions in the case of small perturbations, while no solution exists for high perturbations of the quasilinear term in the reaction.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. Consider the following classical parametric semilinear Dirichlet problem with superlinear subcritical perturbation:

$$\left\{ \begin{array}{l} -\Delta u(z) = \lambda u(z) + \xi(z) u(z)^{r-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{array} \right\} \tag{1}$$

where $1 < r < 2^*$, λ is a real parameter and ξ is a nonnegative notrivial potential. Let $\lambda_1 > 0$ be the first eigenvalue of the Laplace operator in $H_0^1(\Omega)$ and let $\varphi_1 > 0$ denote the corresponding eigenfunction. A direct application of the mountain pass theorem shows that in the coercive case where $\lambda < \lambda_1$, problem (1) has a positive solution. If $\lambda \geq \lambda_1$ (noncoercive case), there is no positive solution of (1); this follows by multiplying with φ_1 and integrating. However, the dual variational method implies that problem (1) has at least a solution for all $\lambda \geq \lambda_1$. The case where ξ is an indefinite potential becomes more complicated, for instance we cannot assert whether problem (1) has positive solutions.

Nonlinear eigenvalue problems arise in many parts of mathematical physics and an understanding of their nature is of practical as well as theoretical importance. Such problems aim to explain a diversity of natural phenomena that have been observed and characterized over the years. For instance, the buckling of the Euler rod, the appearance of Taylor vortices, and the emergence of perturbations in an electric circuit, all have the same cause: a physical parameter crosses a threshold, pressuring the system to assemble itself into a new state that differs significantly from the previous state. Here we refer to the pioneering global bifurcation results established by Crandall and Rabinowitz [4] and Rabinowitz [22].

A deep motivation of the analysis developed in this paper comes from the seminal work by Brezis and Vázquez [2], who established the existence of an "extreme value" λ^* of the bifurcation parameter λ such that a large class of problems with convex and increasing nonlinearity has a smooth positive solution for all $0 < \lambda < \lambda^*$, but no solution exists if $\lambda > \lambda^*$. On the other hand, Garcia Azorero, Peral Alonso and Manfredi [9] proved that for all $0 < \lambda < \lambda^*$, there are at least two solutions. The analysis carried out in [9] is developed in the case of competition phenomena of convex and concave nonlinearities. The present paper is devoted to the analysis of a more general class of parametric Dirichlet problems with indefinite perturbation. We are concerned with the study of the following class of quasilinear elliptic boundary value problems

$$\left\{\begin{array}{l} -\Delta_p^a u(z) = \lambda u(z)^{p-1} + \xi(z)u(z)^{r-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{array}\right\} \qquad (P_\lambda)$$

where $\lambda \geq \hat{\lambda}_1^a > 0$, $1 < p < r < p^*$ and $\hat{\lambda}_1^a$ is the principal eigenvalue of $(-\Delta_p^a, W_0^{1,p}(\Omega))$.

In this equation, $a \in C^{0,1}(\overline{\Omega})$ is a weight function satisfying $0 < \hat{c} \leq a(z)$ for all $z \in \overline{\Omega}$ (recall that $C^{0,1}(\overline{\Omega})$ is the space of all $\mathbb{R}$-valued Lipschitz functions defined on $\overline{\Omega}$). By Δ_p^a we denote the nonautonomous p-Laplace differential operator defined by

$$\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega).$$

The interest in the study of problem (P_λ) is twofold. On the one hand, there are physical motivations, since the non-autonomous differential operator has been applied to describe steady-state solutions of reaction-diffusion problems in biophysics, plasma physics, and chemical reaction analysis. The prototype equation for these models can be written in the form

$$u_t = \Delta_p^a u(z) + \lambda u^{p-1}(z) + \xi(z)u^{r-1}(z).$$

In this framework, the function u (assumed to be positive) generally stands for a concentration, the term $\Delta_p^a u(z)$ corresponds to the diffusion with coefficient $a(z)|Du|^{p-2}$ while $\lambda u^{p-1}(z) + \xi(z)u^{r-1}(z)$ represents the reaction term related to source and loss processes. On the other hand, such differential operators provide a valuable framework for explaining the behavior of highly anisotropic materials whose hardening properties, which are linked to the exponent governing the propagation of the gradient variable, differ considerably with the point, where the modulating coefficient $a(z)$ dictates the geometry of a composite material.

In the reaction of problem (P_λ), λ is a parameter. We are mainly concerned with the case where $\lambda \geq \hat{\lambda}_1^a$, which expresses the fact that the corresponding eigenvalue problem is not coercive. The perturbation $\xi(z)u(z)^{r-1}$ is indefinite, that is, $\xi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ satisfies $\xi^+ \neq 0 \neq \xi^-$. So, we are dealing with an indefinite superlinear perturbation of the eigenvalue problem for $(-\Delta_p^a, W_0^{1,p}(\Omega))$. Our aim is to prove an existence and multiplicity theorem for positive solutions, which is global in the parameter λ (a bifurcation-type theorem).

This problem was first investigated by Brown and Zhang [3] and Ouyang [18] for semilinear equations driven by the Laplacian. Brown and Zhang [3] used the Nehari method, while Ouyang [18] used bifurcation and variational methods. Extensions to equations driven by the autonomous p-Laplacian (that is, $a \equiv 1$), were obtained by Drabek and Pohozaev [7] and by Birindelli and Demengel [1]. Drabek and Pohozaev [7] used the fibering method (see Kuzin and Pohozaev [13]), while Birindelli and Demengel [1] followed a variational approach. Their existence and multiplicity results are not global in $\lambda > 0$. Motivated by the above mentioned pioneering contributions, we develop in this paper an exhaustive bifurcation analysis in the framework of a standard Dirichlet boundary condition. To the best of our knowledge, this is the first analysis carried out for *non-autonomous* quasilinear equations with *indefinite* potential and *noncoercive* perturbation.

2. Mathematical Background and Hypotheses

The main function spaces used in the analysis of problem (P_λ) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \ u = 0 \text{ on } \partial\Omega\}.$$

On account of the Poincaré inequality, we can consider on $W_0^{1,p}(\Omega)$ the following norm:

$$\|u\| = \|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : \ u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\mathrm{int}\, C_+ = \left\{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$ and $\dfrac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$.

Let $a \in C^0(\overline{\Omega})$ with $a(z) \geq \hat{c} > 0$ for all $z \in \overline{\Omega}$ and consider the nonlinear operator $A_p^a : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ defined by

$$\langle A_p^a u, h \rangle = \int_\Omega a(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W_0^{1,p}(\Omega).$$

This operator has the following properties (see Proposition 7.77 of Hu and Papageorgiou [11, p.465]).

Proposition 2.1. *The operator $A_p^a(\cdot)$ is bounded (maps bounded sets to bounded sets), continuous, strictly monotone (thus, maximal monotone, too) and of type $(S)_+$, that is,*
"if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty} \langle A_p^a(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$."

We consider the following nonlinear eigenvalue problem

$$-\Delta_p^a u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \text{ in } \Omega, \ u|_{\partial\Omega} = 0. \tag{2}$$

We say that the real number $\hat{\lambda}$ is an *eigenvalue* of $(-\Delta_p^a, W_0^{1,p}(\Omega))$ if problem (2) has a nontrivial weak solution $\hat{u} \in W_0^{1,p}(\Omega)$, which is known as an *eigenfunction* corresponding to the eigenvalue $\hat{\lambda}$.

We know (see Liu and Papageorgiou [17]) that the following properties hold:

(i) There is a smallest eigenvalue $\hat{\lambda}_1^a(p) > 0$ which is given by

$$\hat{\lambda}_1^a(p) = \inf\left\{\frac{\int_\Omega a(z)|Du|^p dz}{\|u\|_p^p} : \ u \in W_0^{1,p}(\Omega), \, u \neq 0\right\}. \tag{3}$$

(ii) $\hat{\lambda}_1^a(p)$ is simple (that is, if $\hat{u}, \hat{v} \in W_0^{1,p}(\Omega)$ are two eigenfunctions corresponding to $\hat{\lambda}_1^a(p)$, then $\hat{u} = \theta\hat{v}$ with $\theta \in \mathbb{R}$, $\theta \neq 0$), isolated (that is, if σ_p^a denotes the spectrum of (2), then there exists $\varepsilon > 0$ such that $(\hat{\lambda}_1^a(p), \hat{\lambda}_1^a(p) + \varepsilon) \cap \sigma_p^a = \emptyset$); moreover, the eigenfunctions corresponding to $\hat{\lambda}_1^a(p)$ have fixed sign and belong to $\operatorname{int} C_+ \cup (-\operatorname{int} C_+)$.

(iii) If $\hat{\lambda} > \hat{\lambda}_1^a(p)$ is an eigenvalue of (2), then the eigenfunctions corresponding to $\hat{\lambda}$ are nodal (sign-changing).

If $u, v : \Omega \to \mathbb{R}$ are measurable functions and $u(z) \leq v(z)$ for a.a. $z \in \Omega$, then we define

$$\begin{aligned} [u, v] &= \{h \in W_0^{1,p}(\Omega); \ u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\}, \\ \operatorname{int}_{C_0^1(\overline{\Omega})}[u, v] &= \text{interior in } C_0^1(\overline{\Omega}) \text{of } [u, v] \cap C_0^1(\overline{\Omega}), \\ [u) &= \{h \in W_0^{1,p}(\Omega); \ u(z) \leq h(z) \text{ for a.a. } z \in \Omega\}. \end{aligned}$$

We denote by $|\cdot|_N$ the Lebesgue measure on $\mathbb{R}^N$ and when we want to emphasize the domain of the eigenvalue problem, we will write $\hat{\lambda}_1^a(p, \Omega)$. We denote by $\mathrm{Lip}_{loc}(\Omega)$ the space of locally Lipschitz functions on Ω.

Given a measurable function $u : \Omega \to \mathbb{R}$ we write $0 \prec u$ if for all $K \subseteq \Omega$ compact, we have

$$0 < c_K \leq u(z) \text{ for a.a. } z \in K.$$

If X is a Banach space and $\varphi \in C^1(X)$, we denote

$$K_\varphi = \{u \in X;\ \varphi'(u) = 0\} \text{ (the critical set of } \varphi)$$

and we say that $\varphi(\cdot)$ satisfies the *C-condition* if it has the following property:

"Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that
$\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^*,
admits a strongly convergent subsequence."

Our hypotheses on the data of problem (P_λ) are the following.

H_0: $a \in C^{0,1}(\overline{\Omega})$, $a(z) \ge \hat{c} > 0$ for all $z \in \overline{\Omega}$ and $\xi \in \mathrm{Lip}_{loc}(\Omega) \cap L^\infty(\Omega)$ such that $\xi^+ \ne 0 \ne \xi^-$ and if $\Omega_+ = \{z \in \Omega:\ \xi(z) > 0\}$, $\Omega_- = \{z \in \Omega:\ \xi(z) < 0\}$, then $|\Omega\backslash(\Omega_+ \cup \Omega_-)|_N = 0$ and $\int_\Omega \xi(z)\hat{u}_1^r dz < 0$, with $\hat{u}_1$ being the positive L^p-normalized (that is, $\|\hat{u}_1\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1^a(p) > 0$ (we know that $\hat{u}_1 \in \operatorname{int} C_+$).

For $\lambda \ge \hat{\lambda}_1^a(p)$, let $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\psi_\lambda(u) = \frac{1}{p}\int_\Omega a(z)|Du|^p dz - \frac{\lambda}{p}\|u\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega).$$

We write $\psi_1 = \psi_{\lambda_1^a(p)}$.

3. Positive Solutions

We start by considering the following minimization problem:

$$m = \inf\left\{\psi_1(u):\ u \in W_0^{1,p}(\Omega),\ \|u\|_p = 1,\ \int_\Omega \xi(z)|u|^r dz = 0\right\}. \tag{4}$$

Proposition 3.1. *If hypotheses H_0 hold, then $m > 0$.*

Proof. From (3) we see that $m \ge 0$. Suppose that $m = 0$. Then we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$\psi_1(u_n) \downarrow 0,\ \|u_n\|_p = 1,\ \int_\Omega \xi(z)|u_n|^r dz = 0 \text{ for all } n \in \mathbb{N}. \tag{5}$$

From (5) we see that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega),\ u_n \to u \text{ in } L^r(\Omega). \tag{6}$$

The functional ψ_1 is sequentially weakly lower semicontinuous. So, from (6) we have

$$\psi_1(u) \le \liminf_{n\to\infty} \psi_1(u_n) = 0. \tag{7}$$

Moreover, again from (6), we obtain

$$\|u\|_p = 1,\ \int_\Omega \xi(z)|u|^r dz = 0. \tag{8}$$

From (7) and (8) it follows that

$$\begin{aligned}\psi_1(u) &= 0,\\ &\Rightarrow \int_\Omega a(z)|Du|^p dz = \hat{\lambda}_1^a \|u\|_p^p,\\ &\Rightarrow u = \vartheta \hat{u}_1,\ \text{with } \vartheta \neq 0.\end{aligned}$$

Using now (8) we have

$$\int_\Omega \xi(z)\hat{u}_1^r dz = 0,$$

which contradicts hypothesis H_0. Therefore $m > 0$. □

We introduce the following two sets:

$$\begin{aligned}\mathcal{L} &= \{\lambda \geq \hat{\lambda}_1^a(p);\ \text{problem } (P_\lambda) \text{has a positive solution}\},\\ S_\lambda &= \text{ set of positive solutions of } (P_\lambda).\end{aligned}$$

Proposition 3.2. *If hypotheses H_0 hold, then $\mathcal{L} \neq \emptyset$ and for all $\lambda \in \mathcal{L}$ we have $S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. Let $\lambda \geq \hat{\lambda}_1^a(p)$ and consider the following minimization problem

$$\beta_\lambda^* = \inf\left\{\psi_\lambda(u);\ \frac{1}{r}\int_\Omega \xi(z)|u|^r dz = 1,\ u \in W_0^{1,p}(\Omega)\right\}. \tag{9}$$

We first show that if $\varepsilon > 0$ is small and $\lambda \in (\hat{\lambda}_1^a(p), \hat{\lambda}_1^a(p) + \varepsilon)$ then $\beta_\lambda^* > -\infty$. Arguing by contradiction, suppose that for some $\lambda > \hat{\lambda}_1^a(p)$ we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_n) \to -\infty,\ \frac{1}{r}\int_\Omega \xi(z)|u_n|^r dz = 1 \text{ for all } n \in \mathbb{N}. \tag{10}$$

Using (3), we have

$$\frac{1}{p}[\hat{\lambda}_1^a(p) - \lambda]\|u_n\|_p^p \leq \psi_\lambda(u_n) \to -\infty.$$

Since $\lambda > \hat{\lambda}_1^a(p)$, we must have

$$\|u_n\|_p \to \infty. \tag{11}$$

Let $y_n = \frac{u_n}{\|u_n\|_p}$ for $n \in \mathbb{N}$. Then $\|y_n\|_p = 1$ for all $n \in \mathbb{N}$. We can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\psi_\lambda(u_n) \leq 0 \text{ for all } n \geq n_0,\\ &\Rightarrow \psi_\lambda(y_n) \leq 0 \text{ for all } n \geq n_0,\\ &\Rightarrow \hat{c}\,\|Dy_n\|_p^p \leq \lambda \text{ for all } n \geq n_0,\\ &\Rightarrow \{y_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}\end{aligned}$$

We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega),\ y_n \to y \text{ in } L^r(\Omega). \tag{12}$$

Using (12) we have

$$\psi_\lambda(y) \le \liminf_{n\to\infty} \psi_\lambda(y_n) \le 0,$$
$$\frac{1}{r}\int_\Omega \xi(z)|y_n|^r dz = \frac{1}{\|u_n\|_p^r} \text{ for all } n \in \mathbb{N} \text{ (see (10))}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\psi_\lambda(y) \le 0,\ \frac{1}{r}\int_\Omega \xi(z)|y|^r dz = 0. \tag{13}$$

So, if

$$m_\lambda = \inf\left\{\psi_\lambda(v) :\ \frac{1}{r}\int_\Omega \xi(z)|v|^r dz = 0,\ v \in W_0^{1,p}(\Omega)\right\},$$

then from (13), we see that $m_\lambda \le 0$. On the other hand, $m_{\hat{\lambda}_1^a(p)} = m > 0$ and from Proposition 7.18 of Dal Maso [5, p.79], we know that the mapping $\lambda \mapsto m_\lambda$ is continuous on $[\hat{\lambda}_1^a(p), \infty)$. So, we can find $\varepsilon > 0$ such that $m_\lambda > 0$ for all $\lambda \in (\hat{\lambda}_1^a(p), \hat{\lambda}_1^a(p) + \varepsilon)$, a contradiction. We infer that

$$\beta_\lambda^* > -\infty \text{ for all } \lambda \in (\hat{\lambda}_1^a(p), \hat{\lambda}_1^a(p) + \varepsilon).$$

We now consider a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ for problem (9), when $\lambda \in (\hat{\lambda}_1^a(p), \hat{\lambda}_1^a(p) + \varepsilon)$. We have

$$\psi_\lambda(u_n) \downarrow \beta_\lambda^* \text{ and } \frac{1}{r}\int_\Omega \xi(z)|u_n|^r dz = 1 \text{ for all } n \in \mathbb{N}. \tag{14}$$

Claim. $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded.

Arguing by contradiction, assume that at least for a subsequence, we have

$$\|u_n\| \to \infty. \tag{15}$$

Let $y_n = \frac{u_n}{\|u_n\|_p}$ for $n \in \mathbb{N}$. From (14) and (15) it follows that

$$\psi_\lambda(y_n) \to 0 \text{ as } n \to \infty.$$

This implies that $\{y_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ is bounded and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega),\ y_n \to y \text{ in } L^r(\Omega). \tag{16}$$

Then from (16) and the sequential weak lower semicontinuity of $\psi_\lambda(\cdot)$, we have

$$\psi_\lambda(y) \le \liminf_{n\to\infty} \psi_\lambda(y_n) = 0,\ \|y\|_p = 1,\ \tfrac{1}{r}\textstyle\int_\Omega \xi(z)|y|^r dz = 0,$$
$$\Rightarrow m_\lambda \le 0.$$

But from the first part of the proof we have

$$m_\lambda > 0 \text{ for all } \lambda \in (\hat{\lambda}_1^a(p), \hat{\lambda}_1^a(p) + \varepsilon),$$

a contradiction. This proves the Claim.

On account of the Claim, we may assume

$$u_n \xrightarrow{w} \hat{u} \text{ in } W_0^{1,p}(\Omega),\ u_n \to \hat{u} \text{ in } L^r(\Omega). \tag{17}$$

Using (17), we can say that

$$\psi_\lambda(\hat{u}) \leq \liminf_{n\to\infty} \psi_\lambda(u_n) = \beta_\lambda^*, \ \frac{1}{r}\int_\Omega \xi(z)|\hat{u}|^r dz = 1,$$
$$\Rightarrow \psi_\lambda(\hat{u}) = \beta_\lambda^* \text{ and } \frac{1}{r}\int_\Omega \xi(z)|\hat{u}|^r dz = 1 \text{ (see (9))}. \tag{18}$$

Replacing $\hat{u} \in W_0^{1,p}(\Omega)$ with $|\hat{u}| \in W_0^{1,p}(\Omega)$, we see that we may assume that $\hat{u} \geq 0$, $\hat{u} \neq 0$. From (18) and the Lagrange multiplier rule (see [19, p.422]), we can find $\eta \in \mathbb{R}$ such that

$$\langle \psi_\lambda'(\hat{u}), h\rangle = \eta \int_\Omega \xi(z)\hat{u}^{r-1} h dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{19}$$

In (19) we use the test function $h = \hat{u} \in W_0^{1,p}(\Omega)$ and obtain

$$\int_\Omega a(z)|D\hat{u}|^p dz - \lambda\|\hat{u}\|_p^p = \eta \int_\Omega \xi(z)\hat{u}^r dz,$$
$$\Rightarrow p\psi_\lambda(\hat{u}) = \eta r,$$
$$\Rightarrow \eta = \frac{p}{r}\beta_\lambda^* \text{ (see (18))}.$$

From (19) we have

$$-\Delta_p^a \hat{u} - \lambda \hat{u}^{p-1} = \frac{p}{r}\beta_\lambda^* \xi(z)\hat{u}^{r-1} \text{ in } \Omega. \tag{20}$$

Let $\tilde{u} = \left(\frac{p}{r}\beta_\lambda^*\right)^{1/(r-p)} \hat{u} \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
&-\Delta_p^a \tilde{u} - \lambda\tilde{u}^{p-1} \\
&\quad = \left(\frac{p}{r}\beta_\lambda^*\right)^{\frac{p-1}{r-p}} (-\Delta_p^a \hat{u} - \lambda\hat{u}^{p-1}) \\
&\quad = \left(\frac{p}{r}\beta_\lambda^*\right)^{\frac{r-1}{r-p}} \xi(z)\hat{u}^{r-1} \text{ (see (20))} \\
&\quad = \xi(z)\tilde{u}^{r-1} \text{ in } \Omega, \\
&\quad \Rightarrow \tilde{u} \in S_\lambda \text{ for all } \lambda \in (\hat{\lambda}_1^a, \hat{\lambda}_1^a + \varepsilon).
\end{aligned}$$

We have proved that

$$(\hat{\lambda}_1^a, \hat{\lambda}_1^a + \varepsilon) \subset \mathcal{L} \neq \emptyset.$$

Let $u \in S_\lambda$. Then

$$-\Delta_p^a u = \lambda u^{p-1} + \xi(z)u^{r-1} \text{ in } \Omega.$$

Theorem 7.1 of Ladyzhenskaya and Uraltseva [14] implies that $u \in L^\infty(\Omega)$. Then, applying the nonlinear regularity theory of Lieberman [15], we have that

$$u \in C_+\backslash\{0\}.$$

We have

$$-\Delta_p^a u + \|\xi\|_\infty \|u\|_\infty^{r-p} u^{p-1} \geq 0 \text{ in } \Omega.$$

Invoking Lemma 1 of Liu and Papageorgiou [16], we obtain

$$u \in \operatorname{int} C_+.$$

Therefore for all $\lambda \in \mathcal{L}$, $S_\lambda \subseteq \operatorname{int} C_+$. $\square$

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 3.3. *If hypotheses H_0 hold, then $\lambda^* < \infty$.*

Proof. Let $\hat{\Omega}_+$ be a connected component of Ω_+. Let $\hat{\lambda}_1(\hat{\Omega}_+)$ be the principal eigenvalue of $(-\Delta_p^a, W_0^{1,p}(\hat{\Omega}_+))$ and let $\hat{u}_+ \in W_0^{1,p}(\hat{\Omega}_+) \cap L^\infty(\hat{\Omega}_+) \cap C_{loc}^{0,\alpha}(\hat{\Omega}_+)$ be the corresponding L^p-normalized positive eigenfunction for $\hat{\lambda}_1^a(\hat{\Omega}_+)$. Using Proposition 2.4 of Papageorgiou, Vetro and Vetro [21], we have

$$\hat{u}_+(z) > 0 \text{ for all } z \in \Omega. \tag{21}$$

We know that $\hat{\lambda}_1^a < \hat{\lambda}_1^a(\hat{\Omega}_+)$. Let $\lambda > \hat{\lambda}_1^a(\hat{\Omega}_+)$ and suppose that $\lambda \in \mathcal{L}$. Let $u \in S_\lambda \subseteq \operatorname{int} C_+$. Then $u(z) > 0$ for all $z \in \hat{\Omega}_+$. Consider the following function defined on $\hat{\Omega}_+$:

$$R(\hat{u}_+, u) = |D\hat{u}_+|^p - a(z)|Du|^{p-2}(Du, D\left(\frac{\hat{u}_+^p}{u^{p-1}}\right))_{\mathbb{R}^N}.$$

Integrating over $\hat{\Omega}_+$ and using the nonlinear Picone's inequality of Jaros [12], we have

$$\begin{aligned}
0 &\le \int_{\hat{\Omega}_+} R(\hat{u}_+, u)dz \\
&= \|D\hat{u}_+\|^p_{L^p(\hat{\Omega}_+)} - \int_{\hat{\Omega}_+} (-\Delta_p^a u)\frac{\hat{u}_+^p}{u^{p-1}}dz \\
&\quad \text{(using the nonlinear Green's identity, see [19, p.34])} \\
&= \|D\hat{u}_+\|^p_{L^p(\hat{\Omega}_+)} - \int_{\hat{\Omega}_+} [\lambda u^{p-1} + \xi^+(z)u^{r-1}]\frac{\hat{u}_+^p}{u^{p-1}}dz \\
&= \|D\hat{u}_+\|^p_{L^p(\hat{\Omega}_+)} \lambda \|\hat{u}_+\|^p_{L^p(\hat{\Omega}_+)} - \int_{\hat{\Omega}_+} \xi^+(z)u^{r-p}\hat{u}_+^p dz \\
&= -\int_{\hat{\Omega}_+} \xi^+(z)u^{r-p}\hat{u}_+^p dz < 0,
\end{aligned}$$

a contradiction. Therefore $\lambda \notin \mathcal{L}$. We conclude that $\lambda^* \le \hat{\lambda}_1^a(\hat{\Omega}_+) < \infty$. □

Next, we show that $\mathcal{L}$ is connected (an interval).

Proposition 3.4. *If hypotheses H_0 hold, $\lambda \in \mathcal{L}$ and $\mu \in (\hat{\lambda}_1^a, \lambda)$, then $\mu \in \mathcal{L}$ and given $u_\lambda \in S_\lambda$ we can find $u_\mu \in S_\mu$ such that $u_\lambda - u_\mu \in \operatorname{int} C_+$.*

Proof. Since $\lambda \in \mathcal{L}$, we have $S_\lambda \neq \emptyset$. Let $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$. We have

$$\begin{aligned}
-\Delta_p^a u_\lambda &= \lambda u_\lambda^{p-1} + \xi(z)u_\lambda u_\lambda^{r-1} \\
&\ge \mu u_\lambda^{p-1} + \xi(z)u_\lambda u_\lambda^{r-1} \text{ in } \Omega.
\end{aligned} \tag{22}$$

We introduce the Carathéodory function $k_\mu(z, x)$ defined by

$$k_\mu(z, x) = \begin{cases} \mu(x^+)^{p-1} + \xi(z)(x^+)^{r-1} & \text{if } x \le u_\lambda(z) \\ \mu u_\lambda(z)^{p-1} + \xi(z)u_\lambda(z)^{r-1} & \text{if } x > u_\lambda(z). \end{cases} \tag{23}$$

We set $K_\mu(z, x) = \int_0^x k_\mu(z, s)ds$ and consider the C^1-functional $\gamma_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_\mu(u) = \frac{1}{p}\int_\Omega a(z)|Du|^p dz - \int_\Omega K_\mu(z, u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Using hypotheses H_0 and (23), we obtain

$$\gamma_\mu(u) \geq \frac{\hat{c}}{p}\,\|Du\|_p^p - c_1 \text{ for some} c_1 > 0, \text{ all} u \in W_0^{1,p}(\Omega),$$

hence $\gamma_\mu(\cdot)$ is coercive.

Also, using the Sobolev embedding theorem, we see that $\gamma_\mu(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\gamma_\mu(u_\mu) = \inf\{\gamma_\mu(u);\ u \in W_0^{1,p}(\Omega)\}. \tag{24}$$

Recall that $u_\lambda \in \operatorname{int} C_+$. Using Proposition 4.1.22 of [19, p.274], we can find $t \in (0,1)$ small such that

$$0 \leq t\hat{u}_1 \leq u_\lambda \text{ in } \overline{\Omega}. \tag{25}$$

Then we have

$$\gamma_\mu(t\hat{u}_1) \leq \tfrac{t^p}{p}[\hat{\lambda}_1^a - \mu] + \tfrac{\|\xi\|_\infty}{r}\, t^r \|\hat{u}_1\|_r^r$$
$$\text{(see (23), (25) and recall that} \|\hat{u}_1\|_p = 1).$$

Since $\mu > \hat{\lambda}_1^a$, we can write

$$\gamma_\mu(t\hat{u}_1) \leq c_2 t^r - c^3 t^p \quad \text{for some } c_2, c_3 > 0.$$

But $p < r$. So, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} &\gamma_\mu(t\hat{u}_1) < 0,\\ &\Rightarrow \gamma_\mu(u_\mu) < 0 = \gamma_\mu(0) \text{ (see (24))},\\ &\Rightarrow u_\mu \neq 0. \end{aligned}$$

From (24), we have

$$\langle \gamma_\mu'(u_\mu), h\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega),$$
$$\Rightarrow \langle A_p^a(u_\mu), h\rangle = \int_\Omega k_\mu(z, u_\mu) \text{ for all } h \in W_0^{1,p}(\Omega). \tag{26}$$

In (26) we first choose the test function $h = -u_\mu^- \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned} &\hat{c}\,\|Du_\mu^-\|_p^p \leq 0,\\ &\Rightarrow u_\mu \geq 0,\ u_\mu \neq 0. \end{aligned}$$

Next, we choose the test function $(u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} &\langle A_p^a(u_\mu), (u_\mu - u_\lambda)^+\rangle\\ &= \int_\Omega [\mu u_\lambda^{p-1} + \xi(z) u_\lambda^{r-1}](u_\mu - u_\lambda)^+ dz \text{ (see (23))}\\ &\leq \langle A_p^a(u_\lambda), (u_\mu - u_\lambda)^+\rangle \text{ (see (22))}\\ &\Rightarrow u_\mu \leq u_\lambda \text{ (see Proposition 2.1)}. \end{aligned}$$

So, we have proved that

$$u_\mu \in [0, u_\lambda],\ u_\mu \neq 0. \tag{27}$$

From (27), (23) and (26) it follows that

$$u_\mu \in S_\mu \subseteq \operatorname{int} C_+ \text{ and } \mu \in \mathcal{L}.$$

Consider the function $x \mapsto \xi(z)x^{r-1}$, $x \geq 0$. Let $\rho = \|u_\lambda\|_\infty$. Since $\xi \in L^\infty(\Omega)$ and $r > p$, we can find $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \mapsto \xi(z)x^{r-1} + \hat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$. We have

$$\begin{aligned} & -\Delta_p^a u_\lambda + \hat{\xi}_\rho u_\lambda^{p-1} \\ & = \lambda u_\lambda^{p-1} + \xi(z)u_\lambda^{r-1} + \hat{\xi}_\rho u_\lambda^{p-1} \\ & = \mu u_\lambda^{p-1} + (\lambda - \mu)u_\lambda^{p-1} + \xi(z)u_\lambda^{r-1} + \hat{\xi}_\rho u_\lambda^{p-1} \\ & \geq \mu u_\lambda^{p-1} + \xi(z)u_\lambda^{r-1} + \hat{\xi}_\rho u_\lambda^{p-1} \text{ (see (27))} \\ & = -\Delta_p^a u_\mu + \hat{\xi}_\rho u_\rho^{p-1} \text{ in } \Omega. \end{aligned}$$

Since $u_\lambda \in \operatorname{int} C_+$ we see that $0 \prec (\lambda - \mu)u_\lambda^{p-1}$ and so, using Proposition 3.2 of Gasinski and Papageorgiou [10], we infer that

$$u_\lambda - u_\mu \in \operatorname{int} C_+.$$

The proof is now complete. □

According to the above proposition, we have

$$(\hat{\lambda}_1^a, \lambda^*) \subset \mathcal{L} \subset [\hat{\lambda}_1^a, \lambda^*].$$

We will show that for $\lambda \in (\hat{\lambda}_1^a, \lambda^*)$ we have multiplicity of positive solutions. For this purpose, we introduce the energy functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ of problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\int_\Omega a(z)|Du|^p dz - \frac{\lambda}{p}\|u\|_p^p - \frac{1}{r}\int_\Omega \xi(z)|u|^r dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Evidently, $\varphi \in C^1(W_0^{1,p}(\Omega))$.

Proposition 3.5. *If hypotheses H_0 hold and $\lambda \geq \hat{\lambda}_1^a$, then $\varphi_\lambda(\cdot)$ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ such that

$$|\varphi_\lambda(u_n)| \leq c_4 \text{ for some} c_4 > 0, \text{all} n \in \mathbb{N}, \tag{28}$$

$$(1 + \|u_n\|)\varphi'_\lambda(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{as} n \to \infty. \tag{29}$$

From (29) we have

$$\begin{aligned} & \left|\langle A_p^a(u_n), h\rangle - \int_\Omega \lambda|u_n|^{p-2}u_n h dz - \int_\Omega \xi(z)|u_n|^{r-2}u_n h dz\right| \leq \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \\ & \text{for all } h \in W_0^{1,p}(\Omega), \text{with} \varepsilon_n \to 0^+. \end{aligned} \tag{30}$$

In (30) we use the test function $h = u_n \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} & \left|\int_\Omega a(z)|Du_n|^p dz - \lambda\|u_n\|_p^p - \int_\Omega \xi(z)|u_n|^r dz\right| \leq \varepsilon_n \text{ for all } n \in \mathbb{N}, \\ & \Rightarrow \int_\Omega \xi(z)|u_n|^r dz \leq \varepsilon_n + \int_\Omega a(z)|Du_n|^p dz + \lambda\|u_n\|_p^p \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{31}$$

From (28) we have

$$\begin{aligned}
&\frac{r}{p}\int_\Omega a(z)|Du_n|^p dz - \frac{\lambda r}{p}\|u_n\|_p^p \le rc_4\int_\Omega a(z)\xi(z)|u_n|^r dz,\\
&\Rightarrow \frac{r}{p}\int_\Omega a(z)|Du_n|^p dz - \frac{\lambda r}{p}\|u_n\|_p^p\\
&\le rc_4+\varepsilon_n+\int_\Omega a(z)|Du_n|^p dz+\lambda\|u_n\|_p^p \text{ for all } n\in\mathbb{N} (\text{see } (31)),\\
&\Rightarrow \left[\frac{r}{p}-1\right]\left(\int_\Omega a(z)|Du_n|^p dz-\lambda\|u_n\|_p^p\right)\le c_5,\\
&\quad \text{for some } c_5>0, \text{ all } n\in\mathbb{N}.
\end{aligned} \tag{32}$$

Suppose that $\|u_n\|_p\to\infty$ and let $y_n=\frac{u_n}{\|u_n\|_p}$ for all $n\in\mathbb{N}$. As before we may assume that $u_n\ge 0$ for every $n\in\mathbb{N}$ (just replace u_n by $|u_n|$). So, we have

$$\|y_n\|_p=1,\ y_n\ge 0 \ \text{ for all } n\in\mathbb{N}.$$

Multiplying (32) with $\frac{1}{\|u_n\|_p^p}$, we obtain

$$\begin{aligned}
&\left[\frac{r}{p}-1\right]\left(\int_\Omega a(z)|Dy_n|^p dz-\lambda\right)\le\frac{c_5}{\|u_n\|_p^p},\\
&\Rightarrow \{y_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,p}(\Omega) \text{ is bounded (see hypotheses } H_0).
\end{aligned}$$

We may assume that

$$y_n\xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega),\ y_n\to y \text{ in } L^r(\Omega),\ \|y\|_p=1,\ y\ge 0. \tag{33}$$

Multiplying (30) with $\frac{1}{\|u_n\|_p^{p-1}}$, we obtain

$$\langle A_p^a(y_n),h\rangle-\lambda\int_\Omega y_n^{p-1}hdz=\|u_n\|_p^{r-p}\int_\Omega\xi(z)y_n^{r-1}hdz+\varepsilon_n'\|h\|, \text{ with } \varepsilon_n'\to 0^+ \tag{34}$$

We examine relation (34) and we see that the left-hand side is bounded. Since $r>p$ and $\|u_n\|_p\to\infty$, we must have

$$\begin{aligned}
&\int_\Omega\xi(z)y_n^{r-1}hdz\to 0 \text{ for all } h\in W_0^{1,p}(\Omega),\\
&\Rightarrow \int_\Omega\xi(z)y^{r-1}hdz=0 \text{ for all } h\in W_0^{1,p}(\Omega).
\end{aligned} \tag{35}$$

Since $|\Omega\backslash(\Omega_+\cup\Omega_-)|_N=0$ (see hypotheses H_0), from (35) it follows that $y(z)=0$ for a.a. $z\in\Omega$, which contradicts (33). Therefore

$$\{u_n\}_{n\in\mathbb{N}}\subset L^p(\Omega) \text{ is bounded.}$$

But then from (32), we infer that

$$\{u_n\}_{n\in\mathbb{N}}\subset W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n\xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega),\ u_n\to u \text{ in } L^r(\Omega). \tag{36}$$

In (30) we use the test function $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (36). We obtain

$$\begin{aligned}
&\lim_{n\to\infty} \langle A_p^a(u_n), u_n - u\rangle = 0,\\
&\quad \Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 2.1)},\\
&\quad \Rightarrow \varphi_\lambda(\cdot) \text{ satisfies the } C-\text{condition}.
\end{aligned}$$

The proof is now complete. □

Now we can prove the multiplicity result when $\lambda \in (\hat{\lambda}_1^a, \lambda^*)$.

Proposition 3.6. *If hypotheses H_0 hold and $\lambda \in (\hat{\lambda}_1^a, \lambda^*)$, then problem (P_λ) has at least two solutions*

$$u_\lambda, \hat{u}_\lambda \in \operatorname{int} C_+.$$

Proof. Let $\vartheta \in (\lambda, \lambda^*)$. Then $\vartheta \in \mathcal{L}$ and we can find $u_\vartheta \in S_\vartheta \subseteq \operatorname{int} C_+$. On account of Proposition 3.4, we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ such that

$$u_\vartheta - u_\lambda \in \operatorname{int} C_+. \tag{37}$$

Let $\mu \in (\hat{\lambda}_1^a, \lambda)$ and consider the following auxiliary Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta_p^a u(z) = \mu u(z)^{p-1} - \|\xi\|_\infty u(z)^{r-1} \text{ in } \Omega, \\ \quad u|_{\partial\Omega} = 0,\ u > 0. \end{array} \right\} \tag{38}$$

Let $\sigma_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (38) defined by

$$\sigma_\mu(u) = \frac{1}{p}\int_\Omega a(z)|Du|^p dz + \frac{\|\xi\|_\infty}{r}\|u\|_r^r - \frac{\mu}{p}\|u\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega).$$

Evidently, $\sigma_\mu \in C^1(W_0^{1,p}(\Omega))$. Moreover, since $r > p$, we see that $\sigma_\mu(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u}_\mu \in W_0^{1,p}(\Omega)$ such that

$$\sigma_\mu(\overline{u}_\mu) = \inf\{\sigma_\mu(u) :\ u \in W_0^{1,p}(\Omega)\}. \tag{39}$$

As before, we can always replace $\overline{u}_\mu$ by $|\overline{u}_\mu|$ and so we may assume that $\overline{u}_\mu \geq 0$. Since $\mu > \hat{\lambda}_1^a$ and $r > p$, for $t \in (0,1)$ small, we have

$$\begin{aligned}
&\sigma_\mu(t\hat{u}_1) < 0,\\
&\quad \Rightarrow \sigma_\mu(\overline{u}_\mu) < 0 = \sigma_\mu(0) \text{ (see (39))},\\
&\quad \Rightarrow \overline{u}_\mu \neq 0.
\end{aligned}$$

From (39) we have

$$\begin{aligned}
&\langle \sigma_\mu'(\overline{u}_\mu), h\rangle = 0,\\
&\quad \Rightarrow \langle A_p^a(\overline{u}_\mu), h\rangle = \int_\Omega [\mu \overline{u}_\mu^{p-1} - \|\xi\|_\infty \overline{u}_\mu^{r-1}] h dz \text{ for all } h \in W_0^{1,p}(\Omega),\\
&\quad \Rightarrow \overline{u}_\mu \text{ is a solution of problem (38)}.
\end{aligned}$$

As before (see the proof of Proposition 3.2), using the nonlinear regularity theory of Lieberman [16], we inder that

$$\overline{u}_\mu \in \operatorname{int} C_+.$$

From Diaz and Saa [6] (see also Fragnelli, Mugnai and Papageorgiou [8]), we obtain that this positive solution $\overline{u}_\mu$ is unique.

Claim: $u - \overline{u}_\mu \in \operatorname{int} C_+$ for all $u \in S_\lambda \subseteq \operatorname{int} C_+$.

We first show that $\overline{u}_\mu \leq u$ for all $u \in S_\lambda$. To this end, we introduce the Carathéodory function $e_\mu(z,x)$ defined by

$$e_\mu(z,x) = \begin{cases} \mu(x^+)^{p-1} - \|\xi\|_\infty (x^+)^{r-1} & \text{if } x \leq u(z) \\ \mu u(x)^{p-1} - \|\xi\|_\infty u(z)^{r-1} & \text{if } x > u(z). \end{cases} \tag{40}$$

We set $E_\mu(z,x) = \int_0^x e_\mu(z,s)ds$ and consider the C^1-functional $\hat{\sigma_\mu} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_m \hat{u}(u) = \frac{1}{p}\int_\Omega a(z)|Du|^p dz - \int_\Omega E_\mu(z,u)dz \ \text{ for all } u \in W_0^{1,p}(\Omega).$$

Using hypotheses H_0 and (40), we see that

$$\begin{aligned} &\hat{\sigma_\mu}(u) \geq \tfrac{\hat{c}}{p}\|Du\|_p^p - c_6 \text{ for some } c_6 > 0, \text{all } u \in W_0^{1,p}(\Omega), \\ &\quad \Rightarrow \hat{\sigma_\mu}(\cdot) \text{ is coercive.} \end{aligned}$$

Also, by the Sobolev embedding theorem, $\hat{\sigma_\mu}(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\tilde{u}_\mu \in W_0^{1,p}(\Omega)$ such that

$$\hat{\sigma_\mu}(\tilde{u}_\mu) = \inf\{\hat{\sigma_\mu}(v) : \ v \in W_0^{1,p}(\Omega)\}. \tag{41}$$

Recall that $u \in S_\lambda \subseteq \operatorname{int} C_+$. So, using Proposition 4.1.22 of Papageorgiou, Rădulescu and Repovš [19, p.274], we can find $t \in (0,1)$ small such that

$$0 \leq t\hat{u}_1(z) \leq u(z) \text{ for all } z \in \overline{\Omega}.$$

Since $\mu > \hat{\lambda}_1^a$ and $r > p$, taking $t \in (0,1)$ even smaller if necessary, we obtain that

$$\begin{aligned} &\hat{\sigma_\mu}(t\hat{u}_1) < 0 \text{ (see (40))}, \\ &\quad \Rightarrow \hat{\sigma_\mu}(\tilde{u}_\mu) < 0 = \hat{\sigma_\mu}(0) \text{ (see (41))}, \\ &\quad \Rightarrow \tilde{u}_\mu \neq 0. \end{aligned}$$

From (41) we have

$$\begin{aligned} &\langle \hat{\sigma_\mu}'(\tilde{u}_\mu), h\rangle = 0 \text{ for all } u \in W_0^{1,p}(\Omega), \\ &\quad \Rightarrow \langle A_p^a(\tilde{u}_\mu), h\rangle = \int_\Omega e_\mu(z,\tilde{u}_\mu)hdz \text{ for all } u \in W_0^{1,p}(\Omega). \end{aligned} \tag{42}$$

In (42) let $h = -\tilde{u}_\mu^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\hat{c}\,\|D\tilde{u}_\mu^-\|_p^p \leq 0 \text{ (see hypotheses} H_0 \text{ and (40))}, \\ &\quad \Rightarrow \ \tilde{u}_\mu \geq 0, \ \tilde{u}_\mu \neq 0. \end{aligned}$$

Next, in (42) we use the test function $(\tilde{u}_\mu - u)^+ \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned} &\langle A_p^a(\tilde{u}_\mu), (\tilde{u}_\mu - u)^+\rangle \\ &= \int_\Omega [\mu u^{p-1} - \|\xi\|_\infty u^{r-1}](\tilde{u}_\mu - u)^+ dz \text{ (see (40))} \\ &\leq \int_\Omega [\lambda u^{p-1} - \|\xi\|_\infty u^{r-1}](\tilde{u}_\mu - u)^+ dz \\ &\leq \int_\Omega [\lambda u^{p-1} + \xi(z) u^{r-1}](\tilde{u}_\mu - u)^+ dz \\ &= \langle A_p^a(u), (\tilde{u}_\mu - u)^+\rangle \text{ (since } u \in S_\lambda), \\ &\Rightarrow \tilde{u}_\mu \leq u. \end{aligned}$$

So, we have proved that

$$\tilde{u}_\mu \in [0, u],\ \tilde{u}_\mu \neq 0. \tag{43}$$

From (43), (40) and (42), we infer that

$$\begin{aligned} &\tilde{u}_\mu \text{ is a positive solution of (38),} \\ &\quad \Rightarrow \tilde{u}_\mu = \overline{u}_\mu \text{ (uniqueness of the solution),} \\ &\quad \Rightarrow \overline{u}_\mu \leq u \text{ for all } u \in S_\lambda \text{(see (43)).} \end{aligned}$$

Now let $\rho = \|u\|_\infty$ and let $\hat{\xi_\rho} > 0$ be such that for a.a. $z \in \Omega$ the function $x \mapsto \xi(z)x^{r-1} + \hat{\xi}_\rho x^{p-1}$ is nondecreasing (recall that $\xi \in L^\infty(\Omega)$ and $r > p$). We have

$$\begin{aligned} &-\Delta_p^a \overline{u}_\mu + \hat{\xi}\overline{u}_\mu^{p-1} \\ &\quad = \mu\overline{u}_\mu^{p-1} - \|\xi\|_\infty \overline{u}_\mu^{r-1} + \hat{\xi}\overline{u}_\mu^{p-1} \\ &\quad \leq \mu\overline{u}_\mu^{p-1} + \xi(z)\overline{u}_\mu^{r-1} + \hat{\xi}\overline{u}_\mu^{p-1} \\ &\quad = \lambda\overline{u}_\mu^{p-1} - (\lambda - \mu)u_\mu^{p-1} + \xi(z)\overline{u}_\mu^{r-1} + \hat{\xi}\overline{u}_\mu^{p-1} \\ &\quad \leq \lambda u^{p-1} + \xi(z)u^{r-1} + \hat{\xi}u^{p-1} \text{ (see (43))} \\ &\quad = -\Delta_p^a u + \hat{\xi}u^{p-1} \text{ in } \Omega (\text{since} u \in S_\lambda). \end{aligned}$$

Note that $0 \prec (\lambda - \mu)\overline{u}_\mu^{p-1}$ (recall that $\overline{u}_\mu \in \operatorname{int} C_+$). So, using Proposition 3.2 of Gasinski and Papageorgiou [10], we obtain

$$u - \overline{u}_\lambda \in \operatorname{int} C_+ \text{ for all } u \in S_\lambda. \tag{44}$$

This proves the Claim.

Now we introduce the Carathéodory function $\hat{\ell_\lambda}(z, x)$ defined by

$$\hat{\ell_\lambda}(z, x) = \begin{cases} \lambda\overline{u}_\mu(z)^{p-1} + \xi(z)\overline{u}_\mu(z)^{r-1} & \text{if } x < \overline{u}_\mu(z) \\ \lambda x^{p-1} + \xi(z)x^{r-1} & \text{if } \overline{u}_\mu(z) \leq x \leq u_\vartheta(z) \\ \lambda u_\vartheta(z)^{p-1} + \xi(z)u_\vartheta(z)^{r-1} & \text{if } u_\vartheta(z) < x. \end{cases} \tag{45}$$

We set $\hat{L}_\lambda(z,x)=\int_0^x \hat{\ell}_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\beta}_\lambda : W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\hat{\beta}_\lambda(u)=\frac{1}{p}\int_\Omega a(z)|Du|^p dz-\int_\Omega \hat{L}_\lambda(z,u)dz \text{ for all } u\in W_0^{1,p}(\Omega).$$

Also let $\gamma_\lambda : W_0^{1,p}(\Omega)\to\mathbb{R}$ be the C^1-functional introduced in the proof of Proposition 3.4 using (23) (with μ replaced by λ and u_λ by u_ϑ). We see that

$$\hat{\beta}_\lambda|_{[\overline{u}_\mu,u_\vartheta]}=\gamma_\lambda|_{[\overline{u}_\mu,u_\vartheta]}+\hat{\eta_\lambda\}}\text{with } \hat{\eta}_\lambda\in\mathbb{R}. \tag{46}$$

From the proof of Proposition 3.4, we know that u_λ is a global minimizer of $\gamma_\lambda(\cdot)$. Moreover, from (37) and (44), we see that

$$u_\lambda\in \operatorname{int}_{C_0^1(\overline{\Omega})}[\overline{u}_\mu,u_\vartheta]. \tag{47}$$

From (46) and (47) we infer that

$$u_\lambda \text{ is a local } C_0^1(\overline{\Omega}) \text{ -minimizer of } \hat{\beta}_\lambda(\cdot). \tag{48}$$

Let $\ell_\lambda(z,x)$ be the Carathéodory function defined by

$$\ell_\lambda(z,x)=\begin{cases}\lambda\overline{u}_\mu(z)^{p-1}+\xi(z)\overline{u}_\mu(z)^{r-1} & \text{if } x\le\overline{u}_\mu(z)\\ \lambda x^{p-1}+\xi(z)x^{r-1} & \text{if } x>\overline{u}_\mu(z).\end{cases} \tag{49}$$

We set $L_\lambda(z,x)=\int_0^x \ell_\lambda(z,s)ds$ and consider the C^1-functional $\beta_\lambda : W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\beta_\lambda(u)=\frac{1}{p}\int_\Omega a(z)|Du|^p dz-\int_\Omega L_\lambda(z,u)dz \text{ for all } u\in W_0^{1,p}(\Omega).$$

From (45) and (47) we see that

$$\beta_\lambda|_{[\overline{u}_\mu,u_\vartheta]}=\hat{\beta}_\lambda|_{[\overline{u}_\mu,u_\vartheta]}.$$

From (48) we have that

$$\begin{aligned}&u_\lambda \text{ is a local } C_0^1(\overline{\Omega})-\text{minimizer of } \beta_\lambda(\cdot),\\ &\Rightarrow u_\lambda \text{ is a local } W_0^{1,p}(\Omega)-\text{minimizer of } \beta_\lambda(\cdot)\\ &\quad\text{(see [20, Proposition A3]).}\end{aligned} \tag{50}$$

Using (49) and the nonlinear regularity theory, we can easily show that

$$K_{\beta_\lambda}\subseteq[\overline{u}_\mu)\cap \operatorname{int} C_+. \tag{51}$$

So, we may assume that K_{β_λ} is finite. Otherwise, on account of (51) and (49), we see that we already have an infinity of positive solutions for problem (P_λ) and so we are done. Then (50) and Theorem 5.7.6 of [19, p.449] imply that we can find $\rho\in(0,1)$ small such that

$$\beta_\lambda(u_\lambda)=\inf\{\beta_\lambda(u):\ \|u-u_\lambda\|=\rho\}=m_\lambda. \tag{52}$$

Let $\hat{u}_+\in W_0^{1,p}(\Omega)\cap L^\infty(\hat{\Omega}_+)\cap C^{0,\alpha}(\hat{\Omega}_+)$ be as in the proof of Proposition 3.3. We extend this function to all of Ω by setting $u_+(z)=0$ for all $z\in\Omega\backslash\hat{\Omega}_+$. We continue to denote the extended function by $\hat{u}_+$. We have $\hat{u}_+\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Since $r>p$ and $\hat{u}_+(z)>0$ for all $z\in\Omega$, we see that

$$\beta_\lambda(t\hat{u}_+)\to-\infty \quad\text{as } t\to+\infty. \tag{53}$$

Note that

$$\beta_\lambda|_{[\hat{u}_\mu)} = \varphi_\lambda|_{[\hat{u}_\mu)} + \eta_\lambda^* \text{ with } \eta_\lambda^* \in \mathbb{R} \text{ (see (49))}.$$

This equality and Proposition 3.5 imply that

$$\beta_\lambda(\cdot) \text{ satisfies the } C-\text{condition}. \tag{54}$$

Then (52), (53) and (54) permit the use of the mountain pass theorem. So, we can find $\hat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\hat{u}_\lambda \in K_{\beta_\lambda} \subseteq [\overline{u}_\mu) \cap \operatorname{int} C_+ \text{ (see (51))}, \ \beta_\lambda(u_\lambda) < m_\lambda \le \beta_\lambda(\hat{u}_\lambda).$$

From these relations and (49), we conclude that $\hat{u}_\lambda \in \operatorname{int} C_+$ is the second positive solution of (P_λ), distinct from $u_\lambda \in \operatorname{int} C_+$. □

Remark 3.7. It is easy to see that the mapping $\mu \mapsto \overline{u}_\mu$ is nondecreasing, that is,

$$\mu \le \mu' \Rightarrow \overline{u}_\mu \le \overline{u}_{\mu'}.$$

It remains to decide about the admissibility of the two critical parameters $\hat{\lambda}_1^a$ and λ^*.

Proposition 3.8. *If hypotheses H_0 hold, then $\hat{\lambda}_1^a, \lambda^* \in \mathcal{L}$.*

Proof. We first show that $\hat{\lambda}_1^a \in \mathcal{L}$.

As in the proof of Proposition 3.2, we define

$$\beta^*_{\hat{\lambda}_1^a} = \inf\left\{\frac{1}{p}\int_\Omega a(z)|Du|^p dz - \frac{\hat{\lambda}_1^a}{p}\|u\|_p^p : \ u \in W_0^{1,p}(\Omega), \ \frac{1}{p}\int_\Omega \xi(z)|u|^p dz = 1\right\}.$$

Evidently

$$\beta^*_{\hat{\lambda}_1^a} \ge 0.$$

Reasoning as in the proof of Proposition 3.2, we show that $\beta^*_{\hat{\lambda}_1^a}$ is attained, that is, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\beta^*_{\hat{\lambda}_1^a} = \frac{1}{p}\int_\Omega a(z)|D\hat{u}|^p dz - \frac{\hat{\lambda}_1^a}{p}\|\hat{u}\|_p^p, \ \frac{1}{p}\int_\Omega \xi(z)|\hat{u}|^p dz = 1.$$

Clearly we may assume that $\hat{u} \ge 0$.

If $\beta^*_{\hat{\lambda}_1^a} = 0$, then

$$\begin{aligned} &\int_\Omega a(z)|D\hat{u}|^p dz = \hat{\lambda}_1^a \|\hat{u}\|_p^p, \\ &\Rightarrow \hat{u} = \theta\hat{u}_1 \text{ for some } \theta > 0, \\ &\Rightarrow \int_\Omega \xi(z)\hat{u}_1^r dz = \frac{1}{\theta^r} > 0, \end{aligned}$$

which contradicts hypotheses H_0. Therefore

$$\beta^*_{\hat{\lambda}_1^a} > 0.$$

Via the Lagrange multiplier rule, as in the proof of Proposition 3.2, we show that

$$\tilde{u} = \left[\frac{p}{r}\,\beta^*_{\hat{\lambda}_1^a}\right]^{1/(r-p)} \hat{u} \in S_{\hat{\lambda}_1^a},$$
$$\Rightarrow \hat{\lambda}_1^a \in \mathcal{L}.$$

Next, we show that $\lambda^* \in \mathcal{L}$.

Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ be such that $\lambda_n \uparrow \lambda^*$. From the proof of Proposition 3.6, we know that we can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ such that

$$\varphi_{\lambda_n}(u_n) \le 0 \text{ for all } n \in \mathbb{N}. \tag{55}$$

Also, we have

$$\langle \varphi'_{\lambda_n}(u_n), h\rangle = 0 \text{ for all } n \in \mathbb{N}, \text{all} h \in W_0^{1,p}(\Omega). \tag{56}$$

Using (55) and (56) and reasoning as in the proof of Proposition 3.5, we show that

$$\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u^* \text{ in } W_0^{1,p}(\Omega),\ u_n \to u^* \text{ in } L^r(\Omega). \tag{57}$$

We know that

$$\overline{u}_{\lambda_1} \le u_n \quad \text{for all } n \in \mathbb{N},$$
$$\Rightarrow \overline{u}_{\lambda_1} \le u^*,$$
$$\Rightarrow u_* \ne 0. \tag{58}$$

In (56) we use the test function $h = u_n - u^* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (57). We obtain

$$\lim_{n\to\infty} \langle A_p^a(u_n), u_n - u^*\rangle = 0,$$
$$\Rightarrow u_n \to u^* \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 2.1).} \tag{59}$$

Passing to the limit as $n \to \infty$ in (56) and using (59), we have

$$\langle \varphi'_{\lambda_n}(u^*), h\rangle = 0 \text{ for all } u \in W_0^{1,p}(\Omega),$$
$$\Rightarrow u^* \in S_{\lambda^*} \subseteq \operatorname{int} C_+ \text{ (see (58))},$$

hence $\lambda^* \in \mathcal{L}$.

The proof is now complete. □

On account of this proposition, we have

$$\mathcal{L} = [\hat{\lambda}_1^a, \lambda^*].$$

Finally, we can state the following global in $\lambda \ge \hat{\lambda}_1^a$ (noncoercive case) existence and multiplicity theorem for problem (P_λ).

Theorem 3.9. *If hypotheses H_0 hold and $\lambda \ge \hat{\lambda}_1^a$, then there exists $\lambda^* > \hat{\lambda}_1^a$ such that*

(a) *for all* $\lambda \in (\hat{\lambda}_1^a, \lambda^*)$ *problem* (P_λ) *has at least two positive solutions*

$$u_\lambda, \hat{u}_\lambda \in \operatorname{int} C_+;$$

(b) *for* $\lambda = \hat{\lambda}_1^a$ *and for* $\lambda = \lambda^*$*, problem* (P_λ) *has at least one positive solution* $u^* \in \operatorname{int} C_+$*;*

(c) *for all* $\lambda > \lambda^*$ *problem* (P_λ) *has no positive solution.*

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