OPEN ACCESS
Bulletin of Mathematical Sciences
(2023) 2350012 (56 pages)

DOI: 10.1142/S1664360723500121

Multiplicity of positive solutions for the fractional Schrödinger–Poisson system with critical nonlocal term

Xilin Dou* and Xiaoming He†

College of Science, Minzu University of China
Beijing 100081, P. R. China
*xldou816@163.com
†xmhe923@muc.edu.cn

Vicenţiu D. Rădulescu‡

Faculty of Applied Mathematics, AGH University of Kraków
al. Mickiewicza 30, 30-059 Kraków, Poland

Faculty of Electrical Engineering and Communication
Brno University of Technology Technická 3058/10
Brno 61600, Czech Republic

Department of Mathematics, University of Craiova
Street A.I. Cuza No. 13, 200585 Craiova, Romania

Simion Stoilow Institute of Mathematics of the Romanian Academy, Calea Griviţei No. 21, 010702
Bucharest, Romania

School of Mathematics, Zhejiang Normal University
Jinhua, Zhejiang 321004, P. R. China
radulescu@inf.ucv.ro

Received 10 March 2023
Revised 31 July 2023
Accepted 10 August 2023
Published 12 September 2023

Communicated by Neil Trudinger

This paper deals with the following fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u + u - K(x)\phi|u|^{2_s^*-3}u = f_\lambda(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = K(x)|u|^{2_s^*-1}, & x \in \mathbb{R}^3 \end{cases}$$

‡Corresponding author.

with multiple competing potentials and a critical nonlocal term, where $s \in (0,1)$, $q \in (1,2)$ or $q \in (4, 2_s^*)$, and $2_s^* = \frac{6}{3-2s}$ is the fractional critical exponent. By combining the Nehari manifold analysis and the Ljusternik–Schnirelmann category theory, we establish how the coefficient K of the nonlocal critical nonlinearity affects the number of positive solutions. We propose a new relation between the number of positive solutions and the category of the global maximal set of K.

Keywords: Positive solutions; critical nonlocal term; fractional Schrödinger–Poisson system; Ljusternik–Schnirelmann category; variational method.

Mathematics Subject Classification 2020: 35J62, 35J50, 35B65

1. Introduction

The classical Schrödinger–Poisson-type system takes the form

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

This system arises in quantum mechanics and semiconductor theory and it was introduced by Benci–Fortunato [7], where the unknowns u and ϕ represent the wave functions associated with the particle and the electric potentials, while V is an exterior potential and K denotes a nonnegative density charge. The nonlinearity f simulates the interaction effect among many particles. The Schrödinger equation coupled with a Poisson equation is usually used to interpret the phenomenon that a quantum particle interacts with an electromagnetic field [8, 29]. We refer to [1, 7, 10, 12] for more details on the physical background.

It is easily seen that system (1.1) can be transformed into a nonlinear Schrödinger equation with a nonlocal term, for example, see [7, 12]. Briefly, the Poisson equation can be solved by using the Lax–Milgram theorem. For all $u \in H^1(\mathbb{R}^3)$, the unique $\phi_{K,u} \in D^{1,2}(\mathbb{R}^3)$ is given by

$$\phi_{K,u}(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)|u(y)|^2}{|x-y|}dy,$$

which solves equation $-\Delta\phi_{K,u} = K(x)u^2$, and that it can be substituted into the first equation of system (1.1) to obtain that

$$-\Delta u + V(x)u + K(x)\phi_{K,u}u = f(x,u), \quad x \in \mathbb{R}^3.$$

Such equation is variational, and its solutions are critical points of the corresponding energy functional I defined in $H^1(\mathbb{R}^3)$.

In view of this, system (1.1) was extensively investigated in both bounded and unbounded domains under various assumptions on potentials and nonlinearities. Some profound results on the existence of solutions of system (1.1) have been presented in [1, 4, 6, 10, 36, 39, 46, 50] and references therein. We notice that, in

[26, 27], Li, Li and Shi considered positive solutions to the following Schrödinger–Poisson-type system with critical nonlocal term:

$$\begin{cases} -\Delta u + bu + q\phi|u|^3u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

and the existence of positive solutions to (1.2) was obtained by using variational method which does not require usual compactness conditions. Since then, several existence results have been established for the Schrödinger–Poisson systems with critical nonlocal term in the literature. In [32], Liu studied the following generalized Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)|u|^5, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

By using the mountain pass theorem and the concentration-compactness principle, Liu obtained the existence of a positive solution for (1.3). Feng [16] studied the existence of positive solutions to (1.3) with the critical nonlinearity $f(x,u) = |u|^4u + g(u)$, by the modified concentration-compactness principle and Nehari manifold method. Li and He [25] studied the existence and multiplicity of positive solutions for (1.3) and related the number of positive solutions with the topology of the set of the minimum value points of potential $V(x)$, by using the Ljusternik–Schnirelmann theory. Yin, Zhang and Shang [46] studied the existence of positive ground state solution to Eq. (1.3) with vanishing potentials by means of variational approach. Azzollini and d'Avenia1 [5] proved the existence and nonexistence results of positive solutions to a Schrödinger–Poisson system with a critical nonlocal term set on a bounded domain in both the resonance and the nonresonance cases for higher dimensions.

In the setting of the fractional Laplacian, system (1.1) becomes the following fractional Schrödinger–Poisson-type system:

$$\begin{cases} (-\Delta)^s u + V(x)u + K(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t\phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.4}$$

System (1.4) is a fundamental equation in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes [9, 21, 22]. Recently, there is an increasing interesting in the study of solutions for fractional Schrödinger–Poisson systems and we quote some existence results available in the literature. In [48], by using a perturbation approach, Zhang, do Ó and Squassina considered the existence and the asymptotical behaviors of positive solutions to system (1.4) with $V(x) = 0$ and $K(x) = \lambda > 0$, a parameter, and a general subcritical or critical nonlinearity f. Teng [42] analyzed the existence of ground state solutions of (1.4) with $K(x) = 1$ and $f(x,u) = \mu|u|^{q-1}u + |u|^{2_s^*-2}u, q \in (2, 2_s^*)$, by combining Pohozaev–Nehari manifold, arguments of Brezis–Nirenberg type, the monotonicity trick and global compactness lemma. Fan, Feng and Yan [15] studied the existence and multiple solutions for (1.4) via variational methods. Murcia and Siciliano [37] studied the semiclassical state of the following system:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + K(x)\phi u = f(u), & x \in \mathbb{R}^N, \\ \varepsilon^{\theta}(-\Delta)^{\alpha/2}\phi = \gamma_\alpha u^2, & x \in \mathbb{R}^N. \end{cases} \tag{1.5}$$

and established the multiplicity of positive solutions that concentrate on the minima of V by the Ljusternik–Schnirelmann category theory. For more results on multiplicity and concentration of positive solutions of (1.5), we refer to [33, 42, 45, 47] and references therein.

We notice that in the above-mentioned works for the fractional Schrödinger–Poisson systems, the second Poisson equation is subcritical growth. After a bibliography review we find that there are only few papers that deal with fractional Schrödinger–Poisson system with critical nonlocal term. In [19], the second author of this paper studied the existence of ground state solution of the following fractional Schrödinger–Poisson equations:

$$\begin{cases} (-\Delta)^s u + V(x)u - K(x)\phi|u|^{2_s^*-3}u = f(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^s\phi = K(x)|u|^{2_s^*-1}, & x \in \mathbb{R}^3 \end{cases} \tag{1.6}$$

with the nonlinearity $f(x,u) = u^{2_s^*-1} + h(u)$, where h is subcritical growth, in this case, system (1.6) is called doubly critical growth. In [17], Feng studied the existence of nonnegative solutions of (1.6), by employing the mountain pass theorem, concentration-compactness principle and approximation method, and extended the main results of [32] to the fractional Laplacian case.

When $K(x) \equiv 0$, (1.6) simplifies to the following fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^3. \tag{1.7}$$

In [40], Secchi constructed solutions to (1.7) by variational approach in nature, and based on minimization on the Nehari manifold. do Ó, Miyagaki and Squassina [13] investigated (1.7) with critical power nonlinearity $f(x,u) = K(x)f(u) + \lambda|u|^{2_s^*-2}u$, and the involved potentials are allowed for vanishing behavior at infinity. For more existence results for problem (1.7), we refer to [13, 35, 40] and references therein.

Motivated by the aforementioned works, in this paper we are concerned with multiplicity of solutions for the fractional Schrödinger-Poisson system with multiple competing potentials and a critical nonlocal term

$$\begin{cases} (-\Delta)^s u + u - K(x)\phi|u|^{2_s^*-3}u = f_\lambda(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^s\phi = K(x)|u|^{2_s^*-1}, & x \in \mathbb{R}^3, \end{cases} \tag{1.8}$$

where $s \in (0,1)$, $q \in (1,2)$ or $q \in (4, 2_s^*)$. The function $f_\lambda(x)$ is defined by $f_\lambda(x) = \lambda f_+(x) + f_-(x)$, where $\lambda > 0$ is a small parameter and $f_\pm(x) = \pm\max\{\pm f(x), 0\}$.

Before stating the main results, we introduce some assumptions on the continuous functions $f_\lambda(x)$, $K(x)$ as follows:

(H_1) $f_\lambda(x) \in L^{q^*}(\mathbb{R}^3)$, where $q^* = \frac{2_s^*}{2_s^*-q}$.
(H_2) $\lim_{x\to\infty} K(x) = K_\infty \in (0,+\infty)$ and $K(x) \geq K_\infty$ for all $x \in \mathbb{R}^3$.
(H_3) There exist a non-empty compact set $M := \{z \in \mathbb{R}^3; K(z) = \max_{x\in\mathbb{R}^3} K(x) = 1\}$ and a positive number $\rho \in [3-2s, 3)$ such that $K(z) - K(x) = O(|x-z|^\rho)$ as $x \to z$ and uniformly in $z \in M$.
(H_4) $f_\lambda(x) > 0$ for $x \in M$.

Remark 1.1. Let $M_r = \{x \in \mathbb{R}^3;\ \mathrm{dist}(x, M) < r\}$ for $r > 0$. Then according to the conditions (H_3)–(H_4), it follows from the continuity of f_λ and g that there exist positive constants $C_0, r_0 > 0$ such that

$$f_\lambda(x) > 0, \quad x \in M_{r_0} \subset \mathbb{R}^3$$

and

$$K(z) - K(x) \leq C_0|x-z|^\rho \quad \text{for } x \in B_{r_0}(z)$$

uniformly for $z \in M$, where $B_{r_0}(z) = \{x \in \mathbb{R}^3;\ |x-z| < r_0\}$. Furthermore, given the conditions (H_2)–(H_3), it is easy to see that $K_\infty < 1$.

Theorem 1.1. *Let $\frac{1}{2} < s < 1, 1 < q < 2$ and assume that conditions (H_1)–(H_4) are satisfied. Then for each $0 < \delta < r_0$, there exists $\lambda_\delta > 0$ such that if $\lambda \in (0, \lambda_\delta)$, system (1.8) possesses at least $\mathrm{cat}_{M_\delta}(M) + 1$ distinct positive solutions, where cat denotes the Ljusternik–Schnirelmann category.*

In the case $4 < q < 2_s^*$, we use the following condition on f_λ instead of condition (H_1):

$(H_1)'$ $f_\lambda(x) \equiv \lambda f(x)$ and $\lim_{|x|\to\infty} f(x) = f_\infty \in (0,+\infty)$ and $f(x) \geq f_\infty$, $\forall\, x \in \mathbb{R}^3$.

Theorem 1.2. *Let $3/4 < s < 1, 4 < q < 2_s^*$ and assume that conditions $(H_1)', (H_2)$ and (H_3) are satisfied. Then for each $0 < \delta < r_0$, there exists $\lambda_\delta > 0$ such that if $\lambda \in (0, \lambda_\delta)$, system (1.8) possesses at least $\mathrm{cat}_{M_\delta}(M)$ distinct positive solutions.*

Remark 1.2. If we replace $(H_1)'$ by condition (H_1), the conclusion of Theorem 1.2 is still valid. In fact, condition (H_1) implies that $\lim_{|x|\to\infty} f_\lambda(x) = 0$, and it becomes more easier to obtain the compactness of the (PS) sequence of the associated energy function, see the proof in Lemma 2.8. The rest of the proofs is very similar to the proofs of Theorem 1.2.

We summarize the main ingredients and innovations in the proofs of Theorems 1.1 and 1.2 as follows:

(1) Different from the work mentioned in [2, 3, 20, 23, 25, 28, 33, 37, 45, 49], we aim to establish a relationship between the number of positive solutions and the topology of the weighted potential $K(x)$ (not $V(x)$) in the critical nonlocal term. In the proof of Theorem 1.1, one of the main difficulties is to find the minimum of the

associated energy functional constrained on the Nehari manifold. This is because the variational methods used in the literature, such as [20, 25, 32–34, 37, 38, 45, 48], are no longer applicable to the case of growth order $1 < q < 2$. In order to overcome these difficulties, we need to introduce new auxiliary functionals to compare their energy levels to that of the energy functional of system (1.8), and more analytical technologies are involved. As far as we known, the multiplicity of solutions to (1.8) has not been studied in the literature, which is where our purpose and interest are focused.

(2) Since the appearance of a nonlocal critical convolution term in the Schrödinger–Poisson system, it would be natural to consider how the interaction between the nonlocal term and the nonlinear term will affect the existence and concentration of solutions of (1.8). Moreover, when we use the condition $(H_1)'$ instead of (H_1), the resultant difficulties in the proof of Theorem 1.2 become harder to prove the compactness of the (PS) sequence of the associated energy functional. For this reason, we propose a new analytical technique to derive an exact estimate of the (PS) sequence, see Lemma 5.6.

The remainder part of this paper is organized as follows. In Sec. 2, we recall some relevant preliminary results for fractional Sobolev spaces and present the variational setting of the problem, and properties of Nehari manifolds. In Sec. 3, we provide some useful estimates that play a key role in the proof of Theorem 1.1. In Sec. 4, we construct a barycenter map and prove Theorem 1.1 by means of the Ljusternik–Schnirelmann category theory. Section 5 is devoted to the proof of Theorem 1.2.

2. Preliminary Results

In this section, we present some preliminary results on fractional Sobolev spaces; we refer to [35] for more details. For convenience, we use $\to$ (respectively, $\rightharpoonup$) to represent strong (respectively, weak) convergence. We will use $C, C_0, C_1, C_2, \ldots$, to represent various normal numbers, and employ $L^r(\mathbb{R}^N)$, $1 < r < \infty$ to denote the usual vector-valued function of Lebesgue space with the usual L^r norm in $\mathbb{R}^N$, denoted by $\|\cdot\|_r$.

For any $s \in (0,1)$, the fractional space $H^s(\mathbb{R}^3)$ is defined by the completion of $C_c^\infty(\mathbb{R}^3)$ under the norm

$$\|u\| := \|u\|_{H^s} = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^3} |u|^2 dx\right)^{\frac{1}{2}}.$$

Set the homogeneous Sobolev space $D^{s,2}(\mathbb{R}^3)$ as the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|^2_{D^{s,2}} := \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy.$$

The fractional Sobolev best constant is given by

$$S = \inf_{u\in D^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2}{2_s^*}}}, \tag{2.1}$$

and it is attained by the function

$$U_{\varepsilon,z}(x) = \frac{C_s \varepsilon^{\frac{3-2s}{2}}}{\left(\varepsilon^2 + |x-z|^2\right)^{\frac{2-2s}{2}}}, \tag{2.2}$$

for $\varepsilon > 0, z \in \mathbb{R}^3$ and a normalized constant $C_s > 0$. Moreover

$$\|U_{\varepsilon,z}\|^2_{D^{s,2}} = \int_{\mathbb{R}^3} |U_{\varepsilon,z}|^{2_s^*} dx = S^{\frac{3}{2s}}.$$

Next, we recall the following embedding result.

Lemma 2.1 ([35]). *The embedding $H^s(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for any $r \in [2, 2_s^*]$, and is locally compact whenever $r \in [1, 2_s^*)$.*

For each $u \in H^s(\mathbb{R}^3)$, we define the linear operator $L_u : D^{s,2}(\mathbb{R}^3) \to \mathbb{R}$ by

$$L_u(v) = \int_{\mathbb{R}^3} K(x)|u|^{2_s^*-1} v dx.$$

Using the Hölder inequality and the Sobolev inequality, there holds

$$|L_u(v)| \le \max_{x\in\mathbb{R}^3} K(x) \|u\|^{2_s^*-1}_{2_s^*} \|v\|_{2_s^*} \le \|u\|^{2_s^*-1}_{2_s^*} S^{-\frac{1}{2}} \|v\|_{D^{s,2}}, \tag{2.3}$$

which implies that L_u is continuous. Then, by the Lax–Milgram theorem, for given $u \in H^s(\mathbb{R}^3)$, there exists a unique solution $\phi_u \in D^{s,2}(\mathbb{R}^3)$ solving $(-\Delta)^s \phi_u = K(x)|u|^{2_s^*-1}$ in a weak sense. From [14], the function ϕ_u is represented by

$$\phi_u(x) = k_s \int_{\mathbb{R}^3} \frac{K(y)|u(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dy, \quad x \in \mathbb{R}^3, \tag{2.4}$$

where $k_s := \frac{\Gamma(\frac{3-2s}{2})}{2^{2s}\pi^{\frac{3}{2}}\Gamma(s)}$, and ϕ_u has the following properties.

Lemma 2.2. *For each $u \in H^s(\mathbb{R}^3)$, the following properties hold*:

(i) *$\phi_u : H^s(\mathbb{R}^3) \to D^{s,2}(\mathbb{R}^3)$ is continuous and maps bounded sets into bounded sets.*

(ii) *For each $u \in H^s(\mathbb{R}^3)$, one has $\|\phi_u\|_{D^{s,2}} \le S^{-\frac{1}{2}} \|u\|^{2_s^*-1}_{2_s^*}$ and*

$$\int_{\mathbb{R}^3} K(x)\phi_u |u|^{2_s^*-1} dx \le \max_{x\in\mathbb{R}^3} K(x) S^{-\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2_s^*-1}{2_s^*}} \|\phi_u\|_{D^{s,2}}$$
$$\le S^{-2_s^*} \|u\|^{2(2_s^*-1)}_{D^{s,2}}.$$

(iii) *$\phi_{tu} = t^{2_s^*-1}\phi_u$ for all $t \ge 0$.*

(iv) *If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, and $u_n \to u$ a.e. in $\mathbb{R}^3$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\mathbb{R}^3)$; and $\phi_{u_n} - \phi_{u_n-u} - \phi_u \to 0$ in $D^{s,2}(\mathbb{R}^3)$.*

(v) *If* $u_n \to u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi_{u_n} \to \phi_u$ *in* $D^{s,2}(\mathbb{R}^3)$, *and* $\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1} dx \to \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx$.

(vi) *If* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$ *and* $u_n \to u$ *a.e. in* $\mathbb{R}^3$, *then*

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_{u_n-u}|u_n-u|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \to 0.$$

Proof. It is only need to check that (i), (ii), (iv) and (vi) hold.

(i) By the definition of ϕ_u, we have for all $v \in D^{s,2}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}\phi_u(-\Delta)^{\frac{s}{2}}v dx = \int_{\mathbb{R}^3} K(x)|u|^{2_s^*-1}v dx.$$

Thus, by Riesz representation theorem, one has

$$\|L_u\|_{\mathcal{L}(D^{s,2}(\mathbb{R}^3));\mathbb{R})} = \|\phi_u\|_{D^{s,2}(\mathbb{R}^3)}, \quad \forall\, u \in H^s(\mathbb{R}^3).$$

In addition, from (2.3) we get

$$\begin{aligned}
\|\phi_u\|_{D^{s,2}(\mathbb{R}^3)} &= \|L_u\|_{\mathcal{L}(D^{s,2}(\mathbb{R}^3));\mathbb{R})} \\
&= \sup_{v\in D^{s,2}(\mathbb{R}^3);\|v\|_{D^{s,2}}\le 1} |L_u(v)| \\
&\le S^{-\frac{1}{2}}\left(\int_{\mathbb{R}^3}|u|^{2_s^*}dx\right)^{\frac{2_s^*-1}{2_s^*}}\|v\|_{D^{s,2}} \\
&\le S^{-\frac{2_s^*}{2}}\|u\|_{D^{s,2}}^{2_s^*-1} \le S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*-1},
\end{aligned} \tag{2.5}$$

where $\mathcal{L}(D^{s,2}(\mathbb{R}^3);\mathbb{R})$ denotes the set of bounded linear operators from $D^{s,2}(\mathbb{R}^3)$ to $\mathbb{R}$, and hence ϕ_u maps bounded sets into bounded sets. In order to prove the continuity of ϕ_u, we let $\{u_n\} \subset H^s(\mathbb{R}^3)$ being such that $u_n \to u$ in $H^s(\mathbb{R}^3)$. Then $|u_n|^{\frac{2_s^*}{2_s^*-1}} \to |u|^{\frac{2_s^*}{2_s^*-1}}$ in $L^{2_s^*}(\mathbb{R}^3)$. By (2.1) we have, as $n \to \infty$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^3}(-\Delta)^s(\phi_{u_n}-\phi_u)v dx\right| &= |L_{u_n}(v) - L_u(v)| \\
&\le \max_{x\in\mathbb{R}^3} K(x)\int_{\mathbb{R}^3}||u_n|^{2_s^*-1}-|u|^{2_s^*-1}||v|dx \\
&\le S^{-\frac{1}{2}}\||u_n|^{2_s^*-1}-|u|^{2_s^*-1}\|_{\frac{2_s^*}{2_s^*-1}}\|v\|_{D^{s,2}(\mathbb{R}^3)} \to 0.
\end{aligned}$$

Since v is arbitrary, we obtain that $\lim_{n\to\infty}\|\phi_{u_n}-\phi_u\|_{D^{s,2}(\mathbb{R}^3)} = 0$.

(ii) Using Hölder inequality and (2.1), we deduce to

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\phi_u|^2 dx = \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx$$

$$\leq \max_{x\in\mathbb{R}^3} K(x) \left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2_s^*-1}{2_s^*}} \left(\int_{\mathbb{R}^3} |\phi_u|^{2_s^*} dx\right)^{\frac{1}{2_s^*}}$$

$$\leq S^{-\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2_s^*-1}{2_s^*}} \|\phi_u\|_{D^{s,2}}, \tag{2.6}$$

and so

$$\|\phi_u\|_{D^{s,2}} \leq S^{-\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2_s^*-1}{2_s^*}} = S^{-\frac{1}{2}} \|u\|_{2_s^*}^{2_s^*-1}. \tag{2.7}$$

Thus, by (2.6)–(2.7) we have

$$\int_{\mathbb{R}^3} K(x)\phi_u |u|^{2_s^*-1} dx \leq S^{-\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2_s^*-1}{2_s^*}} \|\phi_u\|_{D^{s,2}}$$

$$\leq S^{-1} \|u\|_{2_s^*}^{2(2_s^*-1)} \leq S^{-2_s^*} \|u\|_{D^{s,2}}^{2(2_s^*-1)}. \tag{2.8}$$

(iv) From the Sobolev embedding, one has $u_n \rightharpoonup u \in L^{2_s^*}(\mathbb{R}^3)$ and then $|u_n|^{2_s^*-1} \rightharpoonup |u|^{2_s^*-1}$ in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$. Using [26, Lemma 3.2], we have

$$|u_n|^{2_s^*-1} - |u_n - u|^{2_s^*-1} - |u|^{2_s^*-1} \to 0 \quad \text{in } L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3). \tag{2.9}$$

Therefore, for any $v \in D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$, we get

$$(\phi_{u_n}, v) = \int_{\mathbb{R}^3} K(x)|u_n|^{2_s^*-1} v dx \to \int_{\mathbb{R}^3} K(x)|u|^{2_s^*-1} v dx = (\phi_u, v),$$

which reveals that $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\mathbb{R}^3)$. Since for every $w \in D^{s,2}(\mathbb{R}^3)$.

$$|(\phi_{u_n} - \phi_{v_n} - \phi_u, w)| = \left|\int_{\mathbb{R}^3} K(x)(|u_n|^{2_s^*-1} - |v_n|^{2_s^*-1} - |u|^{2_s^*-1}) w dx\right|$$

$$\leq \max_{x\in\mathbb{R}^3} K(x) \|w\|_{2_s^*} \cdot \| |u_n|^{2_s^*-1} - |v_n|^{2_s^*-1} - |u|^{2_s^*-1} \|_{2_s^*/(2_s^*-1)},$$

then

$$\phi_{u_n} - \phi_{u_n - u} - \phi_u \to 0 \quad \text{in } D^{s,2}(\mathbb{R}^3),$$

which implies the assertion.

(vi) Since $u_n \rightharpoonup u$ in $L^{2_s^*}(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then $v_n \rightharpoonup 0$ in $L^{2_s^*}(\mathbb{R}^3)$ and $v_n \to 0$ a.e. in $\mathbb{R}^3$. By item (iv) we have $\phi_{v_n} \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^3)$. Thus, as $n \to \infty$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_{v_n}|v_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \\
&\quad = \int_{\mathbb{R}^3} K(x)[\phi_{u_n} - \phi_{v_n} - \phi_u]|u_n|^{2_s^*-1}dx \\
&\qquad + \int_{\mathbb{R}^3} K(x)\phi_{v_n}[|u_n|^{2_s^*-1} - |u_n - u|^{2_s^*-1} - |u|^{2_s^*-1}]dx \\
&\qquad + \int_{\mathbb{R}^3} K(x)\phi_{v_n}|u|^{2_s^*-1}dx + \int_{\mathbb{R}^3} K(x)\phi_u[|u_n|^{2_s^*-1} - |u|^{2_s^*-1}]dx \to 0. \qquad \square
\end{aligned}$$

Remark 2.1. When $K(x) \equiv 1$, we recharge (2.4) as

$$\widetilde{\phi}_u(x) = k_s \int_{\mathbb{R}^3} \frac{|u(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dy, \quad x \in \mathbb{R}^3,$$

then the conclusions of Lemma 2.2 still hold, see [19, Lemma 2.1].

The energy functional associated with the system (1.8) is defined by

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}}u|^2 + |u|^2)dx \\
&\quad - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx - \frac{1}{q}\int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx, \qquad (2.10)
\end{aligned}$$

where $u \in H^s(\mathbb{R}^N)$. It is easy to check that $I_\lambda \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$, and on account of I_λ being not bounded from below on $H^s(\mathbb{R}^N)$, we consider the behavior of I_λ on the Nehari manifold as follows:

$$N_\lambda := \{u \in H^s(\mathbb{R}^3)\backslash\{0\}; I'_\lambda(u)u = 0\}.$$

Then $u \in N_\lambda$ if and only if

$$\|u\|^2 - \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx - \int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx = 0. \qquad (2.11)$$

We denote by

$$\Psi_\lambda(u) := \|u\|^2 - \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx - \int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx. \qquad (2.12)$$

Then for $u \in N_\lambda$, we have

$$\begin{aligned}
\Psi'_\lambda(u)u &= 2\|u\|^2 - 2(2_s^*-1)\int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx - q\int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx \\
&= [2 - 2(2_s^*-1)]\|u\|^2 + [2(2_s^*-1) - q]\int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx \\
&= (2-q)\|u\|^2 - [2(2_s^*-1) - q]\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx, \qquad (2.13)
\end{aligned}$$

and the Nehari manifold N_λ can be split into the following three parts:

$$N_\lambda^+ = \{u \in N_\lambda : \Psi'_\lambda(u)u > 0\}, \quad N_\lambda^0 = \{u \in N_\lambda : \Psi'_\lambda(u)u = 0\},$$

and

$$N_\lambda^- = \{u \in N_\lambda : \Psi'_\lambda(u)u < 0\}.$$

Lemma 2.3. *I_λ is coercive and bounded from below on N_λ.*

Proof. Let $u \in N_\lambda$. By Hölder and Young inequalities, it follows from Lemma 2.1 and (2.10) that

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{2(2_s^*-1)} I'_\lambda(u)u \\ &= \left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\|u\|^2 - \left(\frac{1}{q} - \frac{1}{2(2_s^*-1)}\right)\int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\|u\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{2(2_s^*-1)}\right) C\|f_+\|_{q^*} S^{-\frac{q}{2}}\|u\|^q \\ &\geq \left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\|u\|^2 - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\|u\|^2 - C\lambda^{\frac{2}{2-q}} \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\|u\|^2 - C\lambda^{\frac{2}{2-q}}, \end{aligned} \tag{2.14}$$

where C is a positive constant independent of the choice of $u \in H^s(\mathbb{R}^3)$ and $\lambda > 0$. □

Lemma 2.4. *Assume that u_0 is a critical point for I_λ on N_λ and $u_0 \notin N_\lambda^0$. Then we have $I'_\lambda(u_0) = 0$.*

Proof. Since $u_0 \in N_\lambda$, there holds $I'_\lambda(u_0)u_0 = 0$. By the principle of Lagrange multiplier, there exists $\theta \in \mathbb{R}$ such that $I'_\lambda(u_0) = \theta\Psi'_\lambda(u_0)$. Thus,

$$0 = I'_\lambda(u_0)u_0 = \theta\Psi'_\lambda(u_0)u_0.$$

As $u_0 \notin N_\lambda^0$, we infer to $\theta = 0$ and so $I'_\lambda(u_0) = 0$. □

Lemma 2.5. *There exists $\Lambda_1 > 0$ such that $N_\lambda^0 = \emptyset$ for $\lambda \in (0, \Lambda_1)$.*

Proof. Suppose by contradiction that there exists a $u \in N_\lambda^0$. It follows from (2.12) and Lemmas 2.1 and 2.2 that

$$\begin{aligned} [2(2_s^*-1) - 2]\|u\|^2 &= [2(2_s^*-1) - q]\int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx \\ &\leq \lambda[2(2_s^*-1) - q]\|f_+\|_{q^*} S^{-\frac{q}{2}}\|u\|^q, \end{aligned}$$

and

$$(2-q)\|u\|^2 = [2(2_s^*-1)-q]\int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx$$
$$\leq [2(2_s^*-1)-q]S^{-2_s^*}\|u\|^{2(2_s^*-1)}.$$

Consequently, we have

$$C_1 \leq \|u\| \leq \lambda^{\frac{1}{2-q}} C_2,$$

where C_1, $C_2 > 0$ are independent of the choice of $u \in H^s(\mathbb{R}^3)$ and $\lambda > 0$, which implies a contradiction, when λ is small enough. □

By Lemma 2.5, we have $N_\lambda = N_\lambda^+ \cup N_\lambda^-$. Define

$$\alpha_\lambda^+ := \inf_{u\in N_\lambda^+} I_\lambda(u) \quad \text{and} \quad \alpha_\lambda^- := \inf_{u\in N_\lambda^-} I_\lambda(u).$$

Lemma 2.6. *The following two assertions hold true*:

(i) $\alpha_\lambda^+ < 0$.
(ii) *There exists* $\Lambda_2 \in (0, \Lambda_1)$ *such that* $\alpha_\lambda^- > d_0$ *for some* $d_0 > 0$ *and* $\lambda \in (0, \Lambda_2)$. *Moreover,* $\alpha_\lambda^+ = \inf_{u\in N_\lambda} I_\lambda(u)$ *for* $\lambda \in (0, \Lambda_2)$.

Proof. (i) For every $u \in N_\lambda^+$, it follows (2.13) that

$$(2-q)\|u\|^2 > [2(2_s^*-1)-q]\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx,$$

from which we have

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{q}I'_\lambda(u)u \\ &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 + \left(\frac{1}{q}-\frac{1}{2(2_s^*-1)}\right)\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \\ &= \frac{q-2}{2q}\|u\|^2 + \frac{2(2_s^*-1)-q}{2(2_s^*-1)q}\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \\ &< -\frac{(2_s^*-2)[2(2_s^*-1)-q]}{2(2_s^*-1)q}\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx < 0, \end{aligned}$$

and thus

$$\alpha_\lambda^+ < 0.$$

(ii) For any $u \in N_\lambda^-$, it follows (2.13) that

$$(2-q)\|u\|^2 < [2(2_s^*-1)-q]\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \leq [2(2_s^*-1)-q]S^{-2_s^*}\|u\|^{2(2_s^*-1)}$$

and

$$\|u\| \geq \left[\frac{2-q}{2(2_s^*-1)-q}S^{2_s^*}\right]^{\frac{1}{2(2_s^*-2)}}. \tag{2.15}$$

Combining (2.14) and (2.15), we infer to

$$\begin{aligned}
I_\lambda(u) &\geq \left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\|u\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{2(2_s^*-1)}\right) C\|f_+\|_{q^*} S^{\frac{q}{2}}\|u\|^q \\
&= \|u\|^q \left[\left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\|u\|^{2-q} - \lambda\left(\frac{1}{q} - \frac{1}{2(2_s^*-1)}\right) C\|f_+\|_{q^*} S^{\frac{q}{2}}\right] \\
&\geq d_0 > 0,
\end{aligned}$$

for small $\lambda > 0$ and some constant $d_0 > 0$ independent of the choice of $u \in N_\lambda^-$. □

For each $u \in H^s(\mathbb{R}^3)\backslash\{0\}$, we set

$$h(t) := t^{2-q}\|u\|^2 - t^{2(2_s^*-1)-q}\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx$$

for $t \geq 0$. Then $h(0) = 0$, $h(t) > 0$ for small $t > 0$ small, and $h(t) \to -\infty$ as $t \to \infty$. There exists a unique $t_{\max} > 0$ such that

$$h(t_{\max}) = \sup_{x\geq 0} h(t) > 0.$$

It is easy to see that $h(t)$ is increasing on $(0, t_{\max})$ and decreasing on $(t_{\max}, \infty)$.

Lemma 2.7. *For each $u \in H^s(\mathbb{R}^3)$,, there exists $\Lambda_3 \in (0, \Lambda_2)$ such that if $\lambda \in (0, \Lambda_3)$, the following two assertions are true*:

(i) *If $\int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx \leq 0$, then there is a unique $t^- = t^-(u) > t_{\max}$ such that $t^-u \in N_\lambda^-$ and $I_\lambda(tu)$ is increasing on $(0, t^-)$ and decreasing on (t^-, ∞). Moreover, $I_\lambda(t^-u) = \sup_{t\geq 0} I_\lambda(tu)$.*

(ii) *If $\int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max} < t^-(u) = t^-$ such that $t^-u \in N_\lambda^-$, $t^+u \in N_\lambda^+$, $I_\lambda(tu)$ is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^-, ∞). Moreover, $I_\lambda(t^+u) = \inf_{0\leq t\leq t_{\max}} I_\lambda(tu)$; $I_\lambda(t^-u) = \sup_{t\geq t^+} I_\lambda(tu)$.*

Proof. For each $u \in H^s(\mathbb{R}^3)\backslash\{0\}$, by Lemma 2.2(ii) we infer that

$$\begin{aligned}
h(t_{\max}) &= \max_{t\geq 0}\left(t^{2-q}\|u\|^2 - t^{2(2_s^*-1)-q}\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx\right) \\
&= \left(\frac{(2-q)\|u\|^2}{(2(2_s^*-1)-q)\int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx}\right)^{\frac{2-q}{2(2_s^*-2)}}\|u\|^2 \\
&\quad - \left(\frac{(2-q)\|u\|^2}{(2(2_s^*-1)-q)\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx}\right)^{\frac{2(2_s^*-1)-q}{2(2_s^*-2)}}\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx
\end{aligned}$$

$$
\begin{aligned}
&= \|u\|^q \left[\left(\frac{2-q}{2(2_s^*-1)-q} \right)^{\frac{2-q}{2(2_s^*-2)}} - \left(\frac{2-q}{2(2_s^*-1)-q} \right)^{\frac{2(2_s^*-1)-q}{2(2_s^*-2)}} \right] \\
&\quad \times \left(\frac{\|u\|^{2(2_s^*-1)}}{\int_{\mathbb{R}^3} K(x)\phi_u |u|^{2_s^*-1} dx} \right)^{\frac{2-q}{2(2_s^*-2)}} \\
&\geq \|u\|^q \left(\frac{2(2_s^*-2)}{2(2_s^*-1)-q} \right) \left(\frac{2-q}{2(2_s^*-1)-q} \right)^{\frac{2-q}{2(2_s^*-2)}} \left(\frac{\|u\|^{2(2_s^*-1)}}{S^{-2_s^*}\|u\|^{2(2_s^*-1)}} \right)^{\frac{2-q}{2(2_s^*-2)}} \\
&= \|u\|^q \left(\frac{2(2_s^*-2)}{2(2_s^*-1)-q} \right) \left(\frac{2-q}{2(2_s^*-1)-q} \right)^{\frac{2-q}{2(2_s^*-2)}} S^{\frac{2_s^*(2-q)}{2(2_s^*-2)}} > 0.
\end{aligned} \tag{2.16}
$$

Next, we consider the following two cases.

Case 1: $\int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx \leq 0$. In this case, there is a unique $t^- > t_{\max}$ such that $h(t^-) = \int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx$ and $h'(t^-) < 0$, which implies $t^- u \in N_\lambda^-$. Moreover $I_\lambda(tu)$ is increasing on $(0, t^-)$ and decreasing on (t^-, ∞). Hence, we get

$$
I_\lambda(t^- u) = \sup_{t \geq 0} I_\lambda(tu).
$$

Case 2: $\int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx > 0$. In this case, it follows (2.16) that

$$
\begin{aligned}
h(0) = 0 &< \int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx \leq \lambda \|f_+\|_{q^*} S^{-\frac{q}{2}} \|u\|^q \\
&< \|u\|^q \left(\frac{2(2_s^*-2)}{2(2_s^*-1)-q} \right) \left(\frac{2-q}{2(2_s^*-1)-q} \right)^{\frac{2-q}{2(2_s^*-2)}} S^{\frac{2_s^*(2-q)}{2(2_s^*-2)}} \\
&\leq h(t_{\max})
\end{aligned}
$$

for small $\lambda > 0$. There are a unique t^+ and a unique t^- such that $0 < t^+ < t_{\max} < t^-$ and

$$
h(t^+) = \int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx = h(t^-),
$$

and $h'(t^+) > 0 > h'(t^-)$, which implies that $t^- u \in N_\lambda^-$, $t^+ u \in N_\lambda^+$, and $I_\lambda(t^- u) \geq I_\lambda(tu) \geq I_\lambda(t^+ u)$ for each $t \in [t^+, t^-]$. Furthermore, we derive that $I_\lambda(t^+ u) \leq I_\lambda(tu)$ for $t \in [0, t^+]$. In other words, $I_\lambda(tu)$ is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^-, ∞). As a result, we obtain

$$
I_\lambda(t^+ u) = \inf_{0 \leq t \leq t_{\max}} I_\lambda(tu) \quad \text{and} \quad I_\lambda(t^- u) = \sup_{t \geq t^+} I_\lambda(tu).
$$

□

Lemma 2.8. *I_λ satisfies the $(PS)_c$ condition for $c \in (-\infty, \alpha_\lambda^+ + \frac{2s}{3+2s} S^{\frac{3}{2s}}) \backslash \{0\}$.*

Proof. Let $\{u_n\} \subset H^s(\mathbb{R}^3)$ be a $(PS)_c$ sequence for I_λ with $c \in (-\infty, \alpha_\lambda^+ + \frac{2s}{3+2s}S^{\frac{3}{2s}})\backslash\{0\}$. Then we have

$$\begin{aligned} c + 1 + \|u_n\| &\geq I_\lambda(u_n) - \frac{1}{2(2_s^*-1)}I'(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\|u_n\|^2 - \left(\frac{1}{q} - \frac{1}{2(2_s^*-1)}\right)\int_{\mathbb{R}^3} f_\lambda(x)|u_n|^q dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\|u_n\|^2 - \lambda C\|f_+\|_{q^*}S^{-\frac{q}{2}}\|u_n\|^q, \end{aligned}$$

which follows that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Thus there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ for some $u \in H^s(\mathbb{R}^3)$.

According to (H_1), mean value theorem, and Lemma 2.1, it is easy to check that

$$\int_{\mathbb{R}^3} f_\lambda(x)|u_n|^q dx \to \int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx, \quad \text{as } n \to \infty. \tag{2.17}$$

Let $\Psi_n = u_n - u$. By Brézis–Lieb Lemma [43] and Lemma 2.2(vi), we have

(i) $\|\Psi_n\|^2 = \|u_n\|^2 - \|u|^2 + o_n(1)$;
(ii) $\int_{\mathbb{R}^3} K(x)|\Psi_n|^{2_s^*}dx = \int_{\mathbb{R}^3} K(x)|u_n|^{2_s^*}dx - \int_{\mathbb{R}^3} K(x)|u|^{2_s^*}dx + o_n(1)$.

By virtue of Lemmas 2.2 and 2.1 and the weak convergence $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, it is easy to verify that $I_\lambda'(u) = 0$, and thus,

$$\frac{1}{2}\|\Psi_n\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{\Psi_n}|\Psi_n|^{2_s^*-1}dx = c - I_\lambda(u) + o_n(1), \tag{2.18}$$

and

$$o_n(1) = I_\lambda'(u_n)\Psi_n = (I_\lambda'(u_n) - I_\lambda'(u))\Psi_n = \|\Psi_n\|^2 - \int_{\mathbb{R}^3} K(x)\phi_{\Psi_n}|\Psi_n|^{2_s^*-1}dx, \tag{2.19}$$

as $n \to \infty$. We can assume that

$$\|\Psi_n\|^2 \to l \quad \text{and} \quad \int_{\mathbb{R}^3} K(x)\phi_{\Psi_n}|\Psi_n|^{2_s^*-1}dx \to l, \quad \text{as } n \to \infty,$$

for some $l \in [0, +\infty)$. If $l \neq 0$, we know from Lemma 2.2(ii) that $l \geq S^{\frac{3}{2s}}$, and from (2.18)–(2.19) we derive that

$$c = I_\lambda(u) + \frac{1}{2}l - \frac{1}{2(2_s^*-1)}l \geq \alpha_\lambda^+ + \left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)l \geq \alpha_\lambda^+ + \frac{2s}{3+2s}S^{\frac{3}{2s}},$$

which contradicts the definition of c. Therefore, $l = 0$, consequently, $u_n \to u$ in $H^s(\mathbb{R}^3)$. □

The following theorem reveals that a local minimum of I_λ is attainable in N_λ^+, and that it is in fact a positive solution of (1.8).

Theorem 2.9. *For every $\lambda \in (0, \Lambda_3)$, Λ_3 is the same as given in Lemma 2.7, then I_λ has a minimizer u_λ^+ in N_λ^+ which satisfies*:

(i) u_λ^+ *is a positive solution of system* (1.8).
(ii) $I_\lambda(u_\lambda^+) = \alpha_\lambda^+$.
(iii) $I_\lambda(u_\lambda^+) \to 0$ *as* $\lambda \to 0$.
(iv) $\|u_\lambda^+\| \to 0$ *as* $\lambda \to 0$.

Proof. By Lemma 2.3 and Ekeland variational principle [43], we can obtain a $(PS)_{\alpha_\lambda^+}$ sequence for I_λ defined by $\{u_n\} \subset N_\lambda$. According to Lemma 2.8, there exists a subsequence, still denoted by $\{u_n\}$, and $u_\lambda^+ \in H^s(\mathbb{R}^3)$ such that $u_n \to u_\lambda^+$ in $H^s(\mathbb{R}^3)$ when $n \to \infty$. Note that $N_\lambda^0 = \emptyset$, we have $u_\lambda^+ \in N_\lambda^+$ and $I_\lambda(u_\lambda^+) = \alpha_\lambda^+ < 0$. From Lemma 2.4, we know that u_λ^+ is a solution of system (1.8). Following the proof of [38, Proposition 3.1], we can show that u_λ^+ is a positive solution of system (1.8). Therefore, the expected results (i) and (ii) are checked.

By Lemma 2.6 and (2.14), we have

$$0 > I_\lambda(u_\lambda^+) \geq -C\lambda^{\frac{2}{2-q}},$$

which implies $I_\lambda(u_\lambda^+) \to 0$ as $\lambda \to 0^+$, and (iii) follows.

By virtue of $u_\lambda^+ \in N_\lambda^+$ and (2.13), we deduce to

$$\|u_\lambda^+\|^2 < \frac{2(2_s^* - 1) - q}{2(2_s^* - 1) - 2} \int_{\mathbb{R}^3} f_\lambda(x)|u|^q dx \leq \lambda C \|f_+\|_{q^*} S^{-\frac{q}{2}} \|u_\lambda^+\|^q. \tag{2.20}$$

Since I_λ is coercive and bounded from below on N_λ, $\{u_\lambda^+\}_\lambda$ is bounded in $H^s(\mathbb{R}^3)$. By (2.20), we get

$$\|u_\lambda^+\| < C\lambda^{\frac{1}{2-q}},$$

which means $\|u_\lambda^+\| \to 0$ as $\lambda \to 0^+$, and item (iv) follows. □

3. Results on Estimates

In this section, we present some estimates that are useful in the proof of Theorem 1.1. For $b > 0$, we introduce the auxiliary functional

$$I_\infty^b(u) = \frac{1}{2}\|u\|^2 - \frac{b}{2(2_s^* - 1)} \int_{\mathbb{R}^3} K(x)\phi_u |u|^{2_s^*-1} dx,$$

and the associated Nehari manifold

$$N_\infty^b := \{u \in H^s(\mathbb{R}^3)\backslash\{0\} : (I_\infty^b)'(u)u = 0\}.$$

Lemma 3.1. *For each $u \in N_\lambda^-$, the following two assertions are true*:

(i) *There is a unique solution t_u^b such that $t_u^b u \in N_\infty^b$ and*

$$\max_{t \geq 0} I_\infty^b(tu) = I_\infty^b(t_u^b u) = \frac{2s}{3+2s} b^{-\frac{3-2s}{4s}} \left(\frac{\|u\|^{2(2_s^*-1)}}{\int_{\mathbb{R}^3} K(x)\phi_u |u|^{2_s^*-1} dx} \right)^{\frac{3-2s}{4s}}.$$

(ii) *For* $\mu \in (0,1)$, *there is a unique* t_u^1 *that makes* $t_u^1 u \in N_\infty^1$, *and*

$$I_\infty^1(t_u^1 u) \le (1-\mu)^{-\frac{3+2s}{4s}}\left(I_\lambda(u) + \frac{2-q}{2q}\mu^{-\frac{q}{2-q}}\lambda^{\frac{2}{2-q}}C\right).$$

Proof. (i) For any $u \in N_\lambda^-$, we define

$$h(t) := I_\infty^b(tu) = \frac{t^2}{2}\|u\|^2 - \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3} bK(x)\phi_u|u|^{2_s^*-1}dx.$$

Clearly, $h(t) \to -\infty$ as $t \to \infty$, and

$$h'(t) = t\|u\|^2 - t^{22_s^*-3}\int_{\mathbb{R}^3} bK(x)\phi_u|u|^{2_s^*-1}dx,$$

and

$$h''(t) = \|u\|^2 - (22_s^*-3)t^{2(2_s^*-2)}\int_{\mathbb{R}^3} bK(x)\phi_u|u|^{2_s^*-1}dx.$$

It is easy to see that $h'(t_u^b) = 0$, where

$$t_u^b = \left(\frac{\|u\|^2}{\int_{\mathbb{R}^3} bK(x)\phi_u|u|^{2_s^*-1}dx}\right)^{\frac{1}{2(2_s^*-2)}} > 0,$$

and $h''(t_u^b) = -2(2_s^*-2)\|u\|^2 < 0$. Moreover, $t_u^b > 0$ is unique such that $t_u^b u \in N_\infty^b$ and

$$\max_{t\ge 0} I_\infty^b(tu) = I_\infty^b(t_u^b u) = \frac{2s}{3+2s}b^{-\frac{3-2s}{4s}}\left(\frac{\|u\|^{2(2_s^*-1)}}{\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx}\right)^{\frac{3-2s}{4s}}.$$

(ii) For $\mu \in (0,1)$, from (H_1), Lemma 2.1 and Young inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \lambda f_+(x)|t_u^b u|^q dx &\le \lambda S^{-\frac{q}{2}}\|f_+\|_{q^*}\|t_u^b u\|^q \\ &\le \frac{2-q}{2}\left(\lambda\mu^{-\frac{q}{2}}S^{-\frac{q}{2}}\|f_+\|_{q^*}\right)^{\frac{2}{2-q}} + \frac{q}{2}\left(\mu^{\frac{q}{2}}\|t_u^b u\|^q\right)^{\frac{2}{q}} \\ &= \frac{2-q}{2}\mu^{-\frac{q}{2-q}}\lambda^{\frac{2}{2-q}}C + \frac{q}{2}\mu\|t_u^b u\|^2.\end{aligned}$$

Combined with part (i) and $b = \frac{1}{1-\mu}$, we derive that

$$\begin{aligned}I_\lambda(u) &= \max_{t\ge 0} I_\lambda(tu) \\ &\ge I_\lambda\left(t_u^{\frac{1}{1-\mu}} u\right) \\ &\ge \frac{1-\mu}{2}\left\|t_u^{\frac{1}{1-\mu}} u\right\|^2 - \frac{\left(t_u^{\frac{1}{1-\mu}}\right)^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx \\ &\quad - \frac{2-q}{2q}\mu^{-\frac{q}{2-q}}\lambda^{\frac{2}{2-q}}C \\ &= (1-\mu)I_\lambda^{\frac{1}{1-\mu}}\left(t_u^{\frac{1}{1-\mu}} u\right) - \frac{2-q}{2q}\mu^{-\frac{q}{2-q}}\lambda^{\frac{2}{2-q}}C\end{aligned}$$

$$= (1-\mu)\frac{2s}{3+2s}\left(\frac{1}{1-\mu}\right)^{-\frac{3-2s}{4s}}\left(\frac{\|u\|^{2(2_s^*-1)}}{\int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx}\right)^{\frac{3-2s}{4s}}$$

$$- \frac{2-q}{2q}\mu^{-\frac{q}{2-q}}\lambda^{\frac{2}{2-q}}C$$

$$= (1-\mu)^{\frac{3+2s}{4s}} I_\infty^1(t_u^1 u) - \frac{2-q}{2q}\mu^{-\frac{q}{2-q}}\lambda^{\frac{2}{2-q}}C. \qquad \square$$

From [41], we define $v_{\varepsilon,z}(x) = \eta(x-z)U_{\varepsilon,z}(x), z \in M$, where $U_{\varepsilon,z}$ is given in (2.2), and the cut-off function $\eta \in C_0^\infty(\mathbb{R}^3)$ is such that $0 \le \eta \le 1$ in $\mathbb{R}^3$, $\eta(x) = 1$ if $|x| < 1$ and $\eta = 0$ if $|x| > 2$. Then, according to [41, Propositions 21,22], we know that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_{\varepsilon,z}|^2 dx \le S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}), \quad \int_{\mathbb{R}^3} |v_{\varepsilon,z}|^{2_s^*} dx = S^{\frac{3}{2s}} + O(\varepsilon^3), \tag{3.1}$$

$$\int_{\mathbb{R}^3} |v_{\varepsilon,z}|^p dx = \begin{cases} O(\varepsilon^{\frac{3(2-p)+2sp}{2}}) & \text{if } p > \dfrac{3}{3-2s}, \\ O(\varepsilon^{\frac{3(2-p)+2sp}{2}}|\log\varepsilon|) & \text{if } p = \dfrac{3}{3-2s}, \\ O(\varepsilon^{\frac{(3-2s)p}{2}}) & \text{if } p < \dfrac{3}{3-2s}. \end{cases} \tag{3.2}$$

Lemma 3.2. *The following estimate holds*:

$$\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-1}dx \ge S^{\frac{3}{2s}} - O(\varepsilon^{3-2s}).$$

Proof. Since $(-\Delta)^s\phi_{v_{\varepsilon,z}} = K(x)|v_{\varepsilon,z}|^{2_s^*-1}$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} K(x)|v_{\varepsilon,z}|^{2_s^*}dx \\ &\quad = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}\phi_{v_{\varepsilon,z}}(-\Delta)^{\frac{s}{2}}|v_{\varepsilon,z}|dx \\ &\quad \le \frac{1}{2\max_{x\in\mathbb{R}^3} K(x)}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\phi_{v_{\varepsilon,z}}|^2 + \frac{\max_{x\in\mathbb{R}^3} K(x)}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}|v_{\varepsilon,z}||^2dx \\ &\quad \le \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-1}dx + \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}v_{\varepsilon,z}|^2dx, \end{aligned}$$

which shows that

$$\begin{aligned} \int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-1}dx &\ge 2\int_{\mathbb{R}^3} K(x)|v_{\varepsilon,z}|^{2_s^*}dx - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}v_{\varepsilon,z}|^2dx \\ &= 2S^{\frac{3}{2s}} + O(\varepsilon^\rho) - S^{\frac{3}{2s}} - O(\varepsilon^{3-2s}) = S^{\frac{3}{2s}} - O(\varepsilon^{3-2s}), \end{aligned}$$

where we have used (H_3) and (3.1) to get

$$\int_{\mathbb{R}^3} K(x)|v_{\varepsilon,z}|^{2_s^*}dx = S^{\frac{3}{2s}} + O(\varepsilon^\rho),$$

and the fact that $\rho \geq 3-2s$ in (H_3). □

Lemma 3.3 ([18]). *The following statements hold true*:

(i) *If* $s \in (\frac{1}{2}, 1)$, *then for all* $\tau \in (0,1)$,

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)\phi_{u_\lambda^+ + tv_{\varepsilon,z}}|u_\lambda^+ + tv_{\varepsilon,z}|^{2_s^*-1}dx\\ &\quad\geq \int_{\mathbb{R}^3} K(x)\phi_{u_\lambda^+}|u_\lambda^+|^{2_s^*-1}dx + t^{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-1}dx\\ &\quad\quad + Ct^{22_s^*-3}\int_\Omega\int_\Omega \frac{K(x)|v_{\varepsilon,z}(x)|^{2_s^*-1}|v_{\varepsilon,z}(y)|^{2_s^*-2}u_\lambda^+(y)}{|x-y|^{3-2s}}dxdy\\ &\quad\quad + 2(2_s^*-1)t\int_\Omega\int_\Omega \frac{K(x)|u_\lambda^+(x)|^{2_s^*-1}|u_\lambda^+(y)|^{2_s^*-2}v_{\varepsilon,z}(y)}{|x-y|^{3-2s}}dxdy\\ &\quad\quad - O(\varepsilon^{\frac{3+2s}{4}\tau}).\end{aligned}$$

(ii) *There exists a* $C_0 > 0$ *such that*

$$\int_\Omega\int_\Omega \frac{K(x)|v_{\varepsilon,z}(x)|^{2_s^*-1}|v_{\varepsilon,z}(y)|^{2_s^*-2}u_\lambda^+(y)}{|x-y|^{3-2s}}dxdy \geq C_0\varepsilon^{\frac{3-2s}{2}},$$

where $\Omega := \operatorname{supp} v_{\varepsilon,z}$.

Lemma 3.4. *There exists* $\varepsilon_0 > 0$ *sufficiently small such that for* $\varepsilon \in (0, \varepsilon_0)$, *there exists*

$$\sup_{t\geq 0} I_\lambda(u_\lambda^+ + tv_{\varepsilon,z}) < \alpha_\lambda^+ + \frac{2s}{3+2s}S^{\frac{3}{2s}} - \sigma(\varepsilon_0)$$

uniformly distributed in $z \in M$. *In addition, there exists* $t_z^- > 0$ *such that*

$$u_\lambda^+ + t_z^- v_{\varepsilon,z} \in N_\lambda^- \quad \text{for } z \in M.$$

Proof. Note that

$$\lim_{t\to 0} I_\lambda(u_\lambda^+ + tv_{\varepsilon,z}) = \alpha_\lambda^+ < 0 \quad \text{and} \quad \lim_{t\to\infty} I_\lambda(u_\lambda^+ + tv_{\varepsilon,z}) = -\infty,$$

for small $\varepsilon > 0$, there exist a small $t_0 > 0$ and a large $t_1 > 0$ such that

$$I_\lambda(u_\lambda^+ + tv_{\varepsilon,z}) < \alpha_\lambda^+ + \frac{2s}{3+2s}S^{\frac{3}{2s}}, \quad \forall\, t \in (0,t_0] \cup [t_1, +\infty). \tag{3.3}$$

Thus, it suffices to prove for $z \in M$, that

$$I_\lambda(u_\lambda^+ + tv_{\varepsilon,z}) < \alpha_\lambda^+ + \frac{2s}{3+2s} S^{\frac{3}{2s}}, \quad \forall\, t \in [t_0, t_1].$$

For this aim, by Theorem 2.9 and Lemma 3.3, we derive that

$$\begin{aligned}
&I_\lambda(u_\lambda^+ + tv_{\varepsilon,z})\\
&\quad = \frac{1}{2}\left\|u_\lambda^+ + tv_{\varepsilon,z}\right\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{u_\lambda^+ + tv_{\varepsilon,z}}|u_\lambda^+ + tv_{\varepsilon,z}|^{2_s^*-1}dx\\
&\qquad - \frac{1}{q}\int_{\mathbb{R}^3} f_\lambda(x)|u_\lambda^+ + tv_{\varepsilon,z}|^q dx\\
&\quad = I_\lambda(u_\lambda^+) + \frac{t^2}{2}\|v_{\varepsilon,z}\|^2 + t\int_{\mathbb{R}^3}[(-\Delta)^{\frac{s}{2}}u_\lambda^+(-\Delta)^{\frac{s}{2}}v_{\varepsilon,z} + u_\lambda^+ v_{\varepsilon,z}]dx\\
&\qquad - t\int_{\mathbb{R}^3}\left[K(x)\phi_{u_\lambda^+}|u_\lambda^+|^{2_s^*-2}v_{\varepsilon,z} + f_\lambda(x)|u_\lambda^+|^{q-1}v_{\varepsilon,z}\right]dx\\
&\qquad - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\big[\phi_{u_\lambda^+ + tv_{\varepsilon,z}}|u_\lambda^+ + tv_{\varepsilon,z}|^{2_s^*-1} - \phi_{u_\lambda^+}|u_\lambda^+|^{2_s^*-1}\\
&\qquad - 2(2_s^*-1)\phi_{u_\lambda^+}|u_\lambda^+|^{2_s^*-2}(tv_{\varepsilon,z})\big]dx\\
&\qquad - \frac{1}{q}\int_{\mathbb{R}^3} f_\lambda(x)\big[|u_\lambda^+ + tv_{\varepsilon,z}|^q - |u_\lambda^+|^q - q|u_\lambda^+|^{q-1}(tv_{\varepsilon,z})\big]dx\\
&\quad = I_\lambda(u_\lambda^+) + \frac{t^2}{2}\|v_{\varepsilon,z}\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\big[\phi_{u_\lambda^+ + tv_{\varepsilon,z}}|u_\lambda^+ + tv_{\varepsilon,z}|^{2_s^*-1}\\
&\qquad - \phi_{u_\lambda^+}|u_\lambda^+|^{2_s^*-1}\\
&\qquad - 2(2_s^*-1)\phi_{u_\lambda^+}|u_\lambda^+|^{2_s^*-2}(tv_{\varepsilon,z})\big]dx\\
&\qquad - \frac{1}{q}\int_{\mathbb{R}^3} f_\lambda(x)\big[|u_\lambda^+ + tv_{\varepsilon,z}|^q - |u_\lambda^+|^q - q|u_\lambda^+|^{q-1}(tv_{\varepsilon,z})\big]dx\\
&\quad \le \alpha_\lambda^+ + \frac{t^2}{2}\|v_{\varepsilon,z}\|^2_{D^{s,2}} - \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-1}dx\\
&\qquad + C\int_{\mathbb{R}^3}|v_{\varepsilon,z}|^2dx - \frac{Ct^{22_s^*-3}}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-2}u_\lambda^+dx\\
&\qquad - \frac{1}{q}\int_{\mathbb{R}^3} f_\lambda(x)\big[|u_\lambda^+ + tv_{\varepsilon,z}|^q - |u_\lambda^+|^q - q|u_\lambda^+|^{q-1}(tv_{\varepsilon,z})\big]dx\\
&\qquad + O(\varepsilon^{\frac{3+2s}{4}\tau})\\
&\quad := \alpha_\lambda^+ + i(t) - k(t) + C\int_{\mathbb{R}^3}|v_{\varepsilon,z}|^2dx + o(\varepsilon^{\frac{3-2s}{2}}),
\end{aligned} \tag{3.4}$$

where we have chosen $\tau = \frac{2}{2_s^*-1} + \theta < 1$ for a suitable small $\theta > 0$, and

$$i(t) := \frac{t^2}{2}\|v_{\varepsilon,z}\|^2_{D^{s,2}} - \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-1}dx - \frac{Ct^{22_s^*-3}}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-2}u_\lambda^+ dx,$$

$$k(t) := \frac{1}{q}\int_{\mathbb{R}^3} f_\lambda(x)\big[|u_\lambda^+ + tv_{\varepsilon,z}|^q - |u_\lambda^+|^q - q|u_\lambda^+|^{q-1}(tv_{\varepsilon,z})\big]dx.$$

By Lemma 3.3(ii), we get

$$\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-2}u_\lambda^+ dx \geq \iint_{B_2(0)\times B_2(0)} \frac{K(x)|v_\varepsilon(y)|^{2_s^*-1}|v_\varepsilon(x)|^{2_s^*-2}u_\lambda^+(x)}{|x-y|^{3-2s}}dxdy \geq C\varepsilon^{\frac{3-2s}{2}}, \tag{3.5}$$

for $z \in M$. It follows from Lemma 3.2 and (3.1)–(3.2) that

$$\begin{aligned} i(t) &\leq \frac{t^2}{2}\|v_{\varepsilon,z}\|^2_{D^{s,2}} - \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-1}dx - C\varepsilon^{\frac{3-2s}{2}} \\ &\leq \frac{2s}{3+2s}\left(\frac{\|v_{\varepsilon,z}\|^2}{\left(\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-1}dx\right)^{\frac{1}{2_s^*-1}}}\right)^{\frac{3+2s}{4s}} - C\varepsilon^{\frac{3-2s}{2}} \\ &\leq \frac{2s}{3+2s}\left(\frac{S^{\frac{3}{2s}} + O(\varepsilon^{3-2s})}{\left(S^{\frac{3}{2s}} - O(\varepsilon^{3-2s})\right)^{\frac{1}{2_s^*-1}}}\right)^{\frac{3+2s}{4s}} - C\varepsilon^{\frac{3-2s}{2}} \\ &\leq \frac{2s}{3+2s}S^{\frac{3}{2s}} - C\varepsilon^{\frac{3-2s}{2}}, \end{aligned} \tag{3.6}$$

for $z \in M$ and $t \in [t_0, t_1]$.

Considering condition (H_4), Remark 1.1 and the definition of $v_{\varepsilon,z}$, we know that

$$\begin{aligned} k(t) &= \frac{\lambda}{q}\int_{M_{r_0}} f_\lambda(x)\big[|u_\lambda^+ + tv_{\varepsilon,z}|^q - |u_\lambda^+|^q - q|u_\lambda^+|^{q-1}(tv_{\varepsilon,z})\big]dx \\ &\quad + \frac{\lambda}{q}\int_{M_{r_0}^c} f_\lambda(x)\big[|u_\lambda^+ + tv_{\varepsilon,z}|^q - |u_\lambda^+|^q - q|u_\lambda^+|^{q-1}(tv_{\varepsilon,z})\big]dx \\ &= \frac{\lambda}{q}\int_{M_{r_0}} f_\lambda(x)\big[|u_\lambda^+ + tv_{\varepsilon,z}|^q - |u_\lambda^+|^q - q|u_\lambda^+|^{q-1}(tv_{\varepsilon,z})\big]dx \\ &= \frac{\lambda}{q}\int_{M_{r_0}} f_\lambda(x)\left[\int_0^{tv_{\varepsilon,z}} \left(|u_\lambda^+ + s|^{q-1} - |u_\lambda^+|^{q-1}\right)ds\right]dx \\ &\geq 0, \end{aligned} \tag{3.7}$$

for $z \in M$ and $t \in [t_0, t_1]$.

Substituting (3.6) and (3.7) into (3.4), and by $\frac{1}{2} < s < 1$ we infer to

$$\begin{aligned} I_\lambda(u_\lambda^+ + tv_{\varepsilon,z}) &\le \alpha_\lambda^+ + \frac{2s}{3+2s}S^{\frac{3}{2s}} - C\varepsilon^{\frac{3-2s}{2}} + o(\varepsilon^{\frac{3-2s}{2}}) + C\int_{\mathbb{R}^3}|v_{\varepsilon,z}|^2dx \\ &= \alpha_\lambda^+ + \frac{2s}{3+2s}S^{\frac{3}{2s}} - C\varepsilon^{\frac{3-2s}{2}} + o(\varepsilon^{\frac{3-2s}{2}}) + \begin{cases} O(\varepsilon^{2s}) & \text{if } s < \frac{3}{4}, \\ O(\varepsilon^{2s}|\log\varepsilon|) & \text{if } s = \frac{3}{4}, \\ O(\varepsilon^{3-2s}) & \text{if } s > \frac{3}{4} \end{cases} \\ &< \alpha_\lambda^+ + \frac{2s}{3+2s}S^{\frac{3}{2s}}, \end{aligned} \tag{3.8}$$

for $z \in M$ and $t \in [t_0, t_1]$. By (3.3) and (3.8), there exists a small $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, we have

$$\sup_{t\ge 0} I_\lambda(u_\lambda^+ + tv_{\varepsilon,z}) < \alpha_\lambda^+ + \frac{2s}{3+2s}S^{\frac{3}{2s}} - \sigma(\varepsilon_0) \quad \text{for } z \in M.$$

To show that there exists the existence of $t_z^- > 0$ such that

$$u_\lambda^+ + t_z^- v_{\varepsilon,z} \in N_\lambda^- \quad \text{for } z \in M,$$

we define

$$U_1 := \left\{u \in H^s(\mathbb{R}^3)\backslash\{0\} : t^-\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} > 1\right\} \cup \{0\}$$

and

$$U_2 := \left\{u \in H^s(\mathbb{R}^3)\backslash\{0\} : t^-\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} < 1\right\}.$$

Then we claim that N_λ^- split $H^s(\mathbb{R}^3)$ into two disjoint connected components U_1 and U_2. In fact, if $u \in N_\lambda^-$, let $v = \frac{u}{\|u\|}$, then there is a unique $t^-(v) > 0$ such that $t^-(v)v \in N_\lambda^-$ or $t^-\big(\frac{u}{\|u\|}\big)\frac{1}{\|u\|}u \in N_\lambda^-$. Since $u \in N_\lambda^-$, we have $t^-\big(\frac{u}{\|u\|}\big)\frac{1}{\|u\|} = 1$, which means that

$$N_\lambda^- \subset \left\{u \in H^s(\mathbb{R}^3)\backslash\{0\} : t^-\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} = 1\right\}.$$

Conversely, let $u \in H^s(\mathbb{R}^3)\backslash\{0\}$ such that $t^-\big(\frac{u}{\|u\|}\big)\frac{1}{\|u\|} = 1$. Then

$$t^-\left(\frac{u}{\|u\|}\right)\frac{u}{\|u\|} \in N_\lambda^-.$$

Therefore, we get

$$N_\lambda^- = \left\{u \in H^s(\mathbb{R}^3)\backslash\{0\} : t^-\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} = 1\right\}.$$

Since $u_\lambda^+ \in N_\lambda^+$, one has $1 < t^-(u_\lambda^+)$ and $u_\lambda^+ \in U_1$. We claim that

$$0 < t^-\left(\frac{u_\lambda^+ + tv_{\varepsilon,z}}{\|u_\lambda^+ + tv_{\varepsilon,z}\|}\right) < \widetilde{C}$$

for some $\widetilde{C} > 0$ and all $t \geq 0$. Suppose otherwise that there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and

$$t^-\left(\frac{u_\lambda^+ + t_n v_{\varepsilon,z}}{\|u_\lambda^+ + t_n v_{\varepsilon,z}\|}\right) \to \infty \quad \text{as } n \to \infty.$$

Set

$$v_n := \frac{u_\lambda^+ + t_n v_{\varepsilon,z}}{\|u_\lambda^+ + t_n v_{\varepsilon,z}\|}.$$

Recalling that $t^-(v_n)v_n \in N_\lambda^-$, by the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x)\phi_{v_n}|v_n|^{2_s^*-1}dx \\
&\quad = \frac{1}{\|u_\lambda^+ + t_n v_{\varepsilon,z}\|^{2(2_s^*-1)}} \int_{\mathbb{R}^3} K(x)\phi_{u_\lambda^+ + t_n v_{\varepsilon,z}} |u_\lambda^+ + t_n v_{\varepsilon,z}|^{2_s^*-1}dx \\
&\quad = \frac{1}{\|\frac{u_\lambda^+}{t_n} + v_{\varepsilon,z}\|^{2(2_s^*-1)}} \int_{\mathbb{R}^3} K(x)\phi_{u_\lambda^+/t_n + v_{\varepsilon,z}} \left|\frac{u_\lambda^+}{t_n} + v_{\varepsilon,z}\right|^{2_s^*-1} dx \\
&\quad \to \frac{1}{\|v_{\varepsilon,z}\|^{2(2_s^*-1)}} \int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-1}dx
\end{aligned}$$

as $n \to \infty$. Consequently,

$$I_\lambda(t^-(v_n)v_n) \to -\infty, \quad \text{as } n \to \infty,$$

which contradicts the fact that I_λ is bounded below on N_λ. Now we set

$$t_\lambda = \frac{\|u_\lambda^+\| + \sqrt{\widetilde{C} + \|u_\lambda^+\|^2}}{\|v_{\varepsilon,z}\|} + 1.$$

By a simple calculation, we have

$$\begin{aligned}
\|u_\lambda^+ + t_\lambda v_{\varepsilon,z}\|^2 &= \|u_\lambda^+\|^2 + t_\lambda^2\|v_{\varepsilon,z}\|^2 + 2t_\lambda\langle u_\lambda^+, v_{\varepsilon,z}\rangle \\
&\geq \|u_\lambda^+\|^2 + t_\lambda^2\|v_{\varepsilon,z}\|^2 - 2t_\lambda\|u_\lambda^+\|\|v_{\varepsilon,z}\| \\
&> \left[t^-\left(\frac{u_\lambda^+ + tv_{\varepsilon,z}}{\|u_\lambda^+ + tv_{\varepsilon,z}\|}\right)\right]^2,
\end{aligned}$$

which implies that $u_\lambda^+ + tv_{\varepsilon,z} \in U_2$. Therefore, there exists some $0 < t_{\varepsilon,z}^- < t_\lambda$ such that $u_\lambda^+ + t_{\varepsilon,z}^- v_{\varepsilon,z} \in N_\lambda^-$. □

Lemma 3.5. *The following estimates hold true*:

$$\inf_{u\in N^1_\infty} I^1_\infty(u) = \inf_{u\in N^\infty} I^\infty(u) = \frac{2s}{3+2s}S^{\frac{3}{2s}},$$

where

$$I^\infty(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2(2^*_s-1)}\int_{\mathbb{R}^3}\widetilde{\phi}_u|u|^{2^*_s-1}dx,$$

and

$$N^\infty = \{u\in H^s(\mathbb{R}^3)\backslash\{0\};\ (I^\infty)'(u)u = 0\}.$$

Proof. Notice that, for $a>0$ and $b>0$,

$$\max_{t\geq 0}\left(\frac{a}{2}t^2 - \frac{b}{2(2^*_s-1)}t^{2(2^*_s-1)}\right) = \frac{2s}{3+2s}\left(\frac{a}{b^{\frac{1}{2^*_s-1}}}\right)^{\frac{2^*_s-1}{2^*_s-2}}.$$

We deduce from Remark 1.1 and Lemma 2.2 that

$$\begin{aligned}\inf_{u\in N^\infty} I^\infty(u) &= \inf_{u\in H^s(\mathbb{R}^3)\backslash\{0\}}\sup_{t\geq 0} I^\infty(tu)\\ &= \inf_{u\in H^s(\mathbb{R}^3)\backslash\{0\}}\frac{2s}{3+2s}\left(\frac{\|u\|^2}{(\int_{\mathbb{R}^3}\widetilde{\phi}_u|u|^{2^*_s-1}dx)^{\frac{1}{2^*_s-1}}}\right)^{\frac{2^*_s-1}{2^*_s-2}}\\ &\geq \inf_{u\in H^s(\mathbb{R}^3)\backslash\{0\}}\frac{2s}{3+2s}\left(\frac{\|u\|^2_{D^{s,2}}}{(S^{-2^*_s}\|u\|^{2(2^*_s-1)}_{D^{s,2}})^{\frac{1}{2^*_s-1}}}\right)^{\frac{2^*_s-1}{2^*_s-2}}\\ &= \frac{2s}{3+2s}S^{\frac{3}{2s}}. \end{aligned} \tag{3.9}$$

It follows from (3.2) and Lemma 3.2 that

$$\begin{aligned}\sup_{t\geq 0} I^1_\infty(tv_{\varepsilon,z}) &= \frac{2s}{3+2s}\left(\frac{\|v_{\varepsilon,z}\|^2}{(\int_{\mathbb{R}^3}K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2^*_s-1}dx)^{\frac{1}{2^*_s-1}}}\right)^{\frac{2^*_s-1}{2^*_s-2}}\\ &\leq \frac{2s}{3+2s}\left(\frac{S^{\frac{3}{2s}} + O(\varepsilon^{\min\{3-2s,2s\}}|\log\varepsilon|)}{(S^{\frac{3}{2s}} - O(\varepsilon^{3-2s}))^{\frac{1}{2^*_s-1}}}\right)^{\frac{2^*_s-1}{2^*_s-2}}\\ &= \frac{2s}{3+2s}S^{\frac{3}{2s}} + O(\varepsilon^{\min\{3-2s,2s\}}|\log\varepsilon|).\end{aligned}$$

Thus, we have

$$\inf_{u\in N^1_\infty} I^1_\infty(u) \leq \frac{2s}{3+2s}S^{\frac{3}{2s}}, \quad \text{as } \varepsilon\to 0. \tag{3.10}$$

Since $K(x) \leq 1$, from (3.9) and (3.10), we have

$$\begin{aligned}\frac{2s}{3+2s}S^{\frac{3}{2s}} &\leq \inf_{u\in N^\infty} I^\infty(u)\\ &= \inf_{u\in H^s(\mathbb{R}^3)\setminus\{0\}} \sup_{t\geq 0} I^\infty(tu)\\ &\leq \inf_{u\in H^s(\mathbb{R}^3)\setminus\{0\}} \sup_{t\geq 0} I^1_\infty(tu) = \inf_{u\in N^1_\infty} I^1_\infty(u) \leq \frac{2s}{3+2s}S^{\frac{3}{2s}}. \quad (3.11)\end{aligned}$$

□

4. Proof of Theorem 1.1

In this section, we shall apply the Ljusternik–Schnirelmann category theory to study multiple positive solutions of system (1.8) and complete the proof of Theorem 1.1.

Proposition 4.1 ([11]). *Let $\mathcal{M}$ be a $C^{1,1}$ complete Riemannian manifold (modeled on a Hilbert space) and assume that $F \in C^1(\mathbb{R},\mathbb{R})$ bounded from below. Let*

$$-\infty < \inf_{\mathcal{M}} F < a < b < +\infty.$$

Suppose that F satisfies the (PS) condition on the sublevel $\{u \in \mathcal{M}; F(u) \leq b\}$ and that a is not a critical level for F. Then we have

$$\sharp\{u \in F^a; \nabla F(u) = 0\} \geq \mathrm{cat}_{F^a}(F^a),$$

where $F^a \equiv \{u \in \mathcal{M}; F(u) \leq a\}$.

Proposition 4.2 ([11]). *Let Q, Ω^+ and Ω^- be closed sets with $\Omega^- \subset \Omega^+$, and $\phi : Q \to \Omega^+$, $\varphi : \Omega^- \to Q$ be two continuous maps such that $\phi\circ\varphi$ is homotopically equivalent to the embedding $j : \Omega^- \to \Omega^+$. Then*

$$\mathrm{cat}_Q(Q) \geq \mathrm{cat}_{\Omega^+}(\Omega^-).$$

We will apply Propositions 4.1 and 4.2 to study the existence of multiple positive solutions of system (1.8). To argue further, we introduce the following lemma, which is useful in proving the compactness of the (PS) sequences.

Corollary 4.3 ([24]). *Let $\{u_n\} \subset H^s(\mathbb{R}^3)$ be a nonnegative function sequence with $\|u_n\|_{2^*_s} = 1$ and $\|u_n\|^2_{D^{s,2}} \to S$. Then there exists a sequence $\{(x_n, \varepsilon_n)\} \subset \mathbb{R}^3 \times \mathbb{R}^+$ such that*

$$u_n(x) := \frac{1}{S^{\frac{3-2s}{4s}}} u_{\varepsilon_n}(x - x_n) + o_n(1)$$

in $D^{s,2}(\mathbb{R}^3)$, where $u_\varepsilon = U_{\varepsilon,0}$ is defined in (2.2). Moreover, if $x_n \to \overline{x}$ then $\varepsilon_n \to 0$ or it is unbounded.

We define a continuous map $\Phi : H^s(\mathbb{R}^3)\backslash G \to \mathbb{R}^3$ by

$$u \mapsto \Phi(u) := \frac{\iint_{\mathbb{R}^6} \frac{x|u(x)-u_\lambda^+(x)|^{2_s^*-1}|u(y)-u_\lambda^+(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy}{\iint_{\mathbb{R}^6} \frac{|u(x)-u_\lambda^+(x)|^{2_s^*-1}|u(y)-u_\lambda^+(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy},$$

where

$$G = \left\{ u \in H^s(\mathbb{R}^3); \iint_{\mathbb{R}^6} \frac{|u(x)-u_\lambda^+(x)|^{2_s^*-1}|u(y)-u_\lambda^+(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy = 0 \right\}.$$

Lemma 4.4. *For any fixed* $0 < \delta < r_0$, *there exist* $\lambda_\delta > 0$ *and* $\delta_0 > 0$ *such that if* $u \in N_\infty^1$ *with* $I_\infty^1(u) < \frac{2s}{3+2s} S^{\frac{3}{2s}} + \delta_0$ *and* $\lambda \in (0, \lambda_\delta)$, *then* $\Phi(u) \in M_\delta$, *where* M_δ *is defined in Remark* 1.1.

Proof. Suppose by contradiction that there exists a sequence $\{u_n\} \subset N_\infty^1$ such that $I_\infty^1(u_n) < \frac{2s}{3+2s} S^{\frac{3}{2s}} + o_n(1)$ as $n \to \infty$, but

$$\Phi(u_n) \notin M_\delta. \tag{4.1}$$

From $\{u_n\} \subset N_\infty^1$, we have

$$\|u_n\|^2 = \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx + o_n(1). \tag{4.2}$$

Recalling Lemma 2.2 we get

$$\|u_n\|_{D^{s,2}}^2 \le \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \le S^{-1}\|u\|_{2_s^*}^{2(2_s^*-1)} \le S^{-2_s^*}\|u\|_{D^{s,2}}^{2(2_s^*-1)}, \tag{4.3}$$

which implies that

$$\|u_n\|_{D^{s,2}}^2 \ge S^{\frac{3}{2s}}. \tag{4.4}$$

Therefore, we infer to

$$\begin{aligned} o_n(1) + \frac{2s}{3+2s}S^{\frac{3}{2s}} &\ge I_\infty^1(u_n) = I_\infty^1(u_n) - \frac{1}{2(2_s^*-1)}(I_\infty^1)'(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\|u_n\|^2 \\ &\ge \frac{2s}{3+2s}\|u_n\|_{D^{s,2}}^2 \ge \frac{2s}{3+2s}S^{\frac{3}{2s}} + o_n(1), \end{aligned} \tag{4.5}$$

which shows that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Moreover, from (4.2)–(4.5), we see that

$$\|u_n\|_{D^{s,2}}^2 = \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx + o_n(1) = S^{\frac{3}{2s}} + o_n(1). \tag{4.6}$$

Define

$$v_n = \frac{u_n}{\|u_n\|_{2_s^*}},$$

then $\|v_n\|_{2_s^*} = 1$. By $0 < K(x) \le 1$, Lemma 2.2 and Remark 2.1 we have

$$\begin{aligned} S \le \|v_n\|^2_{D^{s,2}} &= \frac{\|u_n\|^2_{D^{s,2}}}{\|u_n\|^2_{2_s^*}} \\ &\le \frac{\|u_n\|^2_{D^{s,2}}}{\left(S\int_{\mathbb{R}^3} \widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx\right)^{\frac{1}{2_s^*-1}}} \\ &\le \frac{\|u_n\|^2_{D^{s,2}}}{\left(S\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx\right)^{\frac{1}{2_s^*-1}}} = \frac{S^{\frac{3}{2s}} + o_n(1)}{\left(S^{\frac{3+2s}{2s}} + o_n(1)\right)^{\frac{1}{2_s^*-1}}} = S + o_n(1). \end{aligned}$$

Therefore,

$$\begin{gathered} \int_{\mathbb{R}^3} \widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx = \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx + o_n(1) \\ \text{and} \quad \|v_n\|^2_{D^{s,2}} = S + o_n(1). \end{gathered} \tag{4.7}$$

It follows from Corollary 4.3 that there is a sequence $\{(x_n, \varepsilon_n)\} \subset \mathbb{R}^3 \times \mathbb{R}^+$ such that

$$v_n(x) := \frac{1}{S^{\frac{3-2s}{4s}}} u_{\varepsilon_n, x_n}(x) + o_n(1). \tag{4.8}$$

In addition, $x_n \to \bar{x}$ or $|x_n| \to \infty$ as $n \to \infty$. We need to consider two situations below.

Case 1. Assume that $|x_n| \to \infty$ as $n \to \infty$. From (4.7) and (4.8) we derive that

$$\begin{aligned} 1 &= \frac{\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx}{\int_{\mathbb{R}^3} \widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx} + o_n(1) = \frac{\int_{\mathbb{R}^3} K(x)\phi_{v_n}|v_n|^{2_s^*-1}dx}{\int_{\mathbb{R}^3} \widetilde{\phi}_{v_n}|v_n|^{2_s^*-1}dx} + o_n(1) \\ &= \frac{\iint_{\mathbb{R}^6} \frac{K(x)K(y)|u_{\varepsilon_n,x_n}(x)|^{2_s^*-1}|u_{\varepsilon_n,x_n}(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy}{\iint_{\mathbb{R}^6} \frac{|u_{\varepsilon_n,x_n}(x)|^{2_s^*-1}|u_{\varepsilon_n,x_n}(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy} + o_n(1) \\ &= \frac{\iint_{\mathbb{R}^6} \frac{K(x+x_n)K(y+x_n)|u_{\varepsilon_n,0}(x)|^{2_s^*-1}|u_{\varepsilon_n,0}(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy}{\iint_{\mathbb{R}^6} \frac{|u_{\varepsilon_n,0}(x)|^{2_s^*-1}|u_{\varepsilon_n,0}(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy} + o_n(1) \\ &= K_\infty^2 + o_n(1) \end{aligned}$$

which contradicts the definition of K_∞.

Case 2. Assume that $x_n \to \bar{x}$ as $n \to \infty$. We infer from Corollary 4.3 that $\varepsilon_n \to 0$ as $n \to \infty$. By (4.7) and (4.8) we have

$$1 = \frac{\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx}{\int_{\mathbb{R}^3} \widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx} + o_n(1) = \frac{\int_{\mathbb{R}^3} K(x)\phi_{v_n}|v_n|^{2_s^*-1}dx}{\int_{\mathbb{R}^3} \widetilde{\phi}_{v_n}|v_n|^{2_s^*-1}dx} + o_n(1)$$

$$= \frac{\iint_{\mathbb{R}^6} \frac{K(x)K(y)|u_{\varepsilon_n,x_n}(x)|^{2_s^*-1}|u_{\varepsilon_n,x_n}(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy}{\iint_{\mathbb{R}^6} \frac{|u_{\varepsilon_n,x_n}(x)|^{2_s^*-1}|u_{\varepsilon_n,x_n}(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy} + o_n(1)$$

$$= \frac{\iint_{\mathbb{R}^6} \frac{K(\sqrt{\varepsilon_n}x+x_n)K(\sqrt{\varepsilon_n}y+x_n)|u_{1,0}(x)|^{2_s^*-1}|u_{1,0}(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy}{\iint_{\mathbb{R}^6} \frac{|u_{1,0}(x)|^{2_s^*-1}|u_{1,0}(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy} + o_n(1)$$

$$= K^2(\bar{x}) + o_n(1),$$

which implies that $\bar{x} \in M$, where $u_{1,0}(x) = u_{\varepsilon,0}(x)$ for $\varepsilon = 1$. Furthermore, we have

$$\Phi(u_n) = \frac{\iint_{\mathbb{R}^6} \frac{x|u_n(x)-u_\lambda^+(x)|^{2_s^*-1}|u_n(y)-u_\lambda^+(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy}{\iint_{\mathbb{R}^6} \frac{|u_n(x)-u_\lambda^+(x)|^{2_s^*-1}|u_n(y)-u_\lambda^+(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy}$$

$$= \frac{\iint_{\mathbb{R}^6} \frac{x|u_n(x)|^{2_s^*-1}|u_n(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy}{\iint_{\mathbb{R}^6} \frac{|u_n(x)|^{2_s^*-1}|u_n(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy} + o_\lambda(1)$$

$$= \frac{\iint_{\mathbb{R}^6} \frac{x|v_n(x)|^{2_s^*-1}|v_n(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy}{\iint_{\mathbb{R}^6} \frac{|v_n(x)|^{2_s^*-1}|v_n(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy} + o_\lambda(1)$$

$$= \frac{\iint_{\mathbb{R}^6} \frac{x|u_{\varepsilon_n,x_n}(x)|^{2_s^*-1}|u_{\varepsilon_n,x_n}(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy}{\iint_{\mathbb{R}^6} \frac{|u_{\varepsilon_n,x_n}(x)|^{2_s^*-1}|u_{\varepsilon_n,x_n}(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy} + o_\lambda(1)$$

$$= \frac{\iint_{\mathbb{R}^6} \frac{\left(x_n+\sqrt{\varepsilon_n}x\right)|u_{1,0}(x)|^{2_s^*-1}|u_{1,0}(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy}{\iint_{\mathbb{R}^6} \frac{|u_{1,0}(x)|^{2_s^*-1}|u_{1,0}(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy} + o_\lambda(1)$$

$$\to \bar{x} \in M$$

as $n \to \infty, \lambda \to 0$, which is a contradiction with (4.1). $\square$

Lemma 4.5. *There exists a small $\lambda_\delta > 0$ such that if $\lambda \in (0, \lambda_\delta)$ and $u \in N_\lambda^-$ with $I_\lambda(u) < \frac{2s}{3+2s} S^{\frac{3}{2s}} + \frac{\delta_0}{2}, \delta_0$ is given in Lemma 4.4. Then we have $\Phi(u) \in M_\delta$.*

Proof. For $u \in N_\lambda^-$ with $I_\lambda(u) < \frac{2s}{3+2s} S^{\frac{3}{2s}} + \frac{\delta_0}{2}$, we can derive from Lemma 3.1(ii) that there exists a unique t_u^1 such that $t_u^1 u \in N_\infty^1$ and

$$I_\infty^1(t_u^1 u) \le (1-\mu)^{-\frac{3+2s}{4s}} \left(I_\lambda(u) + \frac{2-q}{2q} \mu^{-\frac{q}{2-q}} \lambda^{\frac{2}{2-q}} C \right),$$

for any $\mu \in (0, 1)$. Therefore, there exists a sufficiently small $\lambda_\delta > 0$ such that if $\lambda \in (0, \lambda_\delta)$, then

$$I_\infty^1(t_u^1 u) \le \frac{2s}{3+2s} S^{\frac{3}{2s}} + \delta_0. \tag{4.9}$$

From the last inequality and Lemma 4.4, we obtain the expected results. $\square$

Define the notations

$$c_\lambda := \alpha_\lambda^+ + \frac{2s}{3+2s} S^{\frac{3}{2s}} - \sigma(\varepsilon_0) \quad \text{and} \quad N_\lambda^-(c_\lambda) := \{u \in N_\lambda^-; I_\lambda(u) \leq c_\lambda\}.$$

Lemma 4.6. *If u is a critical point of I_λ restricted on N_λ^-, then it is a critical point of I_λ in $H^s(\mathbb{R}^3)$.*

Proof. Let u be a critical point of I_λ on N_λ^-. Then we get

$$I'_\lambda(u) = \tau \Psi'_\lambda(u)$$

for some $\tau \in \mathbb{R}$, where Ψ_λ is defined in (2.11). Since $u \in N_\lambda^-$, we know

$$0 = I'_\lambda(u)u = \tau \Psi'_\lambda(u)u \quad \text{and} \quad \Psi'_\lambda(u)u < 0,$$

which means that $\tau = 0$, i.e. $I'_\lambda(u) = 0$. □

Denote by $I_{N_\lambda^-}$ the restriction of I_λ on N_λ^-.

Lemma 4.7. *$I_{N_\lambda^-}$ satisfies the (PS) condition on $N_\lambda^-(c_\lambda)$.*

Proof. Let $\{u_n\} \subset N_\lambda^-(c_\lambda)$ be a (PS) sequence. There exists a sequence $\{\theta_n\} \subset \mathbb{R}$ such that

$$I'_\lambda(u_n) = \theta_n \Psi'_\lambda(u_n) + o_n(1).$$

Since $u_n \in N_\lambda^-$, we have $\Psi'_\lambda(u_n)u_n < 0$ and so, there exists a subsequence, still denoted by $\{u_n\}$, such that

$$\Psi'_\lambda(u_n)u_n \to l \leq 0, \quad \text{as } n \to \infty.$$

If $l = 0$, combining (2.10) and (2.12), we infer to

$$\begin{aligned} I_\lambda(u_n) &= I_\lambda(u_n) - \frac{1}{q} I'_\lambda(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2 + \left(\frac{1}{q} - \frac{1}{2(2_s^*-1)}\right)\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx \\ &= \frac{q-2}{2q}\|u_n\|^2 + \frac{2(2_s^*-1)-q}{2(2_s^*-1)q}\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx \\ &= \frac{\left[q-2(2_s^*-1)\right](2_s^*-2)}{2(2_s^*-1)q}\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx + o_n(1) \\ &\leq 0, \end{aligned}$$

which leads to a contradiction with $\alpha_\lambda^- > 0$ (see Lemma 2.6(ii)). Thus $l < 0$. By $I'_\lambda(u_n)u_n = 0$, we derive that $\theta_n \to 0$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. Applying Lemma 2.8, we get the desired results. □

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let δ, $\lambda_\delta > 0$ be given as in Lemmas 4.4 and 4.5. To prove that I_λ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $N_\lambda^-(c_\lambda)$ for $\lambda \in (0, \lambda_\delta)$ and $z \in M$, by Lemma 3.4, we define the map

$$h(z) := u_\lambda^+ + t_z^- v_{\varepsilon,z} \in N_\lambda^-(c_\lambda).$$

If follows from Lemma 4.5 that $\Phi(N_\lambda^-(c_\lambda)) \subset M_\delta$ for $\lambda < \lambda_\delta$. Define $\xi : [0,1] \times M \to M_\delta$ by

$$\xi(\theta, z) := \Phi\big(u_\lambda^+ + t_z^- v_{(1-\theta)\varepsilon,z}\big),$$

where $\Phi\big(u_\lambda^+ + t_z^- v_{(1-\theta)\varepsilon,z}\big) \in N_\lambda^-(c_\lambda)$. By a simple calculation, we have $\xi(0,z) = \Phi \circ h(z)$ and $\lim_{\theta\to 1^-} \xi(\theta, z) = z$. Thus, $\Phi \circ h$ is homotopic to the injective $j : M \to M_\delta$. Based on Lemma 4.7, Propositions 4.1 and 4.2, we deduce that $I_{N_\lambda^-(c_\lambda)}$ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $N_\lambda^-(c_\lambda)$. By Lemma 4.6, we know that I_λ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $N_\lambda^-(c_\lambda)$. Hence the system (1.8) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions in $N_\lambda^-(c_\lambda)$. By virtue of Theorem 2.9 and $N_\lambda^+ \cap N_\lambda^- = \emptyset$, we obtain the desired result.

5. Proof of Theorem 1.2

In this section, we study the existence of multiplicity of solutions to (1.8) with $f_\lambda(x) := \lambda f(x)$. In this situation, the energy function associated with the system (1.8) is defined by

$$J_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + u^2)dx - \frac{1}{2(2_s^* - 1)}\int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} f(x)|u|^q dx,$$

for $u \in H^s(\mathbb{R}^3)$.

We define the Nehari manifold for J_λ as

$$M_\lambda := \{u \in H^s(\mathbb{R}^3)\backslash\{0\} : J'_\lambda(u)u = 0\}.$$

Then $u \in M_\lambda$ if and only if

$$\|u\|^2 - \int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx - \lambda\int_{\mathbb{R}^3} f(x)|u|^q dx = 0.$$

For $u \in M_\lambda$, we have

$$\begin{aligned} J_\lambda(u) &= J_\lambda(u) - \frac{1}{q}J'_\lambda(u)u \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 + \left(\frac{1}{q} - \frac{1}{2(2_s^*-1)}\right)\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 > 0, \end{aligned} \tag{5.1}$$

which implies that J_λ is coercive and bounded from below on M_λ.

Let

$$\Upsilon_\lambda(u) := \|u\|^2 - \int_{\mathbb{R}^3} K(x)\phi_u |u|^{2_s^*-1}dx - \lambda \int_{\mathbb{R}^3} f(x)|u|^q dx. \tag{5.2}$$

For $u \in M_\lambda$, we have

$$\begin{aligned}\Upsilon_\lambda'(u)u &= 2\|u\|^2 - 2(2_s^*-1)\int_{\mathbb{R}^3} K(x)\phi_u |u|^{2_s^*-1}dx - q\lambda \int_{\mathbb{R}^3} f(x)|u|^q dx \\ &= -(q-2)\|u\|^2 - \big(2(2_s^*-1)-q\big)\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx < 0. \end{aligned} \tag{5.3}$$

Define

$$\beta_\lambda := \inf_{u\in M_\lambda} J_\lambda(u).$$

Lemma 5.1. *There exists some $d_0 > 0$ such that $\beta_\lambda \geq d_0 > 0$.*

Proof. For any $u \in M_\lambda$, from Lemma 2.2 and (5.3), we infer to

$$\begin{aligned} 2\|u\|^2 &< 2(2_s^*-1)\int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx + q\lambda\int_{\mathbb{R}^3} f(x)|u|^q dx \\ &\leq 2(2_s^*-1)S^{-2_s^*}\|u\|^{2(2_s^*-1)} + \lambda C\|u\|^q, \end{aligned} \tag{5.4}$$

and thus

$$C \leq \|u\|^{2(2_s^*-2)} + \lambda C\|u\|^{q-2}$$

for some choice of $C > 0$ independent of $u \in M_\lambda$. Using (5.1) we can obtain the desired result. □

Lemma 5.2. (i) *For each $u \in H^s(\mathbb{R}^3)\backslash\{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in M_\lambda$ and*

$$J_\lambda(t_u u) = \max_{t\geq 0} J_\lambda(tu).$$

(ii) *Let $\{u_n\} \subset H^s(\mathbb{R}^3)$ be a sequence such that $J_\lambda'(u_n)u_n \to 0$, and $\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx + \lambda\int_{\mathbb{R}^3} f(x)|u_n|^q dx \to a > 0$ as $n \to \infty$. Then, up to a subsequence, there exists $t_n > 0$ such that*

$$t_n u_n \in M_\lambda \quad \textit{and} \quad \lim_{n\to\infty} t_n \to 1.$$

(iii) $M_\lambda = M_\lambda^-$.

Proof. (i) Let $u \in H^s(\mathbb{R}^3)\backslash\{0\}$ be fixed and denote the function $g(t) := J_\lambda(tu)$ on $[0,\infty)$. Thus, $g(0) = 0, g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for t large. On the other hand, clearly,

$$\begin{aligned} g'(t) = 0 &\Leftrightarrow tu \in M_\lambda \\ &\Leftrightarrow \|u\|^2 = t^{2(2_s^*-1)-2}\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx + \lambda t^{q-2}\int_{\mathbb{R}^3} f(x)|u|^q dx. \end{aligned}$$

which implies that the right side is an increasing function of t. Hence, $\max_{t\geq 0} g(t)$ is achieved at a unique $t_u = t(u) > 0$ so that $g'(t_u u) = 0$ and $t_u u \in M_\lambda$.

(ii) We only need to prove that the limit of t_n as $n \to \infty$. Set

$$a_n := \|u_n\|^2; \quad b_n := \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx; \quad d_n := \lambda\int_{\mathbb{R}^3} f(x)|u_n|^q dx.$$

Then $b_n + d_n \to a > 0$. Up to a subsequence, we may suppose that $a_n \to a, b_n \to b$ and $d_n \to d$ as $n \to \infty$, thus $a = b + d$. By item (i), we can found $t_n > 0$ such that $J'_\lambda(t_n u_n)t_n u_n = 0$, i.e.

$$t_n^2\|u_n\|^2 = |t_n|^{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx + \lambda|t_n|^q\int_{\mathbb{R}^3} f(x)|u_n|^q dx.$$

A simple estimate shows that there are positive constants $T_1, T_2 > 0$ such that $T_1 < t_n < T_2$. Assume that $t_n \to t^*$. Passing to the limit in last equality, we obtain

$$a(t^*)^2 = b(t^*)^{2(2_s^*-1)} + d(t^*)^q,$$

which implies that $t^* = 1$ and so $t_n \to 1$ as $n \to \infty$.

(iii) $M_\lambda = M_\lambda^-$ follows directly from (5.3) and $4 < q < 2_s^*$. □

Remark 5.3. From Lemma 5.2, it is easy to verify that

$$\beta_\lambda = \inf_{u\in H^s(\mathbb{R}^3)\setminus\{0\}} \max_{t\geq 0} J_\lambda(tu).$$

Moreover, we have

$$0 < \beta_{\lambda_1} \leq \beta_{\lambda_2} \leq \beta_0$$

for $\lambda_1 \geq \lambda_2 \geq 0$.

We see that the limit problem associated to system (1.8) reads as

$$\begin{cases} (-\Delta)^s u + u = K_\infty^2 \widetilde{\phi}_u|u|^{2_s^*-3}u + \lambda f_\infty|u|^{q-2}u, & \text{in } \mathbb{R}^3, \\ u \in H^s(\mathbb{R}^3), \end{cases} \tag{5.5}$$

and the solution of the system (5.5) corresponds to the critical point of the energy function defined as

$$J_{\lambda,\infty}(u) = \frac{1}{2}\|u\|^2 - \frac{K_\infty^2}{2(2_s^*-1)}\int_{\mathbb{R}^3}\widetilde{\phi}_u|u|^{2_s^*-1}dx - \frac{\lambda f_\infty}{q}\int_{\mathbb{R}^3}|u|^q dx,$$

where $u \in H^s(\mathbb{R}^3)$.

Define

$$M_{\lambda,\infty} := \{u \in H^s\{\mathbb{R}^3\}\setminus\{0\}; J'_{\lambda,\infty}(u)u = 0\}$$

and

$$\beta_{\lambda,\infty} := \inf_{u\in M_{\lambda,\infty}} J_{\lambda,\infty}(u).$$

In order to give a precise description for the (PS) condition of J_λ, we recall the well-known concentration compactness principle of Lions [30, 31], and the vanishing lemma of Secchi [40].

Proposition 5.4 ([30, 31]). *Let $\rho_n(x) \in L^1(\mathbb{R}^N), N \geq 3$ be a nonnegative sequence satisfying*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \rho_n(x)dx = l > 0.$$

Then there exists a subsequence, still denoted by $\{\rho_n(x)\}$, such that one of the following cases occurs:

(i) (*Compactness*) *There exists $y_n \in \mathbb{R}^N$ such that for any $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{B_R(y_n)} \rho_n(x)dx \geq l - \varepsilon, \quad n = 1, 2, \ldots.$$

(ii) (*Vanishing*) *For any fixed $R > 0$, there holds*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n(x)dx = 0.$$

(iii) (*Dichotomy*) *There exists $\alpha \in (0, l)$ such that for any $\varepsilon > 0$, there exist $n_0 \geq 1$ and $\rho_n^{(1)}(x)$, $\rho_n^{(2)}(x) \in L^1(\mathbb{R}^N)$, for $n \geq n_0$ there holds*

$$\|\rho_n - (\rho_n^{(1)} + \rho_n^{(2)})\|_{L^1(\mathbb{R}^N)} < \varepsilon, \quad \left|\int_{\mathbb{R}^N} \rho_n^{(1)}(x)dx - \alpha\right| < \varepsilon,$$
$$\left|\int_{\mathbb{R}^N} \rho_n^{(2)}(x)dx - (l-\alpha)\right| < \varepsilon$$

and

$$\mathrm{dist}(\mathrm{supp}\rho_n^{(1)}, \quad \mathrm{supp}\rho_n^{(2)}) \to \infty, \quad \text{as } n \to \infty.$$

Lemma 5.5 ([40]). *Assume that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and it satisfies*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u(x)|^2 dx = 0,$$

where $R > 0, N \geq 3$. Then $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for every $2 < r < 2_s^*$.

Lemma 5.6. *J_λ satisfies the $(PS)_c$ condition for $c \in (0, \min\{\beta_{\lambda,\infty}, \frac{2s}{3+2s}S^{\frac{3}{2s}}\})$.*

Proof. Let $\{u_n\} \subset H^s(\mathbb{R}^3)$ be a $(PS)_c$ sequence for J_λ with $c \in (0, \min\{\beta_{\lambda,\infty}, \frac{2s}{3+2s}S^{\frac{3}{2s}}\})$. From (5.1), it is easy to check that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Next, we aim to show that $u_n \to u$ for some $u \in H^s(\mathbb{R}^3)$. For this purpose, set

$$\rho_n(x) = |u_n|^2 + K(x)\phi_{u_n}(x)|u_n|^{2_s^*-1} + \lambda f(x)|u_n|^q,$$

which belongs to $L^1(\mathbb{R}^3)$. We may assume that

$$\|\rho_n\|_1 \to l \geq 0, \quad \text{as } n \to \infty.$$

Then, we claim that $l > 0$. Otherwise, we have $\|u_n\| = \int_{\mathbb{R}^3} \rho_n(x)dx + o_n(1) \to 0$ as $n \to \infty$, and then $J_\lambda(u_n) \to 0$ as $n \to \infty$, which clearly contradicts the assumption of $c > 0$.

It is sufficient to prove the compactness of $\{\rho_n\}$ by means of Proposition 5.4.

Firstly, we assume that $\{\rho_n\}$ vanishes. Then, there exists $R > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 dx = 0.$$

By Lemma 5.5, one has $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for $2 < r < 2_s^*$, and so

$$\int_{\mathbb{R}^3} f(x)|u_n|^q dx \to 0, \quad \text{as } n \to \infty.$$

Hence, we have

$$J'_\lambda(u_n)u_n = \|u_n\|^2 - \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx + o_n(1),$$

and

$$J_\lambda(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{2(2_s^*-1)} \int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)|u_n|^{2_s^*-1}dx + o_n(1).$$

Thus, we may assume that

$$\|u_n\|^2 \to l, \quad \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx \to l$$

for some $l > 0$. Furthermore, from Lemma 2.2, we can obtain $l \geq S^{\frac{3}{2s}}$, and so

$$\begin{aligned} c &= J_\lambda(u_n) + o_n(1) \\ &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2(2_s^*-1)} \int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)|u_n|^{2_s^*-1}dx + o_n(1) \\ &= \frac{2s}{3+2s} l + o_n(1) \\ &\geq \frac{2s}{3+2s} S^{\frac{3}{2s}} + o_n(1), \end{aligned}$$

which contradicts the assumption on c.

Secondly, we assume that the dichotomy occurs. Then for any $\varepsilon > 0$, there exist $\alpha \in (0, l)$, $\{x_n\} \subset \mathbb{R}^3$ and $R_\varepsilon > 0$ such that for any $R > R_\varepsilon$ and $\overline{R} > R_\varepsilon$ there exist

$$\liminf_{n\to\infty} \int_{B_R(x_n)} \rho_n(x)dx \geq \alpha - \varepsilon \quad \text{and} \quad \liminf_{n\to\infty} \int_{\mathbb{R}^3\setminus B_{\overline{R}}(x_n)} \rho_n(x)dx \geq (l-\alpha) - \varepsilon.$$

Thus, there exist $\varepsilon_n \to 0$, $R_n \to +\infty$ and $\overline{R}_n = 4R_n$ such that

$$\int_{B_{R_n}(x_n)} \rho_n(x)dx \geq \alpha - \varepsilon_n \quad \text{and} \quad \int_{\mathbb{R}^3\setminus B_{\overline{R}_n}(x_n)} \rho_n(x)dx \geq (l-\alpha) - \varepsilon_n, \tag{5.6}$$

which implies that

$$\int_{B_{4R_n(x_n)}\setminus B_{R_n}(x_n)} \rho_n(x)dx \leq 2\varepsilon_n. \tag{5.7}$$

Let $\xi_0(s) \in C^\infty(\mathbb{R}_+)$ be a cut-off function such that

$$\xi_0(s) = \begin{cases} 0 & \text{if } s \leq 1 \text{ or } s \geq 4, \\ 1 & \text{if } 2 \leq s \leq 3, \end{cases}$$

and $|\xi_0'(s)| \leq 2$. Set $\xi_n(x) = \xi_0(\frac{|(x-x_n)|}{R_n})$. Since $\{u_n\xi_n\}$ is bounded in $H^s(\mathbb{R}^3)$, and by $J_\lambda'(u_n)(u_n\xi_n) = o_n(1)$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n(-\Delta)^{\frac{s}{2}}(u_n\xi_n)dx + \int_{\mathbb{R}^3} |u_n|^2\xi_n dx \\ = \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}\xi_n dx + \lambda\int_{\mathbb{R}^3} f(x)|u_n|^q\xi_n dx + o_n(1).\end{aligned} \tag{5.8}$$

On the other hand, it can be seen from (5.7)

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}\xi_n dx \leq \int_{B_{4R_n}(x_n)\setminus B_{R_n}(x_n)} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx = o_n(1),$$

$$\int_{\mathbb{R}^3} f(x)|u_n|^q\xi_n dx \leq \int_{B_{4R_n}(x_n)\setminus B_{R_n}(x_n)} f(x)|u_n|^q dx = o_n(1),$$

and

$$\int_{\mathbb{R}^3} |u_n|^2\xi_n dx \leq \int_{B_{4R_n}(x_n)\setminus B_{R_n}(x_n)} |u_n|^2 dx = o_n(1). \tag{5.9}$$

It is easy to see that (5.8) and (5.9) imply that

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n(-\Delta)^{\frac{s}{2}}(\xi_n u_n)dx = o_n(1). \tag{5.10}$$

On the other hand, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(\xi_n u_n)dx \\
&\quad=\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)||\xi_n(x)u_n(x)-\xi_n(y)u_n(y)|}{|x-y|^{3+2s}}dxdy \\
&\quad=\iint_{\mathbb{R}^6}\frac{\big(u_n(x)-u_n(y)\big)^2\xi_n(y)}{|x-y|^{3+2s}}dxdy \\
&\qquad+\iint_{\mathbb{R}^6}\frac{\big(u_n(x)-u_n(y)\big)\big(\xi_n(x)-\xi_n(y)\big)u_n(x)}{|x-y|^{3+2s}}dxdy \\
&\quad\le\iint_{\mathbb{R}^6}\frac{\big(u_n(x)-u_n(y)\big)^2\xi_n(y)}{|x-y|^{3+2s}}dxdy \\
&\qquad+C\left(\iint_{\mathbb{R}^6}\frac{u_n^2(x)|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}}dxdy\right)^{\frac{1}{2}}. \qquad (5.11)
\end{aligned}
$$

In the following, we focus on showing that

$$
\lim_{n\to\infty}\iint_{\mathbb{R}^6}\frac{u_n^2(x)|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}}dxdy=0. \qquad (5.12)
$$

To this aim, we decompose the integral region

$$
\begin{aligned}
\mathbb{R}^3\times\mathbb{R}^3 \;=\;& \{(\mathbb{R}^3\backslash B_{4R_n}(x_n))\times(\mathbb{R}^3\backslash B_{4R_n}(x_n))\}\cup\{B_{R_n}(x_n)\times B_{R_n}(x_n)\} \\
&\cup\{(\mathbb{R}^3\backslash B_{4R_n})\times B_{R_n}\}\cup\{B_{R_n}\times(\mathbb{R}^3\backslash B_{4R_n})\} \\
&\cup\{(B_{4R_n}(x_n)\backslash B_{R_n}(x_n))\times\mathbb{R}^3\} \\
&\cup\{B_{R_n}(x_n)\times(B_{4R_n}(x_n)\backslash B_{R_n}(x_n))\} \\
&\cup\{(\mathbb{R}^3\backslash B_{4R_n}(x_n))\times(B_{4R_n}(x_n)\backslash B_{R_n}(x_n))\} \\
:=\;& \mathbb{X}_1\cup\mathbb{X}_2\cup\mathbb{X}_3\cup\mathbb{X}_4\cup\mathbb{X}_5\cup\mathbb{X}_6\cup\mathbb{X}_7.
\end{aligned}
$$

By the definition of ξ_n, we have

$$
\iint_{\mathbb{X}_1}\frac{|u_n(x)|^2|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}}dxdy=\iint_{\mathbb{X}_2}\frac{|u_n(x)|^2|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}}dxdy=0, \qquad (5.13)
$$

and

$$
\iint_{\mathbb{X}_3}\frac{|u_n(x)|^2|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}}dxdy=\iint_{\mathbb{X}_4}\frac{|u_n(x)|^2|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}}dxdy=0. \qquad (5.14)
$$

When $(x,y) \in \mathbb{X}_5 := (B_{4R_n}(x_n)\backslash B_{R_n}(x_n)) \times \mathbb{R}^3$, by $0 \le \xi_0 \le 1$ and $|\nabla \xi_0| \le 2$, we infer to

$$\begin{aligned}
&\int_{B_{4R_n}(x_n)\backslash B_{R_n}(x_n)} \int_{R^3} \frac{|u_n(x)|^2|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \\
&\quad = \int_{B_{4R_n}(x_n)\backslash B_{R_n}(x_n)} dx \int_{\{y\in\mathbb{R}^3;|x-y|\le R_n\}} \frac{|u_n(x)|^2|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}} dy \\
&\qquad + \int_{B_{4R_n}(x_n)\backslash B_{\mathbb{R}_n}(x_n)} dx \int_{\{y\in\mathbb{R}^3;|x-y|> R_n\}} \frac{|u_n(x)|^2|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}} dy \\
&\quad \le CR_n^{-2}|\nabla \xi_n|^2_{L^\infty(\mathbb{R}^3)} \int_{B_{4R_n}(x_n)\backslash B_{R_n}(x_n)} dx \int_{\{y\in\mathbb{R}^3;|x-y|\le R_n\}} \frac{|u_n(x)|^2}{|x-y|^{1+2s}} dy \\
&\qquad + C\int_{B_{4R_n}(x_n)\backslash B_{\mathbb{R}_n}(x_n)} dx \int_{\{y\in\mathbb{R}^3;|x-y|> R_n\}} \frac{|u_n(x)|^2}{|x-y|^{3+2s}} dy \\
&\quad \le CR_n^{-2s} \int_{B_{4R_n}(x_n)\backslash B_{\mathbb{R}_n}(x_n)} |u_n(x)|^2 dx \\
&\qquad + CR_n^{-2s} \int_{B_{4R_n}(x_n)\backslash B_{\mathbb{R}_n}(x_n)} |u_n(x)|^2 dx \\
&\quad = CR_n^{-2s} \int_{B_{4R_n}(x_n)\backslash B_{\mathbb{R}_n}(x_n)} |u_n(x)|^2 dx. \qquad (5.15)
\end{aligned}$$

When $(x,y) \in \mathbb{X}_6 := B_{R_n}(x_n) \times \big(B_{4R_n}(x_n)\backslash B_{R_n}(x_n)\big)$, then we have $|x-y| \le |x-x_n| + |y-x_n| \le 5R_n$, and

$$\begin{aligned}
&\int_{B_{R_n}(x_n)} \int_{B_{4R_n}(x_n)\backslash B_{R_n}(x_n)} \frac{|u_n(x)|^2|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \\
&\quad \le \int_{B_{R_n}(x_n)} \int_{\{y\in B_{4R_n}(x_n)\backslash B_{R_n}(x_n):|x-y|\le R_n\}} \frac{|u_n(x)|^2|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \\
&\qquad + \int_{B_{R_n}(x_n)} \int_{\{y\in B_{4R_n}(x_n)\backslash B_{R_n}(x_n):|x-y|> R_n\}} \frac{|u_n(x)|^2|\xi_n(x)-\xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \\
&\quad \le R_n^{-2}|\nabla \xi_n|^2_{L^\infty(\mathbb{R}^3)} \int_{B_{R_n}(x_n)} dx \int_{\{y\in B_{4R_n}(x_n)\backslash B_{R_n}(x_n):|x-y|\le R_n\}} \frac{|u_n(x)|^2}{|x-y|^{1+2s}} dy \\
&\qquad + C\int_{B_{R_n}(x_n)} dx \int_{\{y\in B_{4R_n}(x_n)\backslash B_{\mathbb{R}_n}(x_n):|x-y|> R_n\}} \frac{|u_n(x)|^2}{|x-y|^{3+2s}} dy \\
&\quad \le CR_n^{-2s} \int_{B_{R_n}(x_n)} |u_n(x)|^2 dx. \qquad (5.16)
\end{aligned}$$

When $(x, y) \in \mathbb{X}_7 := (\mathbb{R}^3 \backslash B_{4R_n}(x_n)) \times (B_{4R_n}(x_n) \backslash B_{R_n}(x_n))$, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^3 \backslash B_{4R_n}(x_n)} \int_{B_{4R_n}(x_n) \backslash B_{R_n}(x_n)} \frac{|u_n(x)|^2 |\xi_n(x) - \xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \\
&\quad = \int_{\mathbb{R}^3 \backslash B_{4R_n}(x_n)} \int_{\{y \in B_{4R_n}(x_n) \backslash B_{R_n}(x_n) : |x-y| \le R_n\}} \frac{|u_n(x)|^2 |\xi_n(x) - \xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \\
&\qquad + \int_{\mathbb{R}^3 \backslash B_{4R_n}(x_n)} \int_{\{y \in B_{4R_n}(x_n) \backslash B_{R_n}(x_n) : |x-y| > R_n\}} \\
&\qquad \times \frac{|u_n(x)|^2 |\xi_n(x) - \xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \\
&\quad := H_1 + H_2. \qquad (5.17)
\end{aligned}$$

Using the mean value theorem, we see that if $(x, y) \in \mathbb{X}_7$ with $|x - y| \le R_n$, then $|x| \le 5R_n$, and hence

$$\begin{aligned}
H_1 &\le R_n^{-2} |\nabla \xi_n|^2_{L^\infty(\mathbb{R}^3)} \int_{B_{5R_n}(x_n)} dx \int_{\{y \in B_{4R_n}(x_n) \backslash B_{R_n}(x_n) : |x-y| \le R_n\}} \frac{|u_n(x)|^2}{|x-y|^{1+2s}} dy \\
&\le C R_n^{-2} \int_{B_{5R_n}(x_n)} |u_n(x)|^2 dx \int_{\{z \in \mathbb{R}^3 ; |z| \le R_n\}} \frac{1}{|z|^{1+2s}} dz \\
&= C R_n^{-2s} \int_{B_{5R_n}(x_n)} |u_n(x)|^2 dx. \qquad (5.18)
\end{aligned}$$

Observe that for any $K > 8$ it holds

$$\begin{aligned}
\mathbb{X}_7 &= \{(\mathbb{R}^3 \backslash B_{4R_n}(x_n)) \times (B_{4R_n}(x_n) \backslash B_{\mathbb{R}_n}(x_n))\} \\
&\subset \{B_{KR_n}(x_n) \times (B_{4R_n}(x_n) \backslash B_{\mathbb{R}_n}(x_n))\} \\
&\quad \cup \{(\mathbb{R}^3 \backslash B_{KR_n}(x_n)) \times (B_{4R_n}(x_n) \backslash B_{\mathbb{R}_n}(x_n))\}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\int_{B_{KR_n}(x_n)} \int_{\{y \in B_{4R_n}(x_n) \backslash B_{R_n}(x_n) ; |x-y| > R_n\}} \frac{|u_n(x)|^2 |\xi_n(x) - \xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \\
&\quad \le C \int_{B_{KR_n}(x_n)} \int_{\{y \in B_{4R_n}(x_n) \backslash B_{R_n}(x_n) ; |x-y| > R_n\}} \frac{|u_n(x)|^2}{|x-y|^{3+2s}} dxdy \\
&\quad \le C \int_{B_{KR_n}(x_n)} |u_n(x)|^2 dx \int_{\{z \in \mathbb{R}^3 ; |z| > R_n\}} \frac{1}{|z|^{3+2s}} dz \\
&\quad = C R_n^{-2s} \int_{B_{KR_n}(x_n)} |u_n(x)|^2 dx. \qquad (5.19)
\end{aligned}$$

And we note that if $(x,y) \in (\mathbb{R}^3 \backslash B_{KR_n}(x_n)) \times (B_{4R_n}(x_n) \backslash B_{\mathbb{R}_n}(x_n))$, then $|x - y| \geq |x - x_n| - |y - x_n| \geq \frac{|x-x_n|}{2} + \frac{K}{2}R_n - 4R_n > \frac{|x-x_n|}{2}$, we derive that

$$\begin{aligned}
&\int_{\mathbb{R}^3 \backslash B_{KR_n}(x_n)} \int_{\{y \in B_{4R_n}(x_n) \backslash B_{\mathbb{R}_n}(x_n); |x-y| > R_n\}} \frac{|u_n(x)|^2 |\xi_n(x) - \xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \\
&\quad < C \int_{\mathbb{R}^3 \backslash B_{KR_n}(x_n)} dx \int_{\{y \in B_{4R_n}(x_n) \backslash B_{\mathbb{R}_n}(x_n); |x-y| > R_n\}} \frac{|u_n(x)|^2}{|x-y|^{3+2s}} dy \\
&\quad \leq CR_n^3 \int_{\mathbb{R}^3 \backslash B_{KR_n}(x_n)} \frac{|u_n(x)|^2}{|x - x_n|^{3+2s}} dx \\
&\quad \leq CR_n^3 \left(\int_{\mathbb{R}^3 \backslash B_{KR_n}(x_n)} |u_n(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \\
&\qquad \times \left(\int_{\mathbb{R}^3 \backslash B_{KR_n}(x_n)} |x - x_n|^{-(3+2s)\frac{2_s^*}{2_s^*-2}} dx \right)^{\frac{2_s^*-2}{2_s^*}} \\
&\quad \leq CK^{-3} \left(\int_{\mathbb{R}^3 \backslash B_{KR_n}(x_n)} |u_n(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}. \qquad (5.20)
\end{aligned}$$

Using (5.19) and (5.20), we infer to

$$H_2 \leq CR_n^{-2s} \int_{B_{KR_n}(x_n)} |u_n(x)|^2 dx + CK^{-3}. \qquad (5.21)$$

Combining (5.13)–(5.21), we have

$$\iint_{\mathbb{R}^6} \frac{|u_n(x)|^2 |\xi_n(x) - \xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \leq CR_n^{-2s} \int_{B_{KR_n}(x_n)} |u_n(x)|^2 dx + CK^{-3}.$$

Consequently, we get

$$\begin{aligned}
&\limsup_{n \to \infty} \iint_{\mathbb{R}^6} \frac{|u_n(x)|^2 |\xi_n(x) - \xi_n(y)|^2}{|x-y|^{3+2s}} dxdy \\
&\quad = \lim_{K \to \infty} \limsup_{n \to \infty} \iint_{\mathbb{R}^6} \frac{|u_n(x)|^2 |\xi_n(x) - \xi_n(y)|^2}{|x-y|^{3+2s}} dxdy = 0, \qquad (5.22)
\end{aligned}$$

and we derive from (5.10)–(5.12) that

$$\begin{aligned}
&\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} (\xi_n u_n) dx \\
&\quad = \iint_{\mathbb{R}^6} \frac{(u_n(x) - u_n(y))(\xi_n(x) u_n(x) - \xi_n(y) u_n(y))}{|x-y|^{3+2s}} dxdy
\end{aligned}$$

$$= \iint_{\mathbb{R}^6} \frac{\big(u_n(x) - u_n(y)\big)^2 \xi_n(y)}{|x-y|^{3+2s}} dxdy + o_n(1)$$
$$= \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \xi_n dx + o_n(1) = o_n(1). \tag{5.23}$$

Now, we define the cut-off function $\eta_0(s) \in C^\infty(\mathbb{R}_+)$ by $\eta_0(s) = 1$ if $s \le 2$, $\eta_0(s) = 0$, if $s \ge 3$, and $|\eta_0'(s)| \le C$. Set $\eta_n(x) = \eta_0(|x - x_n|/R_n)$, and

$$v_n(x) = \eta_n(x) u_n(x), \quad w_n(x) = (1 - \eta_n(x)) u_n(x).$$

From (5.6), we have

$$\int_{\mathbb{R}^3} (|v_n|^2 + K(x)\phi_{v_n} |v_n|^{2_s^*-1} + \lambda f(x) |v_n|^q) dx \ge \alpha - \varepsilon_n, \tag{5.24}$$

$$\int_{\mathbb{R}^3} \Big(|w_n|^2 + K(x)\phi_{w_n} |w_n|^{2_s^*-1} + \lambda f(x) |w_n|^q\Big) dx \ge (l - \alpha) - \varepsilon_n. \tag{5.25}$$

By (5.22) and (5.23), we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} (w_n) dx \right| \\
&\quad = \left| \iint_{\mathbb{R}^6} \frac{[\eta_n(x)u_n(x) - \eta_n(y)u_n(y)][(1-\eta_n(x))u_n(x) - (1-\eta_n(y))u_n(y)]}{|x-y|^{3+2s}} dxdy \right| \\
&\quad = \left| \iint_{\mathbb{R}^6} \frac{\eta_n(x)(1-\eta_n(x))|u_n(x) - u_n(y)|^2}{|x-y|^{3+2s}} dxdy \right. \\
&\qquad - \iint_{\mathbb{R}^6} \frac{\eta_n(x)(u_n(x) - u_n(y))(\eta_n(x) - \eta_n(y))u(y)}{|x-y|^{3+2s}} dxdy \\
&\qquad + \iint_{\mathbb{R}^6} \frac{(u_n(x) - u_n(y))(1-\eta_n(x))(\eta_n(x) - \eta_n(y))u(y)}{|x-y|^{3+2s}} dxdy \\
&\qquad \left. - \iint_{\mathbb{R}^6} \frac{(\eta_n(x) - \eta_n(y))^2 |u(y)|^2}{|x-y|^{3+2s}} dxdy \right| \\
&\quad \le \left| \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \xi_n dx \right| + C \left[\iint_{\mathbb{R}^6} \frac{(\eta_n(x) - \eta_n(y))^2 |u(x)|^2}{|x-y|^{3+2s}} dxdy \right]^{\frac{1}{2}} \\
&\quad = o_n(1),
\end{aligned} \tag{5.26}$$

where the second integral of the last inequality tends to zero as $n \to \infty$, by using [2, Lemma 2.3]. Consequently, we get

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n|^2 dx + o_n(1). \tag{5.27}$$

From (5.7), we have

$$\int_{\mathbb{R}^3} |v_n(x)w_n(x)|dx \le \int_{B_{3R_n(x_n)}\setminus B_{2R_n}(x_n)} |u_n|^2 dx = o_n(1), \tag{5.28}$$

and then

$$\int_{\mathbb{R}^3} |u_n|^2 dx = \int_{\mathbb{R}^3} |v_n|^2 dx + \int_{\mathbb{R}^3} |w_n|^2 dx + o_n(1). \tag{5.29}$$

In the sequel, we need to consider two cases.

Case 1**.** If $\{x_n\}$ is bounded, it follows from conditions $(H_1)'$ and (H_3) that

$$\begin{aligned}\int_{\mathbb{R}^3} (f(x) - f_\infty)|w_n(x)|^q dx &\le \sup_{|x-x_n|\ge 2R_n} |f(x) - f_\infty| C\|u_n\|^q \\ &\le C \sup_{|x-x_n|\ge 2R_n} |f(x) - f_\infty| = o_n(1), \end{aligned}\tag{5.30}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} K(x)\phi_{w_n}|w_n|^{2_s^*-1}dx - K_\infty^2 \int_{\mathbb{R}^3} \widetilde{\phi}_{w_n}|w_n|^{2_s^*-1}dx\right| \\ &\quad = \left|\iint_{\mathbb{R}^6} \frac{K(x)(K(y)-K_\infty)|w_n(x)|^{2_s^*-1}|w_n(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy \right.\\ &\qquad \left. - \iint_{\mathbb{R}^6} \frac{K_\infty(K(x)-K_\infty)|w_n(x)|^{2_s^*-1}|w_n(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy\right| \\ &\quad \le \int_{\mathbb{R}^3} |K(x)-K_\infty|\widetilde{\phi}_{w_n}|w_n|^{2_s^*-1}dx + K_\infty \int_{\mathbb{R}^3} |K(x)-K_\infty|\widetilde{\phi}_{w_n}|w_n|^{2_s^*-1}dx \\ &\quad \le (1+K_\infty) \sup_{|x-x_n|\ge 2R_n} |K(x)-K_\infty| S^{-2_s^*}\|u_n\|^{2(2_S^*-1)} \\ &\quad \le C \sup_{|x-x_n|\ge 2R_n} |K(x)-K_\infty| = o_n(1). \end{aligned}\tag{5.31}$$

It follows that

$$J_\lambda(w_n) = J_{\lambda,\infty}(w_n) + o_n(1). \tag{5.32}$$

Moreover, we can obtain that

$$J'_\lambda(u_n)w_n = J'_{\lambda,\infty}(w_n)w_n + o_n(1) = o_n(1). \tag{5.33}$$

In fact, we have the following inequality:

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-2}w_n(x)dx - \int_{\mathbb{R}^3} K_\infty^2\widetilde{\phi}_{w_n}|w_n|^{2_s^*-1}dx\right| \\ &\quad = \left|\iint_{\mathbb{R}^6} \frac{K(x)K(y)|u_n(y)|^{2_s^*-1}|u_n(x)|^{2_s^*-2}w_n(x)}{|x-y|^{3-2s}}dxdy \right.\\ &\qquad \left. - \iint_{\mathbb{R}^6} \frac{K_\infty^2|w_n(y)|^{2_s^*-1}|w_n(x)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy\right|\end{aligned}$$

$$\begin{aligned}
&\leq \left|\iint_{\mathbb{R}^6} \frac{(K(x)K(y)-K_\infty^2)|w_n(y)|^{2_s^*-1}|w_n(x)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy\right| \\
&\quad + \left|\iint_{\mathbb{R}^6} \frac{\begin{array}{c}K(x)K(y)(|u_n(y)|^{2_s^*-1}|u_n(x)|^{2_s^*-2}w_n(x)\\ -|w_n(y)|^{2_s^*-1}|w_n(x)|^{2_s^*-1})\end{array}}{|x-y|^{3-2s}}dxdy\right| \\
&:= I_1 + I_2.
\end{aligned} \tag{5.34}$$

For item I_1, by (H_2) we have that

$$\begin{aligned}
I_1 &:= \left|\iint_{\mathbb{R}^6} \frac{(K(x)K(y)-K_\infty^2)|w_n(y)|^{2_s^*-1}|w_n(x)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy\right| \\
&= \left|\iint_{\mathbb{R}^6} \frac{K(x)(K(y)-K_\infty)|w_n(y)|^{2_s^*-1}|w_n(x)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy\right. \\
&\quad \left. + \iint_{\mathbb{R}^6} \frac{K_\infty(K(x)-K_\infty)|w_n(y)|^{2_s^*-1}|w_n(x)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy\right| \\
&\leq \left|\int_{\mathbb{R}^3}(K(y)-K_\infty)\phi_{w_n}|w_n(y)|^{2_s^*-1}dy\right| + \left|\int_{\mathbb{R}^3}(K(x)-K_\infty)\phi_{w_n}|w_n(x)|^{2_s^*-1}dx\right| \\
&\leq 2\sup_{|x-x_n|\geq 2R_n}|K(x)-K_\infty|\int_{\mathbb{R}^3}\phi_{w_n}|w_n(x)|^{2_s^*-1}dx \leq C\varepsilon_n\|w_n\|^{2(2_s^*-1)} = o_n(1).
\end{aligned}$$

For item I_2, by (5.7) we deduce to

$$\begin{aligned}
I_2 &:= \left|\iint_{\mathbb{R}^6} \frac{\begin{array}{c}K(x)K(y)(|u_n(y)|^{2_s^*-1}|u_n(x)|^{2_s^*-2}w_n(x)\\ -|w_n(y)|^{2_s^*-1}|w_n(x)|^{2_s^*-1})\end{array}}{|x-y|^{3-2s}}dxdy\right| \\
&= \left|\iint_{\mathbb{R}^6} \frac{K(x)K(y)[(1-\eta_n(x))-(1-\eta_n(y))^{2_s^*-1}(1-\eta_n(x))^{2_s^*-1}]}{|x-y|^{3-2s}}\right. \\
&\quad \left. \times|u_n(x)|^{2_s^*-1}|u_n(y)|^{2_s^*-1}dxdy\right| \\
&= \left|\int_{B_{2R_n}(y_n)}\int_{B_{3R_n}^c(y_n)} \frac{K(x)K(y)[(1-\eta_n(x))-(1-\eta_n(y))^{2_s^*-1}(1-\eta_n(x))^{2_s^*-1}]}{|x-y|^{3-2s}}\right. \\
&\quad \left. \times|u_n(x)|^{2_s^*-1}|u_n(y)|^{2_s^*-1}dxdy + o_n(1)\right| \\
&\leq \frac{C}{R_n^{3-2s}}\|u_n\|_{2_s^*-1}^{2(2_s^*-1)} \leq \frac{C}{R_n^{3-2s}}\|u_n\|^{2(2_s^*-1)} + o_n(1) \to 0
\end{aligned}$$

as $n \to \infty$, here we have used the fact that $|x - y_n| \geq 3R_n, |y - y_n| \leq 2R_n \Rightarrow |x - y| \geq R_n$, the assumptions on the functions K, η and the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^3)$.

Similarly, according to (5.7) and (5.30), we can obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} f(x)|u_n|^{q-1} w_n dx - \int_{\mathbb{R}^3} f_\infty |w_n|^q dx\right| \\
&\quad = \left|\int_{\mathbb{R}^3} \left(f(x)(|u_n|^{q-1} w_n - |w_n|^q)\right) dx + \int_{\mathbb{R}^3} (f(x) - f_\infty)|w_n|^q dx\right| \\
&\quad \leq \sup_{|x - x_n| \geq R_n} |f(x) - f_\infty| \int_{\mathbb{R}^3} |w_n|^q dx \\
&\qquad + \int_{\mathbb{R}^3} \left[(1 - \eta_n(x)) - (1 - \eta_n(x))^{q-1}\right] f(x)|u_n|^q dx \\
&\quad \leq \sup_{|x - x_n| \geq R_n} |f(x) - f_\infty| \|u_n\|_q^q + \int_{B_{3R_n}(x_n) \setminus B_{2R_n}(x_n)} f(x)|u_n|^q dx + o_n(1) \\
&\quad = o_n(1). \qquad (5.35)
\end{aligned}$$

Using (5.7) and (5.26), we deduce that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} w_n dx - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n|^2 dx\right| \\
&\quad = \left|\iint_{\mathbb{R}^6} \frac{(u_n(x) - u_n(y))(w_n(x) - w_n(y))}{|x - y|^{3+2s}} dxdy \right. \\
&\qquad \left. - \iint_{\mathbb{R}^6} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{3+2s}} dxdy\right| \\
&\quad = \left|\iint_{\mathbb{R}^6} \frac{(w_n(x) - w_n(y))(v_n(x) - v_n(y))}{|x - y|^{3+2s}} dxdy\right| \\
&\quad = \left|\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} w_n (-\Delta)^{\frac{s}{2}} v_n dx\right| = o_n(1), \qquad (5.36)
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} (u_n(x) w_n(x) - |w_n(x)|^2) dx\right| &= \left|\int_{\mathbb{R}^3} |u_n(x)|^2 [1 - \eta_n(x)] \eta_n(x) dx\right| \\
&\leq \int_{B_{3R_n}(x_n) \setminus B_{2R_n(x_n)}} u_n^2 dx \\
&= o_n(1). \qquad (5.37)
\end{aligned}$$

It follows from (5.34)–(5.37) that

$$J'_{\lambda,\infty}(w_n) w_n = J'_\lambda(u_n) w_n = o_n(1). \qquad (5.38)$$

By a similar argument as Lemma 5.2, there exists $t_n \to 1$ such that $t_n w_n \in M_{\lambda,\infty}$ and

$$\begin{aligned}
\beta_{\lambda,\infty} &\le J_{\lambda,\infty}(t_n w_n) \\
&= J_{\lambda,\infty}(t_n w_n) - \frac{1}{q} J'_{\lambda,\infty}(t_n w_n)(t_n V_n) \\
&= \left(\frac{1}{2} - \frac{1}{q}\right) \|t_n w_n\|^2 + \left(\frac{1}{q} - \frac{1}{2(2_s^* - 1)}\right) \int_{\mathbb{R}^3} K_\infty^2 \phi_{t_n w_n} |t_n v_n|^{2_s^*-1} dx \\
&= \left(\frac{1}{2} - \frac{1}{q}\right) \|w_n\|^2 + \left(\frac{1}{q} - \frac{1}{2(2_s^* - 1)}\right) \int_{\mathbb{R}^3} K_\infty^2 \phi_{w_n} |w_n|^{2_s^*-1} dx + o_n(1) \\
&< \left(\frac{1}{2} - \frac{1}{q}\right) \|u_n\|^2 + \left(\frac{1}{q} - \frac{1}{2(2_s^* - 1)}\right) \int_{\mathbb{R}^3} K_\infty^2 \phi_{u_n} |u_n|^{2_s^*-1} dx + o_n(1) \\
&= J_\lambda(u_n) - \frac{1}{q} J'_\lambda(u_n)(u_n) + o_n(1) \\
&= c + o_n(1) \\
&< \beta_{\lambda,\infty},
\end{aligned} \tag{5.39}$$

which raises a contradiction.

Case 2. If $\{x_n\}$ is unbounded, then we have $|x_n| \to +\infty$ as $n \to \infty$. According to condition (H_3), for any $\varepsilon > 0$, there exists an $R' > 0$ such that

$$|K(x) - K_\infty| < \varepsilon, \quad \text{for all } |x| \ge R'.$$

Let $3R_n = |x_n| - R'$. Then $R_n \to +\infty$ and $B_{3R_n}(x_n) \subset \mathbb{R}^3 \backslash B_{R'}(0)$ as $n \to \infty$. Consequently, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} K(x) \phi_{v_n} |v_n|^{2_s^*-1} dx - K_\infty^2 \int_{\mathbb{R}^3} \widetilde{\phi}_{v_n} |v_n|^{2_s^*-1} dx \right| \\
&\quad = \left| \iint_{\mathbb{R}^6} \frac{K(x)(K(y) - K_\infty)|v_n(x)|^{2_s^*-1} |v_n(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy \right. \\
&\qquad \left. - \iint_{\mathbb{R}^6} \frac{K_\infty(K(x) - K_\infty)|v_n(x)|^{2_s^*-1} |v_n(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dxdy \right| \\
&\quad \le \int_{\mathbb{R}^3} |K(x) - K_\infty| \widetilde{\phi}_{v_n} |v_n|^{2_s^*-1} dx + K_\infty \int_{\mathbb{R}^3} |K(x) - K_\infty| \widetilde{\phi}_{v_n} |v_n|^{2_s^*-1} dx \\
&\quad = \int_{B_{3R_n}(x_n)} |K(x) - K_\infty| \widetilde{\phi}_{v_n} |u_n|^{2_s^*-1} |\eta_n(x)|^{2_s^*-1} dx \\
&\qquad + K_\infty \int_{B_{3R_n}(x_n)} |K(x) - K_\infty| \widetilde{\phi}_{v_n} |u_n|^{2_s^*-1} |\eta_n(x)|^{2_s^*-1} dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{B^c_{R'}(0)} |K(x)-K_\infty|\widetilde{\phi}_{v_n}|u_n|^{2^*_s-1}|\eta_n(x)|^{2^*_s-1}dx \\
&\quad + K_\infty \int_{B^c_{R'}(0)} |K(x)-K_\infty|\widetilde{\phi}_{v_n}|u_n|^{2^*_s-1}|\eta_n(x)|^{2^*_s-1}dx \\
&\leq C\varepsilon \int_{B^c_{R'}(0)} \widetilde{\phi}_{u_n}|u_n|^{2^*_s-1}dx \\
&\leq C\varepsilon S^{-2^*_s}\|u_n\|^{2(2^*_S-1)} \leq C\varepsilon.
\end{aligned} \tag{5.40}
$$

Similarly, from condition $(H_1)'$, we can derive

$$
\left|\int_{\mathbb{R}^3} \big(f(x)-f_\infty\big)|v_n(x)|^q dx\right| \leq C\varepsilon.
$$

Consequently, one has

$$
J_\lambda(v_n) = J_{\lambda,\infty}(v_n) + o_n(1). \tag{5.41}
$$

Moreover, we can argue as in Case 1 to show that

$$
J'_\lambda(u_n)v_n = J'_{\lambda,\infty}(v_n)v_n + o_n(1) = o_n(1) \tag{5.42}
$$

and there exists $t_n \to 1$ such that $t_n v_n \in M_{\lambda,\infty}$. Furthermore, we get

$$
\begin{aligned}
\beta_{\lambda,\infty} &\leq J_{\lambda,\infty}(t_n v_n) \\
&= J_{\lambda,\infty}(t_n v_n) - \frac{1}{q}J'_{\lambda,\infty}(t_n v_n)(t_n v_n) \\
&= \left(\frac{1}{2}-\frac{1}{q}\right)\|t_n v_n\|^2 + \left(\frac{1}{q}-\frac{1}{2(2^*_s-1)}\right)\int_{\mathbb{R}^3} K^2_\infty \widetilde{\phi}_{t_n v_n}|t_n v_n|^{2^*_s-1}dx \\
&= \left(\frac{1}{2}-\frac{1}{q}\right)\|v_n\|^2 + \left(\frac{1}{q}-\frac{1}{2(2^*_s-1)}\right)\int_{\mathbb{R}^3} K^2_\infty \widetilde{\phi}_{v_n}|v_n|^{2^*_s-1}dx + o_n(1) \\
&< \left(\frac{1}{2}-\frac{1}{q}\right)\|u_n\|^2 + \left(\frac{1}{q}-\frac{1}{2(2^*_s-1)}\right)\int_{\mathbb{R}^3} K^2_\infty \widetilde{\phi}_{u_n}|u_n|^{2^*_s-1}dx + o_n(1) \\
&= J_\lambda(u_n) - \frac{1}{q}J'_\lambda(u_n)(u_n) + o_n(1) \\
&= c + o_n(1) \\
&< \beta_{\lambda,\infty},
\end{aligned} \tag{5.43}
$$

which yields a contradiction.

Therefore, the dichotomy cannot occur and $\{\rho_n(x)\}$ is compact. That is, there exists $\{x_n\} \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$, there is an $R > 0$ such that

$$
\int_{B^c_R(x_n)} \rho_n(x)dx < \varepsilon, \tag{5.44}
$$

where $B^c_R(x_n) = \mathbb{R}^3 \backslash B_R(x_n)$.

We claim that $\{x_n\}$ is bounded. Otherwise, we suppose that $|x_n| \to +\infty$ when $n \to \infty$. Let $|x_n| \geq R + R'$ for large n. Then we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx - K_\infty^2 \int_{\mathbb{R}^3} \widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx\right| \\
&\quad = \left|\iint_{\mathbb{R}^6} \frac{K(x)(K(y)-K_\infty)|u_n(x)|^{2_s^*-1}|u_n(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy \right. \\
&\qquad \left. - \iint_{\mathbb{R}^6} \frac{K_\infty(K(x)-K_\infty)|u_n(x)|^{2_s^*-1}|u_n(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy\right| \\
&\quad \leq \int_{\mathbb{R}^3} |K(y)-K_\infty|\phi_{u_n}|u_n|^{2_s^*-1}dy + K_\infty \int_{\mathbb{R}^3} |K(x)-K_\infty|\widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx \\
&\quad = \int_{B_{R(x_n)}} |K(x)-K_\infty|\widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx \\
&\qquad + \int_{B^c_{R(x_n)}(x_n)} |K(x)-K_\infty|\widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx \\
&\qquad + K_\infty \int_{B_{R(x_n)}} |K(x)-K_\infty|\widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx \\
&\qquad + K_\infty \int_{B^c_{R(x_n)}} |K(x)-K_\infty|\widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx \\
&\quad \leq C\varepsilon \int_{B^c_{R'}(0)} \widetilde{\phi}_{u_n}|u_n|^{2_s^*-1}dx + C\int_{B^c_{R(x_n)}(x_n)} \phi_{u_n}|u_n|^{2_s^*-1}dx \\
&\quad \leq C_1\varepsilon + C_2\varepsilon \leq C_3\varepsilon. \qquad (5.45)
\end{aligned}$$

In a similar way, we infer to

$$\int_{\mathbb{R}^3} \big(f(x) - f_\infty\big)|u_n|^q dx \leq C\varepsilon.$$

Hence,

$$J_\lambda(u_n) = J_{\lambda,\infty}(u_n) + o_n(1) \quad \text{and} \quad J'_\lambda(u_n)u_n = J'_{\lambda,\infty}(u_n)u_n + o_n(1) = o_n(1).$$

Moreover, there exists $t_n \to 1$ such that $t_n u_n \in M_{\lambda,\infty}$ and

$$c = J_\lambda(u_n) + o_n(1) = J_{\lambda,\infty}(t_n u_n) + o_n(1) \geq \beta_{\lambda,\infty} + o_n(1),$$

which contradicts the definition of c.

Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$, then, there exists $u \in H^s(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$. From Lemma 2.1 and (5.44), as $n \to \infty$, we infer to

$$\int_{\mathbb{R}^3} |u_n|^2 dx \to \int_{\mathbb{R}^3} |u|^2 dx, \quad \int_{\mathbb{R}^3} f(x)|u_n|^q dx \to \int_{\mathbb{R}^3} f(x)|u|^q dx.$$

Set $\Psi_n := u_n - u$. Then by Brézis–Lieb Lemma, Lemma 2.2(vi), we have that

$$\|\Psi_n\|^2 = \|u_n\|^2 - \|u\|^2 + o_n(1),$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{\Psi_n}|\Psi_n|^{2_s^*-1}dx = \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx + o_n(1).$$

According to the weak convergence of $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ and Lemmas 2.1 and 2.2, it is standard to verify $J'_\lambda(u) = 0$, and hence,

$$\frac{1}{2}\|\Psi_n\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{\Psi_n}|\Psi_n|^{2_s^*-1}dx = c - J_\lambda(u) + o_n(1), \quad (5.46)$$

and

$$\begin{aligned} o_n(1) &= J_\lambda(u_n)\Psi_n \\ &= (J'_\lambda(u_n) - J'_\lambda(u))\Psi_n \\ &= \|\Psi_n\|^2 - \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-3}u_n\Psi_n dx + \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-3}u\Psi_n dx. \end{aligned} \quad (5.47)$$

Note that $\Psi_n := u_n - u \rightharpoonup 0$ in $L^{2_s^*}(\mathbb{R}^3)$ and $K(x)\phi_u|u|^{2_s^*-3}u \in L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-3}u\Psi_n dx = o_n(1).$$

Moreover, one has

$$\begin{aligned} &\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-3}u_n\Psi_n dx \\ &\quad = \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \\ &\qquad - \int_{\mathbb{R}^3} K(x)[\phi_{u_n} - \phi_u]|u_n|^{2_s^*-3}u_n u dx \\ &\qquad - \int_{\mathbb{R}^3} K(x)\phi_u[|u_n|^{2_s^*-3}u_n - |u|^{2_s^*-3}u]u dx. \end{aligned} \quad (5.48)$$

Since the sequence $\{(\phi_{u_n} - \phi_u)|u_n|^{2_s^*-3}u_n\}$ is bounded in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$, and

$$\phi_{u_n} \to \phi_u, \quad |u_n|^{2_s^*-3}u_n \to |u|^{2_s^*-3}u \quad \text{a.e. in } \mathbb{R}^3,$$

by [44, Proposition 5.4.7] we have

$$\int_{\mathbb{R}^3} K(x)[\phi_{u_n} - \phi_u]|u_n|^{2_s^*-3}u_n u dx = o_n(1). \quad (5.49)$$

Analogously, we derive to

$$\int_{\mathbb{R}^3} K(x)\phi_u[|u_n|^{2_s^*-3}u_n - |u|^{2_s^*-3}u]u dx = o_n(1). \quad (5.50)$$

Combining (5.49) and (5.50), we conclude

$$\begin{aligned}\int_{\mathbb{R}^3} & K(x)\phi_{u_n}|u_n|^{2_s^*-3}u_n\Psi_n dx \\ &= \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx + o_n(1) \\ &= \int_{\mathbb{R}^3} K(x)\phi_{\Psi_n}|\Psi_n|^{2_s^*-1}dx + o_n(1). \end{aligned} \tag{5.51}$$

Thus, by (5.47)–(5.51) we obtain

$$\|\Psi_n\|^2 - \int_{\mathbb{R}^3} K(x)\phi_{\Psi_n}|\Psi_n|^{2_s^*-1}dx = o_n(1). \tag{5.52}$$

Let

$$\|\Psi_n\|^2 \to \ell \quad \text{and} \quad \int_{\mathbb{R}^3} K(x)\phi_{\Psi_n}|\Psi_n|^{2_s^*-1}dx \to \ell, \quad \text{as } n \to \infty, \tag{5.53}$$

for some $\ell \in [0, +\infty)$. If $\ell = 0$, we have directly $u_n \to u$ in $H^s(\mathbb{R}^3)$. If $\ell \neq 0$, then from Lemma 2.2 we have

$$o_n(1) + \|\Psi_n\|^2 = \int_{\mathbb{R}^3} K(x)\phi_{\Psi_n}|\Psi_n|^{2_s^*-1}dx \leq S^{-2_s^*}\|\Psi_n\|^{2(2_s^*-1)},$$

which indicates $\ell \geq S^{\frac{3}{2s}}$.

On the other hand, by virtue of $J'_\lambda(u) = 0$, we get

$$\begin{aligned} J_\lambda(u) &= J_\lambda(u) - \frac{1}{q}J'_\lambda(u)u \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 + \left(\frac{1}{q} - \frac{1}{2(2_s^*-1)}\right)\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \geq 0. \end{aligned}$$

Thus, from (5.53) we obtain

$$\begin{aligned} c &= \frac{1}{2}\|\Psi_n\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{\Psi_n}|\Psi_n|^{2_s^*-1}dx + J_\lambda(u) + o_n(1) \\ &= \frac{1}{2}\ell - \frac{1}{2(2_s^*-1)}\ell + o_n(1) \\ &\geq \frac{2s}{3+2s}S^{\frac{3}{2s}} + o_n(1) \end{aligned} \tag{5.54}$$

which contradicts the definition of c. Therefore, $\ell = 0$, i.e. $u_n \to u$ in $H^s(\mathbb{R}^3)$. □

In the following, we present some useful estimates which are crucially used in the proof of Theorem 1.2.

Lemma 5.7. *There exists $\lambda_0 > 0$ such that if $\lambda \in (0, \lambda_0)$, then*

$$\frac{2s}{3+2s}S^{\frac{3}{2s}} < \beta_{\lambda,\infty}.$$

Proof. Suppose by contradictory that there exists a sequence $\{\lambda_n\} \to 0$ such that

$$\beta_{\lambda_n,\infty} \le \frac{2s}{3+2s} S^{\frac{3}{2s}}, \quad \text{as } n \to \infty.$$

For each $\lambda_n > 0$. It follows from [14] that there is $u_n \in H^s(\mathbb{R}^3)$ such that

$$J_{\lambda_n,\infty(u_n)} = \alpha_{\lambda_n,\infty} \quad \text{and} \quad J'_{\lambda_n,\infty}(u_n)u_n = 0. \tag{5.55}$$

Hence, we have

$$\begin{aligned}\frac{2s}{3+2s} S^{\frac{3}{2s}} &\ge \beta_{\lambda_n,\infty} = J_{\lambda_n,\infty}(u_n) - \frac{1}{q} I'_{\lambda_n,\infty}(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2 + \left(\frac{1}{q} - \frac{1}{2(2_s^*-1)}\right)\int_{\mathbb{R}^3} K_\infty^2 \phi_{u_n} |u_n|^{2_s^*-1} dx \\ &\ge C\|u_n\|^2,\end{aligned}$$

this means that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. By Lemma 2.1, we infer to

$$\int_{\mathbb{R}^3} \lambda_n f_\infty |u_n|^q dx \le \lambda_n C \|u_n\|^q \to 0, \quad \text{as } n \to \infty. \tag{5.56}$$

Using (5.55) and (5.56), we can assume

$$\|u_n\|^2 \to \ell_1 \quad \text{and} \quad \int_{\mathbb{R}^3} K_\infty^2 \phi_{u_n} |u_n|^{2_s^*-1} dx \to \ell_1, \quad \text{as } n \to \infty,$$

for some $\ell_1 \in [0, +\infty)$. If $\ell_1 = 0$, then

$$\beta_{0,\infty} = 0.$$

This is not possible because $J_{0,\infty}$ has a mountain pass geometry. If $\ell_1 \ne 0$, then from $J'_{\lambda_n,\infty}(u_n)(u_n) = o_n(1)$, by Remark 1.1 we get

$$o_n(1) + \|u_n\|^2 = \int_{\mathbb{R}^3} K_\infty^2 \widetilde{\phi}_{u_n} |u_n|^{2_s^*-1} dx \le S^{-2_s^*} \|u_n\|^{2(2_s^*-1)}$$

which shows that $\ell_1 \ge (K_\infty^2)^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}$. In view of $K_\infty < 1$ and (5.56), we derive to

$$\begin{aligned}\frac{2s}{3+2s} S^{\frac{3}{2s}} &\ge \beta_{\lambda,\infty} = J_{\lambda_n,\infty}(u_n) - \frac{1}{2(2_s^*-1)} J'_{\lambda_n,\infty}(u_n)u_n + o_n(1) \\ &= \frac{2s}{3+2s}\|u_n\|^2 + o_n(1) \\ &\ge \frac{2s}{3+2s}(K_\infty^2)^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}} + o_n(1) > \frac{2s}{3+2s} S^{\frac{3}{2s}} + o_n(1),\end{aligned}$$

which leads to a contradiction. □

Lemma 5.8. *There exist a small $\varepsilon_0 > 0$ and $\sigma(\varepsilon_0) > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\sup_{t\geq 0} J_\lambda(tv_{\varepsilon,z}) < \frac{2s}{3+2s} S^{\frac{3}{2s}} - \sigma(\varepsilon_0) \quad \text{uniformly for } z \in M,$$

where $v_{\varepsilon,z}$ is defined by $v_{\varepsilon,z} = \eta(x-z)U_\varepsilon(x-z)$ given in Lemma 3.4. In addition, there exists $t_z > 0$ such that

$$t_z v_{\varepsilon,z} \in M_\lambda \quad \text{for } z \in M.$$

Proof. Since

$$\lim_{t\to 0} J_\lambda(tv_{\varepsilon,z}) = 0 \quad \text{and} \quad \lim_{t\to\infty} J_\lambda(tv_{\varepsilon,z}) = -\infty,$$

for $z \in M$ and small $\varepsilon > 0$, there exist small $t_0 > 0$ and large $t_1 > 0$ such that

$$J_\lambda(tv_{\varepsilon,z}) < \frac{2s}{3+2s} S^{\frac{3}{2s}}, \quad \text{for } t \in (0, t_0] \cup [t_1, +\infty). \tag{5.57}$$

Now, we show that

$$J_\lambda(tv_{\varepsilon,z}) < \frac{2s}{3+2s} S^{\frac{3}{2s}},$$

for $z \in M$ and $t \in [t_0, t_1]$. For this, by (3.1), Lemma 3.2 and Theorem 2.9 we have

$$\begin{aligned}
J_\lambda(tv_{\varepsilon,z}) &= \frac{t^2}{2}\|v_{\varepsilon,z}\|^2 - \frac{t^{2(2_S^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_S^*-1}dx - \frac{\lambda t^q}{q}\int_{\mathbb{R}^3} f(x)|v_{\varepsilon,z}|^q dx \\
&\leq \frac{2s}{3+2s}\left(\frac{\|v_{\varepsilon,z}\|^2_{D^{s,2}}}{\left(\int_{\mathbb{R}^3} K(x)\phi_{v_{\varepsilon,z}}|v_{\varepsilon,z}|^{2_s^*-1}dx\right)^{\frac{1}{2_s^*-1}}}\right)^{\frac{3+2s}{4s}} \\
&\quad + \frac{t^2}{2}\int_{\mathbb{R}^3}|v_{\varepsilon,z}|^2dx - \lambda C\int_{\mathbb{R}^3}|v_{\varepsilon,z}|^q dx \\
&\leq \frac{2s}{3+2s}\left(\frac{S^{\frac{3}{2s}} + O(\varepsilon^{3-2s})}{\left(S^{\frac{3}{2s}} - O(\varepsilon^{3-2s})\right)^{\frac{1}{2_s^*-1}}}\right)^{\frac{3+2s}{4s}} \\
&\quad + C\int_{\mathbb{R}^3}|v_{\varepsilon,z}|^2dx - \lambda C\int_{\mathbb{R}^3}|v_{\varepsilon,z}|^q dx \\
&= \frac{2s}{3+2s}S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + \begin{cases} O(\varepsilon^{2s}) & \text{if } 2 > \dfrac{3}{3-2s}, \\ O(\varepsilon^{2s}|\log\varepsilon|) & \text{if } 2 = \dfrac{3}{3-2s}, \\ O(\varepsilon^{3-2s}) & \text{if } 2 < \dfrac{3}{3-2s}. \end{cases} \\
&\quad - \lambda C\varepsilon^{\frac{3(2-q)+2sq}{2}},
\end{aligned} \tag{5.58}$$

for $z \in M$ and small $\varepsilon > 0$. Next, we separate three cases:

Case 1: $s < \frac{3}{4} \Leftrightarrow 2 > \frac{3}{3-2s}$. In this case, we get $3 - 2s > 2s > \frac{3(2-q)+2sq}{2}$ and

$$O(\varepsilon^{3-2s}) + O(\varepsilon^{2s}) - \lambda C\varepsilon^{\frac{3(2-q)+2sq}{2}} < 0 \quad \text{for } \varepsilon \text{ small enough.}$$

Case 2: $s = \frac{3}{4} \Leftrightarrow 2 = \frac{3}{3-2s}$. In this case, we choose $\lambda = \varepsilon^{\frac{1}{2}}$, by a simple calculation, we infer that

$$O(\varepsilon^{\frac{3}{2}}) + O(\varepsilon^{\frac{3}{2}}|\log\varepsilon|) - \lambda C\varepsilon^{\frac{3(2-q)+2sq}{2}} < 0 \quad \text{if } \varepsilon \text{ small enough.}$$

Case 3: $s > \frac{3}{4} \Leftrightarrow 2 < \frac{3}{3-2s}$. In this case, we have $\frac{3(2-q)+2sq}{2} < 3 - 2s$ due to $q > 4 > \frac{4s}{3-2s}$, and so,

$$O(\varepsilon^{3-2s}) + O(\varepsilon^{3-2s}) - \lambda C\varepsilon^{\frac{3(2-q)+2sq}{2}} < 0 \quad \text{as } \varepsilon \text{ sufficiently enough.}$$

Combining Cases 1–3 and (5.58), there exists a small $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$,

$$\sup_{t\geq 0} J_\lambda(tv_{\varepsilon,z}) < \frac{2s}{3+2s}S^{\frac{3}{2s}} - \sigma(\varepsilon_0),$$

for $z \in M$. In addition, from Lemma 5.2, there exists $t_z^- > 0$ such that $t_z^- v_{\varepsilon,z} \in M_\lambda^-$ for each $z \in M$. □

Next, we define a continuous map $\Theta : H^s(\mathbb{R}^3)\backslash\{0\} \to \mathbb{R}^3$ as

$$H^s(\mathbb{R}^3) \ni u \mapsto \Theta(u) := \frac{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{x|u(x)|^{2_s^*-1}|u(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy}{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{2_s^*-1}|u(y)|^{2_s^*-1}}{|x-y|^{3-2s}}dxdy}.$$

Then we have the following conclusion.

Lemma 5.9. *For each $0 < \delta < r_0$, there exist $\overline{\lambda}_\delta, \overline{\delta}_0 > 0$ such that if $u \in N_\infty^1$ with $I_\infty^1(u) < \frac{2s}{3+2s}S^{\frac{3}{2s}} + \overline{\delta_0}$, and $\lambda \in (0, \overline{\lambda}_\delta)$, then $\Theta(u) \in M_\delta$.*

The proof of Lemma 5.8 is similar to that of Lemma 4.4, and we omitted it.

Lemma 5.10. *There exists a small $\overline{\lambda}_\delta > 0$ such that if $\lambda \in (0, \overline{\lambda}_\delta)$ and $u \in M_\lambda$ with $J_\lambda(u) < \frac{2s}{3+2s}S^{\frac{3}{2s}} + \frac{\overline{\delta}_0}{2}$, where $\overline{\delta}_0$ is given in Lemma 5.9. Then $\Phi(u) \in M_\delta$.*

Proof. For $u \in M_\lambda$ with $J_\lambda(u) < \frac{2s}{3+2s} + \frac{\bar{\delta}_0}{2}$, we have from (5.1) and Lemma 5.1 that

$$0 < \|u\| \leq C. \tag{5.59}$$

And there exists a unique $t_u > 0$ such that $t_u u \in N_\infty^1$. We claim that $t_u \leq C$ for some $C > 0$ independent of u. Otherwise, there exists a sequence $\{u_n\} \subset M_\lambda$

satisfying $t_u u_n \in N^1_\infty$ and $\{t_{u_n}\} \to \infty$ as $n \to \infty$, as a result we have

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2^*_s-1}dx = t_{u_n}^{2(2-2^*_s)}\|u_n\|^2 \to 0, \quad \text{as } n \to \infty. \tag{5.60}$$

Since $u_n \in M_\lambda$, from (5.59) and (5.60) we have

$$\begin{aligned}\|u_n\|^2 &= \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2^*_s-1}dx + \lambda \int_{\mathbb{R}^3} f(x)|u_n|^q dx \\ &\le \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2^*_s-1}dx + \lambda C\|u_n\|^q \to 0,\end{aligned}$$

as $n \to \infty$ and $\lambda \to 0$, which contradicts Lemma 5.1.

On the other hand, by $u \in M_\lambda$, we have

$$\begin{aligned}\frac{2s}{3+2s}S^{\frac{3}{2s}} + \frac{\overline{\delta_0}}{2} &\ge J_\lambda(u) = \sup_{t\ge 0} J_\lambda(tu) \\ &\ge J_\lambda(t_u u) \\ &= I^1_\infty(t_u u) - \frac{\lambda}{q}\int_{\mathbb{R}^3}(f(x) - f_\infty)|u_n|^q dx \\ &\ge I^1_\infty(t_u u) - \lambda C\|u_n\|^q.\end{aligned} \tag{5.61}$$

Then

$$I^1_\infty(t_u u) \le \frac{2s}{3+2s}S^{\frac{3}{2s}} + \frac{\overline{\delta_0}}{2} + \lambda C\|u_n\|^q. \tag{5.62}$$

By (5.59) and (5.62), there exists $\overline{\lambda}_\delta > 0$ such that for $\lambda \in (0, \overline{\lambda}_\delta)$, we have

$$I^1_\infty(t_u u) \le \frac{2s}{3+2s}S^{\frac{3}{2s}} + \overline{\delta}_0.$$

By virtue of Lemma 5.9, we get that $\Phi(u) = \Phi(t_u u) \in M_\delta$. □

Set

$$\overline{c}_\lambda := \frac{2s}{3+2s}S^{\frac{3}{2s}} - \sigma(\varepsilon_0) \quad \text{and} \quad M_\lambda(\overline{c}_\lambda) := \{u \in M_\lambda; J_\lambda(u) \le \overline{c}_\lambda\}.$$

Lemma 5.11. *If u is a critical point of J_λ restricted on M_λ, then it is a critical point of J_λ in $H^s(\mathbb{R}^3)$.*

The proof is almost similar to Lemma 2.4, and we omit it here.

Lemma 5.12. *$J_\lambda|_{M_\lambda}$ satisfies the (PS) condition on $M_\lambda(\overline{c}_\lambda)$, where $J_\lambda|_{M_\lambda}$ denotes the restriction of J_λ on M_λ.*

Proof. Let $\{u_n\} \subset M_\lambda(\overline{c}_\lambda)$ be a (PS) sequence. Then, there exists a sequence $\{\theta_n\} \subset \mathbb{R}$ such that

$$J'_\lambda(u_n) = \theta_n \Upsilon'_\lambda(u_n) + o_n(1).$$

From (5.3), we know that $\Upsilon'_\lambda(u_n)u_n < 0$, and so, there exists a subsequence, still denoted by $\{u_n\}$, such that

$$\Upsilon'_\lambda(u_n)u_n \to l \leq 0, \quad \text{as } n \to \infty.$$

If $l = 0$, by (5.3) we have

$$\begin{aligned} 0 \leftarrow \Upsilon'_\lambda(u_n)u_n &= -(q-2)\|u_n\|^2 - (2(2^*_s - 1) - q)\int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)|u_n|^{2^*_s - 1}dx \\ &\leq -(q-2)\|u_n\|^2 \leq 0. \end{aligned}$$

Therefore,

$$\|u_n\| \to 0, \quad \text{as } n \to \infty,$$

which yields a contradiction with Lemma 5.1. Hence $l < 0$. Since $J'_\lambda(u_n)u_n = 0$, we derive that $\{\theta_n\} \to 0$ and $J'_\lambda(u_n) \to 0$, as $n \to \infty$. According to Lemma 5.6, we have the desired conclusion. □

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let δ, $\overline{\lambda}_\delta > 0$ be given as in Lemmas 5.9 and 5.10. For each $z \in M$, let $G(z) = t_z v_{\varepsilon,z}$. From Lemma 5.8, it belongs to $M_\lambda(\overline{c}_\lambda)$. It follows from Lemma 5.10 that $\Theta(M_\lambda(\overline{c}_\lambda)) \subset M_\delta$ for $\lambda < \lambda_\delta$. Define the map $\zeta : [0,1] \times M \to M_\delta$ by

$$[0,1] \times M \ni (\theta, z) \mapsto \zeta(\theta, z) = \Theta(t_z v_{(1-\theta)\varepsilon, z}) \in M^-_\lambda(\overline{c}_\lambda).$$

By a direct calculation, we have $\zeta(0,z) = \Theta \circ G(z)$ and $\lim_{\theta \to 1^-} \zeta(\theta, z) = z$. Thus, $\Theta \circ G$ is homotopic to the injective $j : M \to M_\delta$. By virtue of Lemma 5.12, Proposition 4.1 and 4.2, we obtain that $J_{M^-_\lambda(\overline{c}_\lambda)}$ has at least $\operatorname{cat}_{M_\delta}(M)$ critical points in $M^-_\lambda(\overline{c}_\lambda)$. By Lemma 5.11, we know that J_λ has at least $\operatorname{cat}_{M_\delta}(M)$ critical points in $M^-_\lambda(\overline{c}_\lambda)$. Thus, the system (1.8) has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions in $H^s(\mathbb{R}^3)$. □

Acknowledgments

The authors would like to thank the anonymous reviewers for their careful reading of the manuscript and giving valuable comments. This work is supported by NSFC (11771468, 11971027, 12171497). The research of V. D. Rădulescu was supported by the grant “Nonlinear Differential Systems in Applied Sciences” of the Romanian Ministry of Research, Innovation and Digitization, within PNRR-III-C9-2022-I8 (Grant No. 22).

ORCID

Xilin Dou https://orcid.org/0009-0008-7619-9999
Xiaoming He https://orcid.org/0000-0001-5846-4009
Vicenţiu D. Rădulescu https://orcid.org/0000-0003-4615-5537

References

[1] A. Ambrosetti, On Schrödinger-Poisson systems, *Milan J. Math.* **76** (2008) 257–274.

[2] V. Ambrosio and T. Isernia, Multiplicity and concentration results for some nonlinear Schrödinger equations with the fractional p-Laplacian, *Discrete Contin. Dyn. Syst.* **38** (2018) 5835–5881.

[3] V. Ambrosio, T. Isernia and V. D. Rădulescu, Concentration of positive solutions for a class of fractional p-Kirchhoff type equations, *Proc. Roy. Soc. Edinburgh Sect. A* **151**(2) (2021) 601–651.

[4] C. O. Alves, M. A. S. Souto and S. H. Soares, Schrödinger–Poisson equations without Ambrosetti–Rabinowitz condition, *J. Math. Anal. Appl.* **377** (2011) 584–592.

[5] A. Azzollini, P. d'Avenia and G. Vaira, Generalized Schrödinger–Newton system in dimension $N \geq 3$: Critical case, *J. Math. Anal. Appl.* **449** (2017) 531–552.

[6] V. Benci and G. Cerami, Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$ in $\mathbb{R}^N$, *J. Funct. Anal.* **88** (1990) 90–117.

[7] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger–Maxwell equation, *Topol. Methods Nonlinear Anal.* **11** (1998) 283–293.

[8] V. Benci and D. Fortunato, Solitary waves of the nonlinear Klein–Gordon equation coupled with the Maxwell equations, *Notices Amer. Math. Soc.* **51** (2004) 1336–1347.

[9] L. A. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007) 1245–1260.

[10] G. Cerami and G. Vaira, Positive solutions for some non-autonomous Schrödinger–Poisson systems, *J. Differential Equations* **248** (2010) 521–543.

[11] S. Cingolani and M. Lazzo, Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions, *J. Differential Equations* **160** (2000) 118–138.

[12] T. D'Aprile and D. Mugnai, Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations, *Proc. Roy. Soc. Edinburgh Sect. A* **134** (2004) 893–906.

[13] J. M. do Ó, O. H. Miyagaki and M. Squassina, Critical and subcritical fractional problems with vanishing potentials, *Commun. Contemp. Math.* **18** (2016) 1550063.

[14] X. Dou and X. He, Ground states for critical fractional Schrödinger–Poisson systems with vanishing potentials, *Math. Methods Appl. Sci.* **45** (2022) 9089–9110.

[15] H. Fan, Z. Feng and X. Yan, Positive solutions of fractional Schrödinger–Poisson systems involving critical nonlinearities with potential, *Differential Integral Equations* **35** (2022) 1–56.

[16] X. Feng, Existence and concentration of ground state solutions for doubly critical Schrödinger–Poisson-type systems, *Z. Angew. Math. Phys.* **71**(5) (2020), Paper No. 154, 25 pp.

[17] X. Feng, Nontrivial solution for Schrödinger–Poisson equations involving the fractional Laplacian with critical exponent, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM* **115**(1) (2021), Paper No. 10, 19 pp.

[18] D. Goel and K. Sreenadh, Kirchhoff equations with Hardy–Littlewood–Sobolev critical nonlinearity, *Nonlinear Anal.* **186** (2019) 162–186.

[19] X. He, Positive solutions for fractional Schrödinger–Poisson systems with doubly critical exponents, *Appl. Math. Lett.* **120** (2021) 107190.

[20] X. He and W. Zou, Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities, *Calc. Var. Partial Differential Equations* **55** (2016) 1–39.

[21] N. Laskin, Fractional quantum mechanics and Lévy path integrals, *Phys. Lett. A* **268** (2000) 298–305.

[22] N. Laskin, Fractional Schrödinger equation, *Phys. Rev. E.* **66** (2002) 56–108.

[23] C. Lei, V. D. Rădulescu and B. Zhang, Groundstates of the Schrödinger–Poisson–Slater equation with critical growth, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM* **117**(3) (2023), Paper No. 128, 34 pp.

[24] K. Li, Bound state solutions for non-autonomous fractional Schrödinger–Poisson equations with critical exponent, preprint (2019), arXiv:1905.11643 [math.AP].

[25] N. Li and X. He, Existence and multiplicity results for some Schrödinger–Poisson system with critical growth, *J. Math. Anal. Appl.* **488** (2020) 124071.

[26] F. Li, Y. Li and J. Shi, Existence of positive solutions to Schrödinger–Poisson type systems with critical exponent, *Commun. Contemp. Math.* **16** (2014) 1450036.

[27] F. Li, Y. Li and J. Shi, Existence and multiplicity of positive solutions to Schrödinger–Poisson type systems with critical nonlocal term, *Calc. Var. Partial Differential Equations* **56** (2017) 134.

[28] Y. Li, B. Zhang and X. Han, Existence and concentration behavior of positive solutions to Schrödinger–Poisson–Slater equations, *Adv. Nonlinear Anal.* **12**(1) (2023), Paper No. 20220293, 23 pp.

[29] E. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, Vol. 14 (American Mathematical Society, Providence, RI, 2001).

[30] P. L. Lions, The concentration-compactness principle in the calculus of variation. The locally compact case. Part I, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **1** (1984) 223–283.

[31] P. L. Lions, The concentration-compactness principle in the calculus of variation. The locally compact case. Part II *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984) 109–145.

[32] H. Liu, Positive solutions of an asymptotically periodic Schrödinger–Poisson system with critical exponent, *Nonlinear Anal. Real World Appl.* **32** (2016) 198–212.

[33] Z. Liu and J. Zhang, Multiplicity and concentration of positive solutions for the fractional Schrödinger–Poisson systems with critical growth, *ESAIM: Control Optim. Calc. Var.* **23** (2017) 1515–1542.

[34] H. Luo and X. Tang, Ground state and multiple solutions for the fractional Schrödinger–Poisson system with critical Sobolev exponent, *Nonlinear Anal. Real World Appl.* **42** (2018) 24–52.

[35] G. Molica Bisci, V. D. Rădulescu and R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics and its Applications, Vol. 162 (Cambridge University Press, Cambridge, 2016).

[36] D. Mugnai, Non-existence results for the coupled Klein–Gordon–Maxwell equations, *Adv. Nonlinear Stud.* **4** (2004) 307–322.

[37] E. Murcia and G. Siciliano, Positive semiclassical states for a fractional Schrödinger–Poisson system, *Differential Integral Equations* **30** (2017) 231–258.

[38] S. Qu and X. He, On the number of concentrating solutions of a fractional Schrödinger–Poisson system with doubly critical growth, *Anal. Math. Phys.* **12**(2) (2022), Paper No. 59, 49 pp.

[39] D. Ruiz, The Schrödinger–Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.* **237** (2006) 655–674.

[40] S. Secchi, Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$, *J. Math. Phys.* **54** (2013) 031501, doi:10.1063/1.4793990.

[41] R. Servadei and E. Valdinoci, The Brezis–Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.* **367** (2015) 67–102.

[42] K. Teng, Existence of ground state solutions for the nonlinear fractional Schrödinger–Poisson system with critical Sobolev exponent, *J. Differential Equations* **261** (2016) 3061–3106.

[43] M. Willem, *Minimax Theorems* (Birkhäuser, Boston, 1996).

[44] M. Willem, *Functional Analysis*, *Fundamentals and Applications*, Cornerstones (Birkhäuser/Springer, New York, 2013).

[45] Z. Yang, Y. Yu and F. Zhao, Concentration behavior of ground state solutions for a fractional Schrödinger–Poisson system involving critical exponent, *Commun. Contemp. Math.* **21**(6) (2019) 1850027.

[46] R. Yin, J. Zhang and X. Shang, Positive ground state solutions for Schrödinger–Poisson system with critical nonlocal term in $\mathbb{R}^3$, *Math. Methods Appl. Sci.* **43** (2020) 8736–8752.

[47] Y. Yu, F. Zhao and L. Zhao, The concentration behavior of ground state solutions for a fractional Schrödinger–Poisson system, *Calc. Var. Partial Differential Equations* **56** (2017), Art. 116, 25 pp.

[48] J. Zhang, J. M. do Ó and M. Squassina, Fractional Schrödinger–Poisson systems with a general subcritical or critical nonlinearity, *Adv. Nonlinear Stud.* **16** (2016) 15–30.

[49] W. Zhang, J. Zhang and V. D. Rădulescu, Concentrating solutions for singularly perturbed double phase problems with nonlocal reaction, *J. Differential Equations* **347** (2023) 56–103.

[50] L. Zhao and F. Zhao, Positive solutions for Schrödinger–Poisson equations with a critical exponent, *Nonlinear Anal.* **70** (2009) 2150–2164.