Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2023) 117:128
https://doi.org/10.1007/s13398-023-01457-z

ORIGINAL PAPER

Groundstates of the Schrödinger–Poisson–Slater equation with critical growth

Chunyu Lei[1] · **Vicenţiu D. Rădulescu**[2,3,4,5] · **Binlin Zhang**[5,6]

Received: 29 December 2022 / Accepted: 18 May 2023 / Published online: 6 June 2023

Abstract

In this paper, we study the existence of ground state solutions for the following Schrödinger–Poisson–Slater equation:

$$-\Delta u + (|x|^{\alpha-n} * |u|^2)u = \mu|u|^{p-2}u + |u|^{2^*-2}u, \text{ in } \mathbb{R}^n,$$

where $n \geqslant 3$, $\alpha \in (0, n)$. By combining the Nehari–Pohozaev method with compactness arguments, we first obtain a positive ground state solution for above equation. Then we establish several qualitative properties of the ground state solutions.

Keywords Schrödinger–Poisson–Slater equation · Ground state solution · Coulomb–Sobolev inequality · Critical exponents

Mathematics Subject Classification 35J20 · 35A23 · 35Q55 · 35J61

1 Introduction and main results

The Schrödinger equation is central in quantum mechanics and it plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future

✉ Binlin Zhang
zhangbinlin2012@163.com

Chunyu Lei
leichygzu@sina.cn

Vicenţiu D. Rădulescu
radulescu@inf.ucv.ro

1 School of Mathematical Sciences, Guizhou Minzu University, Guiyang 550025, People's Republic of China

2 Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland

3 Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania

4 Brno University of Technology, Technická 3058/10, 61600 Brno, Czech Republic

5 School of Mathematics, Zhejiang Normal University, Jinhua 321004, People's Republic of China

6 College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, People's Republic of China

behaviour of a dynamical system. The linear Schrödinger equation is a central tool of quantum mechanics, which provides a thorough description of a particle in a non-relativistic setting. Schrödinger's linear equation is

$$\Delta\psi + \frac{8\pi^2 m}{\hbar^2}\,(E - V(x))\,\psi = 0\,,$$

where ψ is the Schrödinger wave function, m is the mass of the particle, $\hbar$ denotes Planck's renormalized constant, E is the energy, and V stands for the potential energy. Schrödinger also established the classical derivation of his equation, based upon the analogy between mechanics and optics, and closer to de Broglie's ideas. He developed a perturbation method, inspired by the work of Lord Rayleigh in acoustics, but he also proved the equivalence between his wave mechanics and Heisenberg's matrix.

The nonlocal version of this equation was first studied in the 1928 pioneering paper by Gamow [12], who proved the *tunneling effect*, which lead to the construction of the electronic microscope and the correct study of the alpha radioactivity. The notion of "solution" used by him was not explicitly mentioned in the paper but it is coherent with the notion of weak solution introduced several years later by other authors such as Leray, Sobolev and Schwartz. This is the Schrödinger–Poisson–Slater, which was considered in [28] as a simplification of the Hartree–Fock model.

In this paper, we study the following Schrödinger–Poisson–Slater equation:

$$-\Delta u + (|x|^{\alpha-n} * |u|^2)u = \mu|u|^{p-2}u + |u|^{2^*-2}u, \ \text{in } \mathbb{R}^n, \tag{1.1}$$

where $n \geqslant 3$, $\alpha \in (0, n)$, $2^* = 2n/(n-2)$, $p > 0$ and $\mu > 0$ is a parameter. In Eq. (1.1), $|x|^{\alpha-n} * |u|^2$ is known as the repulsive Coulomb potential, which makes the usual Sobolev space $H^1(\mathbb{R}^n)$ not to be a good framework for Eq. (1.1). The following Coulomb–Sobolev space is the suitable working space (cf. [19])

$$X^{1,\alpha} := \{v \in \mathcal{D}^{1,2}(\mathbb{R}^n);\, L(v) < \infty\},$$

where

$$L(v) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)|^2|v(y)|^2}{|x-y|^{n-\alpha}}dxdy,$$

is the so-called Coulomb energy of the wave. It is well known that every solution to Eq. (1.1) is a critical point of the energy functional $J : X^{1,\alpha} \to \mathbb{R}$, given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{4}L(u) - \frac{\mu}{p}\int_{\mathbb{R}^n} |u|^p dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u|^{2^*} dx.$$

In [4, 18], the authors studied the following Coulomb–Sobolev inequality

$$\|\phi\|_r^r \leqslant C\|\nabla\phi\|_2^{\frac{r(n+\alpha)-4n}{4+\alpha-n}}\,[L(\phi)]^{\frac{2n-r(n-2)}{2(4+\alpha-n)}} \tag{1.2}$$

with $n \neq 4+\alpha$, where

$$\begin{cases} r \in [\frac{2(4+\alpha)}{2+\alpha}, \infty), & n = 2; \\ r \in [\frac{2(4+\alpha)}{2+\alpha}, \frac{2n}{n-2}], & 3 \leqslant n < 4+\alpha; \\ r \in [\frac{2n}{n-2}, \frac{2(4+\alpha)}{2+\alpha}], & n > 4+\alpha. \end{cases} \tag{1.3}$$

The best constant of (1.2) is helpful to estimate the lower bound of the Coulomb energy. To obtain the best constant, one can consider the minimization problem

$$\inf_{\phi\neq 0} \frac{\|\nabla\phi\|_2^{\frac{r(n+\alpha)-4n}{4+\alpha-n}} [L(\phi)]^{\frac{2n-r(n-2)}{2(4+\alpha-n)}}}{\|\phi\|_r^r}. \tag{1.4}$$

In [2], the authors proved that the minimization problem (1.4) is attained under the assumption (1.3).

To find solutions of the minimization problem (1.4), it is natural to study the following Euler–Lagrange equation

$$-\Delta u + (|x|^{\alpha-n} * |u|^2)u = \mu|u|^{r-2}u, \text{ in } \mathbb{R}^n. \tag{1.5}$$

From a physical standpoint, Eq. (1.5) appeared in various physical frameworks, such as plasma, semiconductor physics and the Hartree–Fock theory (cf. [6, 11, 18] and the references therein). In particular, when $\alpha = 2$ and $n = 3$, the motivation in the study of problem (1.5) originates from the Slater approximation of the exchange term in the Hartree–Fock model, we refer to [28]. In this setting, $r = \frac{5}{3}$ is an important exponent in problem (1.5). Of course, other exponents have been employed in various approximations, see [5, 6] for more details related to these models and their variants. From the mathematical point of view, there is a series of analytical results on the Schrödinger–Poisson systems in the literature, see [1, 9, 10, 13–17, 20, 23–25, 27, 31] and so on. Especially, Ianni and Ruiz [14] focused on the following version of the Schrödinger–Poisson–Slater equation:

$$-\Delta u + \left(|u|^2 * \frac{1}{4\pi|x|}\right)u = \mu|u|^{r-2}u, \text{ in } \mathbb{R}^3.$$

With the aid of the monotonicity trick, a positive ground state solution was obtained when $3 < r < 6$. Following the ideas in [29], the authors [16] considered the higher-dimensional version of the Schrödinger–Poisson–Slater equation (1.5) where r belongs to the intervals in (1.3). Under the assumption $2(\alpha+4)/(2+\alpha) < r < 2n/(n-2)$ when $n < 4+\alpha$, or $2n/(n-2) < r < 2(\alpha+4)/(2+\alpha)$ when $n > 4+\alpha$, they obtained a ground state solution of the Nehari–Pohozaev type. In 2019, Liu et al. [24] investigated the following Schrödinger–Poisson–Slater type equation with critical growth:

$$-\Delta u + \left(|u|^2 * \frac{1}{|4\pi x|}\right)u = \mu|u|^{r-2}u + |u|^4 u, \text{ in } \mathbb{R}^3. \tag{1.6}$$

where $\mu > 0$. When $r \in (3, 6)$, they obtained the existence of positive ground state solutions by combining a new perturbation method and the Mountain–Pass theorem.

Based on the work of [24], we are also concerned with the existence of ground state solutions (which are not limited to radial structure) of higher-dimensional equation (1.1). In addition, we want to establish some necessary and sufficient condition for ground state solutions obtained.

Before stating our main results, we introduce the following functionals and sets, respectively:

$$T(u) = \int_{\mathbb{R}^n} |\nabla u|^2 dx,$$

$$Q(u) = \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{p(n+\alpha)-4n}{p(4+\alpha-n)}\mu \int_{\mathbb{R}^n} |u|^p dx - \int_{\mathbb{R}^n} |u|^{2^*} dx,$$

$$A = \left\{u \in X^{1,\alpha} : u \neq 0 \text{ and } K(u) := \langle J'(u), u\rangle = 0\right\},$$

$$G = \{u \in A : J(u) \leqslant J(v) \text{ for all } v \in A\},$$
$$M = \{u \in X^{1,\alpha} : u \neq 0 \text{ and } Q(u) = 0\}.$$

Set

$$\mathcal{M} = \{u \in u \in X^{1,\alpha}\backslash\{0\} : \ I(u) = 0\},$$

where

$$I(u) = \frac{q-nb}{2}\int_{\mathbb{R}^n}|\nabla u|^2dx + \frac{q-nb}{4}L(u) - \frac{p-nb}{p}\mu\int_{\mathbb{R}^n}|u|^pdx - \frac{2^*-nb}{2^*}\int_{\mathbb{R}^n}|u|^{2^*}dx,$$

with $b = 2/(2+\alpha)$, $q = (8+2\alpha)/(2+\alpha)$. A function $u \in A$ is called a bound state of (1.1). A function $u \in G$ is named a ground state of (1.1).

Now our first main result in this paper can be stated as follows.

Theorem 1.1 *Let $q = (8+2\alpha)/(2+\alpha)$. Then Eq. (1.1) has a positive ground state solution on $\mathcal{M}$ under one of the following cases:*

(i) *When $n = 3$, $q < p < 4$ for μ enough large;*
(ii) *When $n = 3$, $4 < p < 6$ for any $\mu > 0$;*
(iii) *When $3 < n < 4+\alpha$, $q < p < 2^*$ for any $\mu > 0$.*

In the case of $n > 4+\alpha$, it seems difficult to find ground state solutions of (1.1) in space $X^{1,\alpha}$. So we restrict $X^{1,\alpha}$ in radial space $X^{1,\alpha}_{\text{rad}} := \{u \in X^{1,\alpha} : u(x) = u(|x|)\}$. Set

$$\widetilde{\mathcal{M}} = \left\{u \in X^{1,\alpha}_{\text{rad}}\backslash\{0\} : \ \widetilde{I}(u) = 0\right\},$$

where

$$\widetilde{I}(u) = \frac{-q-n\widetilde{b}}{2}\int_{\mathbb{R}^n}|\nabla u|^2dx + \frac{-q-n\widetilde{b}}{4}L(u) - \frac{-p-n\widetilde{b}}{p}\mu\int_{\mathbb{R}^n}|u|^pdx - \frac{-2^*-n\widetilde{b}}{2^*}\int_{\mathbb{R}^n}|u|^{2^*}dx,$$

5 with $\widetilde{b} = -2/(2+\alpha)$. Then we obtain the following main results:

Theorem 1.2 *Let $q = (8+2\alpha)/(2+\alpha)$. Assume that $2^* < p < q$ with $n > 4+\alpha$. Then Eq. (1.1) has a positive ground state solution on $\widetilde{\mathcal{M}}$ for any $\mu > 0$.*

Theorem 1.3 *Under one of the following assumptions:*

(1) *$n = 3$, $2+8/(3+\alpha) < p < 6$, $0 < \alpha < 1$ and $\mu > 0$;*
(2) *$n = 3$, $2+8/(3+\alpha) < p < 6$, $1 \leqslant \alpha < 3$ and μ is enough large;*
(3) *$4 \leqslant n < 4+\alpha$, $2+8/(n+\alpha) < p < 2n/(n-2)$ and $\mu > 0$.*

Then

(i) *A and G are nonempty.*
(ii) *$u \in G$ if and only if u solves the minimization problem*

$$\begin{cases} u \in M, \\ J(u) = \min\{J(w) : w \in M\}. \end{cases} \tag{1.7}$$

In radial space $X^{1,\alpha}_{\text{rad}}$, we also have the following result:

Theorem 1.4 *Assume $\mu > 0$ and $2n/(n-2) < p < 2+8/(n+\alpha)$ with $n > 4+\alpha$. Then*

(i) *A and G are nonempty.*

(ii) *$u \in G$ if and only if u solves the minimization problem*

$$\begin{cases} u \in \mathfrak{M}, \\ J(u) = \min\{J(w) : w \in \mathfrak{M}\}, \end{cases}$$

where $\mathfrak{M} = \left\{ u \in X^{1,\alpha}_{rad} : u \neq 0 \text{ and } Q(u) = 0 \right\}$.

Remark 1.1 When $n = 3$ and $\alpha = 2$ in (i)–(ii) of Theorem 1.1, we recover Theorem 1.1 in [24] by using the Nehari–Pohozaev manifold method instead of using a perturbation method and the mountain pass theorem. More importantly, our approach can handle higher-dimensional case of the Schrödinger–Poisson–Slater equation. Moreover, we also discover a sufficient and necessary condition for ground state solutions obtained.

Remark 1.2 Let us discuss some subtle difficulties during our research process:

(1) The lack of compactness. A general tool that could be useful to overcome the lack of compactness is the Schwartz rearrangement map $u \to u^*$. The following properties are well known (see [16]):

$$\int_{\mathbb{R}^n} |\nabla u^*|^2 dx \leqslant \int_{\mathbb{R}^n} |\nabla u|^2 dx;$$
$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^2|u(y)|^2}{|x-y|^{n-\alpha}} dxdy \leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u^*(x)|^2|u^*(y)|^2}{|x-y|^{n-\alpha}} dxdy;$$
$$\int_{\mathbb{R}^n} |u|^p dx = \int_{\mathbb{R}^n} |u^*|^p dx, \quad 1 \leqslant p < +\infty.$$

For (1.1), however, if we use the Schwartz rearrangement technique, it is impossible to overcome the lack of compactness due to the nonlocal term.

(2) Since $q = (8 + 2\alpha)/(2 + \alpha)$ and $q < 4$, it is difficult to prove the boundedness of (PS) sequence of J. In the present paper we have opted to use the Nehari–Pohozaev manifold $\mathcal{M}$ to avoid the difficulty. However, we would like to point out that this method produces a new difficulty, that is, proving that the minimization sequence in $\mathcal{M}$ (or in M) of J is a Palais–Smale sequence.

(3) As we mentioned in the case $n > 4 + \alpha$, it seems difficult to estimate the threshold value of the energy functional J because of the following fact:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_\varepsilon(x)|^2|u_\varepsilon(y)|^2}{|x-y|^{n-\alpha}} dxdy \to +\infty, \quad \int_{\mathbb{R}^n} |u_\varepsilon|^p dx \to +\infty$$

as $\varepsilon \to 0$, where u_ε is as in Lemma 2.5 below. However, in radial space $X^{1,\alpha}_{\text{rad}}$, the following embedding results

$$X^{1,\alpha}_{\text{rad}} \hookrightarrow L^{2^*}_{\text{loc}}(\mathbb{R}^n), \ X^{1,\alpha}_{\text{rad}} \hookrightarrow L^p(\mathbb{R}^n) \ (2^* < p < (8+2\alpha)/(2+\alpha))$$

are compact. Employing the above embedding results, we can establish a ground state of (1.1). Furthermore, we do not need to estimate the threshold value of the energy functional J even if problem (1.1) satisfies critical growth.

2 Proof of Theorem 1.1

We firstly establish the following result.

Lemma 2.1 *The functional J is unbounded from below.*

Proof Let $u \in X^{1,\alpha}$, and $u_t = tu(t^b x)$, $b = 2/(2+\alpha)$, $t > 0$. By the standard scaling we have

$$\int_{\mathbb{R}^n} |\nabla u_t|^2 dx = t^{q-nb} \int_{\mathbb{R}^n} |\nabla u|^2 dx, \quad L(u_t) = t^{q-nb} L(u),$$

and

$$\int_{\mathbb{R}^n} |u_t|^p dx = t^{p-nb} \int_{\mathbb{R}^n} |u|^p dx, \quad \int_{\mathbb{R}^n} |u_t|^{2^*} dx = t^{2^*-nb} \int_{\mathbb{R}^n} |u|^{2^*} dx.$$

Hence,

$$\begin{aligned} J(u_t) &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_t|^2 dx + \frac{1}{4} L(u_t) - \frac{\mu}{p}\int_{\mathbb{R}^n} |u_t|^p dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_t|^{2^*} dx \\ &= \frac{t^{q-nb}}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{t^{q-nb}}{4} L(u) - \frac{\mu t^{p-nb}}{p}\int_{\mathbb{R}^n} |u|^p dx - \frac{t^{2^*-nb}}{2^*}\int_{\mathbb{R}^n} |u|^{2^*} dx. \end{aligned}$$

Noting that $p > q$, we see that $J(u_t) \to -\infty$ as $t \to +\infty$. □

By calculations, we can easily get the following lemma.

Lemma 2.2 *Let a_1, a_2, a_3 be positive constants, and $f(t) = a_1 t^{q-nb} - a_2 t^{p-nb} - a_3 t^{2^*-nb}$ for $t \geqslant 0$. Then f has a unique critical point, corresponding to its maximum.*

Assume that u is a critical point of J. Write $u_t = tu(t^b x)$ with $b = 2/(2+\alpha)$ and $t > 0$. Clearly, $\varphi(t) := J(u_t)$ is positive for small t and tends to $-\infty$ as $t \to +\infty$. By Lemma 2.2, φ has a unique critical point which corresponds to its maximum. Since u is a critical point of J, the maximum of $\varphi(t)$ should be achieved at $t = 1$ and $\varphi'(1) = 0$. Therefore, define the manifold $\mathcal{M}$ as (2.1) Obviously, $\mathcal{M} \neq \emptyset$. Indeed, for given any $v \neq 0$, Lemma 2.2 shows that there exists $t > 0$ such that $u_t^v \in \mathcal{M}$. Moreover, the curve $\Gamma = \{u_t\}$ intersects the manifold $\mathcal{M}$ and $J|_\Gamma$ attains its maximum along Γ at the point u. If u is a mountain pass type solution of problem (1.1), it is natural to look for the minima of J on $\mathcal{M}$. In addition, For any nontrivial critical point u of J, it is standard to prove the following Pohozaev identity

$$\mathcal{P}(u) = \frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{n+\alpha}{4} L(u) - \frac{\mu n}{p}\int_{\mathbb{R}^n} |u|^p dx - \frac{n-2}{2}\int_{\mathbb{R}^n} |u|^{2^*} dx = 0.$$

It is clear that $I(u) = \langle J'(u), u\rangle - b\mathcal{P}(u)$ with $b = 2/(2+\alpha)$. Therefore, $\mathcal{M}$ is called the Nehari–Pohozaev manifold here. If u is a nontrivial solution of (1.1), then $u \in \mathcal{M}$.

Moreover, we have the following result.

Lemma 2.3 *$\mathcal{M}$ is a C^1-manifold and every critical point of J in $\mathcal{M}$ is a critical point of J.*

Proof We proceed in four steps.

Step 1. We claim $0 \notin \partial\mathcal{M}$.

For $u \in X^{1,\alpha}\backslash\{0\}$, by the Gagliardo–Nirenberg's inequality, there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} |u|^p dx \leqslant C\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{\frac{p(n+\alpha)-4n}{2(4+\alpha-n)}} (L(u))^{\frac{2n-p(n-2)}{2(4-n+\alpha)}}.$$

Therefore, by Sobolev inequality, there holds

$$J(u) \geqslant \frac{1}{2}\|\nabla u\|^2 - C\mu\|\nabla u\|^{\frac{p(n+\alpha)-4n}{4+\alpha-n}} - C\|\nabla u\|^{2^*}$$

for some $C > 0$. We see that there exist $r, \rho > 0$ (ρ enough small), such that $J(u) \geqslant r$ for $\|u\| = \rho$. Thus, $0 \notin \partial\mathcal{M}$.

Step 2. We claim $\inf_{\mathcal{M}} J > 0$.

For any $u \in \mathcal{M}$, there holds

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{4}L(u) - \frac{\mu}{p}\int_{\mathbb{R}^n} |u|^p dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u|^{2^*} dx \\ &= \frac{1}{p}\frac{p-q}{q-nb}\int_{\mathbb{R}^n} |u|^p dx + \frac{1}{2^*}\frac{2^*-q}{q-nb}\int_{\mathbb{R}^n} |u|^{2^*} dx \\ &> 0. \end{aligned} \tag{2.1}$$

Step 3. We claim that $\mathcal{M}$ is a C^1-manifold.

By the implicit function theorem, it only need $I'(u) \neq 0$ for any $u \in \mathcal{M}$. We prove it by argument of contradiction. Namely, suppose that $I'(u) = 0$ for some $u \in \mathcal{M}$. Thus, in a weak sense there holds

$$-\Delta u + (|x|^{\alpha-n} * |u|^2)u = \mu\frac{p-nb}{q-nb}|u|^{p-2}u + \frac{2^*-nb}{q-nb}|u|^{2^*-2}u. \tag{2.2}$$

Multiplying (2.2) by u and integrating, we have

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx + L(u) - \mu\frac{p-nb}{q-nb}\int_{\mathbb{R}^n} |u|^p dx - \frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n} |u|^{2^*} dx = 0. \tag{2.3}$$

The Pohozaev identity corresponding to Eq. (2.2) is

$$\frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{n+\alpha}{4}L(u) - \mu\frac{p-nb}{q-nb}\frac{n}{p}\int_{\mathbb{R}^n} |u|^p dx - \frac{n}{2^*}\frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n} |u|^{2^*} dx = 0. \tag{2.4}$$

It follows from $I(u) = 0$ that

$$\frac{n}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{n}{4}L(u) - \mu\frac{n}{p}\frac{p-nb}{q-nb}\int_{\mathbb{R}^n} |u|^p dx - \frac{n}{2^*}\frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n} |u|^{2^*} dx = 0.$$

Therefore, by (2.3) that

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx = \frac{\alpha}{4}L(u).$$

Multiplying (2.3) by $\frac{1}{p}$ and applying $I(u) = 0$, we have

$$\left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^n} |\nabla u|^2 dx + \left(\frac{1}{4} - \frac{1}{p}\right)L(u) = \left(\frac{1}{2^*} - \frac{1}{p}\right)\frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n} |u|^{2^*} dx.$$

Applying the relation $\int_{\mathbb{R}^n} |\nabla u|^2 dx = \frac{\alpha}{4}L(u)$ to the above equation, we obtain

$$\frac{p(2+\alpha) - 2\alpha - 8}{8q}L(u) = \left(\frac{1}{2^*} - \frac{1}{p}\right)\frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n} |u|^{2^*} dx.$$

Since $(8 + 2\alpha)/(2 + \alpha) < p < 2^*$, we have

$$p(2+\alpha) - 2\alpha - 8 > 0, \quad \frac{1}{2^*} - \frac{1}{p} < 0.$$

Therefore, we reach a contradiction. Thus, $\mathcal{M}$ is a C^1-manifold.

Step 4. We claim that every critical point of J on $\mathcal{M}$ is a critical point of J in $X^{1,\alpha}$.

Assume that u is a critical point of J on $\mathcal{M}$, there exists a Lagrange multiplier λ such that $J'(u) = \lambda I'(u)$. It can be written, in a weak sense, as

$$\begin{aligned}
&-\Delta u + (|x|^{\alpha-n} * |u|^2)u - \mu|u|^{p-2}u - |u|^{2^*-2}u \\
&= \lambda\left[-\Delta u + (|x|^{\alpha-n} * |u|^2)u - \mu\frac{p-nb}{q-nb}|u|^{p-2}u - \frac{2^*-nb}{q-nb}|u|^{2^*-2}u\right]
\end{aligned}$$

That is,

$$-(1-\lambda)\,\Delta u + (1-\lambda)\,(|x|^{\alpha-n} * |u|^2)u = \mu\left(1-\frac{p-nb}{q-nb}\lambda\right)|u|^{p-2}u + \left(1-\frac{2^*-nb}{q-nb}\lambda\right)|u|^{2^*-2}u. \tag{2.5}$$

We see that $\lambda \neq 1$, it remains now to prove that $\lambda = 0$. Denote

$$\mathcal{B} = \mu\int_{\mathbb{R}^n}|u|^p dx, \ \ \mathcal{C} = \int_{\mathbb{R}^n}|u|^{2^*}dx.$$

We can establish the following equations

$$\begin{cases}
\dfrac{1}{2}T + \dfrac{1}{4}L(u) - \dfrac{1}{p}\dfrac{p-nb}{q-nb}\mathcal{B} - \dfrac{1}{2^*}\dfrac{2^*-nb}{q-nb}\mathcal{C} = 0, \\
T + L(u) - \dfrac{1}{1-\lambda}\left(1-\dfrac{p-nb}{q-nb}\lambda\right)\mathcal{B} - \dfrac{1}{1-\lambda}\left(1-\dfrac{2^*-nb}{q-nb}\lambda\right)\mathcal{C} = 0, \\
\dfrac{n-2}{2}T + \dfrac{n+\alpha}{4}L(u) - \dfrac{n}{p}\dfrac{1}{1-\lambda}\left(1-\dfrac{p-nb}{q-nb}\lambda\right)\mathcal{B} - \dfrac{1}{1-\lambda}\left(1-\dfrac{2^*-nb}{q-nb}\lambda\right)\dfrac{n}{2^*}\mathcal{C} = 0,
\end{cases} \tag{2.6}$$

where the second equation follows by multiplying (2.5) by u and integrating, and the third equality is the Pohozaev identity corresponding to Eq. (2.5).

It follows from the first and the third equations in (2.6) that

$$\begin{aligned}
\frac{1}{4}L(u) = &\left[\frac{1}{2(1-\lambda)}\left(1-\frac{p-nb}{q-nb}\lambda\right) - \frac{1}{p}\frac{p-nb}{q-nb}\right]\mathcal{B} \\
&+\left[\frac{1}{2(1-\lambda)}\left(1-\frac{2^*-nb}{q-nb}\lambda\right) - \frac{1}{2^*}\frac{2^*-nb}{q-nb}\right]\mathcal{C}.
\end{aligned} \tag{2.7}$$

Applying the the first and the second equations in (2.6), we have

$$\begin{aligned}
\frac{n-4-\alpha}{4}L(u) &= \frac{n-2}{2(1-\lambda)}\left(1-\frac{p-nb}{q-nb}\lambda\right)\mathcal{B} - \frac{n}{p}\frac{1}{1-\lambda}\left(1-\frac{p-nb}{q-nb}\lambda\right)\mathcal{B} \\
&\quad+\frac{n-2}{2(1-\lambda)}\left(1-\frac{2^*-nb}{q-nb}\lambda\right)\mathcal{C} - \frac{1}{1-\lambda}\left(1-\frac{2^*-nb}{q-nb}\lambda\right)\frac{n}{2^*}\mathcal{C} \\
&= \frac{1}{1-\lambda}\frac{(n-2)p-2n}{2p}\left(1-\frac{p-nb}{q-nb}\lambda\right)\mathcal{B}.
\end{aligned}$$

Consequently,

$$\frac{1}{4}L(u) = \frac{1}{(1-\lambda)(n-4-\alpha)}\frac{(n-2)p-2n}{2p}\left(1-\frac{p-nb}{q-nb}\lambda\right)\mathcal{B}. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\begin{aligned}&\left[\frac{1}{2(1-\lambda)}\left(1-\frac{2^*-nb}{q-nb}\lambda\right)-\frac{1}{2^*}\frac{2^*-nb}{q-nb}\right]\mathcal{C}\\&=\left[\frac{p-nb}{p(q-nb)}-\frac{1}{2(1-\lambda)}\left(1-\frac{p-nb}{q-nb}\lambda\right)\right.\\&\left.\quad+\frac{1}{(1-\lambda)(n-4-\alpha)}\frac{(n-2)p-2n}{2p}\left(1-\frac{p-nb}{q-nb}\lambda\right)\right]\mathcal{B}.\end{aligned}$$

That is,

$$\begin{aligned}&\left[\frac{1}{2}\left(1-\frac{2^*-nb}{q-nb}\lambda\right)-\frac{1}{2^*}\frac{2^*-nb}{q-nb}(1-\lambda)\right]\mathcal{C}\\&=\left[\frac{p-nb}{p(q-nb)}(1-\lambda)-\frac{1}{2}\left(1-\frac{p-nb}{q-nb}\lambda\right)\right.\\&\left.\quad+\frac{1}{n-4-\alpha}\frac{(n-2)p-2n}{2p}\left(1-\frac{p-nb}{q-nb}\lambda\right)\right]\mathcal{B}.\end{aligned}\tag{2.9}$$

Noting that

$$\frac{2^*-nb}{q-nb}=\frac{n}{n-2}.$$

On one hand, we have

$$\frac{1}{2}\left(1-\frac{2^*-nb}{q-nb}\lambda\right)-\frac{1}{2^*}\frac{2^*-nb}{q-nb}(1-\lambda)=\frac{1}{2}\left(1-\frac{n}{n-2}\lambda\right)-\frac{1}{2^*}\frac{n}{n-2}(1-\lambda)=-\frac{1}{n-2}\lambda.$$

On the other hand, there holds

$$\begin{aligned}&\left[\frac{p-nb}{p(q-nb)}(1-\lambda)-\frac{1}{2}\left(1-\frac{p-nb}{q-nb}\lambda\right)\right.\\&\left.\quad+\frac{1}{n-4-\alpha}\frac{(n-2)p-2n}{2p}\left(1-\frac{p-nb}{q-nb}\lambda\right)\right]\\&=\frac{p-nb}{p(q-nb)}-\frac{1}{2}+\frac{1}{n-4-\alpha}\frac{(n-2)p-2n}{2p}\\&\quad+\left[-\frac{p-nb}{p(q-nb)}+\frac{p-nb}{2(q-nb)}-\frac{1}{n-4-\alpha}\frac{(n-2)p-2n}{2p}\frac{p-nb}{q-nb}\right]\lambda.\end{aligned}$$

Note that

$$q-nb=\frac{8+2\alpha}{2+\alpha}-\frac{2n}{2+\alpha}=\frac{2(4+\alpha-n)}{2+\alpha}.$$

By computing, there hold

$$\frac{p-nb}{p(q-nb)}-\frac{1}{2}+\frac{1}{n-4-\alpha}\frac{(n-2)p-2n}{2p}=0,$$

and

$$-\frac{p-nb}{p(q-nb)}+\frac{p-nb}{2(q-nb)}-\frac{1}{n-4-\alpha}\frac{(n-2)p-2n}{2p}\frac{p-nb}{q-nb}=\frac{p-nb}{q-nb}\frac{8+2\alpha-p(2+\alpha)}{2p(n-4-\alpha)}.$$

From the above information and (2.9), we obtain

$$\frac{p-nb}{q-nb}\frac{8+2\alpha-p(2+\alpha)}{2p(n-4-\alpha)}\lambda\mathcal{B}=-\frac{1}{n-2}\lambda\mathcal{C}.$$

Since $n<4+\alpha$ and $(8+2\alpha)/(2+\alpha)<p<2^*$, we have

$$p-nb>0,\quad q-nb>0,$$

so that

$$\frac{p-nb}{q-nb}\frac{8+2\alpha-p(2+\alpha)}{2p(n-4-\alpha)}>0.$$

Consequently, we conclude that

$$\lambda\equiv 0.$$

Therefore, we obtain $J'(u)=0$ for $n\geqslant 3$, i.e., u is a critical point of J. The proof is complete.

□

Lemma 2.4 *Assume $q<p<2^*$ with $n<4+\alpha$, and u is the ground state related to problem (1.1). Then*

$$J(u)=\inf\{J(v):v\in\mathcal{M}\}. \tag{2.10}$$

Proof We proceed in two steps.

By Lemma 2.3, the minimizing problem (2.10) is well defined. Thus, we denote

$$d:=\inf\{J(v):v\in\mathcal{M}\}.$$

Step 1. $J(u)\geqslant d$.

We only prove that $I(u)=0$ for $u\in A$. Since u is a ground state related to problem (1.1), then

$$\langle J'(u),u\rangle=0=\mathcal{P}(u).$$

Consequently

$$I(u)=\langle J'(u),u\rangle-\frac{2}{2+\alpha}\mathcal{P}(u)=0.$$

This implies that $u\in\mathcal{M}$. By the definition of d, we have

$$J(u)\geqslant d.$$

Step 2. $J(u)\leqslant d$ for any $u\in A$.

For any $v\in\mathcal{M}$, if $K(v)=0$, noting that u is the ground state solution, then we have

$$J(v)\geqslant J(u).$$

By the arbitrariness of v, we have $d\geqslant J(u)$. We are done.

If $K(v)\neq 0$, setting $v_\lambda=\lambda v(\lambda^b x)$, where $b=\frac{2}{2+\alpha}$. From Lemma 2.1, we have

$$K(v_\lambda)=\lambda^{q-nb}(T(v)+L(v))-\lambda^{p-nb}\mu\int_{\mathbb{R}^n}|v|^p dx-\lambda^{2^*-nb}\int_{\mathbb{R}^n}|v|^{2^*}dx.$$

Set

$$H(\lambda):=\frac{K(v_\lambda)}{\lambda^{q-nb}}.$$

Since $q < p < 2^*$, we see that

$$\lim_{\lambda\to 0} H(v_\lambda) = T(v) + L(v) > 0, \quad \lim_{\lambda\to +\infty} H(v_\lambda) = -\infty.$$

This implies that there exists $\lambda_0 > 0$, such that

$$H(v_{\lambda_0}) = 0,$$

or equivalently,

$$K(v_{\lambda_0}) = 0.$$

Moreover, it follows from $K(v_{\lambda_0}) = 0$ that

$$J(v_{\lambda_0}) \geqslant J(u).$$

On the other hand, in view of Lemma 2.1,

$$\begin{aligned}\partial_\lambda J(v_\lambda) &= \frac{q-nb}{2}\lambda^{q-nb-1}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{q-nb}{4}\lambda^{q-nb-1}L(u)\\ &\quad -\frac{\mu(p-nb)}{p}\lambda^{p-nb-1}\int_{\mathbb{R}^n}|u|^p dx - \frac{2^*-nb}{2^*}\lambda^{2^*-nb-1}\int_{\mathbb{R}^n}|u|^{2^*}dx\\ &= \frac{1}{\lambda}I(v_\lambda).\end{aligned}$$

Now, we define

$$\begin{aligned}f(\lambda) := I(v_\lambda) &= \frac{q-nb}{2}\lambda^{q-nb}\int_{\mathbb{R}^n}|\nabla u|^2 dx + \frac{q-nb}{4}\lambda^{q-nb}L(u)\\ &\quad -\frac{\mu(p-nb)}{p}\lambda^{p-nb}\int_{\mathbb{R}^n}|u|^p dx - \frac{2^*-nb}{2^*}\lambda^{2^*-nb}\int_{\mathbb{R}^n}|u|^{2^*}dx.\end{aligned}$$

Assume that there exists λ_1 such that $f(\lambda_1) = 0$. It follows $v \in \mathcal{M}$ and $f(\lambda_1) = 0$ that

$$\begin{cases}\lambda_1^{q-nb}[\frac{q-nb}{2}\int_{\mathbb{R}^n}|\nabla v|^2 dx + \frac{q-nb}{4}L(v)] - \frac{\mu(p-nb)}{p}\lambda_1^{p-nb}\int_{\mathbb{R}^n}|v|^p dx - \frac{2^*-nb}{2^*}\lambda_1^{2^*-nb}\int_{\mathbb{R}^n}|v|^{2^*}dx = 0,\\ \frac{q-nb}{2}\int_{\mathbb{R}^n}|\nabla v|^2 dx + \frac{q-nb}{4}L(v) - \frac{\mu(p-nb)}{p}\int_{\mathbb{R}^n}|v|^p dx - \frac{2^*-nb}{2^*}\int_{\mathbb{R}^n}|v|^{2^*}dx = 0.\end{cases}$$

Then

$$\frac{p-nb}{p}\left(\lambda_1^{q-nb} - \lambda_1^{p-nb}\right)\mu\int_{\mathbb{R}^n}|v|^p dx = \frac{2^*-nb}{2^*}\left(\lambda_1^{2^*-nb} - \lambda_1^{q-nb}\right)\int_{\mathbb{R}^n}|v|^{2^*}dx.$$

Noting that

$$q < p < 2^*.$$

Once $\lambda_1 > 1$, we derive

$$\lambda_1^{q-nb} - \lambda_1^{p-nb} < 0, \quad \lambda_1^{2^*-nb} - \lambda_1^{q-nb} > 0,$$

which is an absurd. If $\lambda_1 < 1$, then

$$\lambda_1^{q-nb} - \lambda_1^{p-nb} > 0, \quad \lambda_1^{2^*-nb} - \lambda_1^{q-nb} < 0,$$

we reach a contradiction. Therefore, the equation $f(\lambda) = 0$ admits a unique positive solution $\lambda = 1$. As a result, we obtain

$$\begin{cases}\partial_\lambda J(v_\lambda) > 0, & \text{for all } \lambda \in (0,1),\\ \partial_\lambda J(v_\lambda) < 0, & \text{for all } \lambda \in (1,+\infty).\end{cases}$$

We thus get that $J(v_\lambda) < J(v)$ for any $\lambda > 0$ and $\lambda \neq 1$. In particular, we have

$$J(v_{\lambda_0}) \leqslant J(v).$$

Thus,

$$J(u) \leqslant J(v_{\lambda_0}) \leqslant J(v)$$

for $v \in \mathcal{M}$. Taking the infimum over v, one has

$$J(u) \leqslant d.$$

From the above information, we establish the relation $J(u) = d$. The proof is complete. □

Denote

$$U(x) = \frac{[n(n-2)]^{\frac{n-2}{4}}}{(1+|x|^2)^{\frac{n-2}{2}}}, \quad U_\varepsilon(x) = \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{4}}}{(\varepsilon^2+|x|^2)^{\frac{n-2}{2}}}, \ x \in \mathbb{R}^n, \ \varepsilon > 0.$$

U (and U_ε) satisfies the limit equation

$$\Delta U + U^{2^*-1} = 0, \quad U > 0 \text{ in } \mathbb{R}^n.$$

Choose $\eta \in C_0^\infty(B_\delta(x_0), [0,1])$ where $B_\delta(x_0) \subset \Omega$ such that $\eta(x) = 1$ near $x = x_0$ and $u(x) \geqslant m > 0$ for $x \in B_\delta(x_0)$. Denote $u_\varepsilon = U_\varepsilon \eta$.

Lemma 2.5 *Assume* $(8+2\alpha)/(2+\alpha) < p < 2^*$ *with* $n < 4+\alpha$. *Then*

$$\sup_{t \geqslant 0} J(tu_\varepsilon(t^b x)) \leqslant \frac{1}{n} S^{\frac{n}{2}}$$

for sufficient small $\varepsilon > 0$, *where* $b = 2/(2+\alpha)$.

Proof From Lemma 1.1 in [8], we have

$$\begin{cases} \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 dx = \int_{\mathbb{R}^n} |\nabla U|^2 dx + O(\varepsilon^{n-2}) = S^{\frac{n}{2}} + O(\varepsilon^{n-2}), \\ \int_{\mathbb{R}^n} u_\varepsilon^{2^*} dx = \int_{\mathbb{R}^n} U^{2^*} dx + O(\varepsilon^n) = S^{\frac{n}{2}} + O(\varepsilon^n), \end{cases}$$

and

$$\int_{\mathbb{R}^n} u_\varepsilon^t dx = \begin{cases} c\varepsilon^{\frac{(n-2)t}{2}}, & t < \frac{n}{n-2}, \\ c\varepsilon^{\frac{n}{2}} |\ln \varepsilon|, & t = \frac{n}{n-2}, \\ c\varepsilon^{n-\frac{(n-2)t}{2}}, & \frac{n}{n-2} < t < 2^*. \end{cases} \tag{2.11}$$

Since $\lim_{t\to 0^+} J(tu_\varepsilon(t^b x)) = 0$ and $\lim_{t\to+\infty} J(tu_\varepsilon(t^b x)) \to -\infty$ as $t \to \infty$, there exists a $T_\varepsilon > 0$ such that $\sup_{t \geqslant 0} J(tu_\varepsilon(t^b x)) = I(T_\varepsilon u_\varepsilon(T_\varepsilon x))$. Moreover, we can obtain that there exist $t_1, t_2 > 0$ (independent of ε, μ), such that

$$t_1 \leqslant T_\varepsilon \leqslant t_2 < +\infty.$$

Notice that

$$L(u) \leqslant C|u|^4_{\frac{4n}{n+\alpha}}$$

for some $C > 0$ (which is implied by the Hardy–Littlewood–Sobolev inequality). Consequently,

$$\begin{aligned}\sup_{t \geqslant 0} J(tu_\varepsilon(t^b x)) &\leqslant \sup_{t \geqslant 0} \left\{ \frac{t^{q-nb}}{2} S^{\frac{n}{2}} - \frac{t^{2^*-nb}}{2^*} S^{\frac{n}{2}} \right\} \\ &\quad + \frac{t_2^{q-nb}}{4} L(u_\varepsilon) - \frac{\mu t_1^{p-nb}}{p} \int_{\mathbb{R}^n} u_\varepsilon^p dx + O(\varepsilon^{n-2}) \\ &\leqslant \frac{1}{n} S^{\frac{n}{2}} + O(\varepsilon^{n-2}) + C \left(\int_{\mathbb{R}^n} u_\varepsilon^{\frac{4n}{n+\alpha}} dx \right)^{\frac{n+\alpha}{n}} - C\mu \int_{\mathbb{R}^n} u_\varepsilon^p dx.\end{aligned} \tag{2.12}$$

(i) When $n = 3$. It follows from (2.11) that

$$\left(\int_{\mathbb{R}^3} u_\varepsilon^{\frac{12}{3+\alpha}} dx \right)^{\frac{3+\alpha}{3}} \leqslant C\varepsilon^2 + C\varepsilon^{1+\alpha}.$$

If $q = (8 + 2\alpha)/(2 + \alpha) < p < 6$, then we have

$$\tfrac{8+2\alpha}{2+\alpha} - 3 = \tfrac{2-\alpha}{2+\alpha} \begin{cases} < 0, & 2 < \alpha < 3, \\ = 0, & \alpha = 2, \\ > 0, & 0 < \alpha < 2. \end{cases}$$

This implies that

$$\int_{\mathbb{R}^3} u_\varepsilon^p dx = \begin{cases} C\varepsilon^{\frac{p}{2}}, & q < p < 3, \\ C\varepsilon^{3-\frac{p}{2}}, & p > 3. \end{cases} \tag{2.13}$$

If $q < p < 4$, we have $q/2 > 1$ and $3 - q/2 > 1$. Hence, it follows from (2.12) and (2.13) that

$$\begin{aligned}\sup_{t \geqslant 0} J(tu_\varepsilon(t^b x)) &= \frac{1}{3} S^{\frac{3}{2}} + O(\varepsilon) + C \left(\int_{\mathbb{R}^3} u_\varepsilon^{\frac{12}{3+\alpha}} dx \right)^{\frac{3+\alpha}{3}} - C\mu \int_{\mathbb{R}^3} u_\varepsilon^p dx \\ &\leqslant \frac{1}{3} S^{\frac{3}{2}} + C\varepsilon - \mu(C\varepsilon^{\frac{p}{2}} + C\varepsilon^{3-\frac{p}{2}}) \\ &< \frac{1}{3} S^{\frac{3}{2}}\end{aligned}$$

provided ε enough small and $\mu \geqslant \mu_*$ (μ_* suitable large).

If $4 < p < 6$, we see that

$$\sup_{t \geqslant 0} J(tu_\varepsilon(t^b x)) < \frac{1}{3} S^{\frac{3}{2}}$$

provided ε enough small and $\mu > 0$.

(ii) When $4 \leqslant n < 4 + \alpha$, on one hand, there holds

$$\frac{8 + 2\alpha}{2 + \alpha} - \frac{n}{n - 2} = \frac{6n + n\alpha - 16 - 4\alpha}{(2 + \alpha)(n - 2)} > 0.$$

Then, it follows from (2.11) that

$$\int_{\mathbb{R}^n} u_\varepsilon^p dx = C\varepsilon^{n-\frac{(n-2)p}{2}}.$$

On the other hand, since $n \geqslant 4$ and $\alpha < n$, we have

$$\frac{4n}{n+\alpha} - \frac{n}{n-2} = \frac{n(3n-8-\alpha)}{(n+\alpha)(n-2)} > 0.$$

Thus

$$\left(\int_{\mathbb{R}^n} u_\varepsilon^{\frac{4n}{n+\alpha}} dx\right)^{\frac{n+\alpha}{n}} = C(\varepsilon^{n-\frac{2n(n-2)}{n+\alpha}})^{\frac{n+\alpha}{n}} = C\varepsilon^{4+\alpha-n}.$$

Therefore, noting that

$$\frac{2(2n-4-\alpha)}{n-2} - \frac{8+2\alpha}{2+\alpha} = \frac{2\alpha(n-4-\alpha)}{(n-2)(2+\alpha)} < 0,$$

which implies that

$$p > \frac{8+2\alpha}{2+\alpha} > \frac{2(2n-4-\alpha)}{n-2}.$$

Consequently,

$$n - \frac{(n-2)p}{2} < 4 + \alpha - n.$$

In addition, since $n \geqslant 4$, we have

$$\frac{n}{n-2} - \frac{8+2\alpha}{2+\alpha} = \frac{16-6n-(n-4)\alpha}{(n-2)(2+\alpha)} < 0.$$

This implies that

$$p > \frac{8+2\alpha}{2+\alpha} > \frac{n}{n-2} \geqslant \frac{4}{n-2}.$$

As a result,

$$n - \frac{(n-2)p}{2} < n-2.$$

$$\begin{aligned}\sup_{t \geqslant 0} J(tu_\varepsilon(t^b x)) &\leqslant \frac{1}{n}S^{\frac{n}{2}} + O(\varepsilon^{n-2}) + C\left(\int_{\mathbb{R}^n} u_\varepsilon^{\frac{4n}{n+\alpha}} dx\right)^{\frac{n+\alpha}{n}} - C\mu\int_{\mathbb{R}^n} u_\varepsilon^p dx \\ &\leqslant \frac{1}{n}S^{\frac{n}{2}} + O(\varepsilon^{n-2}) + C\varepsilon^{4+\alpha-n} - C\mu\varepsilon^{n-\frac{(n-2)p}{2}} \\ &< \frac{1}{n}S^{\frac{n}{2}}\end{aligned}$$

for ε enough small and $\mu > 0$. The proof is complete. □

Let us define $\mathcal{F}: X^{1,\alpha} \to \mathbb{R}$ as

$$\mathcal{F}(u) := \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^2|u(y)|^2}{|x-y|^{n-\alpha}} dxdy.$$

Lemma 2.6 *Let $\{u_m\} \subset \mathcal{M}$ be a minimizing sequence of J, and $d < \frac{1}{n}S^{\frac{n}{2}}$. Then $\{u_m\}$ is a bounded $(PS)_d$ sequence for J. Moreover, there exist a subsequence of $\{u_m\}$, still denoted itself, a number $k \in \mathrm{N} \cup \{0\}$ and a finite sequence*

$$(v^0, v^1, \ldots, v^k) \subset X^{1,\alpha},\ v^i \not\equiv 0,\ for\ i > 0$$

of critical points problem (1.1) *and* k *sequences* $\{\xi_m^1\}, \ldots, \{\xi_m^k\} \subset \mathbb{R}^n$, *such that as* $m \to +\infty$,

$$\|u_m - v^0 - \sum_{i=1}^{k} v(\cdot - \xi_m^i)\| \to 0,$$
$$|\xi_m^i| \to +\infty, \ \ |\xi_m^i - \xi_m^j| \to +\infty, \ i \neq j,$$
$$J(v^0) + \sum_{i=1}^{k} J(v^i) = d = \inf_{\mathcal{M}} J. \tag{2.14}$$

Proof *Step 1*. Let $\{u_m\} \subset \mathcal{M}$ be a minimizing sequence of J in $\mathcal{M}$, that is, $J(u_m) \to \inf_{\mathcal{M}} J$ as $m \to \infty$. We claim that $\{u_m\}$ is a (PS) sequence of J. In fact, by the Lemma 2.4, we can obtain that $\{u_m\}$ is also a (PS) sequence of J in A. By the Ekeland variational principle (see Theorem 8.5 in [30]), there exists $\{\lambda_m\} \subset \mathbb{R}$ such that

$$J(u_m) \to \inf_{\mathcal{M}} J,$$
$$J'(u_m) - \lambda_m K'(u_m) \to 0 \text{ as } m \to \infty.$$

Then

$$0 = \langle J'(u_m), u_m\rangle = \lambda_m \langle K'(u_m), u_m\rangle + o(1).$$

Since $\langle K'(u_m), u_m\rangle \neq 0$, we find

$$\lim_{m\to\infty} \lambda_m = 0.$$

Thus we obtain that $\{u_m\} \subset \mathcal{M}$ is (PS) sequence of J. Namely

$$J(u_m) \to \inf_{\mathcal{M}} J, \quad J'(u_m) \to 0 \text{ as } m \to \infty. \tag{2.15}$$

It follows from $J'(u_m) \to 0$ and $I(u_m) = 0$ that

$$\mathcal{P}(u_m) \to 0 \ \text{ as } m \to \infty. \tag{2.16}$$

By (2.1), there holds

$$\frac{1}{p}\frac{p-q}{q-nb}\mu \int_{\mathbb{R}^n} |u_m|^p dx + \frac{1}{2^*}\frac{2^*-q}{q-nb}\int_{\mathbb{R}^n} |u_m|^{2^*} dx \to \inf_{\mathcal{M}} J \ (m \to \infty),$$

which leads to

$$\int_{\mathbb{R}^n} |u_m|^p dx < +\infty, \quad \int_{\mathbb{R}^n} |u_m|^{2^*} dx < +\infty.$$

By $I(u_m) = 0$, that is,

$$\frac{q-nb}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2 dx + \frac{q-nb}{4} L(u_m) - \frac{\mu(p-nb)}{p}\int_{\mathbb{R}^n} |u_m|^p dx - \frac{2^*-nb}{2^*}\int_{\mathbb{R}^n} |u_m|^{2^*} dx = 0,$$

we obtain that $\{u_m\}$ is bounded in $X^{1,\alpha}$.

Step 2. Since $\{u_m\}$ is bounded in $X^{1,\alpha}$, we can find a subsequence of u_m denoted by itself such that $u_m \rightharpoonup v^0$ weakly in $X^{1,\alpha}$ when $m \to \infty$. It follows from $J'(u_m) \to 0$ $(m \to \infty)$ that

$$\int_{\mathbb{R}^n} \nabla v^0 \nabla \varphi dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v^0(x)|^2 v^0(y)\varphi(y)}{|x-y|^{n-\alpha}} dxdy \\ -\mu \int_{\mathbb{R}^n} |v^0|^{p-2} v^0 \varphi dx - \int_{\mathbb{R}^n} |v^0|^{2^*-2} v^0 \varphi dx = 0$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then v^0 is a critical point for J. Thereby,

$$\langle J'(v^0), v^0 \rangle = 0, \quad and \quad \mathcal{P}(v^0) = 0.$$

Denote $u_m^1 := u_m - v^0$, then $u_m^1 \rightharpoonup 0$ in $X^{1,\alpha}$ when $m \to \infty$. By the Brézis–Lieb lemma (cf. [2, 7]), we have

$$\int_{\mathbb{R}^n} |u_m^1|^{2^*} dx = \int_{\mathbb{R}^n} |u_m|^{2^*} dx - \int_{\mathbb{R}^n} |v^0|^{2^*} dx + o(1), \tag{2.17}$$

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx = \int_{\mathbb{R}^n} |\nabla u_m|^2 dx - \int_{\mathbb{R}^n} |\nabla v^0|^2 dx + o(1), \tag{2.18}$$

$$\int_{\mathbb{R}^n} |u_m^1|^p dx = \int_{\mathbb{R}^n} |u_m|^p dx - \int_{\mathbb{R}^n} |v^0|^p dx + o(1), \tag{2.19}$$

and

$$L(u_m^1) = L(u_m) - L(v^0) + o(1) \tag{2.20}$$

when $m \to \infty$. Hence, from the above information, we obtain

$$\begin{cases} J(u_m^1) = J(u_m) - J(v^0) + o(1) \to \inf_{\mathcal{M}} J - J(v^0), \\ J'(u_m^1) \to 0, \end{cases} \tag{2.21}$$

and

$$\begin{aligned} &\mathcal{P}(u_m) \\ &= \frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2 dx + \frac{n+\alpha}{4} L(u_m) - \mu \frac{n}{p}\int_{\mathbb{R}^n} |u_m|^p dx - \frac{n}{2^*}\int_{\mathbb{R}^n} |u_m|^{2^*} dx \\ &= \frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx + \frac{n+\alpha}{4} L(u_m^1) - \mu \frac{n}{p}\int_{\mathbb{R}^n} |u_m^1|^p dx - \frac{n}{2^*}\int_{\mathbb{R}^n} |u_m^1|^{2^*} dx \\ &\quad + \frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla v^0|^2 dx + \frac{n+\alpha}{4} L(v^0) - \mu \frac{n}{p}\int_{\mathbb{R}^n} |v^0|^p dx - \frac{n}{2^*}\int_{\mathbb{R}^n} |v^0|^{2^*} dx + o(1) \\ &= \frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx + \frac{n+\alpha}{4} L(u_m^1) - \mu \frac{n}{p}\int_{\mathbb{R}^n} |u_m^1|^p dx - \frac{n}{2^*}\int_{\mathbb{R}^n} |u_m^1|^{2^*} dx + o(1), \end{aligned}$$

when $m \to \infty$. In view of (2.16), we have

$$\begin{aligned} \mathcal{P}(u_m^1) &= \frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx + \frac{n+\alpha}{4} L(u_m^1) - \mu \frac{n}{p}\int_{\mathbb{R}^n} |u_m^1|^p dx - \frac{n}{2^*}\int_{\mathbb{R}^n} |u_m^1|^{2^*} dx \\ &= o(1). \end{aligned} \tag{2.22}$$

If

$$u_m^1 \to 0 \quad in \ X^{1,\alpha} \quad (m \to \infty),$$

we are done. Indeed, by the Coulomb–Sobolev inequality,

$$\int_{\mathbb{R}^n} |u_m^1|^p dx \to 0, \quad \int_{\mathbb{R}^n} |u_m^1|^{2^*} dx \to 0, \quad \text{as } m \to \infty.$$

By (2.1), we have

$$J(v^0) = \inf_{\mathcal{M}} J > 0, \quad \mathcal{F}[u_m] \to \mathcal{F}[v^0] \ (m \to \infty).$$

Observe that in this case $v^0 \not\equiv 0$ and $v^0 \in \mathcal{M}$.

If

$$u_m^1 \nrightarrow 0 \quad in \ X^{1,\alpha} \quad (m \to \infty).$$

Recall (2.15) and that v^0 is a solution, then when $m \to \infty$,

$$\begin{aligned}\langle J'(u_m), u_m\rangle &= \mathcal{F}[u_m] - \mu \int_{\mathbb{R}^n} |u_m|^p dx - \int_{\mathbb{R}^n} |u_m|^{2^*} dx \\ &\to 0 = \mathcal{F}[v^0] - \mu \int_{\mathbb{R}^n} |v^0|^p dx - \int_{\mathbb{R}^n} |v^0|^{2^*} dx.\end{aligned}$$

Next, the argument is divided into two cases

$$Case\ 1. \quad \lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m|^p dx = \int_{\mathbb{R}^n} |v^0|^p dx;$$

$$Case\ 2. \quad \lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m|^p dx \neq \int_{\mathbb{R}^n} |v^0|^p dx.$$

In *Case* 1, when $m \to \infty$, it follows from (2.22) that

$$\frac{n-2}{2} \int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx + \frac{n+\alpha}{4} L(u_m^1) - \frac{n}{2^*} \int_{\mathbb{R}^n} |u_m^1|^{2^*} dx = o(1). \tag{2.23}$$

In addition, by (2.15) and (2.17)–(2.20), we see that as $m \to \infty$,

$$\begin{aligned}&\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx + \int_{\mathbb{R}^n} |\nabla v^0|^2 dx + L(u_m^1) + L(v^0) \\ &- \int_{\mathbb{R}^n} |u_m^1|^{2^*} dx - \int_{\mathbb{R}^n} |v^0|^{2^*} dx - \mu \int_{\mathbb{R}^n} |v^0|^p dx = o(1).\end{aligned}$$

The equality combined with $\langle J'(v^0), v^0\rangle = 0$ gives that

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx + L(u_m^1) - \int_{\mathbb{R}^n} |u_m^1|^{2^*} dx = o(1) \ \ (m \to \infty). \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$\frac{4+\alpha-n}{4} L(u_m^1) = o(1) \ \ (m \to \infty).$$

Consequently

$$L(u_m^1) = o(1) \ \ (m \to \infty).$$

Namely

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx - \int_{\mathbb{R}^n} |u_m^1|^{2^*} dx = o(1). \tag{2.25}$$

Set

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx = l.$$

Then $l = 0$. Otherwise, if $l > 0$. By using (2.25) and Sobolev inequality, we obtain

$$l \geqslant S^{\frac{n}{2}}.$$

Hence, we obtain

$$\begin{aligned}
J(u_m) &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2 dx + \frac{1}{4}L(u_m) - \frac{\mu}{p}\int_{\mathbb{R}^n} |u_m|^p dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m|^{2^*} dx + o(1)\\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v^0|^2 dx + \frac{1}{4}L(u_m^1) + \frac{1}{4}L(v^0) - \frac{\mu}{p}\int_{\mathbb{R}^n} |u_m^1|^p dx\\
&\quad - \frac{\mu}{p}\int_{\mathbb{R}^n} |v^0|^p dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |v^0|^{2^*} dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m^1|^{2^*} dx + o(1)\\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m^1|^{2^*} dx + J(v^0) + o(1)\\
&= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx + J(v^0) + o(1)\\
&= \frac{1}{n} l + J(v^0) + o(1)\\
&\geqslant \frac{1}{n} S^{\frac{n}{2}} + J(v^0) + o(1)
\end{aligned}$$

when $m \to \infty$. Thus we have

$$\frac{1}{n} S^{\frac{n}{2}} + J(v^0) \leqslant J(u_m) \to d < \frac{1}{n} S^{\frac{n}{2}}.$$

This implies that

$$J(v^0) < 0.$$

Consequently

$$v^0 \not\equiv 0.$$

Since v^0 is a solution of (1.1), thus

$$\langle J'(v^0), v^0 \rangle = 0, \quad \text{and} \quad \mathcal{P}(v^0) = 0.$$

This combining $v^0 \not\equiv 0$ imply

$$v^0 \in \mathcal{M}.$$

Therefore,

$$J(v^0) \geqslant \inf_{\mathcal{M}} J = d > 0.$$

We reach a contradiction. Therefore,

$$l \equiv 0.$$

As a result,

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx \to 0, \quad \int_{\mathbb{R}^n} |u_m^1|^{2^*} dx \to 0$$

as $m \to \infty$. Thus we obtain $J(v^0) = \inf_{\mathcal{M}} J > 0$, which implies that $v^0 \neq 0$. Thus, by the strong maximum principle, v^0 is a positive ground state solution of (1.1), and Theorem 1.1 is proved.

In Case 2, since $X^{1,\alpha} \hookrightarrow L^p_{loc}(\mathbb{R}^n)$ is compact, there exist $\delta_1 > 0$, $\{\xi^1_m\} \subset \mathbb{R}^n$ such that

$$\int_{B_1} |u^1_m(x + \xi^1_m)|^p dx \geqslant \delta_1 > 0. \tag{2.26}$$

This and (2.26) lead to $|\xi^1_m| \to +\infty$ $(m \to \infty)$.

Step 3. Define $v^1_m := u^1_m(\cdot + \xi^1_m)$. Obviously, it is a bounded (PS) sequence at level $\inf_{u\in\mathcal{M}} J - J(v^0)$ (recall (2.21)). Up to a subsequence, we may assume that

$$u^1_m \rightharpoonup v^1 \quad in\ X^{1,\alpha} \quad (m \to \infty),$$

and v^1 is a solution of (1.1). By (2.26) we also have that $v^1 \neq 0$, and $v^1 \in \mathcal{M}$.

Define

$$u^2_m := u^1_m - v^1(\cdot - \xi^1_m).$$

Then

$$u^2_m \rightharpoonup 0 \quad in\ X^{1,\alpha} \quad (m \to \infty).$$

Arguing as in *Step* 2 and taking into account (2.21), we obtain that when $m \to \infty$,

$$\begin{cases} \mathcal{F}[u^2_m] = \mathcal{F}[u^1_m] - \mathcal{F}[v^1] + o(1) = \mathcal{F}[u_m] - \mathcal{F}[v^0] - \mathcal{F}[v^1] + o(1), \\ J(u^2_m) = J(u^1_m) - J(v^1) = J(u_m) - J(v^0) - J(v^1) + o(1), \\ J'(u^2_m) \to 0, \\ \mathcal{P}(u^2_m) \to 0. \end{cases} \tag{2.27}$$

If $u^2_m \to 0$ $(m \to \infty)$ in $X^{1,\alpha}$ we are done. In fact, when $m \to \infty$, $u^1_m \to v^1(\cdot - \xi^1_m)$ in $X^{1,\alpha}$. As in Step 2, we can obtain that v^1 is a nontrivial solution of (1.1) with $J(v^1) = \inf_{u\in\mathcal{M}} J > 0$.

If $u^2_m \not\to 0$ $(m \to \infty)$ in $X^{1,\alpha}$. Similarly, if $u^2_m \to 0$ $(m \to \infty)$ in $L^p(\mathbb{R}^n)$, we are done and $J(v^1) = \inf_{u\in\mathcal{M}} J > 0$. If $u^2_m \nrightarrow 0$ $(m \to \infty)$ in $L^p(\mathbb{R}^n)$, we may assume the existence of $\{\xi^2_m\} \subset \mathbb{R}^n$ such that

$$\int_{B_1} |u^2_m(x + \xi^2_m)|^p dx \geqslant \delta_2 \ \text{ for some } \delta_2 > 0.$$

Since

$$u^2_m \rightharpoonup 0, \quad \text{and} \quad u^2_m(\cdot + \xi^1_m) \rightharpoonup 0 \text{ in } X^{1,\alpha}\ (m \to \infty),$$

we deduce that when $m \to \infty$,

$$|\xi^2_m| \to +\infty, \quad |\xi^2_m - \xi^1_m| \to +\infty.$$

Therefore, up to a subsequence, we may assume that

$$u^2_m(\cdot + \xi^2_m) \rightharpoonup v^2(\cdot) \quad \text{in } X^{1,\alpha}\ (m \to \infty),$$

and v^2 is a nontrivial solution of (1.1). We now define

$$u^3_m := u^2_m - v^2(\cdot - \xi^2_m).$$

Iterating the above procedure we construct sequences $\{u^j_m\}_j$ and $\{\xi^j_m\}_j$, in the following way

$$
\begin{aligned}
&u_m^{j+1} := u_m^j - v^j(\cdot - \xi_m^j),\\
&\mathcal{F}[u_m^j] = \mathcal{F}[u_m] - \sum_{i=0}^{j-1} \mathcal{F}[v^i] + o(1) \ \ (m \to \infty),\\
&J(u_m^j) = J(u_m) - \sum_{i=0}^{j-1} J(v^i) + o(1) \ \ (m \to \infty),\\
&J'(v^i) = 0, \ \text{for } i \geqslant 0.
\end{aligned}
$$

Noting that $\mathcal{F}[u_m]$ is bounded and

$$\frac{1}{2}\mathcal{F}[v^i] \geqslant \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v^i|^2 dx + \frac{1}{4}L(v^i) > \inf_{\mathcal{M}} J,$$

which implies that the iteration must stop by at most finite steps. Namely, there exists some positive integer k, such that $u_m^k \to 0$ $(m \to \infty)$ in $X^{1,\alpha}$. The proof is finished. □

Proof of Theorem 1.1 We apply Lemma 2.6 to obtain

$$\sum_{i=0}^{k} J(v^i) = \inf_{\mathcal{M}} J. \tag{2.28}$$

Since v^i $(i = 1, \ldots, k)$ is a solution of Eq. (1.1), we have $J'(v^i) = 0$ and $\mathcal{P}(v^i) = 0$ for $i = 0, 1, \ldots, k$. This implies that $v^i \in \mathcal{M}$, and thus $J(v^i) \geqslant \inf_{\mathcal{M}} J$ for $i = 1, \ldots, k$. Applying (2.28) and noting that Step 2 in Lemma 2.3, there are two possibilities: either $v^0 \neq 0$ and $k = 0$, or $v^0 = 0$ and $k = 1$. In the first case, $u_m(\cdot + \xi_m^1) \to v^0(\cdot)$ $(m \to \infty)$ in $X^{1,\alpha}$ (by (2.14)) and $v^0 \in A$ is a solution of Eq. (1.1) (by Step 4 in Lemma 2.3) with $J(v^0) = \inf_{\mathcal{M}} J$ (by (2.28)), and so $v^0 \in G$ is a positive ground state solution of (1.1). In the latter, $u_m(\cdot + \xi_m^1) \to v^1(\cdot)$ in $X^{1,\alpha}$ as $m \to \infty$ (by (2.14)) and $v^1 \in G$ is a positive ground state solution of Eq. (1.1) with $J(v^1) = \inf_{\mathcal{M}} J$ (by (2.28)). The proof is ended. □

3 Proof of Theorem 1.2

Lemma 3.1 *The functional J is unbounded from below.*

Proof For $n > 4 + \alpha$, let $u \in X^{1,\alpha}_{\mathrm{rad}}$, and $u_t = t^{-1}u(t^{\widetilde{b}}x)$, $\widetilde{b} = -\frac{2}{2+\alpha}$, $t > 0$. By the standard scaling we have

$$\int_{\mathbb{R}^n} |\nabla u_t|^2 dx = t^{-q-n\widetilde{b}} \int_{\mathbb{R}^n} |\nabla u|^2 dx, \quad L(u_t) = t^{-q-n\widetilde{b}} L(u),$$

and

$$\int_{\mathbb{R}^n} |u_t|^p dx = t^{-p-n\widetilde{b}} \int_{\mathbb{R}^n} |u|^p dx, \quad \int_{\mathbb{R}^n} |u_t|^{2^*} dx = t^{-2^*-n\widetilde{b}} \int_{\mathbb{R}^n} |u|^{2^*} dx.$$

Hence,

$$
\begin{aligned}
J(u_t) &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_t|^2 dx + \frac{1}{4}L(u_t) - \frac{\mu}{p}\int_{\mathbb{R}^n} |u_t|^p dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_t|^{2^*} dx\\
&= \frac{t^{-q-n\widetilde{b}}}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{t^{-q-n\widetilde{b}}}{4}L(u) - \frac{\mu t^{-p-N\widetilde{b}}}{p}\int_{\mathbb{R}^n} |u|^p dx - \frac{t^{-2^*-n\widetilde{b}}}{2^*}\int_{\mathbb{R}^n} |u|^{2^*} dx.
\end{aligned}
$$

Since $2^* < q$ when $n > 4 + \alpha$, we see that $J(u_t) \to -\infty$ as $t \to +\infty$. □

Similar to Lemma 2.2, since $-q - n\widetilde{b} < -p - n\widetilde{b} < -2^* - n\widetilde{b}$, we have the following lemma.

Lemma 3.2 *Let*

$$\varphi(t) := t^{-q-n\widetilde{b}}\left[\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2dx + \frac{1}{4}L(u)\right] - \frac{\mu t^{-p-n\widetilde{b}}}{p}\int_{\mathbb{R}^n}|u|^p dx - \frac{t^{-2^*-n\widetilde{b}}}{2^*}\int_{\mathbb{R}^n}|u|^{2^*}dx$$

for $u \in \widetilde{\mathcal{M}}$ and $t \geqslant 0$. Then φ has a unique positive critical point, corresponding to its maximum.

It is clear that $\widetilde{I}(u) = -\langle J'(u), u\rangle - \widetilde{b}\mathcal{P}(u)$ with $\widetilde{b} = 2/(2+\alpha)$.

Firstly, similar to the proofs of Lemmas 2.3 and 2.4, the conclusions of Lemmas 2.3 and 2.4 hold true when $2^* < p < (8+2\alpha)/(2+\alpha)$ with $n > 4+\alpha$.

Secondly, compared with Theorem 1.1, we do not need to estimate the functional threshold for the case $n > 4+\alpha$. We just need the (PS) sequence to be strongly convergent in $L^{2^*}_{\text{loc}}(\mathbb{R}^n)$. Indeed, since $2^* < p < (8+2\alpha)/(2+\alpha)$ with $n > 4+\alpha$, the the embedding

$$X^{1,\alpha}_{\text{rad}} \hookrightarrow L^{2^*}_{\text{loc}}(\mathbb{R}^n)$$

is compact, we obtain the following result.

Lemma 3.3 *Let $\{u_m\} \subset \widetilde{\mathcal{M}}$ be a minimizing sequence of J. Then $\{u_m\}$ is a bounded (PS) sequence for J. Moreover, there exist a subsequence of $\{u_m\}$, still denoted itself, a number $k \in \mathrm{N} \cup \{0\}$ and a finite sequence*

$$(v^0, v^1, \ldots, v^k) \subset X^{1,\alpha}_{\text{rad}},\ v^i \not\equiv 0,\ \ for\ i > 0$$

of critical points problem (1.1) and k sequences $\{\xi^1_m\}, \ldots, \{\xi^k_m\} \subset \mathbb{R}^n$, such that as $m \to +\infty$,

$$\begin{aligned}
&\|u_m - v^0 - \sum_{i=1}^k v(\cdot - \xi^i_m)\| \to 0,\\
&|\xi^i_m| \to +\infty,\ \ |\xi^i_m - \xi^j_m| \to +\infty,\ i \neq j,\\
&J(v^0) + \sum_{i=1}^k J(v^i) = \inf_{\widetilde{\mathcal{M}}} J.
\end{aligned}$$

Proof We divide the proof into three steps. *Step 1*. Let $\{u_m\} \subset \widetilde{\mathcal{M}}$ be a minimizing sequence of J in $\widetilde{\mathcal{M}}$, that is, $J(u_m) \to \inf_{\widetilde{\mathcal{M}}} J$ as $m \to \infty$. We claim that $\{u_m\}$ is a (PS) sequence of J. In fact, by the Lemma 2.4, we can obtain that $\{u_m\}$ is also a (PS) sequence of J in A. By the Ekeland variational principle, there exists $\{\lambda_m\} \subset \mathbb{R}$ such that

$$\begin{aligned}
&J(u_m) \to \inf_{\widetilde{\mathcal{M}}} J,\\
&J'(u_m) - \lambda_m K'(u_m) \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Then

$$0 = \langle J'(u_m), u_m\rangle = \lambda_m\langle K'(u_m), u_m\rangle + o(1).$$

Since $\langle K'(u_m), u_m\rangle \neq 0$, we find

$$\lim_{m\to\infty}\lambda_m = 0.$$

Thus we obtain that $\{u_m\} \subset \widetilde{\mathcal{M}}$ is (PS) sequence of J. Namely

$$J(u_m) \to \inf_{\widetilde{\mathcal{M}}} J, \quad J'(u_m) \to 0 \text{ as } m \to \infty.$$

It follows from $J'(u_m) \to 0$ and $\widetilde{I}(u_m) = 0$ that

$$\mathcal{P}(u_m) \to 0 \ \text{ as } m \to \infty.$$

Since $\{u_m\} \subset \widetilde{\mathcal{M}}$, we have

$$\frac{1}{p}\frac{q-p}{-q-n\widetilde{b}}\int_{\mathbb{R}^n}|u_m|^p dx + \frac{1}{2^*}\frac{q-2^*}{-q-n\widetilde{b}}\int_{\mathbb{R}^n}|u_m|^{2^*}dx \to \inf_{\widetilde{\mathcal{M}}} J \ (m \to \infty),$$

leading to

$$\int_{\mathbb{R}^n}|u_m|^p dx < +\infty, \quad \int_{\mathbb{R}^n}|u_m|^{2^*}dx < +\infty.$$

By $\widetilde{I}(u_m) = 0$, that is,

$$\begin{aligned}&\frac{-q-n\widetilde{b}}{2}\int_{\mathbb{R}^n}|\nabla u_m|^2 dx + \frac{-q-n\widetilde{b}}{4}L(u_m) - \frac{\mu(-p-n\widetilde{b})}{p}\int_{\mathbb{R}^n}|u_m|^p dx\\ &\quad -\frac{-2^*-n\widetilde{b}}{2^*}\int_{\mathbb{R}^n}|u_m|^{2^*}dx = 0.\end{aligned}$$

Then

$$\frac{-q-n\widetilde{b}}{2}\int_{\mathbb{R}^n}|\nabla u_m|^2 dx + \frac{-q-n\widetilde{b}}{4}L(u_m) \leqslant C < +\infty.$$

This implies that $\{u_m\}$ is bounded in $X^{1,\alpha}_{\mathrm{rad}}$.

Step 2. Since $\{u_m\}$ is bounded in $X^{1,\alpha}_{\mathrm{rad}}$, up to a subsequence, we may assume that $u_m \rightharpoonup v^0$ weakly in $X^{1,\alpha}_{\mathrm{rad}}$ when $m \to \infty$. It follows from $J'(u_m) \to 0$ $(m \to \infty)$ that

$$\begin{aligned}&\int_{\mathbb{R}^n}\nabla v^0\nabla\varphi dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v^0(x)|^2 v^0(y)\varphi(y)}{|x-y|^{n-\alpha}}dxdy\\ &-\mu\int_{\mathbb{R}^n}|v^0|^{p-2}v^0\varphi dx - \int_{\mathbb{R}^n}|v^0|^{2^*-2}v^0\varphi dx = 0\end{aligned}$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then v^0 is a critical point for J. Thereby,

$$\langle J'(v^0), v^0\rangle = 0, \quad \textit{and} \quad \mathcal{P}(v^0) = 0.$$

Denote $u_m^1 := u_m - v^0$, then $u_m^1 \rightharpoonup 0$ in $X^{1,\alpha}_{\mathrm{rad}}$ when $m \to \infty$. By the Brezis-Lieb lemma, we have

$$\begin{aligned}\int_{\mathbb{R}^n}|u_m^1|^{2^*}dx &= \int_{\mathbb{R}^n}|u_m|^{2^*}dx - \int_{\mathbb{R}^n}|v^0|^{2^*}dx + o(1),\\ \int_{\mathbb{R}^n}|\nabla u_m^1|^2 dx &= \int_{\mathbb{R}^n}|\nabla u_m|^2 dx - \int_{\mathbb{R}^n}|\nabla v^0|^2 dx + o(1),\end{aligned}$$

and

$$L(u_m^1) = L(u_m) - L(v^0) + o(1)$$

when $m \to \infty$. Hence, from the above information, we obtain

$$\begin{cases} J(u_m^1) = J(u_m) - J(v^0) + o(1) \to \inf_{\widetilde{\mathcal{M}}} J - J(v^0), \\ J'(u_m^1) \to 0. \end{cases}$$

If

$$u_m^1 \to 0 \quad in\ X_{\text{rad}}^{1,\alpha} \quad (m \to \infty),$$

then Theorem 1.2 is proved. Indeed, by the Coulomb–Sobolev inequality,

$$\int_{\mathbb{R}^n} |u_m^1|^p dx \to 0, \qquad \int_{\mathbb{R}^n} |u_m^1|^{2^*} dx \to 0, \quad \text{as } m \to \infty.$$

By (2.1), we have

$$J(v^0) = \inf_{\widetilde{\mathcal{M}}} J > 0, \ \ \mathcal{F}[u_m] \to \mathcal{F}[v^0]\ (m \to \infty).$$

Observe that in this case $v^0 \not\equiv 0$ and $v^0 \in \widetilde{\mathcal{M}}$.

If

$$u_m^1 \nrightarrow 0 \quad in\ X_{\text{rad}}^{1,\alpha} \quad (m \to \infty).$$

Next, the argument is divided into two cases

$$\begin{aligned} &Case\ 1. \ \lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m|^{2^*} dx = \int_{\mathbb{R}^n} |v^0|^{2^*} dx; \\ &Case\ 2. \ \lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m|^{2^*} dx \neq \int_{\mathbb{R}^n} |v^0|^{2^*} dx. \end{aligned}$$

In *Case* 1, since the the embedding

$$X_{\text{rad}}^{1,\alpha} \hookrightarrow L^p(\mathbb{R}^n)$$

is compact for $2^* < p < q$. Therefore,

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m^1|^p dx = 0.$$

It follows from $\langle J'(u_m), u_m \rangle = o(1)$ that

$$\begin{aligned} &\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx + \int_{\mathbb{R}^n} |\nabla v^0|^2 dx + L(u_m^1) + L(v^0) \\ &\quad - \int_{\mathbb{R}^n} |v^0|^{2^*} dx - \mu \int_{\mathbb{R}^n} |v^0|^p dx = o(1). \end{aligned}$$

The equality combined with $\langle J'(v^0), v^0 \rangle = 0$ gives that

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2 dx + L(u_m^1) = o(1) \ \ (m \to \infty).$$

Consequently,

$$u_m \to v^0 \quad in\ X_{\text{rad}}^{1,\alpha} \quad (m \to \infty).$$

This implies that

$$J(v^0) = \inf_{\widetilde{\mathcal{M}}} J,$$

and

$$v^0 \in \widetilde{\mathcal{M}}.$$

Thus, v^0 is a positive ground state solution of (1.1), and Theorem 1.2 is proved.

In *Case* 2, since $X^{1,\alpha}_{\mathrm{rad}} \hookrightarrow L^{2^*}_{\mathrm{loc}}(\mathbb{R}^n)$ is compact, there exist $\delta_1 > 0$, $\{\xi^1_m\} \subset \mathbb{R}^n$ such that

$$\int_{B_1} |u^1_m(x + \xi^1_m)|^p dx \geqslant \delta_1 > 0. \tag{3.1}$$

This leads to $|\xi^1_m| \to +\infty$ $(m \to \infty)$.

Step 3. Define $v^1_m := u^1_m(\cdot + \xi^1_m)$. Obviously, it is a bounded (PS) sequence at level $\inf_{u\in\widetilde{\mathcal{M}}} J - J(v^0)$. Up to a subsequence, we may assume that

$$u^1_m \rightharpoonup v^1 \quad in\ X^{1,\alpha}_{\mathrm{rad}} \quad (m \to \infty),$$

and v^1 is a solution of (1.1). By (3.1) we also have that $v^1 \neq 0$, and $v^1 \in \widetilde{\mathcal{M}}$.

Define

$$u^2_m := u^1_m - v^1(\cdot - \xi^1_m).$$

Then

$$u^2_m \rightharpoonup 0 \quad in\ X^{1,\alpha} \quad (m \to \infty).$$

Arguing as in *Step* 2, there holds that when $m \to \infty$,

$$\begin{cases} \mathcal{F}[u^2_m] = \mathcal{F}[u^1_m] - \mathcal{F}[v^1] + o(1) = \mathcal{F}[u_m] - \mathcal{F}[v^0] - \mathcal{F}[v^1] + o(1), \\ J(u^2_m) = J(u^1_m) - J(v^1) = J(u_m) - J(v^0) - J(v^1) + o(1), \\ J'(u^2_m) \to 0. \end{cases}$$

If $u^2_m \to 0$ $(m \to \infty)$ in $X^{1,\alpha}_{\mathrm{rad}}$ we are done. In fact, when $m \to \infty$, $u^1_m \to v^1(\cdot - \xi^1_m)$ in $X^{1,\alpha}$. As in Step 2, we can obtain that v^1 is a ground state solution of (1.1) with $J(v^1) = \inf_{u\in\widetilde{\mathcal{M}}} J > 0$.

If $u^2_m \not\to 0$ $(m \to \infty)$ in $X^{1,\alpha}_{\mathrm{rad}}$. Similarly, if $u^2_m \to 0$ $(m \to \infty)$ in $L^{2^*}(\mathbb{R}^n)$, we are done and $J(v^1) = \inf_{u\in\widetilde{\mathcal{M}}} J > 0$. If $u^2_m \nrightarrow 0$ $(m \to \infty)$ in $L^{2^*}(\mathbb{R}^n)$, we may assume the existence of $\{\xi^2_m\} \subset \mathbb{R}^n$ such that

$$\int_{B_1} |u^2_m(x + \xi^2_m)|^{2^*} dx \geqslant \delta_2 \ \text{ for some } \delta_2 > 0.$$

Since

$$u^2_m \rightharpoonup 0, \quad \text{and} \quad u^2_m(\cdot + \xi^1_m) \rightharpoonup 0 \text{ in } X^{1,\alpha}_{\mathrm{rad}} \ (m \to \infty),$$

we deduce that when $m \to \infty$,

$$|\xi^2_m| \to +\infty, \quad |\xi^2_m - \xi^1_m| \to +\infty.$$

Therefore, up to a subsequence, we may assume that

$$u^2_m(\cdot + \xi^2_m) \rightharpoonup v^2(\cdot) \quad \text{in } X^{1,\alpha}_{\mathrm{rad}} \ (m \to \infty),$$

and v^2 is a nontrivial solution of (1.1). We now define

$$u^3_m := u^2_m - v^2(\cdot - \xi^2_m).$$

Iterating the above procedure we construct sequences $\{u_m^j\}_j$ and $\{\xi_m^j\}_j$, in the following way

$$\begin{aligned}
&u_m^{j+1} := u_m^j - v^j(\cdot - \xi_m^j),\\
&\mathcal{F}[u_m^j] = \mathcal{F}[u_m] - \sum_{i=0}^{j-1} \mathcal{F}[v^i] + o(1) \ \ (m \to \infty),\\
&J(u_m^j) = J(u_m) - \sum_{i=0}^{j-1} J(v^i) + o(1) \ \ (m \to \infty),\\
&J'(v^i) = 0, \ \text{for } i \geqslant 0.
\end{aligned}$$

Noting that $\mathcal{F}[u_m]$ is bounded and

$$\frac{1}{2}\mathcal{F}[v^i] \geqslant \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v^i|^2 dx + \frac{1}{4}L(v^i) > \inf_{\widetilde{\mathcal{M}}} J,$$

which implies that the iteration must stop by at most finite steps. Namely, there exists some positive integer k, such that $u_m^k \to 0$ $(m \to \infty)$ in $X_{\text{rad}}^{1,\alpha}$. The proof is finished. □

Proof of Theorem 1.2 By Lemma 3.3,

$$\sum_{i=0}^{k} J(v^i) = \inf_{\widetilde{\mathcal{M}}} J.$$

Similar to the proof of Theorem 1.1, we obtain that v^0 or v^1 is the positive ground state solution of (1.1). The proof is now complete. □

4 Proof of Theorem 1.3

Similar to Lemma 2.4, we have the following relation.

Lemma 4.1 *Let* $2 + 8/(n+\alpha) < p < 2n/(n-2)$ *when* $4 + \alpha > n$, *and* u *be the ground state related to problem* (1.1). *Then*

$$J(u) = \inf\{J(v) : v \in M\}. \tag{4.1}$$

Proof We proceed in three steps.

Step 1. We claim that the minimizing problem (4.1) is well defined. Indeed, let $v \in M$, it follows that

$$\begin{aligned}
J(v) &= J(v) - \tfrac{1}{2}Q(v)\\
&= \frac{1}{4}L(v) + \frac{1}{p}\left(\frac{p(n+\alpha)-4n}{2(4-n+\alpha)} - 1\right)\mu\int_{\mathbb{R}^n} |u|^p dx\\
&= \frac{1}{4}L(u) + \frac{p(n+\alpha)-8-2\alpha-2n}{2p(4-n+\alpha)}\mu\int_{\mathbb{R}^n} |u|^p dx + \frac{1}{n}\int_{\mathbb{R}^n} |u|^{2^*} dx\\
&> 0.
\end{aligned}$$

Thus, we denote

$$\widetilde{d} := \inf\{J(v) : v \in M\}.$$

Step 2. $J(u) \geqslant \widetilde{d}$ for $u \in \ A$.

We shall prove that $u \in A \Rightarrow u \in M$. Since u is a ground state related to problem (1.1), we have

$$\begin{cases} \int_{\mathbb{R}^n} |\nabla u|^2 dx + L(u) - \mu \int_{\mathbb{R}^n} |u|^p dx - \int_{\mathbb{R}^n} |u|^{2^*} dx = 0, \\ \frac{n-2}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{n+\alpha}{4} L(u) - \frac{n}{p} \mu \int_{\mathbb{R}^n} |u|^p dx - \frac{n}{2^*} \int_{\mathbb{R}^n} |u|^{2^*} dx = 0, \end{cases}$$

such that

$$\frac{n-4-\alpha}{4} \int_{\mathbb{R}^n} |\nabla u|^2 dx = \frac{n-4-\alpha}{4} \int_{\mathbb{R}^n} |u|^{2^*} dx + \frac{4n - p(n+\alpha)}{4p} \mu \int_{\mathbb{R}^n} |u|^p dx.$$

Consequently

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx = \frac{p(n+\alpha) - 4n}{p(4+\alpha-n)} \mu \int_{\mathbb{R}^n} |u|^p dx + \int_{\mathbb{R}^n} |u|^{2^*} dx.$$

This implies that $Q(u) = 0$, namely $u \in M$, and so

$$J(u) \geqslant \widetilde{d}.$$

Step 3. $J(u) \leqslant \widetilde{d}$ for any $u \in A$.

For any $v \in M$, if $K(v) = 0$, noting that u is the ground state solution, then we have

$$J(v) \geqslant J(u).$$

By the arbitrariness of v, we have $\widetilde{d} \geqslant J(u)$. We are done.

If $K(v) \neq 0$, setting $v_\lambda = \lambda^{\frac{n+\alpha}{4+\alpha-n}} v(\lambda^{\frac{4}{4+\alpha-n}} x)$. An easy computation yields that

$$\int_{\mathbb{R}^n} |\nabla u_\lambda|^2 dx = \lambda^{\frac{2n+2\alpha}{4-n+\alpha}} \int_{\mathbb{R}^n} |\nabla u|^2 \lambda^{\frac{8}{4-n+\alpha}} \cdot \frac{1}{\lambda^{\frac{4n}{4-n+\alpha}}} dx = \lambda^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx,$$

$$L(u_\lambda) = \lambda^{\frac{4n+4\alpha}{4-n+\alpha}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u^2(x) u^2(y)}{|x-y|^{n-\alpha}} \lambda^{\frac{4(n-\alpha)}{4-n+\alpha}} \cdot \frac{1}{\lambda^{\frac{8n}{4-n+\alpha}}} dx dy = L(u).$$

Thus

$$K(v_\lambda) = \lambda^2 T(v) + L(v) - \lambda^{\frac{p(n+\alpha)-4n}{4-n+\alpha}} \mu \int_{\mathbb{R}^n} |v|^p dx - \lambda^{\frac{2^*(n+\alpha)-4n}{4-n+\alpha}} \int_{\mathbb{R}^n} |v|^{2^*} dx,$$

We see that

$$\lim_{\lambda \to 0} K(v_\lambda) = L(v) > 0, \quad \lim_{\lambda \to +\infty} K(v_\lambda) = -\infty.$$

This implies that there exists $\lambda_0 > 0$, such that

$$K(v_{\lambda_0}) = 0.$$

Moreover, it follows from $K(v_{\lambda_0}) = 0$ that

$$J(v_{\lambda_0}) \geqslant J(u).$$

On the other hand, by some basic calculations, we have

$$\begin{aligned} \partial_\lambda J(v_\lambda) &= \lambda T(v) - \frac{p(n+\alpha) - 4n}{p(4+\alpha-n)} \lambda^{\frac{p(n+\alpha)-4n}{4+\alpha-n} - 1} \mu \int_{\mathbb{R}^n} |v|^p dx \\ &\quad - \frac{2^*(n+\alpha) - 4n}{2^*(4+\alpha-n)} \lambda^{\frac{2^*(n+\alpha)-4n}{4+\alpha-n} - 1} \int_{\mathbb{R}^n} |v|^{2^*} dx \\ &= \lambda T(v) - \frac{p(n+\alpha) - 4n}{p(4+\alpha-n)} \lambda^{\frac{p(n+\alpha)-4n}{4+\alpha-n} - 1} \mu \int_{\mathbb{R}^n} |v|^p dx - \lambda^{\frac{2^*(n+\alpha)-4n}{4+\alpha-n} - 1} \int_{\mathbb{R}^n} |v|^{2^*} dx \\ &= \frac{1}{\lambda} Q(v_\lambda). \end{aligned}$$

Now, we define

$$\widetilde{f}(\lambda) := Q(v_\lambda) = \lambda^2 T(v) - \frac{p(n+\alpha)-4n}{p(4+\alpha-n)}\lambda^{\frac{p(n+\alpha)-4n}{4+\alpha-n}}\mu\int_{\mathbb{R}^n}|v|^p dx - \lambda^{\frac{2^*(n+\alpha)-4n}{4+\alpha-n}}\int_{\mathbb{R}^n}|v|^{2^*}dx.$$

Assume that there exists λ_1 such that $\widetilde{f}(\lambda_1)=0$. It follows $v\in M$ and $\widetilde{f}(\lambda_1)=0$ that

$$\begin{cases}\lambda_1^2 T(v) - \frac{p(n+\alpha)-4n}{p(4+\alpha-n)}\lambda_1^{\frac{p(n+\alpha)-4n}{4+\alpha-n}}\mu\int_{\mathbb{R}^n}|v|^p dx - \lambda_1^{\frac{2^*(n+\alpha)-4n}{4+\alpha-n}}\int_{\mathbb{R}^n}|v|^{2^*}dx = 0,\\ T(v) = \frac{p(n+\alpha)-4n}{p(4+\alpha-n)}\mu\int_{\mathbb{R}^n}|v|^p dx + \int_{\mathbb{R}^n}|v|^{2^*}dx.\end{cases}$$

Then

$$\frac{p(n+\alpha)-4n}{p(4+\alpha-n)}\left(\lambda_1^2 - \lambda_1^{\frac{p(n+\alpha)-4n}{4+\alpha-n}}\right)\mu\int_{\mathbb{R}^n}|v|^p dx = \left(\lambda_1^{\frac{2^*(n+\alpha)-4n}{4+\alpha-n}} - \lambda_1^2\right)\int_{\mathbb{R}^n}|v|^{2^*}dx.$$

Noting that

$$\frac{p(n+\alpha)-4n}{4+\alpha-n} - 2 = \frac{p(n+\alpha)-8-2\alpha-2n}{4+\alpha-n} > 0,$$

and

$$\frac{2^*(n+\alpha)-4n}{4+\alpha-n} - 2 = 2^* - 2 > 0,$$

If $\lambda_1 > 1$, then

$$\lambda_1^2 - \lambda_1^{\frac{p(n+\alpha)-4n}{4+\alpha-n}} < 0, \quad \lambda_1^{\frac{2^*(n+\alpha)-4n}{4+\alpha-n}} - \lambda_1^2 > 0,$$

which is a contradiction. If $\lambda_1 < 1$, then

$$\lambda_1^2 - \lambda_1^{\frac{p(n+\alpha)-4n}{4+\alpha-n}} > 0, \quad \lambda_1^{\frac{2^*(n+\alpha)-4n}{4+\alpha-n}} - \lambda_1^2 < 0,$$

we reach a contradiction. Therefore, the equation $\widetilde{f}(\lambda)=0$ has a unique positive solution $\lambda = 1$. As a result, we obtain

$$\begin{cases}\partial_\lambda J(v_\lambda) > 0, & \text{for all } \lambda\in(0,1),\\ \partial_\lambda J(v_\lambda) < 0, & \text{for all } \lambda\in(1,+\infty).\end{cases}$$

We thus get that $J(v_\lambda) < J(v)$ for any $\lambda > 0$ and $\lambda \neq 1$. In particular, we have

$$J(v_{\lambda_0}) \leqslant J(v).$$

Thus,

$$J(u) \leqslant J(v_{\lambda_0}) \leqslant J(v)$$

for $v\in M$. Taking the infimum over v, one has

$$J(u) \leqslant \widetilde{d}.$$

From the above information, we establish the relation $J(u) = \widetilde{d}$. The proof is complete. □

Lemma 4.2 *Every critical point of J in M is a critical point of J in $X^{1,\alpha}$.*

Proof Assume that u is a critical point of J in M, there exists a Lagrange multiplier λ such that $J'(u) = \lambda Q'(u)$. It can be written, in a weak sense, as

$$-\Delta u + (|x|^{\alpha-n} * |u|^2)u - \mu|u|^{p-2}u - |u|^{2^*-2}u = \lambda(-2\Delta u - \frac{p(n+\alpha)-4n}{4+\alpha-n}\mu|u|^{p-2}u - 2^*|u|^{2^*-2}u).$$

That is,

$$-(1-2\lambda)\Delta u + (|x|^{\alpha-n} * |u|^2)u = (1-\lambda\frac{p(n+\alpha)-4n}{4+\alpha-n})\mu|u|^{p-2}u + (1-2^*\lambda)|u|^{2^*-2}u. \quad (4.2)$$

It remains now to prove that $\lambda = 0$. Denote

$$\mathbf{A} := \mu\int_{\mathbb{R}^n} |u|^p dx, \quad \mathbf{B} := \int_{\mathbb{R}^n} |u|^{2^*} dx.$$

By (4.2), we can establish the following equations we can establish the following equations

$$\begin{cases} T(u) = \frac{p(n+\alpha)-4n}{p(4+\alpha-n)}\mathbf{A} + \mathbf{B}, \\ (1-2\lambda)T(u) + L(u) = (1-\lambda\frac{p(n+\alpha)-4n}{4+\alpha-n})\mathbf{A} + (1-2^*\lambda)\mathbf{B}, \\ \frac{n-2}{2}(1-2\lambda)T(u) + \frac{n+\alpha}{4}L(u) = \frac{n}{p}(1-\lambda\frac{p(n+\alpha)-4n}{4+\alpha-n})\mathbf{A} + \frac{n}{2^*}(1-2^*\lambda)\mathbf{B}, \end{cases} \quad (4.3)$$

where the second equation follows by multiplying (4.2) by u and integrating, and the third equality is the Pohozaev identity corresponding to Eq. (4.2).

From (4.3), we can obtain

$$\begin{cases} L(u) = [1-\lambda\frac{p(n+\alpha)-4n}{4+\alpha-n} - \frac{p(n+\alpha)-4n}{p(4+\alpha-n)}(1-2\lambda)]\mathbf{A} + (2-2^*)\lambda\mathbf{B}, \\ \frac{n+\alpha}{4}L(u) = [\frac{n}{p}(1-\lambda\frac{p(n+\alpha)-4n}{4+\alpha-n}) - \frac{n-2}{2}(1-2\lambda)\frac{p(n+\alpha)-4n}{p(4+\alpha-n)}]\mathbf{A} + \frac{n-2}{2}(2-2^*)\lambda\mathbf{B}. \end{cases}$$

From the above relations, we have

$$(L_{n,p,\alpha}\lambda + H_{n,p,\alpha})\mathbf{A} - \frac{4+\alpha-n}{2}(2^*-2)\lambda\mathbf{B} = 0, \quad (4.4)$$

where

$$\begin{aligned} L_{n,p,\alpha} := &\frac{pn(n+\alpha)-4n^2}{p(4+\alpha-n)} - \frac{p(n+\alpha)(n-2)-4n(n-2)}{p(4+\alpha-n)} \\ &- \frac{p(n+\alpha)^2-4n(n+\alpha)}{4(4+\alpha-n)} + \frac{2p(n+\alpha)^2-8n(n+\alpha)}{4p(4+\alpha-n)}, \\ H_{n,p,\alpha} := &\frac{(n+\alpha)[p(4-2n)+4n]}{4p(4+\alpha-n)} - \frac{n}{p} + \frac{(n-2)[p(n+\alpha)-4n]}{2p(4+\alpha-n)}. \end{aligned}$$

By computing,

$$\begin{aligned} H_{n,p,\alpha} &= \frac{p(n+\alpha)(4-2n)+4n(n+\alpha)-4n(4+\alpha-n)+2p(n-2)(n+\alpha)-8n(n-2)}{4p(4+\alpha-n)} \\ &= 0. \\ L_{n,p,\alpha} &= \frac{4pn(n+\alpha)-16n^2-4p(n+\alpha)(n-2)+16n(n-2)-p^2(n+\alpha)^2}{4p(4+\alpha-n)} \\ &\quad + \frac{4np(n+\alpha)+2p(n+\alpha)^2-8n(n+\alpha)}{4p(4+\alpha-n)} \\ &= \frac{2p(n+\alpha)^2-p^2(n+\alpha)^2+8np(n+\alpha)-4p(n+\alpha)(n-2)-8n(4+n+\alpha)}{4p(4+\alpha-n)} \\ &= \frac{p(n+\alpha)(6n+2\alpha+8-p(n+\alpha))-8n(4+n+\alpha)}{4p(4+\alpha-n)} \\ &\doteq \frac{-\chi^2+(6n+2\alpha+8)\chi-8n(4+n+\alpha)}{4p(4+\alpha-n)}, \end{aligned}$$

where $\chi = p(n+\alpha)$. Consider that the equation

$$f(\chi) = -\chi^2 + (6n + 2\alpha + 8)\chi - 8n(4 + n + \alpha).$$

We claim that $f(\chi) < 0$ when $\chi > 8 + 2\alpha + 2n$ with $n < 4 + \alpha$. Indeed,

$$\Delta = (6n + 2\alpha + 8)^2 - 32n(4 + n + \alpha) = 4(n - \alpha - 4)^2.$$

Consequently, it follows from $f(\chi) = 0$ that

$$\chi = \frac{-(6n + 2\alpha + 8) \pm 2\sqrt{(n - \alpha - 4)^2}}{-2}.$$

Observing that for $n < 4 + \alpha$, we have

$$\chi = p(n+\alpha) = 8 + 2\alpha + 2n, \quad \text{or} \quad \chi = p(n+\alpha) = 4n.$$

Noting that

$$2 + \frac{8}{n+\alpha} < p < \frac{2n}{n-2},$$

we see that

$$f(\chi) < 0$$

for $\chi > 8 + 2\alpha + 2n$, namely $p > 2 + \frac{8}{n+\alpha}$. Consequently, it follows from (4.4) that

$$\left[L_{n,p,\alpha}\mathbf{A} - \frac{4+\alpha-n}{2}(2^* - 2)\mathbf{B} \right] < 0$$

for $2 + \frac{8}{n+\alpha} < p < \frac{2n}{n-2}$. Thus, again from (4.4), we obtain

$$\lambda \equiv 0.$$

Thereby

$$J'(u) = 0.$$

The proof is thus complete. □

Lemma 4.3 *Assume that* $2 + 8/(n+\alpha) < p < 2^*$ *with* $n < 4 + \alpha$, *then*

$$\sup_{t \geqslant 0} J\left(t^{\frac{n+\alpha}{4+\alpha-n}} u_\varepsilon(t^{\frac{4}{4+\alpha-n}} x)\right) < \frac{1}{n} S^{\frac{n}{2}}.$$

Proof *Case 1*: $2 + 8/(n+\alpha) < p < 2^*$ with $n < 4 + \alpha$. Since

$$\lim_{t \to 0^+} J(t^{\frac{n+\alpha}{4+\alpha-n}} u_\varepsilon(t^{\frac{4}{4+\alpha-n}} x)) = 0$$

and

$$\lim_{t \to +\infty} J(t^{\frac{n+\alpha}{4+\alpha-n}} u_\varepsilon(t^{\frac{4}{4+\alpha-n}} x)) \to -\infty$$

as $t \to \infty$, there exists a $T_\varepsilon > 0$ such that

$$\sup_{t \geqslant 0} J(tu_\varepsilon(t^b x)) = J(T_\varepsilon^{\frac{n+\alpha}{4+\alpha-n}} u_\varepsilon(T_\varepsilon^{\frac{4}{4+\alpha-n}} x)).$$

Moreover, we can obtain that there exist $\widetilde{t_1}, \widetilde{t_2} > 0$ (independent of ε, μ), such that

$$\widetilde{t_1} \leqslant T_\varepsilon \leqslant \widetilde{t_2} < +\infty.$$

Case 2: $n = 3$ and $2 + 8/(3+\alpha) < p < 6$. We find

$$2 + \frac{8}{3+\alpha} - 3 = \frac{5-\alpha}{3+\alpha} > 0.$$

$$\begin{aligned}
\sup_{t \geqslant 0} J(t^{\frac{3+\alpha}{1+\alpha}} u_\varepsilon(t^{\frac{4}{1+\alpha}} x)) &\leqslant \sup_{t \geqslant 0} \left(\frac{t^2}{2} \int_\Omega |\nabla u_\varepsilon|^2 dx - \frac{1}{6} t^{\frac{6(3+\alpha)-12}{1+\alpha}} \int_\Omega u_\varepsilon^6 dx \right) \\
&\quad + \frac{1}{4} \left(\int_{\mathbb{R}^3} u_\varepsilon^{\frac{12}{3+\alpha}} dx \right)^{\frac{3+\alpha}{3}} - C\mu \int_{\mathbb{R}^3} u_\varepsilon^p dx \\
&\leqslant \sup_{t \geqslant 0} \left(\frac{t^2}{2} S^{\frac{3}{2}} - \frac{t^6}{6} S^{\frac{3}{2}} \right) + C\varepsilon + C\varepsilon^2 + C\varepsilon^{1+\alpha} - C\mu\varepsilon^{3-\frac{p}{2}} \\
&\leqslant \frac{1}{3} S^{\frac{3}{2}} + C\varepsilon - C\mu\varepsilon^{3-\frac{p}{2}}.
\end{aligned}$$

If $0 < \alpha < 1$, we see that

$$4 < 2 + \frac{8}{3+\alpha} < p < 6.$$

Consequently,

$$3 - \frac{p}{2} < 1,$$

and so

$$\sup_{t \geqslant 0} J(t^{\frac{3+\alpha}{1+\alpha}} u_\varepsilon(t^{\frac{4}{1+\alpha}} x)) < \frac{1}{3} S^{\frac{3}{2}}$$

for ε enough small. If $1 \leqslant \alpha < 3$. Let μ be suitable large, we also get

$$\sup_{t \geqslant 0} J(t^{\frac{3+\alpha}{1+\alpha}} u_\varepsilon(t^{\frac{4}{1+\alpha}} x)) < \frac{1}{3} S^{\frac{3}{2}}.$$

Case 3: $2 + 8/(n+\alpha) < p < 2^*$ with $4 \leqslant n < 4 + \alpha$. Noting that

$$\begin{aligned}
2 + \frac{8}{n+\alpha} - \frac{n}{n-2} &= \frac{n^2 - 16 + 4n + \alpha(n-4)}{(n+\alpha)(n-2)} > 0, \\
n - \frac{n-2}{2} p - (n-2) &< n - \frac{n-2}{2} \left(2 + \frac{8}{n+\alpha} \right) - (n-2) \\
&= -(n-4) - (n-2) \frac{4}{n+\alpha} \\
&< 0,
\end{aligned}$$

and

$$p > 2 + \frac{8}{n+\alpha} > \frac{8+2\alpha}{2+\alpha} > \frac{2(2n-4-\alpha)}{n-2}.$$

Consequently,

$$n - \frac{(n-2)p}{2} < 4 + \alpha - n.$$

So

$$\begin{aligned}
&\sup_{t\geqslant 0} J(t^{\frac{n+\alpha}{4+\alpha-n}}u_\varepsilon(t^{\frac{4}{4+\alpha-n}}x))\\
&\leqslant \sup_{t\geqslant 0}\left(\frac{t^2}{2}\int_\Omega |\nabla u_\varepsilon|^2dx-\frac{1}{2^*}t^{\frac{2^*(n+\alpha)-4n}{4+\alpha-n}}\int_\Omega u_\varepsilon^{2^*}dx\right)+\frac{1}{4}\left(\int_{\mathbb{R}^n}u_\varepsilon^{\frac{4n}{n+\alpha}}dx\right)^{\frac{n+\alpha}{n}}\\
&\quad -C\mu\textstyle\int_{\mathbb{R}^3}u_\varepsilon^p dx\\
&\leqslant \sup_{t\geqslant 0}\left(\frac{t^2}{2}\int_\Omega |\nabla u_\varepsilon|^2dx-\frac{1}{2^*}t^{2^*}\int_\Omega u_\varepsilon^{2^*}dx\right)+C\varepsilon^{4+\alpha-n}-C\mu\varepsilon^{n-\frac{(n-2)p}{2}}\\
&\leqslant \frac{1}{n}S^{\frac{n}{2}}+C\varepsilon^{n-2}+C\varepsilon^{4+\alpha-n}-C\mu\varepsilon^{n-\frac{p(n-2)}{2}}\\
&<\frac{1}{n}S^{\frac{n}{2}}
\end{aligned}$$

provided ε enough small. The proof is complete. □

Proof of Theorem 1.3 We proceed in three steps.

Step 1. (1.7) has nonzero solutions. Since $M\neq\emptyset$, J has a minimizing sequence $\{u_m\}$ by Lemma 4.1. In particular, $Q(u_m)=0$ and $J(u_m)\to\widetilde{d}$. Furthermore, note that

$$\widetilde{d}\leftarrow J(u_m)-\tfrac{1}{2}Q(u_m)=\frac{1}{4}L(u_m)+\frac{p(n+\alpha)-8-2\alpha-2n}{2p(4-n+\alpha)}\mu\int_{\mathbb{R}^n}|u_m|^pdx+\frac{1}{n}\int_{\mathbb{R}^n}|u_m|^{2^*}dx,$$

which implies that

$$\frac{1}{4}L(u_m)+\int_{\mathbb{R}^n}|u_m|^pdx+\frac{1}{n}\int_{\mathbb{R}^n}|u_m|^{2^*}dx\leqslant C<+\infty,$$

for some $C>0$. Since

$$T(u_m)=\frac{p(n+\alpha)-4n}{p(4+\alpha-n)}\mu\int_{\mathbb{R}^n}|u_m|^pdx+\int_{\mathbb{R}^n}|u_m|^{2^*}dx<+\infty,$$

we can obtain that $\{u_m\}$ is bounded in $X^{1,\alpha}$. Similar to Lemma 2.6, we know that $\{u_m\}$ is also a (PS) sequence of J in A.

From $J'(u_m)\to 0$ as $m\to\infty$. Similarly to Lemma 2.6, there exist $k\in \mathrm{N}\cup\{0\}$ and a finite sequence

$$(v_0,v_1,\ldots,v_k)\subset X^{1,\alpha},\ v_i\not\equiv 0,\ \ for\ i>0$$

of solutions of problem (1.1) and k sequences $\{\xi_m^1\},\ldots,\{\xi_m^k\}\subset\mathbb{R}^n$, such that as $m\to+\infty$,

$$\left\|u_m-v_0-\sum_{i=1}^k v_i(\cdot-\xi_m^i)\right\|_{X^{1,\alpha}}\to 0;$$

$$|\xi_m^i|\to+\infty,\ \ |\xi_m^i-\xi_m^j|\to+\infty,\ i\neq j; \tag{4.5}$$

$$\sum_{i=0}^k J(v_i)=\widetilde{d}\ \ \text{(by Lemma 4.1)}. \tag{4.6}$$

Since v_i $(i=0,1,\ldots,k)$ is a solution of Eq. (1.1), we have $J'(v_i)=0$ for $i=0,1,\ldots,k$. This implies that $v_i\in A$, and thus by Lemma 4.1, we have

$$J(v^i)\geqslant\widetilde{d}\geqslant\inf_G J(u)>0\ \ \text{for } i=0,1,\ldots,k.$$

Applying (4.6), there are two possibilities: either $v_0 \neq 0$ and $k = 0$, or $v_0 = 0$ and $k = 1$. In the first case, $u_m(\cdot + \xi_m^1) \to v_0(\cdot)$ in $X^{1,\alpha}$ (by (4.5)) and v_0 is a solution of Eq. (1.7) with $J(v_0) = \widetilde{d}$ (by (4.6)). In the latter, $u_m(\cdot + \xi_m^1) \to v_1(\cdot)$ in $X^{1,\alpha}$ as $m \to \infty$ (by (4.5)) and v_1 is a solution of Eq. (1.7) with $J(v_1) = \widetilde{d}$ (by (4.6)). Hence we prove *Step 1*.

Step 2. Every solution of (1.7) satisfies Eq. (1.1). Consider any solution u of (1.7). For $\sigma > 0$, let $u(x) = \sigma^{\frac{2}{p-2}} u_\sigma(\sigma x)$, we have

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx = \frac{1}{\sigma^{\frac{p(n-2)-2n}{p-2}}} \int_{\mathbb{R}^n} |\nabla u_\sigma|^2 dx,$$

and

$$\int_{\mathbb{R}^n} |u|^p dx = \frac{1}{\sigma^{\frac{p(n-2)-2n}{p-2}}} \int_{\mathbb{R}^n} |u_\sigma|^p dx.$$

Thus

$$Q(u_\sigma) = \sigma^{\frac{p(n-2)-2n}{p-2}} Q(u) = 0,$$

and so $u_\sigma \in M$. Since $u = u_1$ satisfies (1.7), we deduce that $f(\sigma) = J(u_\sigma)$ satisfies $f'(1) = 0$. By using the property $u_\sigma \in M$, we have

$$f'(1) = \langle J'(u), u\rangle_{(X^{1,\alpha})^{-1}, X^{1,\alpha}},$$

where J' is the gradient of the C^1 functional J, i.e.,

$$J'(u) = -\Delta u + \left(\int_{\mathbb{R}^n} \frac{|u(y)|^2}{|x-y|^{n-\alpha}} dy \right) u - \mu |u|^{p-2} u - |u|^{2^*-2} u.$$

Notice that

$$Q'(u) = -2\Delta u - \frac{p(n+\alpha) - 4n}{4 - n + \alpha} \mu |u|^{p-2} u - 2^* |u|^{*-2} u.$$

It follows from $u \in M$ that

$$\begin{aligned} &\langle Q'(u), u\rangle_{(X^{1,\alpha})^{-1}, X^{1,\alpha}} \\ &= 2 \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{p(n+\alpha) - 4n}{4 - n + \alpha} \mu \int_{\mathbb{R}^n} |u|^p dx - 2^* \int_{\mathbb{R}^n} |u|^{2^*} dx - pQ(u) \\ &= (2 - p) \int_{\mathbb{R}^n} |\nabla u|^2 dx - (2^* - p) \int_{\mathbb{R}^n} |u|^{2^*} dx \\ &< 0. \end{aligned}$$

Finally, since u solves (1.7), there exists a Lagrange multiplier λ such that

$$J'(u) = \lambda Q'(u).$$

Thus,

$$0 = \langle J'(u), u\rangle_{(X^{1,\alpha})^{-1}, X^{1,\alpha}} = \lambda \langle Q'(u), u\rangle_{(X^{1,\alpha})^{-1}, X^{1,\alpha}}.$$

Noting that $\langle Q'(u), u\rangle_{(X^{1,\alpha})^{-1}, X^{1,\alpha}} < 0$, we obtain $\lambda = 0$. Consequently, $J'(u) = 0$, which means that u solves Eq. (1.1).

Step 3. Conclusion. Consider

$$l = \min\{J(u) : u \in A\}. \tag{4.7}$$

Let $u \in G$ be such that $J(u) = l$. From *Step* 2 in Lemma 4.1, we have $u \in M$. So $J(u) \geqslant \widetilde{d}$. In particular

$$l = J(u) \geqslant \widetilde{d}. \tag{4.8}$$

Consider now a solution v of (1.7). Since $J(v) = \widetilde{d}$ and $v \in A$ (by *Step* 2), it follows from (4.7) that $\widetilde{d} \geqslant l$. Combining with (4.8), we obtain $\widetilde{d} = l$. The equivalence of the two problems follows easily. This completes the proof. □

Proof of Theorem 1.4 Similar to the Step 1 in the proof of Theorem 1.3, we can prove that $\{u_m\}$ is bounded in $X^{1,\alpha}_{\text{rad}}$. Since the embedding

$$X^{1,\alpha}_{\text{rad}} \hookrightarrow L^{2^*}_{\text{loc}}(\mathbb{R}^n)$$

is compact, we are able to overcome the lack of compactness by the method of Lemma 3.3. Thus, the same result as in Theorem 1.3 can be obtained.

Acknowledgements C. Lei is supported by Science and Technology Foundation of Guizhou Province (No.ZK[2022]199)). B. Zhang was supported by National Natural Science Foundation of China (No. 11871199 and No. 12171152) and Cultivation Project of Young and Innovative Talents in Universities of Shandong Province. The research of Vicenţiu D. Rădulescu was supported by a grant of the Romanian Ministry of Research, Innovation and Digitization, CNCS/CCCDI-UEFISCDI, project number PCE 137/2021, within PNCDI III.

Data Availability The manuscript has no associated data.

References

1. Ambrosetti, A., Ruiz, D.: Multiple bounded states for Schrödinger–Poisson problem. Commun. Contemp. Math. **10**, 391–404 (2008)
2. Bellazzini, J., Frank, R.L., Visciglia, N.: Maximizers for Gagliardo–Nirenberg inequalities and related non-local problems. Math. Ann. **360**, 653–673 (2014)
3. Bellazzini, J., Ghimenti, M., Mercuri, C., Moroz, V., Schaftingen, J.V.: Sharp Gagliardo–Nirenberg inequalities in fractional Coulomb–Sobolev spaces. Trans. Am. Math. Soc. **370**, 285–8310 (2018)
4. Bellazzini, J., Ghimenti, M., Ozawa, T.: Sharp lower bounds for Coulomb energy. Math. Res. Lett. **23**, 621–632 (2016)
5. Benci, V., Fortunato, D.: An eigenvalue problem for the Schrödinger–Maxwell equations. Top. Methods Nonlinear Anal. **11**, 283–293 (1998)
6. Bokanowski, O., López, J.L., Soler, J.: On an exchange interaction model for the quantum transport; the Schrödinger–Poisson–Slater term, M^3AS **13**, 1397–1412 (2003)
7. Brézis, H., Lieb, E.: A relation between pointwise convergence of functions and convergence of functionals. Proc. Am. Math. Soc. **88**, 486–490 (1983)
8. Brézis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent. Commun. Pure. Appl. Math. **36**, 437–477 (1983)
9. Chen, S., Tang, X.H.: On the planar Schrödinger–Poisson system with the axially symmetric potential. J. Differ. Equ. **268**, 945–976 (2020)
10. Chen, S., Fiscella, A., Pucci, P., Tang, X.: Semiclassical ground state solutions for critical Schrödinger–Poisson systems with lower perturbations. J. Differ. Equ. **268**, 2672–2716 (2020)
11. D'Aprile, T., Wei, J.: On bound states concentrating on spheres for the Maxwell–Schrödinger equation. SIAM J. Math. Anal. **37**, 321–342 (2005)
12. Gamow, G.: Zur Quantentheorie des Atomkernes. Z. Phys. **51**(204), 204–212 (1928)
13. Georgiev, V., Prinari, F., Visciglia, N.: On the radiality of constrained minimizers to the Schrödinger–Poisson–Slater energy. Ann. Inst. H. Poincaré Anal. Non Linéaire **29**, 369–376 (2012)
14. Ianni, I., Ruiz, D.: Ground and bound states for a static Schrödinger–Poisson–Slater problem. Commun. Contemp. Math. **14**, 1250003 (2012)
15. Ji, C., Zhang, Y., Rădulescu, V.: Multi-bump solutions for the magnetic Schrödinger–Poisson system with critical growth. Electron. J. Qual. Theory Differ. Equ. **21**, 1–30 (2022)

16. Lei, C., Lei, Y.: On existence of ground states of an equation of the Schrödinger–Poisson–Slater type. Comptes Rendus Math. **359**, 219–227 (2021)
17. Li, Y., Zhang, B., Han, X.: Existence and concentration behavior of positive solutions to Schrödinger–Poisson–Slater equations. Adv. Nonlinear Anal. **12**, 20220293 (2023)
18. Lieb, E.: Coherent states as a tool for obtaining rigorous bounds. In: Feng, D. H., Klauder, J. & Strayer, M. R., (eds.) Proceedings of the Symposium on Coherent States, Past, Present and Future, Oak Ridge. World Scientific, Singapore, pp. 267–278 (1994)
19. Lions, P.L.: Some remarks on Hartree equation. Nonlinear Anal. **5**, 1245–1256 (1981)
20. Lions, P.L.: Solutions of Hartree–Fock equations for Coulomb systems. Commun. Math. Phys. **109**, 33–97 (1987)
21. Lions, P. L.: The concentration-compactness principle in the calculus of variations. The locally compact case. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**, 109–145 (1985) and **1**, 223–283 (1985)
22. Lions, P. L.: The concentration-compactness principle in the calculus of variations. The limit case. Rev. Mat. Iberoamericana **1**, 145–201 (1985) and **2**, 45–121 (1985)
23. Liu, J., Ji, C.: Concentration results for a magnetic Schrödinger–Poisson system with critical growth. Adv. Nonlinear Anal. **27**, 775–798 (2020)
24. Liu, Z., Zhang, Z., Huang, S.: Existence and nonexistence of positive solutions for a static Schrödinger–Poisson–Slater equation. J. Differ. Equ. **266**, 5912–5941 (2019)
25. Ruiz, D., Vaira, G.: Cluster solutions for the Schrödinger–Poinsson–Slater problem around a local minimum of potential. Rev. Mat. Iberoamericana **27**, 253–271 (2011)
26. Ruiz, D.: On the Schrödinger–Poisson–Slater system: Behavior of minimizers, radial and nonradial cases. Arch. Ration. Mech. Anal. **198**, 349–368 (2010)
27. Siciliano, G.: Multiple positive solutions for a Schrödinger–Poisson–Slater system. J. Math. Anal. Appl. **365**, 288–299 (2010)
28. Slater, J.C.: A simplification of the Hartree–Fock method. Phys. Rev. **81**, 385–390 (1951)
29. Tang, X., Chen, S.: Ground state solutions of Nehari–Pohozaev type for Schrödinger–Poisson problems with general potentials. Discret. Contin. Dyn. Syst. **37**, 4973–5002 (2017)
30. Willem, M.: Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications, vol. 24. Birkäuser, Boston (1996)
31. Zhang, J., Xu, Z., Zou, W.: Standing waves for nonlinear Schrödinger equations involving critical growth. J. Lond. Math. Soc. **90**, 827–844 (2014)