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Results in Mathematics

Global Existence and Multiplicity for Nonlinear Robin Eigenvalue Problems

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Abstract. We consider a parametric problem driven by the p-Laplacian with Robin boundary condition. We assume that the reaction can change sign and we prove an existence and multiplicity theorem which is global with respect to the parameter (a bifurcation-type theorem).

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1. Introduction

Bifurcation phenomena arise in many parts of mathematical physics and an understanding of their nature is of practical as well as theoretical importance. Bifurcation theory aims to explain a diversity of natural phenomena that have been observed and characterized over the years. For instance, the buckling of the Euler rod, the appearance of Taylor vortices, and the emergence of perturbations in an electric circuit, all have the same cause: a physical parameter crosses a threshold, pressuring the system to assemble itself into a new state that differs significantly from the previous state. Here we refer to the pioneering global bifurcation results established by Crandall and Rabinowitz [4] and Rabinowitz [18]. We also refer to the seminal papers by Brezis et al. [2], Brezis and Vázquez [3], and Garcia Azorero et al. [6] in the framework of bifurcation problems with Dirichlet boundary condition.

Brezis et al. [2] established the existence of an "extreme value" λ_* of the bifurcation parameter λ such that a large class of problems with convex and increasing nonlinearity has a smooth positive solution for all $0 < \lambda < \lambda_*$, but no solution exists if $\lambda > \lambda_*$. On the other hand, Garcia Azorero et al. [6]

proved that for all $\lambda < \lambda_*$ there are at least two solutions. The analysis carried out in [6] is developed in the case of competition phenomena of convex and concave nonlinearities.

The analysis developed in this paper corresponds to a logistic equation with reaction of super-diffusive type. Indeed, if $f(z,x) = f(x) = (x^+)^{\tau-1} - (x^+)^{r-1}$ with $p < \tau < r$, which is the prototype super-diffusive reaction, then this $f(\cdot)$ function satisfies our hypotheses H_1 (see Sect. 2). Logistic equations are important in models of mathematical biology and describe the steady state of a biological population in the presence of constant rates of reproduction and mortality (see Gurtin and MacCamy [9]). Such equations were studied by Afrouzi and Brown [1] (semilinear equidiffusive problems) and Takeuchi [19] (nonlinear problems). The wotk of Takeuchi [19] revealed that super-diffusive equations exhibit bifurcation phenomena for large values of the parameter λ. This is in contrast with the situation described above for equations with the competing effects of concave and convex nonlinearities.

Motivated by the above mentioned pioneering contributions, we develop in this paper an *exhaustive bifurcation analysis* in the framework of a general *Robin boundary condition*. To the best of our knowledge, this is the first analysis carried out for logistic equations with super-diffusive reaction and nonlinear Robin boundary condition.

1.1. Statement of the Problem

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear parametric Robin problem (eigenvalue problem)

$$\begin{cases} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = \lambda f(z,u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega,\ u \geqslant 0,\ \lambda > 0. \end{cases} \tag{P_λ}$$

In this problem, Δ_p $(1 < p < \infty)$ denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega).$$

Also, $\lambda > 0$ is a parameter and $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the function $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, the mapping $x \mapsto f(z,x)$ is continuous) which exhibits $(p-1)$-sublinear growth as $x \to +\infty$. Our conditions are general and incorporate as a special case, the so-called superdiffusive reaction of the p-logistic equation.

Our aim is to prove an existence and multiplicity result for the positive solutions of (P_λ), which is global in the parameter $\lambda > 0$ (a "bifurcation-type" result). In this way we can have a precise picture of the set of positive solutions of (P_λ) as the parameter λ moves in the open half-axis $\mathring{\mathbb{R}}_+ = (0,+\infty)$. Our work here complements the recent one by Gasiński-Papageorgiou [7] where a related problem was studied when $f(z,\cdot)$ is $(p-1)$-superlinear. We will see that the situation is different. Our main result here will show that the "bifurcation" occurs at large values of the parameter, while in Gasiński-Papageorgiou [7] with the superlinear reaction, the "bifurcation" occurs at small values of $\lambda > 0$. We should also mention the work of Papageorgiou-Qin-Rădulescu [13], where the authors examine an analogous eigenvalue problem for nonlinear Robin equations driven by the (p, q)-Laplacian. The emphasis there is on the existence of nodal (sign-changing) solutions and consequently the tools and methods used are different. Finally, we also refer to the recent works of [21–23] for the qualitative and asymptotic analysis of solutions to double phase problems.

We mention that in the boundary condition, $\dfrac{\partial u}{\partial n_p}$ denotes the conormal derivative of $u \in W^{1,p}(\Omega)$ corresponding to the p-Laplacian. This is interpreted using the nonlinear Green's identity (see Papageorgiou-Rădulescu-Repovš [16, p.35]). So, according to this identity, if $u \in C^1(\bar{\Omega})$, then exists a unique element

$$\frac{\partial u}{\partial n_p} \in W^{-1/p',p'}(\partial\Omega) \; \left(\frac{1}{p} + \frac{1}{p'} = 1\right),$$

which by extension we denote by

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\mathbb{R}^N} = |Du|^{p-2}\frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Our goal is to study the existence, nonexistence and multiplicity of positive solutions as the parameter $\lambda > 0$ varies. More precisely, we prove a bifurcation-type result for large values of the parameter $\lambda > 0$ (bifurcation near $+\infty$). So, we establish the existence of a critical parameter value $\lambda_* > 0$ such that for every $\lambda > \lambda_*$ problem (P_λ) admits at least two positive solutions, when for $\lambda = \lambda_*$ problem (P_λ) has at least one positive solution and finally for all $\lambda \in (0, \lambda_*)$ problem (P_λ) has no positive solutions.

An important role in our analysis is played by the regularity theory of Lieberman [10], who established regularity up to the boundary (global regularity) for solutions of equations driven by a broad class of nonhomogeneous differential operators, which includes as a special case the p-Laplacian. The results of Lieberman [10] extend local regularity results of DiBenedetto [5] and Tolksdorf [20].

2. Mathematical Background and Hypotheses

The main spaces used in the analysis of problem (P_λ) are the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\bar{\Omega})$ and the boundary Lebesgue spaces $L^\tau(\partial\Omega)$, $1 \leqslant \tau \leqslant \infty$.

With $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ given by

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\bar{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C^1(\bar{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \bar{\Omega}\right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Then using $\sigma(\cdot)$ we can define in the usual way the "boundary" Lebesgue spaces $L^\tau(\partial\Omega)$, $1 \leqslant \tau \leqslant \infty$. Recall that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}).$$

The trace map extends the notion of boundary values to all Sobolev functions. The trace map is compact into $L^\tau(\partial\Omega)$ for all $\tau \in [1, \frac{(N-1)p}{N-p})$ if $p < N$ and into $L^\tau(\partial\Omega)$ for all $\tau \in [1, \infty)$ if $N \leqslant p$. In the sequel, for the sake of simplicity, we drop the use of $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

We will also use another open cone in $C^1(\bar{\Omega})$, namely

$$D_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}.$$

Consider the operator $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z \text{ for all } u, h \in W^{1,p}(\Omega).$$

This operator has the following properties (see, Gasiński-Papageorgiou [8, p.279]).

Proposition 1. *The operator $A(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

$$\text{“} u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0 \Rightarrow u_n \to u \text{ in } W^{1,p}(\Omega)\text{''}.$$

If $u : \Omega \to \mathbb{R}$ is a measurable function, then $u^\pm(z) = \max\{\pm u(z), 0\}$. If $u \in W^{1,p}(\Omega)$, then $u^\pm \in W^{1,p}(\Omega)$. Also, we have

$$u = u^+ - u^- \text{ and } |u| = u^+ + u^-.$$

If X is a Banach space and $\varphi \in C^1(X)$, then we say that $\varphi(\cdot)$ satisfies the "C-condition", if it has the following property:

> "Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that
> $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^*,
> admits a strongly convergent subsequence.''

By K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Also, p^* denotes the Sobolev critical exponent corresponding to p. So, we have

$$p^* = \begin{cases} \frac{Np}{N-p}, & \text{if } p < N, \\ +\infty, & \text{if } N \leqslant p. \end{cases}$$

Now we introduce our hypotheses on the potential function $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$.

H_0 : $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\tau}(\partial\Omega)$ with $0 < \tau < 1$, $\xi \geqslant 0$, $\beta \geqslant 0$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Remark 1. These hypotheses incorporate in our framework the Neumann problem (case $\beta \equiv 0$).

Let $\gamma : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p \mathrm{d}z + \int_{\partial\Omega} \beta(z)|u|^p \mathrm{d}\sigma$$

for all $u \in W^{1,p}(\Omega)$.

Using Mugnai-Papageorgiou [11, Lemma 4.11] (case $\beta \equiv 0$) and Gasiński-Papageorgiou [7, Lemma2.4] (case $\xi \equiv 0$), we have

$$\gamma(u) \geqslant c_0\|u\|^p \text{ for some } c_0 > 0, \text{all } u \in W^{1,p}(\Omega). \tag{1}$$

We consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u + \xi(z)|u|^{p-2}u = \widehat{\lambda}|u|^{p-2}u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem has a smallest eigenvalue $\widehat{\lambda}_1 > 0$ which is isolated, simple and

$$\widehat{\lambda}_1 = \inf\left\{\frac{\gamma(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0\right\}. \tag{2}$$

The infimum in (2) is realized on the corresponding one-dimensional eigenspace. For details, see Papageorgiou-Rădulescu [14].

Finally, we mention that by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Our conditions on the reaction function $f(z,x)$ are the following:

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leqslant a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)$, $p \leqslant r < p^*$;

(ii) $\limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leqslant 0$ uniformly for a.a. $z \in \Omega$;

(iii) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(iv) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$,

$$x \mapsto f(z,x) + \widehat{\xi}_\rho |x|^{p-1}$$

is nondecreasing on $[0,\rho]$ and there exists $\tau > p$ such that

$$y - x \geqslant s > 0 \Rightarrow \frac{f(z,x)}{x^{\tau-1}} - \frac{f(z,y)}{y^{\tau-1}} \geqslant \widetilde{\eta}_s > 0 \text{ for a.a. } z \in \Omega;$$

(v) there exists $\tilde{u} \in L^r(\Omega)$ such that $\int_\Omega F(z,\tilde{u})\mathrm{d}z > 0$ with $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$.

Remark 2. Since we look for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$. Evidently these conditions incorporate the case of a $(p-1)$-sublinear reaction which is sign-changing.

Example 1. The following function satisfies hypotheses H_1. For the sake of simplicity, we drop the z-dependence

$$f(x) = \begin{cases} (x^+)^{r-1}, & \text{if } x \leqslant 1 \\ x^{s-1}\ln x + x^{\tau-1}, & \text{if } 1 < x \end{cases} \text{ with } 1 < s \leqslant p < \tau < r.$$

The main result of this work is the following global bifurcation-type theorem for problem (P_λ).

Theorem 1. *If hypotheses H_0 and H_1 hold, then there exists $\lambda_* > 0$ such that*

(a) *for all $\lambda > \lambda_*$, problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in \operatorname{int} C_+$, $u_0 \neq \hat{u}$;*
(b) *for $\lambda = \lambda_*$, problem (P_λ) has at least one positive solution $u_* \in \operatorname{int} C_+$;*
(c) *for all $\lambda \in (0,\lambda_*)$, problem (P_λ) has no positive solutions.*

3. Positive Solutions

We introduce the following two sets:

$$\mathscr{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a positive solution}\},$$

and let S_λ be the set of positive solutions of (P_λ).

Also, let $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\gamma(u) - \int_\Omega \lambda F(z,u^+)\mathrm{d}z \text{ for all } u \in W^{1,p}(\Omega).$$

Evidently, $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$.

Proposition 2. *If hypotheses H_0 and H_1 hold, then $\mathscr{L} \neq \emptyset$ and for every $\lambda > 0$ in $\mathscr{L}$, $S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. On account of hypothesis H_1-(iii), given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\epsilon}{p}x^p \text{ for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta. \tag{3}$$

Let $u \in C^1(\bar{\Omega})$ with $\|u\|_{C^1(\bar{\Omega})} \leqslant \delta$. Then

$$\begin{aligned} \varphi_\lambda(u) &= \frac{1}{p}\gamma(u) - \int_\Omega \lambda F(z,u^+)\mathrm{d}z \\ &\geqslant \frac{1}{p}\gamma(u) - \frac{\lambda\epsilon}{p}\|u^+\|_p^p \quad (\text{see}(3)) \\ &\geqslant \frac{1}{p}\left[c_0 - \frac{\lambda\epsilon}{\hat{\lambda}_1}\right]\|u\|^p, \quad (\text{see}(1),(2)). \end{aligned}$$

If $\epsilon \in (0, \frac{\widehat{\lambda}_1 c_0}{\lambda})$, then we have

$$\begin{aligned} &\varphi_\lambda(u) > 0 \text{ for all } u \in C^1(\bar{\Omega}), 0 < \|u\|_{C^1(\bar{\Omega})} \leqslant \delta, \\ &\quad \Rightarrow \; u = 0 \text{ is a local } C^1(\bar{\Omega})\text{-minimizer of } \varphi_\lambda(\cdot), \\ &\quad \Rightarrow \; u = 0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda(\cdot), \end{aligned}$$

(see Papageorgiou-Rădulescu [15, Proposition 2.12]).

We may assume that K_{φ_λ} is finite or otherwise we already have an infinity of positive solutions of (P_λ) and so we are done. Then using [16, Theorem 5.7.6, p.449], we can find $\rho \in (0,1)$ small such that

$$\varphi_\lambda(0) = 0 < \inf\left\{\varphi_\lambda(u) : \|u\| = \rho\right\} = m_\lambda. \tag{4}$$

Hypothesis H_1-(i) implies that the integral functional

$$u \mapsto \int_\Omega F(z,u)\mathrm{d}z$$

is continuous on $L^r(\Omega)$. From hypothesis H_1-(v), we have

$$\int_\Omega F(z,\tilde{u})\mathrm{d}z > 0.$$

Then the continuity of the integral functional and the density of $W^{1,p}(\Omega)$ in $L^r(\Omega)$ imply that we can find $\bar{u} \in W^{1,p}(\Omega)$, $\bar{u} \geqslant 0$ (recall $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$) such that

$$\int_\Omega F(z,\bar{u})\mathrm{d}z > 0.$$

So, choosing $\lambda > 0$ big, we have

$$\varphi_\lambda(\bar{u}) < 0. \tag{5}$$

Hypotheses H_1-(i) and H_1-(ii) imply that given $\epsilon > 0$, we can find $c_\epsilon > 0$ such that

$$F(z,x) \leqslant \frac{\epsilon}{p}x^p + c_\epsilon \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{6}$$

Then for $u \in W^{1,p}(\Omega)$, we have

$$\varphi_\lambda(u) \geqslant \frac{1}{p}\left[c_0 - \frac{\lambda\epsilon}{\widehat{\lambda}_1}\right]\|u\|^p - c_\epsilon|\Omega|_N \text{ (see(1), (2), (6))}.$$

Choosing $\epsilon \in (0, \frac{\widehat{\lambda}_1 c_0}{\lambda})$, we see that $\varphi_\lambda(\cdot)$ is coercive. Hence by [16, Proposition 5.1.15, p.369], we have that

$$\varphi_\lambda(\cdot) \text{ satisfies the } C\text{-condition.} \tag{7}$$

We can always take $\rho < \|\bar{u}\|$. Then from (4), (5), (7) and recalling that $\rho < \|\bar{u}\|$, we can apply the mountain pass theorem and find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$u_\lambda \in K_{\varphi_\lambda} \text{ and } m_\lambda \leqslant \varphi_\lambda(u_\lambda) \text{ (see(4))}. \tag{8}$$

From (4) and (8), we see that $u_\lambda \neq 0$ and we have

$$\langle A(u_\lambda), h\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h \mathrm{d}z + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda \mathrm{d}\sigma = \int_\Omega \lambda f(z, u_\lambda^+)h\mathrm{d}z. \tag{9}$$

for all $h \in W^{1,p}(\Omega)$.

In (9), we use the test function $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\gamma(u_\lambda^-) = 0,\\
\Rightarrow\ & c_0\|u_\lambda^-\| \leqslant 0 \quad \text{(see(1))}\\
\Rightarrow\ & u_\lambda \geqslant 0, u_\lambda \neq 0 \quad \text{(recall that } \lambda > 0 \text{ is big)}\\
\Rightarrow\ & \mathscr{L} \neq \emptyset.
\end{aligned}$$

From Proposition 2.10 of Papageorgiou-Rădulescu [15], we have that $u_\lambda \in L^\infty(\Omega)$. Then Theorem 2 of Lieberman [10] implies that $u_\lambda \in C_+\backslash\{0\}$. Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis H_1-(iv). We have

$$\begin{aligned}
&-\Delta_p u_\lambda + \left[\xi(z) + \widehat{\xi}_\rho\right]u_\lambda^{p-1} \geqslant 0 \text{ in } \Omega,\\
&\Rightarrow\ u_\lambda \in \operatorname{int} C_+ \quad \text{(by the maximum principle, see [8, p.841])}.
\end{aligned}$$

Therefore, we conclude that for all $\lambda > 0$, $S_\lambda \subseteq \operatorname{int} C_+$. □

Next, we establish a structural property for the set $\mathscr{L}$. We show that $\mathscr{L}$ is an upper half line.

Proposition 3. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathscr{L}$ and $\theta > \lambda$, then $\theta \in \mathscr{L}$.*

Proof. Since $\lambda \in \mathscr{L}$, we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ (see Proposition 2). Let $t \in (0, 1)$ be such that $t^{\tau-p}\theta = \lambda$ (recall that by hypothesis H_1-(iv), $\tau > p$). We have

$$\begin{aligned}
&-\Delta_p u_\lambda + \xi(z)u_\lambda^{p-1} = \lambda f(z, u_\lambda) = t^{\tau-p}\theta f(z, u_\lambda),\\
&\Rightarrow\ t^{p-1}\left[-\Delta_p u_\lambda + \xi(z)u_\lambda^{p-1}\right] = t^{\tau-1}\theta f(z, u_\lambda).
\end{aligned} \tag{10}$$

We set $\underline{u}_\lambda = tu_\lambda \in \operatorname{int} C_+$. On account of hypothesis H_1-(iv), we have

$$-\Delta_p \underline{u}_\lambda + \xi(z)\underline{u}_\lambda^{p-1} \leqslant \theta f(z, \underline{u}_\lambda),\ \frac{\partial \underline{u}_\lambda}{\partial n_p} + \beta(z)\underline{u}_\lambda^{p-1} = 0. \tag{11}$$

We introduce the Carathéodory function $g(z,x)$ defined by

$$g(z,x) = \begin{cases} f(z, \underline{u}_\lambda(z)), & \text{if } x \leqslant \underline{u}_\lambda(z), \\ f(z,x), & \text{if } \underline{u}_\lambda(z) < x. \end{cases} \tag{12}$$

Let $G(z,x) = \int_0^x g(z,s)\mathrm{d}s$ and consider the C^1-functional $\psi_\theta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\theta(u) = \frac{1}{p}\gamma(u) - \int_\Omega \theta G(z,u)\mathrm{d}z \text{ for all } u \in W^{1,p}(\Omega).$$

As we did for $\varphi_\lambda(\cdot)$ in the proof of Proposition 2, using hypotheses H_1-(i), H_1-(ii) and (12), we show that $\psi_\theta(\cdot)$ is coercive. Also, by the Sobolev embedding theorem and the compactness of the trace map, we see that $\psi_\theta(\cdot)$ is sequentially weakly lower semicontinuous. Hence by the Weierstrass-Tonelli theorem, we can find $u_\theta \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \psi_\theta(u_\theta) &= \inf\left\{\psi_\theta(u) : u \in W^{1,p}(\Omega)\right\}, \\ &\Rightarrow\ \psi_\theta'(u_\theta) = 0, \\ &\Rightarrow\ \langle A(u_\theta), h\rangle + \int_\Omega \xi(z)|u_\theta|^{p-2}u_\theta h\mathrm{d}z + \int_{\partial\Omega} \beta(z)|u_\theta|^{p-2}u_\theta h\mathrm{d}\sigma, \\ &= \int_\Omega \theta g(z,u_\theta)h\mathrm{d}z \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{13}$$

In (13), we choose $h = (\underline{u}_\lambda - u_\theta)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(u_\theta), (\underline{u}_\lambda - u_\theta)^+\rangle + \int_\Omega \xi(z)|u_\theta|^{p-2}u_\theta(\underline{u}_\lambda - u_\theta)^+\mathrm{d}z \\ &\quad + \int_{\partial\Omega} \beta(z)|u_\theta|^{p-2}u_\theta(\underline{u}_\lambda - u_\theta)^+\mathrm{d}\sigma \\ &= \int_\Omega \theta f(z, \underline{u}_\lambda)(\underline{u}_\lambda - u_\theta)^+\mathrm{d}z \quad (\text{see}(12)) \\ &\geqslant \langle A(\underline{u}_\lambda), (\underline{u}_\lambda - u_\theta)^+\rangle + \int_\Omega \xi(z)\underline{u}_\lambda^{p-1}(\underline{u}_\lambda - u_\theta)^+\mathrm{d}z \\ &\quad + \int_{\partial\Omega} \beta(z)\underline{u}_\lambda^{p-1}(\underline{u}_\lambda - u_\theta)^+\mathrm{d}\sigma(\text{see}(11)) \\ &\Rightarrow\ \underline{u}_\lambda \leqslant u_\theta \quad (\text{see Proposition (1)}\ and\, recall\, that\, \xi \geqslant 0, \beta \geqslant 0), \\ &\Rightarrow\ u_\theta \in S_\theta \subseteq \operatorname{int} C_+(\text{see}(12),(13)). \end{aligned}$$

So, $\theta \in \mathscr{L}$. □

We set $\lambda_* = \inf \mathscr{L}$.

Proposition 4. *If hypotheses H_0 and H_1 hold, then $\lambda_* > 0$.*

Proof. Arguing by contradiction, suppose that $\lambda_* = 0$. Consider $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathscr{L}$ such that $\lambda_n \downarrow 0$ and let $u_n = u_{\lambda_n} \in S_{\lambda_n} \subseteq \operatorname{int} C_+$, $n \in \mathbb{N}$. We have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h \mathrm{d}z + \int_{\partial\Omega} \beta(z) u_n^{p-1} h \mathrm{d}\sigma = \lambda_n \int_\Omega f(z, u_n) h \mathrm{d}z, \tag{14}$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$. Hypotheses H_1-(i) and H_1-(ii) imply that given $\epsilon > 0$, we can find $c_\epsilon > 0$ such that

$$f(z, x) \leqslant \epsilon x^{p-1} + c_\epsilon \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{15}$$

In (14), we choose the test function $h = u_n \in W^{1,p}(\Omega)$ and then use (15), we obtain

$$\gamma(u_n) \leqslant \lambda_n \left[\epsilon \|u_n\|^p + c_\epsilon |\Omega|_N\right] \text{ for all } n \in \mathbb{N},$$
$$\Rightarrow \left[c_0 - \lambda_1 \epsilon\right] \|u_n\|^p \leqslant c_\epsilon |\Omega|_N \quad (\text{since } \lambda_n \leqslant \lambda_1 \text{ for all } n \in \mathbb{N}).$$

Choosing $\epsilon \in (0, \frac{c_0}{\lambda_1})$, we infer that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

Proposition 2.10 of Papageorgiou-Rădulescu [15] implies that

$$u_n \in L^\infty(\Omega) \text{ and } \|u_n\|_\infty \leqslant c_1 \text{ for some } c_1 > 0, \text{ all } n \in \mathbb{N}.$$

Then Theorem 2 of Lieberman [10] says that there exist $\alpha \in (0, 1)$ and $c_2 > 0$ such that

$$u_n \in C^{1,\alpha}(\bar\Omega), \|u_n\|_{C^{1,\alpha}(\bar\Omega)} \leqslant c_2 \text{ for all } n \in \mathbb{N}.$$

We know that $C^{1,\alpha}(\bar\Omega) \hookrightarrow C^1(\bar\Omega)$ compactly. So by passing to a subsequence if necessary, we can say that

$$u_n \to u_* \text{ in } C^1(\bar\Omega). \tag{16}$$

If in (14) we pass to the limit as $n \to \infty$ and use (16), then

$$\langle A(u_*), h\rangle + \int_\Omega \xi(z) u_*^{p-1} h \mathrm{d}z + \int_{\partial\Omega} \beta(z) u_*^{p-1} h \mathrm{d}\sigma = 0,$$

for all $h \in W^{1,p}(\Omega)$ (recall that $\lambda_n \downarrow 0$).

Choosing $h = u_* \in W^{1,p}(\Omega)$, we have

$$\gamma(u_*) = 0,$$
$$\Rightarrow c_0 \|u_*\|^p \leqslant 0 \text{ (see(1)) and so } u_* = 0.$$

Therefore, we have

$$u_n \to 0 \text{ in } C^1(\bar\Omega) \text{ as } n \to \infty \quad (\text{see } (16)). \tag{17}$$

According to hypothesis H_1-(iii), given $\epsilon > 0$, we can find $\delta = \delta_\epsilon > 0$ such that

$$f(z,x) \leqslant \epsilon x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta. \tag{18}$$

Then (17) and (18) imply that we can find $n_0 \in \mathbb{N}$ such that

$$0 < \lambda_n \leqslant 1 \text{ and } f(z,u_n(z))u_n(z) \leqslant \epsilon u_n(z)^p \text{ for a.a. } z \in \Omega, \text{ all } n \geqslant n_0. \tag{19}$$

In (14) we use the test function $h = u_n \in W^{1,p}(\Omega)$, we have

$$\begin{aligned} &\gamma(u_n) \leqslant \lambda_n \int_\Omega f(z,u_n)u_n \mathrm{d}z \leqslant \epsilon \|u_n\|^p \text{ for all } n \geqslant n_0, \quad (\text{see}(19)), \\ &\quad \Rightarrow \ [c_0 - \epsilon]\|u_n\|^p \leqslant 0 \text{ for all } n \geqslant n_0. \end{aligned}$$

Choosing $\epsilon \in (0, c_0)$, we see that $u_n = 0$ for all $n \geqslant n_0$, a contradiction. Therefore, $\lambda_* > 0$. □

So far, we have

$$(\lambda_*, \infty) \subseteq \mathscr{L} \subseteq [\lambda_*, \infty). \tag{20}$$

Now we show that for $\lambda > \lambda_*$, we have multiplicity of positive solutions.

Proposition 5. *If hypotheses H_0 and H_1 hold, and $\lambda > \lambda_*$, then problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in int C_+$, $u_0 \neq \hat{u}$.*

Proof. Let $\eta \in (\lambda_*, \lambda)$, then $\eta \in \mathscr{L}$ (see (20)) and so we can find $u_\eta \in S_\eta \subseteq \operatorname{int} C_+$. Reasoning as in the proof of Proposition 3, let $t \in (0,1)$ be such that $t^{\tau-p}\lambda = \eta$. We set $\underline{u}_\eta = tu_\eta \in \operatorname{int} C_+$, we have

$$\begin{aligned} &-\Delta_p \underline{u}_\eta + \xi(z)\underline{u}_\eta^{p-1} \\ &\quad = t^{p-1}\left[-\Delta_p u_\eta + \xi(z)u_\eta^{p-1}\right] \\ &\quad = t^{p-1}\eta f(z,u_\eta) \ \ (\text{recall that } u_\eta \in S_\eta) \\ &\quad = t^{p-1}t^{\tau-p}\lambda f(z,u_\eta) \ \ (\text{recall that } \eta = t^{\tau-p}\lambda) \\ &\quad \leqslant \lambda f(z,\underline{u}_\eta) \ \ (\text{see hypothesis } H_1\text{-(iv) and recall } t \in (0,1)). \end{aligned} \tag{21}$$

We introduce the Carathéodory functions $k(z,x)$ defined by defined by

$$k(z,x) = \begin{cases} f(z,\underline{u}_\eta(z)), & \text{if } x \leqslant \underline{u}_\eta(z), \\ f(z,x), & \text{if } \underline{u}_\eta(z) < x. \end{cases} \tag{22}$$

We set $K(z,x) = \int_0^x k(z,s)\mathrm{d}s$ and consider the C^1-functional $\widehat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\gamma(u) - \int_\Omega \lambda K(z,u)\mathrm{d}z \text{ for all } u \in W^{1,p}(\Omega).$$

As before, using (1), (22) and hypothesis H_1-(ii), we infer that $\widehat{\varphi}_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\widehat{\varphi}_\lambda(u_0) = \inf\left[\widehat{\varphi}_\lambda(u) : u \in W^{1,p}(\Omega)\right], \\ &\quad \Rightarrow \langle \widehat{\varphi}'_\lambda(u_0), h\rangle = 0 \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{23}$$

Choosing $h = (\underline{u}_\eta - u_0)^+ \in W^{1,p}(\Omega)$ in (23) and arguing as in the proof of Proposition 3, using (22) and (21), we obtain

$$\begin{aligned} &\underline{u}_\eta \leqslant u_0, \\ &\Rightarrow u_0 \in S_\lambda \subseteq \operatorname{int} C_+ \quad (\text{see}(24), (22) and (23)). \end{aligned} \tag{24}$$

Let $\rho = \|u\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis H_1-(iv). We have

$$\begin{aligned} &-\Delta_p \underline{u}_\eta + [\xi(z) + \lambda\widehat{\xi}_\rho]\underline{u}_\eta^{p-1} \\ &\quad \leqslant \lambda f(z, \underline{u}_\eta) + \lambda\widehat{\xi}_\rho \underline{u}_\eta^{p-1} \quad (\text{see}(21)) \\ &\quad \leqslant \lambda f(z, u_0) + \lambda\widehat{\xi}_\rho u_0^{p-1} \quad (\text{see}(24) \text{ and hypothesis } H_1\text{-(iv)}) \\ &\quad = -\Delta_p u_0 + [\xi(z) + \lambda\widehat{\xi}_\rho]u_0^{p-1} \quad (\text{since } u_0 \in S_\lambda). \end{aligned} \tag{25}$$

Let $s = \min_{\bar{\Omega}}(1-t)u_\eta > 0$ (recall $u_\eta \in \operatorname{int} C_+$). Since $t \in (0,1)$, by hypothesis H_1-(iv), we have

$$\frac{f(z, \underline{u}_\eta)}{t^{\tau-1}} - f(z, u_\eta) \geqslant \widehat{\eta}_s > 0. \tag{26}$$

From (25), (26) and Proposition A4 of Papageorgiou-Rădulescu-Zhang [17], we infer that

$$u_0 - \underline{u}_\eta \in D_+. \tag{27}$$

Let

$$[\underline{u}_\eta) = \left\{u \in W^{1,p}(\Omega) : \underline{u}_\eta(z) \leqslant u(z) \text{ for a.a. } z \in \Omega\right\}.$$

From (22), we see that

$$\varphi_\lambda|_{[\underline{u}_\eta)} = \widehat{\varphi}_\lambda|_{[u_\eta)} + \mu_0 \text{ with } \mu_0 \in \mathbb{R}. \tag{28}$$

From (28) and (27), we see that

$$\begin{aligned} &u_0 \text{ is a local } C^1(\bar{\Omega})\text{-minimizer of } \varphi_\lambda, \\ &\quad \Rightarrow u_0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda, \end{aligned} \tag{29}$$

(see Papageorgiou-Rădulescu)[8, p.2.12].

Also, from the proof of Proposition 2, we know that

$$u = 0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda. \tag{30}$$

We may assume that

$$\varphi_\lambda(0) = 0 \leqslant \varphi_\lambda(u_0). \tag{31}$$

The analysis is similar if the opposite inequality holds using this time (30) instead of (29).

The nonlinear regularity theory and the nonlinear maximum principle imply that

$$K_{\varphi_\lambda} \subseteq \operatorname{int} C_+ \cup \{0\}.$$

So, we may assume that K_{φ_λ} is finite or otherwise we already have an infinity of positive smooth solutions of (P_λ) and so we are done. Using the finiteness of K_{φ_λ}, (29) and [16, Theorem 5.7.6, p.449], we see that we can find $\rho \in (0, 1)$ small such that

$$\varphi_\lambda(0) = 0 \leqslant \varphi_\lambda(u_0) < \inf\left\{\varphi_\lambda(u) : \|u - u_0\| = \rho\right\} = m_\lambda,\ \|u_0\| > \rho. \tag{32}$$

Recall that φ_λ is coercive (see the proof of Proposition 2). Hence by Proposition 5.1.15 of [16, p.369], we have

$$\varphi_\lambda(\cdot) \text{ satisfies the } C\text{-condition.} \tag{33}$$

Then (31), (32) and the mountain pass theorem imply that there exists $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{u} \in K_{\varphi_\lambda}, m_\lambda \leqslant \varphi_\lambda(\hat{u}), \\ \Rightarrow\ &\hat{u} \in S_\lambda \subseteq \operatorname{int} C_+, \hat{u} \neq u_0 \quad \text{(see (32))}. \end{aligned}$$

The proof is now complete. □

It remains to check what can be said about the critical parameter value $\lambda_* > 0$.

Proposition 6. *If hypotheses H_0 and H_1 hold, then $\lambda_* \in \mathscr{L}$.*

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathscr{L}$ such that $\lambda_n \downarrow \lambda_*$. Let $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$, $n \in \mathbb{N}$. As in the proof Proposition 4, we show that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

From this and the nonlinear regularity theory (see the proof of Proposition 4), imply that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq C^1(\bar{\Omega}) \text{ is relatively compact.}$$

So, we may assume that

$$u_n \to u_* \text{ in } C^1(\bar{\Omega}) \text{ as } n \to \infty.$$

If $u_* = 0$, then following the argument in the proof of Proposition 4 and using hypothesis H_1-(iii) (see (18)), we reach a contradiction. Therefore $u_* \neq 0$ and

$$\langle A(u_*), h\rangle + \int_\Omega \xi(z) u_*^{p-1} h \mathrm{d}z + \int_{\partial\Omega} \beta(z) u_*^{p-1} h \mathrm{d}\sigma = \lambda_* \int_\Omega f(z, u_*) h \mathrm{d}z$$
$$\text{for all } h \in W^{1,p}(\Omega) \quad \text{(see Proposition (1))},$$
$$\Rightarrow u_* \in S_{\lambda_*} \subseteq \operatorname{int} C_+ \text{ and } \lambda_* \in \mathscr{L}.$$

This completes the proof. □

Finally, we can say that

$$\mathscr{L} = [\lambda_*, +\infty)$$

and we have completed the proof of the Theorem.

Remark 3. Reviewing the proofs of the propositions, we see that we have used repeatedly the $(p-1)$-homogeneity of the p-Laplacian. So, our approach here cannot be used to problems driven by a nonhomogeneous differential operator. So, it is an interesting open problem whether the global multiplicity result of this paper can be extended to anisotropic equations or to double phase equations (see [12]). Clearly, a different approach is needed.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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