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NONLINEAR ELLIPTIC PROBLEMS WITH SUPERLINEAR REACTION AND PARAMETRIC CONCAVE BOUNDARY CONDITION

BY

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ABSTRACT

We study parametric nonlinear elliptic boundary value problems driven by the p-Laplacian with convex and concave terms. The convex term appears in the reaction and the concave in the boundary condition (source). We study the existence and nonexistence of positive solutions as the parameter $\lambda > 0$ varies. For the semilinear problem ($p = 2$), we prove a bifurcation type result. Finally, we show the existence of nodal (sign changing) solutions.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric boundary value problem:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) = f(z, u(z)) \text{ in } \Omega, & 1 < p < \infty, \\ \dfrac{\partial u}{\partial n_p} = \lambda\beta(z)u(z)^{q-1} \text{ on } \partial\Omega, & u > 0. \end{cases}$$

In this equation, by Δ_p we denote the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega).$$

Also $\frac{\partial u}{\partial n_p}$ denotes the generalized normal derivative corresponding to the p-Laplacian and defined by $\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\mathbb{R}^N}$, with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The reaction $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \longmapsto f(z, x)$ is measurable and for almost all $z \in \Omega$, $x \longmapsto f(z, x)$ is continuous), which is $(p-1)$-superlinear near $+\infty$. In the boundary condition, $\lambda > 0$ is a parameter, $\beta \in L^\infty(\Omega)_+$, $\beta \neq 0$ and $1 < q < p$. So, problem (P_λ) is an alternative version of the well-known "concave-convex" problem (problem with competing nonlinearities) in which a "convex" (superlinear) reaction $f(z, x)$ is coupled with a "sublinear" parametric source term. The original "concave-convex" problem had both the competing nonlinearities in the reaction, which had the form $\lambda x^{q-1} + x^{r-1}$ for all $x \geqslant 0$, with $\lambda > 0$ being the parameter and

$$1 < q < p < r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leqslant p, \end{cases}$$

where p^* is the critical Sobolev exponent.

The study of such problems was launched with the pioneering works of Garcia Azorero and Peral [10], and Ambrosetti, Brezis and Cerami [2]. In the first paper, among other results, the critical case is considered for small values of the parameter. Ambrosetti, Brezis and Cerami [2] investigated the following semilinear Dirichlet problem:

$$-\Delta u(z) = \lambda u(z)^{q-1} + u(z)^{r-1} \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,\ u > 0,$$

with $1 < q < 2 < r < 2^*$. They proved bifurcation-type results describing the set of positive solutions as the parameter $\lambda > 0$ varies. Their work was extended to equations driven by the p-Laplacian, by Garcia Azorero, Manfredi and Peral Alonso [8] and Guo and Zhang [13]. Further extensions with more general reactions can be found in Filippakis, Kristaly and Papageorgiou [6] and Iannizzotto and Papageorgiou [15]. We also refer to Boccardo, Escobedo and Peral [4], who studied the branch of minimal solutions without growth hypotheses. Problems in which the competing nonlinearities come from both the reaction (the convex term) and the source (the concave term) were first considered by Garcia Azorero, Peral and Rossi [9] for semilinear problems with a reaction of the form $f(x) = x^{r-1}$ for all $x \geqslant 0$, where $1 < 2 < r < 2^*$. Semiliear problems with a more general reaction were studied recently by Furtado and Ruviaro [7]. Generalizations to p-Laplacian equations with a reaction of the form $f(x) = x^{r-1}$ for all $x \geqslant 0$, where $1 < p < q < p^*$, can be found in the work of Sabina de Lis [21]. We stress that in all of the aforementioned works, the differential operator (left-hand side of the equation) has the form $-\Delta_p u + u^{p-1}$ (with $p = 2$ in [7], [9]). This operator is coercive and this facilitates the analysis. In contrast, in problem (P_λ) the differential operator is not coercive.

In Section 3, for problem (P_λ), we prove a theorem concerning the existence and nonexistence of positive solutions, depending on the value of the parameter $\lambda > 0$. We also show the existence of a minimal positive solution $\underline{u}_\lambda$ and investigate the properties of the map $\lambda \longmapsto \underline{u}_\lambda$. If $p = 2$ (semilinear problem), then we prove a bifurcation result describing in a more precise way the existence and multiplicity of positive solutions as the parameter $\lambda > 0$ varies. It is an interesting open problem whether such a bifurcation result is also possible for the p-Laplacian equation. In Section 4 we prove the existence of nodal (sign changing) solutions.

In the next section, for easy reference we recall the main mathematical tools which we will use in the sequel and fix our notation.

2. Mathematical background

In what follows X is a Banach space and X^* is its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the "C-condition" if the following is true:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$, such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ. It is needed because the ambient space is not in general locally compact (since X in general is infinite dimensional). With this compactness-type condition on φ, one can prove a deformation theorem which leads to a minimax theory for the critical values of φ. One of the main results in this theory, is the so-called "mountain pass theorem" due to Ambrosetti and Rabinowitz [3]. Here we state it in a slightly more general form (see Gasinski and Papageorgiou [11, p. 648]).

Theorem 1: *Assume that $\varphi \in C^1(X)$ satisfies the C-condition, $u_0, u_1 \in X$, $||u_1 - u_0|| < p < 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : ||u - u_0|| = \rho] = m_\rho$$

and $c = \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} \ \varphi(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}.$$

Then $c \geqslant m_\rho$ and c is a critical value of φ.

In the study of problem (P_λ), in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0$ for all $z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

By $||\cdot||$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$. We recall that

$$||u|| = [||u||_p^p + ||Du||_p^p]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. With this measure, we can define the Lebesgue spaces $L^p(\partial\Omega)$ $(1 \leqslant p \leqslant \infty)$. From the trace theorem, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, with the property that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in C^1(\overline{\Omega})$. The trace map is compact into $L^q(\partial\Omega)$ with $1 \leqslant q < \frac{Np-p}{N-p}$ and we have

$$\operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)\Big(\frac{1}{p} + \frac{1}{p'} = 1\Big), \quad \ker \gamma_0 = W_0^{1,p}(\Omega).$$

In the sequel, for notational simplicity, we drop the use of the trace map γ_0. Every Sobolev function defined on $\partial\Omega$ is understood in the sense of traces.

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z,x)| \leqslant a_0(z)(1 + |x|^{r-1}) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^*$. We set $F_0(z,x) = \int_0^x f_0(z,s)ds$. Let $\beta \in L^\infty(\partial\Omega)$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}||Du||_p^p - \frac{1}{q}\int_{\partial\Omega} \beta(z)|u(z)|^q d\sigma - \int_\Omega F_0(z,u(z))dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From Papageorgiou and Rădulescu [19], we have the following result.

Proposition 2: *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \text{ for all } h \in C^1(\overline{\Omega}) \text{ with } ||h||_{C^1(\overline{\Omega})} \leqslant \rho_0,$$

then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \text{ for all } h \in W^{1,p}(\Omega) \text{ with } ||h|| \leqslant \rho_1.$$

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

From Papageorgiou and Kyritsi [18, p. 314], we have:

Proposition 3: *The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$ in $W^{1,p}(\Omega)$.*

Finally, let us fix our notation. For $\varphi \in C^1(X)$, by K_φ we denote the set of critical points of φ, that is, $K_\varphi = \{u \in X : \varphi'(u) = 0\}$. Also, if $x \in \mathbb{R}$, then $x^\pm = \max\{0, \pm x\}$. Given $u \in W^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$ and we have

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Also, if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function), then

$$N_h(u)(\cdot) = h(\cdot, u(\cdot))$$

(the Nemytskii map corresponding to h). Evidently, the mapping

$$z \longmapsto N_h(u)(z) = f(z, u(z))$$

is measurable.

3. Positive solutions

In this section, we study the existence and nonexistence of positive solutions for problem (P_λ) as $\lambda > 0$ varies. We also prove the existence of a minimal positive solution $\underline{u}_\lambda$ and examine the properties of the map $\lambda \longmapsto \underline{u}_\lambda$.

The hypotheses on the reaction $f(z,x)$ are the following:

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for almost all $z \in \Omega$, $f(z,0) = 0$ and

(i) $|f(z,x)| \leqslant a(z)(1 + x^{r-1})$ for almost all $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)_+$, $p < r < p^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then there exists $\eta > p$ and $M > 0$ such that

$$0 < \eta F(z,x) \leqslant f(z,x)x \quad \text{for almost all } z \in \Omega, \text{ all } x \geqslant M,$$
$$\operatorname{essinf} F(\cdot, M) > 0;$$

(iii) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for almost all $z \in \Omega$ and for all $\tau > 0$, $f(z,x) \geqslant \mu_\tau > 0$ for almost all $z \in \Omega$, all $x \geqslant \tau$;

(iv) if $p = 2$, then for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for almost all $z \in \Omega$, the map $x \longmapsto f(z,x) + \xi_\rho x$ is nondecreasing on $[0,\rho]$.

Remark 1: Since we are interested in positive solutions and the above hypotheses concern the positive semi-axis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that $f(z,x) = 0$ for almost all $z \in \Omega$, all $x \leqslant 0$. Hypothesis H_1(ii) is the well-known Ambrosetti-Rabinowitz condition (AR-condition for short) (see [3]). It implies that

$$c_1 x^\eta \leqslant F(z,x) \text{ for almost all } z \in \Omega, \text{ all } x \geqslant M, \text{ some } c_1 > 0 \tag{1}$$

(see, for example, Papageorgiou and Kyritsi [18, p. 424]). It is an interesting open problem whether we can replace the AR-condition by a more general superlinearity condition, like the one employed by Gasinski and Papageorgiou [12] and Iannizzotto and Papageorgiou [15]. The noncoercivity of the differential operator together with the boundary term $\lambda\beta(z)x^{q-1}$ for all $(z,x) \in \partial\Omega \times \mathbb{R}_+$ raise serious technical difficulties and make the use of more general superlinearity conditions problematic. Hypothesis H_1(iv) is satisfied if, for almost all $z \in \Omega$, $f(z,\cdot) \in C^1(\mathbb{R})$ and $f'_x(z,x)$ is locally $L^\infty(\Omega)$-bounded.

Example 1: The following functions satisfy hypotheses H_1:

$$f_1(x) = x^{r-1} \text{ for all } x \geqslant 0, \text{ with } p < r < p^*,$$

$$f_2(x) = \begin{cases} 0 & \text{if } x < 0, \\ cx^{s-1} - x^{\tau-1} & \text{if } 0 \leqslant x \leqslant 1, \\ (c-1)x^{\eta-1} & \text{if } 1 < x, \end{cases} \quad \text{with } c > 1,\ p < s < \tau \text{ and } p < \eta < p^*,$$

$$f_3(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^{\tau-1} & \text{if } 0 \leqslant x \leqslant 1, \\ x^{\eta-1} & \text{if } 1 < x, \end{cases} \quad \text{with } p < \tau, \eta \text{ and } \eta < p^*.$$

The hypotheses on the boundary term are:

$\widehat{H}$: $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1]$, $\beta \geqslant 0$, $\beta \neq 0$ and $1 < q < p$.

We introduce the following Carathéodory function:

$$\hat{f}(z,x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ f(z,x) + x^{p-1} & \text{if } 0 < x. \end{cases} \tag{2}$$

Let

$$\widehat{F}(z,x) = \int_0^x \hat{f}(z,s)ds$$

and, for every $\lambda > 0$, we consider the C^1-functional $\hat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p - \frac{\lambda}{q}\int_{\partial\Omega} \beta(z)u^+(z)^q d\sigma - \int_\Omega \widehat{F}(z,u(z))dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proposition 4: *If hypotheses* H_1(i), (ii), (iii) *and* $\widehat{H}$ *hold, then the functional* $\hat{\varphi}_\lambda$ *satisfies the* C*-condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$|\hat{\varphi}_\lambda(u_n)| \leqslant M_1 \text{ for some } M_1 > 0, \text{ all } n \geqslant 1, \tag{3}$$

$$(1 + ||u_n||)\hat{\varphi}'_\lambda(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{4}$$

From (4) we have

$$
(5)\quad
\begin{aligned}
&|\langle \hat{\varphi}'_\lambda(u_n), h\rangle| \leqslant \frac{\epsilon_n ||h||}{1+||u_n||} \text{ for all } h \in W^{1,p}(\Omega), \text{ with } \epsilon_n \to 0^+,\\
&\Rightarrow \Big| \langle A(u_n), h\rangle + \int_\Omega |u_n|^{p-2}u_n h dz\\
&\qquad - \lambda \int_{\partial\Omega} \beta(z)(u_n^+)^{q-1} h d\sigma - \int_\Omega \hat{f}(z,u_n) h dz \Big|\\
&\leqslant \frac{\epsilon_n ||h||}{1+||u_n||} \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \geqslant 1.
\end{aligned}
$$

In (5) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then

$$
(6)\quad
\begin{aligned}
&\frac{1}{p}||Du_n^-||_p^p + \frac{1}{p}||u_n^-||_p^p \leqslant \epsilon_n \text{ for all } n \geqslant 1 \text{ (see (2))},\\
&\Rightarrow u_n^- \to 0 \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty.
\end{aligned}
$$

From (3), (6) and hypothesis H_1(i), we have

$$
(7)\quad
\begin{aligned}
&||Du_n^+||_p^p - \frac{\lambda p}{q}\int_{\partial\Omega} \beta(z)(u_n^+)^q d\sigma - \int_\Omega pF(z,u_n^+)dz \leqslant M_2\\
&\qquad \text{for some } M_2 > 0, \text{ all } n \geqslant 1.
\end{aligned}
$$

Also, in (5) we choose $h = u_n^+ \in W^{1,p}(\Omega)$ and obtain

$$
(8)\quad
\begin{aligned}
&-||Du_n^+||_p^p + \lambda \int_{\partial\Omega} \beta(z)(u_n^+)^q d\sigma + \int_\Omega f(z,u_n^+)u_n^+ dz \leqslant \epsilon_n\\
&\qquad \text{for all } n \geqslant 1 \text{ (see (4))}.
\end{aligned}
$$

Adding (7) and (8), we obtain

$$
(9)\quad
\begin{aligned}
&\int_\Omega [f(z,u_n^+)u_n^+ - pF(z,u_n^+)]dz \leqslant M_3 + \lambda\Big(\frac{p}{q}-1\Big)\int_{\partial\Omega} \beta(z)(u_n^+)^q d\sigma\\
&\qquad \text{for some } M_3 > 0, \text{ all } n \geqslant 1,\\
&\qquad \leqslant c_2(1+||u_n^+||^q) \text{ for some } c_2 > 0, \text{ all } n \geqslant 1\\
&\qquad \text{(use the trace theorem)},\\
&\Rightarrow \int_\Omega [f(z,u_n^+)u_n^+ - \eta F(z,u_n^+)]dz + (\eta - p)\int_\Omega F(z,u_n^+)dz\\
&\qquad \leqslant c_2(1+||u_n^+||^q) \text{ for all } n \geqslant 1,\\
&\Rightarrow (\eta - p)\int_\Omega F(z,u_n^+)dz \leqslant c_2(1+||u_n^+||^q) \text{ for all } n \geqslant 1\\
&\qquad \text{(see hypothesis } H_1\text{(ii))}.
\end{aligned}
$$

From (1) and hypothesis H_1(i), we have

$$c_1 x^\eta - c_3 \leqslant F(z,x) \text{ for almost all } z \in \Omega,\ \text{all } x \geqslant 0,\ \text{some } c_3 > 0. \tag{10}$$

Using (10) in (9), we obtain

$$\begin{aligned} ||u_n^+||_p^p \leqslant & c_4(1 + ||u_n^+||^q)^{p/\eta} \text{ for some } c_4 > 0,\ \text{all } n \geqslant 1 \\ \leqslant & c_4(1 + ||u_n^+||^{qp/\eta}) \text{ (recall } p < \eta) \\ \leqslant & c_5(1 + ||u_n^+||^q) \text{ for some } c_5 > 0,\ \text{all } n \geqslant 1. \end{aligned} \tag{11}$$

From (3), (6) and hypothesis H_1(i) we have

$$\begin{aligned} \frac{\eta}{p}||Du_n^+||_p^p - \frac{\lambda\eta}{q}\int_{\partial\Omega}\beta(z)(u_n^+)^q d\sigma - \int_\Omega \eta F(z,u_n^+)dz \leqslant M_4 \\ \text{for some } M_4 > 0,\ \text{all } n \geqslant 1. \end{aligned} \tag{12}$$

Adding (8) and (12), we obtain

$$\begin{aligned} & \Big(\frac{\eta}{p} - 1\Big)||Du_n^+||_p^p + \int_\Omega [f(z,u_n^+)u_n^+ - \eta F(z,u_n^+)]dz \\ & \qquad \leqslant M_5 + \lambda(\frac{\eta}{q} - 1)\int_{\partial\Omega}\beta(z)(u_n^+)^q d\sigma \\ & \qquad\qquad \text{for some } M_5 > 0,\ \text{all } n \geqslant 1, \\ & \Rightarrow ||Du_n^+||_p^p \leqslant c_6(1 + ||u_n^+||^q) \text{ for some } c_6 > 0,\ \text{all } n \geqslant 1 \\ & \quad \text{(see hypotheses } H_1\text{(i), (ii), recall that } p < \eta \text{ and use the trace theorem)} \end{aligned} \tag{13}$$

From (11) and (13) and recalling that $u \longmapsto ||u||_\eta + ||Du||_p$ is an equivalent norm on $W^{1,p}(\Omega)$ (see, for example, Gasinski and Papageorgiou [11, p. 227]), we infer that

$$||u_n^+||^p \leqslant c_7(1 + ||u_n^+||^q) \text{ for some } c_7 > 0,\ \text{all } n \geqslant 1.$$

Since $q < p$ (see hypotheses $\widehat{H}$), we conclude that

$$\{u_n^+\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{14}$$

From (6) and (14) it follows that $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded. Using the Sobolev embedding theorem and the trace theorem and by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty. \tag{15}$$

In (5) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (15). Then

$$\begin{aligned}
&\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0, \\
&\qquad\Rightarrow u_n \to u \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see Proposition 3)}, \\
&\qquad\Rightarrow \hat{\varphi}_\lambda \text{ satisfies the } C\text{-condition.}
\end{aligned}$$

This completes the proof. ∎

Proposition 5: *If hypotheses H_1(i), (ii), (iii) and $\widehat{H}$ hold, then there exists $\lambda_+ > 0$ such that for every $\lambda \in (0, \lambda_+)$, there exists $\rho_\lambda > 0$ for which we have*

$$\inf[\hat{\varphi}_\lambda(u) : ||u|| = \rho_\lambda] = \hat{m}_\lambda > 0 = \hat{\varphi}_\lambda(0).$$

Proof. Hypotheses H_1(i), (iii) imply that given $\epsilon > 0$, we can find $c_8 = c_8(\epsilon) > 0$ such that

$$F(z,x) \leqslant \epsilon x^p + c_8 x^r \text{ for almost all } z \in \Omega, \text{ all } x \geqslant 0. \tag{16}$$

Then for every $u \in W^{1,p}(\Omega)$, we have

$$\hat{\varphi}_\lambda(u) \geqslant \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p - \lambda c_9||u||^q - c_{10}||u||^r - \epsilon||u||^p - \frac{1}{p}||u||^p \text{ (see (2))}. \tag{17}$$

Since $q < p < r$, we have

$$||u||^p \leqslant \lambda ||u||^q + c_{11}||u||^r \text{ for some } c_{11} = c_{11}(\lambda) > 0.$$

Returning to (17) and choosing $\epsilon > 0$ small, we have

$$\begin{aligned}
\hat{\varphi}_\lambda(u) \geqslant & c_{12}||u||^p - \lambda c_{13}||u||^q - c_{14}||u||^r \\
& \text{with } c_{12} = c_{12}(\epsilon) > 0,\ c_{13} = c_9 + \frac{1}{p} > 0,\ c_{14} = c_{10} + \frac{1}{p} > 0 \\
= & [c_{12} - (\lambda c_{13}||u||^{q-p} + c_{14}||u||^{r-p})]||u||^p.
\end{aligned} \tag{18}$$

Let $\vartheta_\lambda(t) = \lambda c_{13} t^{q-p} + c_{14} t^{r-p}$ for all $t > 0$. Evidently $\vartheta_\lambda \in C^1(0, +\infty)$ and since $q < p < r$, we have

$$\vartheta_\lambda(t) \to +\infty \text{ as } t \to 0^+ \text{ and as } t \to +\infty.$$

So we can find $t_0 \in (0,+\infty)$ such that

$$\begin{aligned}\vartheta_\lambda(t_0) &= \inf_{t>0} \vartheta_\lambda(t),\\ &\Rightarrow \vartheta'_\lambda(t_0) = 0,\\ &\Rightarrow \lambda(p-q)c_{13} = (r-p)c_{14}t_0^{r-q},\\ &\Rightarrow t_0 = t_0(\lambda) = \Big[\frac{\lambda(p-q)c_{13}}{(r-p)c_{14}}\Big]^{\frac{1}{r-q}}.\end{aligned}$$

Evidently, we have

$$\vartheta_\lambda(t_0) \to 0^+ \text{ as } \lambda \to 0^+.$$

So from (18) we see that we can find $\lambda_+ > 0$ such that

$$\vartheta_\lambda(t_0) < c_{12} \text{ for all } \lambda \in (0,\lambda_+).$$

Therefore

$$\hat{\varphi}_\lambda(u) \geqslant \hat{m}_\lambda > 0 = \hat{\varphi}_\lambda(0) \text{ for all } u \in W^{1,p}(\Omega) \text{ with } ||u|| = \rho_\lambda = t_0(\lambda). \qquad \blacksquare$$

An immediate consequence of the AR-condition (see (1)) is the following proposition.

Proposition 6: *If hypotheses H_1(i), (ii), (iii) and $\widehat{H}$ hold $\lambda > 0$ and $u \in \operatorname{int} C_+$, then*

$$\hat{\varphi}_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty.$$

We introduce the following sets:

$$\begin{aligned}\mathcal{L} =&\{\lambda > 0: \text{ problem } (P_\lambda) \text{ admits a positive solution}\},\\ S(\lambda) =& \text{ set of positive solutions for problem } (P_\lambda).\end{aligned}$$

Proposition 7: *If hypotheses H_1(i), (ii), (iii) and $\widehat{H}$ hold, then $\mathcal{L} \neq \varnothing$ and, for every $\lambda > 0$, $S(\lambda) \subseteq \operatorname{int} C_+$.*

Proof. Let $\lambda_+ > 0$ be as in Proposition 5. We fix $\lambda \in (0,\lambda_+)$. Then Propositions 5 and 6 imply that the functional $\hat{\varphi}_\lambda$ satisfies the mountain pass geometry. This fact, together with Proposition 4, permit the use of Theorem 1 (the mountain pass theorem). So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$u_0 \in K_{\hat{\varphi}_\lambda} \quad \text{and} \quad \hat{\varphi}_\lambda(0) = 0 < \hat{m}_\lambda \leqslant \hat{\varphi}_\lambda(u_0). \tag{19}$$

From (19) we see that $u_0 \neq 0$ and

$$\begin{aligned} &\hat{\varphi}'_\lambda(u_0) = 0 \\ (20) \quad &\Rightarrow \langle A(u_0), h\rangle + \int_\Omega |u_0|^{p-2}u_0 h dz = \lambda \int_\Omega \beta(z)(u_0^+)^{q-1} h d\sigma + \int_\Omega \hat{f}(z, u_0) h dz \\ &\qquad \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

In (20) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Using (2), we have

$$\begin{aligned} &||Du_0^-||_p^p + ||u_0^-||_p^p = 0, \\ &\Rightarrow u_0 \geqslant 0, \ u_0 \neq 0. \end{aligned}$$

So, (20) becomes

$$\begin{aligned} (21) \quad &\langle A(u_0), h\rangle = \lambda \int_{\partial\Omega} \beta(z) u_0^{q-1} h d\sigma + \int_\Omega f(z, u_0) h dz \\ &\qquad \text{for all } h \in W^{1,p}(\Omega) \text{ (see (2))}. \end{aligned}$$

In what follows, by $\langle \cdot, \cdot \rangle_0$ we denote the duality brackets for the pair

$$(W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*, \, W_0^{1,p}(\Omega)) \quad \Big(\frac{1}{p} + \frac{1}{p'} = 1\Big).$$

From the representation theorem for the elements of the dual space

$$W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$$

(see, for example, Gasinski and Papageorgiou [11, p. 212]), we have

$$\Delta_p u_0 \in W^{-1,p'}(\Omega).$$

Then integration by parts gives

$$(22) \qquad \langle A(u_0), h\rangle = \langle -\Delta_p u_0, h\rangle_0 \text{ for all } h \in W_0^{1,p}(\Omega) \subseteq W^{1,p}(\Omega).$$

We return to (21) and use (22). Recall that $\ker \gamma_0 = W_0^{1,p}(\Omega)$. So we have

$$\begin{aligned} &\langle -\Delta_p u_0, h\rangle_0 = \int_\Omega f(z, u_0) h dz \text{ for all } h \in W_0^{1,p}(\Omega) \\ (23) \quad &\Rightarrow -\Delta_p u_0(z) = f(z, u_0(z)) \text{ for almost all } z \in \Omega \\ &\qquad \text{(recall } L^{r'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)). \end{aligned}$$

Hypothesis H_1(i) implies that $f(\cdot, u_0(\cdot)) \in L^{r'}(\Omega)$. Since $W_0^{1,r}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ continuously and densely (recall $p < r$), we have $W^{-1,p'}(\Omega) \hookrightarrow W^{-1,r'}(\Omega)$ continuously and densely (see, for example, Gasinski and Papageorgiou [11, p. 141]).

Then because of (23) we can apply the nonlinear Green's identity (see, for example, Gasinski and Papageorgiou [11, p. 210]) and have

$$\langle A(u_0), h\rangle + \int_\Omega (\Delta_p u_0) h dz = \left\langle \frac{\partial u_0}{\partial n_p}, h \right\rangle_{\partial\Omega} \text{ for all } h \in W^{1,r}(\Omega) \hookrightarrow W^{1,p}(\Omega) \quad \text{(see (23))}. \tag{24}$$

Here, by $\langle \cdot, \cdot \rangle_{\partial\Omega}$ we denote the duality brackets for the pair

$$(W^{-\frac{1}{p'},p'}(\partial\Omega), \quad W^{\frac{1}{p'},p}(\partial\Omega)).$$

If we use (21) and (23) in (24), we obtain

$$\lambda \int_{\partial\Omega} \beta(z) u_0^{q-1} h d\sigma = \left\langle \frac{\partial u_0}{\partial n_p}, h \right\rangle_{\partial\Omega} \text{ for all } h \in W^{1,p}(\Omega) \quad \text{recall } W^{1,r}(\Omega) \text{ is dense in } W^{1,p}(\Omega)). \tag{25}$$

Recall that

$$\gamma_0(W^{1,p}(\Omega)) = W^{\frac{1}{p'},p}(\partial\Omega).$$

So from (25) it follows that

$$\frac{\partial u_0}{\partial n_p} = \lambda\beta(z) u_0^{q-1} \text{ on } \partial\Omega,$$

$$\Rightarrow u_0 \in S(\lambda) \text{ and so } (0, \lambda_+) \subseteq \mathcal{L}, \text{ hence } \mathcal{L} \neq \varnothing.$$

From Winkert [22] we have $u_0 \in L^\infty(\Omega)$ and then Theorem 2 of Lieberman [16] implies that $u_0 \in C_+ \backslash \{0\}$.

Let $\rho = ||u_0||_\infty$. Hypotheses H_1(i), (iii) imply that we can find $\hat{x}_\rho > 0$ such that

$$f(z, x) + \hat{\xi}_\rho x^{p-1} \geqslant 0 \text{ for almost all } z \in \Omega, \text{ all } x \in [0, \rho]. \tag{26}$$

From (23) we obtain

$$\Delta_p u_0(z) \leqslant \hat{\xi}_\rho u_0(z)^{p-1} \text{ for almost all } z \in \Omega,$$

$$\Rightarrow u_0 \in \operatorname{int} C_+ \text{ (by the nonlinear maximum principle, see [11, p. 738]).}$$

The above argument shows that

$$S(\lambda) \subseteq \operatorname{int} C_+ \text{ for all } \lambda > 0$$

(of course, if $\lambda \notin \mathcal{L}$, then $S(\lambda) = \varnothing$). ∎

Next we prove a useful structural property of the set $\mathcal{L}$, which shows that $\mathcal{L}$ is an interval.

Proposition 8: *If hypotheses H_1(i), (ii), (iii) and $\widehat{H}$ hold, $\lambda \in \mathcal{L}$ and $\nu \in (0,\lambda)$, then $\nu \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 7). We introduce the following Carathéodory functions:

$$k(z,x) = \begin{cases} f(z,x) + (x^+)^{p-1} & \text{if } x \leqslant u_\lambda(z) \\ f(z,u_\lambda(z)) + u_\lambda(z)^{p-1} & \text{if } u_\lambda(z) < x \end{cases} \quad \text{for all } (z,x)\in\Omega\times\mathbb{R}, \tag{27}$$

$$\gamma_\nu(z,x) = \begin{cases} \nu\beta(z)(x^+)^{q-1} & \text{if } x \leqslant u_\lambda(z) \\ \nu\beta(z)u_\lambda(z)^{q-1} & \text{if } u_\lambda(z) < x \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{28}$$

Let $K(z,x) = \int_0^x k(z,s)ds$ and $\Gamma_\nu(z,x) = \int_0^x \gamma_\nu(z,s)ds$. We consider the C^1-functional $\hat{\psi}_\nu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\nu(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p - \int_{\partial\Omega} \Gamma_\nu(z,u(z))d\sigma - \int_\Omega K(z,u(z))dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (27) and (28), it is clear that $\hat{\psi}_\nu(\cdot)$ is coercive. Also, using the Sobolev embedding theorem and the compactness of the trace map, we have that $\hat{\psi}_\nu$ is sequentially weakly lower semicontinuous. So the Weierstrass theorem implies that we can find $u_\nu \in W^{1,p}(\Omega)$ such that

$$\hat{\psi}_\nu(u_\nu) = \inf[\hat{\psi}_\nu(u) : u \in W^{1,p}(\Omega)]. \tag{29}$$

Let $m_\lambda = \min_{\overline{\Omega}} u_\lambda > 0$ (recall that $u_\lambda \in \operatorname{int} C_+$). Because of hypothesis H_1(iii), given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) \in (0, m_\lambda)$ such that

$$F(z,x) \geqslant -\frac{\epsilon}{p}x^p \text{ for almost all } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{30}$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small such that $tu(z) \in (0,\delta]$ for all $z \in \overline{\Omega}$. Then we have

$$\hat{\psi}_\nu(tu) \leqslant \frac{t^p}{p}||Du||_p^p - \frac{\nu t^q}{q}\int_{\partial\Omega} \beta(z)u^q d\sigma + \frac{\epsilon t^p}{p}||u||_p^p \quad \text{(see (27), (28) and (30))}. \tag{31}$$

Since $q < p$, from (31) we see that by choosing $t \in (0,1)$ even smaller if necessary, we can have

$$\begin{aligned}&\hat{\psi}_\nu(tu) < 0,\\ &\qquad \Rightarrow \hat{\psi}_\nu(u_\nu) < 0 = \hat{\psi}_\nu(0) \text{ (see (29)), hence } u_\nu \neq 0.\end{aligned}$$

From (29) we have

$$\begin{aligned}&\hat{\psi}'_\nu(u_\nu) = 0,\\ &\Rightarrow \langle A(u_\nu), h\rangle + \int_\Omega |u_\nu|^{p-2}u_\nu h dz = \int_{\partial\Omega} \gamma_\nu(z, u_\nu) h d\sigma + \int_\Omega k(z, u_\nu) dz \\ &\qquad\qquad \text{for all } h \in W^{1,p}(\Omega).\end{aligned} \tag{32}$$

In (32) first we choose $h = -u_\nu^- \in W^{1,p}(\Omega)$. We obtain

$$\begin{aligned}&||Du_\nu^-||_p^p + ||u_\nu^-||_p^p = 0 \text{ (see (27), (28))},\\ &\qquad \Rightarrow u_\nu \geqslant 0,\ u_\nu \neq 0.\end{aligned}$$

Next in (32) we choose $(u_\nu - u_\lambda)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}&\langle A(u_\nu), (u_\nu - u_\lambda)^+\rangle + \int_\Omega u_\nu^{p-1}(u_\nu - u_\lambda)^+ dz\\ &\quad = \int_{\partial\Omega} \nu\beta(z) u_\lambda^{q-1}(u_\nu - u_\lambda)^+ d\sigma + \int_\Omega f(z, u_\lambda)(u_\nu - u_\lambda)^+ dz\\ &\qquad + \int_\Omega u_\lambda^{p-1}(u_\nu - u_\lambda)^+ dz \text{ (see (27),(28))}\\ &\quad \leqslant \int_{\partial\Omega} \lambda\beta(z) u_\lambda^{q-1}(u_\nu - u_\lambda)^+ d\sigma + \int_\Omega f(z, u_\lambda)(u_\nu - u_\lambda)^+ dz\\ &\qquad + \int_\Omega u_\lambda^{p-1}(u_\nu - u_\lambda)^+ dz \text{ (since } \nu < \lambda \text{ and see } \widehat{H})\\ &\quad = \langle A(u_\lambda), (u_\nu - u_\lambda)^+\rangle + \int_\Omega u_\lambda^{p-1}(u_\nu - u_\lambda)^+ dz,\\ &\qquad \Rightarrow \langle A(u_\nu) - A(u_\lambda), (u_\nu - u_\lambda)^+\rangle + \int_\Omega (u_\nu^{p-1} - u_\lambda^{p-1})(u_\nu - u_\lambda)^+ dz \leqslant 0,\\ &\qquad \Rightarrow |\{u_\nu > u_\lambda\}|_N = 0, \text{ hence } u_\nu \leqslant u_\lambda.\end{aligned}$$

So we have proved that

$$u_\nu \in [0, u_\lambda] = \{u \in W^{1,p}(\Omega) : 0 \leqslant u(z) \leqslant u_\lambda(z) \text{ for almost all } z \in \Omega\}.$$

Using (27) and (28), equation (32) becomes

$$\begin{aligned}\langle A(u_\nu), h\rangle = \nu \int_{\partial\Omega} \beta(z) u_\nu^{q-1} h d\sigma + \int_\Omega f(z, u_\nu) h dz,\\ &\Rightarrow u_\nu \in S(\nu) \subseteq \operatorname{int} C_+ \text{ (see the proof of Proposition 7)},\\ &\Rightarrow \nu \in \mathcal{L} \text{ and so } (0, \lambda] \subseteq \mathcal{L}.\end{aligned}$$

The proof is now complete. ∎

Remark 2: As a consequence of Proposition 8, we see that $\mathcal{L}$ is an interval.

An interesting byproduct of the above proof is the following corollary.

Corollary 9: *If hypotheses H_1(i), (ii), (iii) and $\widehat{H}$ hold,*

$$\lambda \in \mathcal{L}, \quad u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+ \quad \textit{and} \quad \nu \in (0, \lambda),$$

then there exists $u_\nu \in S(\nu) \subseteq \operatorname{int} C_+$ such that $u_\nu \leqslant u_\lambda$.

In the semilinear case ($p = 2$), we can improve the above corollary by bringing into play hypothesis H_1(iv). We will need this result in order to produce a second positive solution for problem (P_λ) when $\lambda \in (0, \lambda^* = \sup \mathcal{L})$.

Proposition 10: *If $p = 2$ (semilinear problem), hypotheses H_1 and $\widehat{H}$ hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ and $\nu \in (0, \lambda)$, then there exists $u_\nu \in S(\nu) \subseteq \operatorname{int} C_+$ such that $u_\lambda - u_\nu \in \operatorname{int} C_+$.*

Proof. From Corollary 9, we already have a solution $u_\nu \in S(\nu) \subseteq \operatorname{int} C_+$ such that

$$u_\nu \leqslant u_\lambda. \tag{33}$$

Let $\rho = ||u_\lambda||_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H_1(iv)$. Then

$$\begin{aligned}-\Delta_p u_\nu(z) + \xi_\rho u_\nu(z) &= f(z, u_\nu(z)) + \xi_\rho u_\nu(z)\\ &\leqslant f(z, u_\lambda(z)) + \xi_\rho u_\lambda(z) \text{ (see (33) and hypothesis } H_1\text{(iv))}\\ &= -\Delta u_\lambda(z) + \xi_\rho u_\lambda(z) \text{ for almost all } z \in \Omega \text{ (since } u_\lambda \in S(\lambda)),\\ \Rightarrow \Delta(u_\lambda - u_\nu)(z) &\leqslant \xi_\rho (u_\lambda - u_\nu)(z) \text{ for almost all } z \in \Omega,\\ \Rightarrow u_\lambda - u_\nu &\in \operatorname{int} C_+ \text{ (from the maximum principle, see [11, p. 738])}. \quad ∎\end{aligned}$$

Remark 3: In the nonlinear case ($p \in (1, \infty)$), it is this strong comparison result that we are missing in order to have a bifurcation-type theorem. It is an interesting open problem whether Proposition 10 is still valid when $1 < p < \infty$.

Such a result will lead to a bifurcation-type theorem for the general nonlinear problem (P_λ).

Let $\lambda^* = \sup \mathcal{L}$. We will show that $\lambda^* < \infty$. To this end we will need some preparation. Note that hypotheses H_1(i), (ii), (iii) imply that there exists $c_{15} > 0$ such that

$$f(z,x) \geqslant -c_{15}x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } x \geqslant 0. \tag{34}$$

We consider the following auxiliary parameter nonlinear problem:

$$\left\{ \begin{array}{ll} -\Delta_p u(z) + c_{15}u(z)^{p-1} = 0 & \text{in } \Omega,\ 1 < p < \infty, \\ \dfrac{\partial u}{\partial n_p} = \lambda\beta(z)u(z)^{q-1} & \text{on } \partial\Omega,\ u > 0. \end{array} \right\} \tag{35}$$

For this problem we have the following existence and uniqueness result (see also Sabina de Lis [21, p. 472]).

Proposition 11: *If hypotheses $\widehat{H}$ hold, then for every $\lambda > 0$ problem* (35) *admits a unique solution $\tilde{u}_\lambda \in \operatorname{int} C_+$,*

$\tilde{u}_\lambda = \lambda^{\frac{1}{p-q}}\tilde{u}_1,\ \tilde{u}_\lambda \to 0$ *in* $C^1(\overline{\Omega})$ *as* $\lambda \to 0^+$ *and* $\tilde{u}_\lambda \leqslant u$ *for all* $u \in S(\lambda) \subseteq \operatorname{int} C_+$.

Proof. Let $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\psi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{c_{15}}{p}||u||_p^p - \frac{\lambda}{q}\int_{\partial\Omega} \beta(z)u^+(z)^q d\sigma \text{ for all } u \in W^{1,p}(\Omega).$$

Since $q < p$, the functional ψ_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So we can find $(\tilde{u})_\lambda \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda(\tilde{u}_\lambda) = \inf[\psi_\lambda(u) : u \in W^{1,p}(\Omega)]. \tag{36}$$

Let $u \in \operatorname{int} C_+$ and $t > 0$. We have

$$\begin{aligned} \psi_\lambda(tu) =& \frac{t^p}{p}||Du||_p^p + \frac{c_{15}t^p}{p}||u||_p^p - \frac{\lambda}{q}t^q \int_{\partial\Omega} \beta(z)u(z)d\sigma \\ =& \frac{t^p}{p}[||Du||_p^p + c_{15}||u||_p^p] - t^q\frac{\lambda}{q}c_{16} \end{aligned} \tag{37}$$

where $c_{16} = c_{16}(u) = \int_{\partial\Omega} \beta(z)u(z)d\sigma > 0$ (see hypotheses $\widehat{H}$). Since $q < p$, choosing $t \in (0,1)$ small, from (37) we have

$$\begin{aligned} &\psi_\lambda(tu) < 0,\\ &\quad \Rightarrow \psi_\lambda(\tilde{u}_\lambda) < 0 = \psi_\lambda(0),\\ &\quad \Rightarrow \tilde{u}_\lambda \neq 0. \end{aligned}$$

From (36) we have

$$\begin{aligned} &\psi'_\lambda(\tilde{u}_\lambda) = 0,\\ &\quad \Rightarrow \langle A(\tilde{u}_\lambda), h\rangle + c_{15}\int_\Omega |\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda h dz = \lambda \int_{\partial\Omega} \beta(z)(\tilde{u}_\lambda^+)^{q-1} h d\sigma \\ &\qquad\qquad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{38}$$

In (38) we choose $h = -\tilde{u}_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &||D\tilde{u}_\lambda^-||_p^p + c_{15}||\tilde{u}_\lambda^-||_p^p = 0,\\ &\quad \Rightarrow \tilde{u}_\lambda \geqslant 0,\ \tilde{u}_\lambda \neq 0. \end{aligned}$$

Therefore $\tilde{u}_\lambda$ is a positive solution of problem (35) (see the proof of Proposition 7) and, as before, the nonlinear regularity theory (see Lieberman [16]) and the nonlinear maximum principle (see [11, p. 738]) imply $\tilde{u}_\lambda \in \operatorname{int} C_+$.

Next we prove the uniqueness of this positive solution. So we consider the integral functional $J_\lambda : L^1(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \begin{cases} \dfrac{1}{p}||Du^{1/p}||_p^p - \dfrac{\lambda}{q}\displaystyle\int_{\partial\Omega} \beta(z)u(z)^{q/p} d\sigma & \text{if } u \geqslant 0,\ u^{1/p} \in W^{1,p}(\Omega),\\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom} J_\lambda = \{u \in W^{1,p}(\Omega) : J_\lambda(u) < \infty\}$. From Lemma 1 of Diaz and Saa [5] we see that $u \longmapsto \frac{1}{p}||Du||^{1/p}||_p^p$ is convex. Since $q < p$, the map

$$u \longmapsto -\frac{\lambda}{q}\int_{\partial\Omega} \beta(z)u^{q/p} d\sigma$$

is convex. Therefore J_λ is convex and by Fatou's lemma it is also lower semicontinuous.

Suppose $\tilde{u}_\lambda, \tilde{v}_\lambda$ are two positive solutions of (35). From the first part of the proof we have $\tilde{u}_\lambda, \tilde{v}_\lambda \in \operatorname{int} C_+$. Hence $\tilde{u}_\lambda^p, \tilde{v}_\lambda^p \in \operatorname{dom} J_\lambda$. Also, if $h \in C^1(\overline{\Omega})$, then for all $t \in (-1,1)$ with $|t|$ small we have

$$\tilde{u}_\lambda^p + th \in \operatorname{dom} J_\lambda \quad \text{and} \quad \tilde{v}_\lambda^p + th \in \operatorname{dom} J_\lambda.$$

Moreover, via the chain rule and Green's identity, we have

$$\begin{aligned} J'_\lambda(\tilde{u}_\lambda^p)(h) =& \frac{1}{p}\int_\Omega \frac{-\Delta_p \tilde{u}_\lambda}{\tilde{u}_\lambda^{p-1}} h dz, \\ J'_\lambda(\tilde{v}_\lambda^p)(h) =& \frac{1}{p}\int_\Omega \frac{-\Delta_p \tilde{v}_\lambda}{\tilde{v}_\lambda^{p-1}} h dz \text{ for all } h \in C^1(\overline{\Omega}). \end{aligned}$$

The convexity of J_λ implies the monotonicity of J'_λ. Hence

$$\begin{aligned} 0 \leqslant & \int_\Omega \Big(\frac{-\Delta_p \tilde{u}_\lambda}{\tilde{u}_\lambda^{p-1}} + \frac{\Delta_p \tilde{v}_\lambda}{\tilde{v}_\lambda^{p-1}}\Big)(\tilde{u}_\lambda^{p-1} - \tilde{v}_\lambda^{p-1}) dz \\ =& \int_\Omega c_{15}(\tilde{v}_\lambda - \tilde{u}_\lambda)(\tilde{u}_\lambda^{p-1} - \tilde{v}_\lambda^{p-1}) dz \text{ (see (35))}, \\ & \Rightarrow \tilde{u}_\lambda = \tilde{v}_\lambda. \end{aligned}$$

This proves the uniqueness of the positive solution $\tilde{u}_\lambda \in \operatorname{int} C_+$ of problem (35).

Clearly $\tilde{u}_\lambda = \lambda^{\frac{1}{p-q}} \tilde{u}_1$ for all $\lambda > 0$.

Let $\lambda_n \to 0^+$ and let $\tilde{u}_n = \tilde{u}_{\lambda_n} \in \operatorname{int} C_+$ be the corresponding positive solution of (35). Then from Lieberman [16, Theorem 2] we can find $\alpha \in (0,1)$ and $c_{16} > 0$ such that

$$\tilde{u}_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad ||\tilde{u}_n||_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_{16} \text{ for all } n \geqslant 1. \tag{39}$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we obtain

$$\tilde{u}_n \to \tilde{u} \text{ in } C^1(\overline{\Omega}). \tag{40}$$

We have

$$\begin{aligned} \langle A(\tilde{u}_n), h\rangle + c_{15}\int_\Omega \tilde{u}_n^{p-1} dz =& \lambda_n \int_{\partial\Omega} \beta(z)\tilde{u}_n^{q-1} h d\sigma \\ & \text{for all } h \in W^{1,p}(\Omega), \text{ all } n \geqslant 1, \\ \Rightarrow \langle A(\tilde{u}), h\rangle + c_{15}\int_\Omega \tilde{u}^{p-1} h dz =& 0 \text{ (see (40))}. \end{aligned}$$

Let $h = \tilde{u} \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} ||D\tilde{u}||_p^p + c_{15}||\tilde{u}||_p^p = 0, \\ \Rightarrow \tilde{u} = 0. \end{aligned}$$

So we conclude that $\tilde{u}_\lambda \to 0$ in $C^1(\overline{\Omega})$ as $\lambda \to 0^+$.

Finally, let $u \in S(\lambda)$ and consider the following Carathéodory functions:

$$e(z,x) = \begin{cases} (c_{15}-1)(x^+)^{p-1} & \text{if } x \leqslant u(z), \\ (c_{15}-1)u(z)^{p-1} & \text{if } u(z) < x, \end{cases}$$

(41) and

$$d_\lambda(z,x) = \begin{cases} \lambda\beta(z)(x^+)^{q-1} & \text{if } x \leqslant u(z), \\ \lambda\beta(z)u(z)^{q-1} & \text{if } u(z) < x. \end{cases}$$

We set $E(z,x) = \int_0^x e(z,s)ds$ and $D_\lambda(z,x) = \int_0^x d_\lambda(z,s)ds$ and consider the C^1-functional $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \int_\Omega E(z,u(z))dz - \int_{\partial\Omega} D_\lambda(z,u(z))d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

It is clear from (41) that ψ_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda(\tilde{u}) = \inf[\psi_\lambda(u) : u \in W^{1,p}(\Omega)]. \tag{42}$$

Since $q < p$, as before we have that $\psi_\lambda(\tilde{u}) < 0 = \psi_\lambda(0)$, hence $\tilde{u} \neq 0$. From (42) we have

$$\begin{aligned} &\psi'_\lambda(\tilde{u}) = 0, \\ &\Rightarrow \langle A(\tilde{u}), h\rangle + \int_\Omega |\tilde{u}|^{p-2}\tilde{u}h dz + \int_\Omega e(z,\tilde{u})h dz = \int_{\partial\Omega} \lambda\beta(z)\tilde{u}^{q-1}h d\sigma \\ &\qquad \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

Choosing $h = -\tilde{u}^- \in W^{1,p}(\Omega)$ and $h = (\tilde{u}-u)^+ \in W^{1,p}(\Omega)$, using (34) and reasoning as in the proof of Proposition 8, we show that

$$\begin{aligned} &\tilde{u} \in [0,u],\ \tilde{u} \neq 0 \\ &\Rightarrow \tilde{u} \in \operatorname{int} C_+ \text{ is a positive solution of (35)}, \\ &\Rightarrow \tilde{u} = \tilde{u}_\lambda \leqslant u. \qquad \blacksquare \end{aligned}$$

Now let $\hat{a} \in L^\infty(\Omega)$ with $\operatorname{essinf}_\Omega \hat{a} > 0$ and $\hat{b} \in L^\infty(\partial\Omega)$. We consider the following nonlinear eigenvalue problem:

$$\left\{ \begin{array}{ll} -\Delta_p u(z) = \vartheta\hat{a}(z)|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} = \hat{b}(z)|u(z)|^{p-2}u(z) & \text{on } \partial\Omega. \end{array} \right\} \tag{43}$$

PROPOSITION 12: *Problem* (43) *has a similar eigenvalue θ_1 with positive eigenfunctions. No other eigenvalue has positive eigenfunctions and θ_1 is simple (that is, if u_1, u_2 are both eigenfunctions for ϑ_1, then $u_1 = \xi u_2$ with $\xi \in \mathbb{R}\backslash\{0\}$).*

Proof. Let $M = \{u \in W^{1,p}(\Omega) : \int_\Omega \hat{a}(z)|u(z)|^p d\sigma = 1\}$. The Sobolev embedding theorem implies that M is weakly closed in $W^{1,p}(\Omega)$.

Let $j : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$j(u) = \frac{1}{p}||Du||_p^p - \frac{1}{p}\int_{\partial\Omega} \hat{b}(z)|u(z)|^p d\sigma \text{ for all } u \in W^{1,p}(\Omega). \tag{44}$$

From Ehrling's inequality (see [18, p. 695] and [21]), we have that, given $\epsilon > 0$, we can find $c_\epsilon > 0$ such that

$$\int_{\partial\Omega} |u|^p d\sigma \leqslant \epsilon||Du||_p^p + c_\epsilon||u||_p^p. \tag{45}$$

Also note that, if $\hat{m} = \operatorname{essinf}_\Omega \hat{a} > 0$, then

$$1 = \int_\Omega \hat{a}(z)|u|^p dz \geqslant \hat{m}||u||_p^p \text{ for all } u \in M. \tag{46}$$

From (44), (45) and (46) it follows that $j|_M$ is coercive. Also, by the Sobolev embedding theorem and the compactness of the trace map, we see that $j(\cdot)$ is sequentially weakly lower semicontinuous. Since M is weakly closed, we can find $\hat{u}_1 \in M$ such that

$$\vartheta_1 = j(\hat{u}_1) = \inf[j(u) : u \in M].$$

Replacing $\hat{u}_1$ by $|\hat{u}_1| \in M$, we see that $j(|\hat{u}_1|) = \theta_1$ and so, without any loss of generality, we may assume that $\hat{u}_1 \geqslant 0$ and of course $\hat{u}_1 \neq 0$ since $\hat{u}_1 \in M$.

From the Lagrange multiplier rule (see, for example, Papageorgiou and Kyritsi [18, p. 361]), we have

$$\langle A(\hat{u}_1), h\rangle - \int_{\partial\Omega} \hat{b}(z)\hat{u}_1^{p-1} h d\sigma = \eta_1 \int_\Omega \hat{a}(z)\hat{u}_1^{p-1} h dz \text{ for all } h \in W^{1,p}(\Omega)$$

$$\Rightarrow -\Delta_p \hat{u}_1(z) = \eta_1 \hat{a}(z)\hat{u}_1(z)^{p-1} \text{ for almost all } z \in \Omega,\ \frac{\partial \hat{u}_1}{\partial n_p} = \hat{b}(z)\hat{u}_1^{p-1} \text{ on } \partial\Omega$$

(see the proof of Proposition 7).

As before, the nonlinear regularity theory and the nonlinear maximum principle imply $\hat{u}_1 \in \operatorname{int} C_+$.

That every other eigenvalue $\vartheta > \vartheta_1$ has nodal (sign changing) eigenfunctions and that ϑ_1 is simple, follows exactly as in Gasinski and Papageorgiou [11, pp. 741, 743]. ∎

Now we are ready to show the finiteness of $\lambda^* = \sup \mathcal{L}$.

Proposition 13: *If hypotheses* H_1(i), (ii), (iii) *and* $\widehat{H}$ *hold, then* $\lambda^* < \infty$.

Proof. Let $\lambda \in \mathcal{L}$. We show that we can find $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 7). From Proposition 11, we know that

$$\tilde{u}_\lambda \leqslant u_\lambda. \tag{47}$$

We have

$$\left\{\begin{array}{l} -\Delta_p u_\lambda(z) = f(z, u_\lambda(z)) = \dfrac{f(z, u_\lambda(z))}{u_\lambda(z)^{p-1}} u_\lambda(z)^{p-1} \text{ for almost all } z \in \Omega, \\ \dfrac{\partial u_\lambda}{\partial n_p} = \lambda\beta(z) u_\lambda^{q-1} = \lambda\beta(z) u_\lambda^{q-p} u_\lambda^{p-1} \text{ on } \partial\Omega. \end{array}\right\} \tag{48}$$

We set $\hat{a}(z) = \frac{f(z,u_\lambda(z))}{u_\lambda(z)^{p-1}}$ and $\hat{b}(z) = \lambda\beta(z)u_\lambda(z)^{q-p}$. Then

$$\begin{aligned} &\hat{a} \in L^\infty(\Omega) \text{ and } \operatorname{essinf}_\Omega \hat{a} \geqslant \frac{\mu_{\tau_\lambda}}{||u_\lambda||_\infty^{p-1}} \text{ with } \tau_\lambda = \min_{\overline{\Omega}} u_\lambda > 0 \\ &\qquad\qquad (\text{recall } u_\lambda \in \operatorname{int} C_+ \text{ and see } H_1(\text{iii})), \\ &\hat{b} \in L^\infty(\Omega) \text{ (see hypotheses } \widehat{H} \text{ and recall } u_\lambda \in \operatorname{int} C_+). \end{aligned}$$

So problem (48) has the form of problem (43). Since $u_\lambda \in \operatorname{int} C_+$ solves problem (48), according to Proposition 12 we must have $\vartheta_1 = 1$. Moreover, from the proof of Proposition 12 we have

$$1 \leqslant \frac{||Du||_p^p - \int_{\partial\Omega} \hat{b}(z)|u|^p d\sigma}{\int_\Omega \hat{a}(z)|u|^p dz} \text{ for all } u \in W^{1,p}(\Omega). \tag{49}$$

From (1) and hypotheses H_1(ii), (iii) we see that we can find $c_{17} > 0$ such that

$$\begin{aligned} &f(z,x) \geqslant c_{17} x^{\eta-1} \text{ for almost all } z \in \Omega, \text{ all } z \geqslant \tau_\lambda > 0 \\ &\qquad\qquad (\text{recall that } \tau_\lambda = \min_{\overline{\Omega}} u_\lambda \text{ and that } u_\lambda \in \operatorname{int} C_+). \end{aligned} \tag{50}$$

Then we have

$$\begin{aligned}\hat{a}(z) = \frac{f(z, u_\lambda(z))}{u_\lambda(z)^{p-1}} &\geqslant c_{17} u_\lambda(z)^{\eta-p} \text{ (see (50))} \\ &\geqslant c_{17} \tilde{u}_\lambda(z)^{\eta-p} \text{ (see (47))} \\ &\geqslant \lambda^{\frac{\eta-p}{p-q}} c_{17} \tilde{u}_\lambda(z)^{\eta-p} \\ &\qquad \text{for almost all } z \in \Omega \text{ (see Proposition 11).}\end{aligned} \tag{51}$$

We return to (49) and use (51). Then since $\hat{b} \geqslant 0$, we have

$$\begin{aligned}\lambda^{\frac{\eta-p}{p-q}} &\leqslant \frac{||Du||_p^p + ||u||_p^p}{\int_\Omega c_{17} \tilde{u}_1^{\eta-p} |u|^p dz} \text{ for all } u \in W^{1,p}(\Omega), \\ \Rightarrow \lambda^{\frac{\eta-p}{p-q}} &\leqslant \gamma_1,\end{aligned} \tag{52}$$

where $\gamma_1 > 0$ is the principle eigenvalue of

$$\left\{ \begin{array}{l} -\Delta_p u(z) + |u(z)|^{p-2} u(z) = \gamma c_{17} \tilde{u}_1(z)^{\eta-p} |u(z)|^{p-2} u(z) \text{ in } \Omega \\ \dfrac{\partial u}{\partial n_p} = 0. \end{array} \right\}$$

(see Mugnai and Papageorgiou [17]). Since $\lambda \in \mathcal{L}$ is arbitrary, from (52) we conclude that $\lambda^* < \infty$. ∎

Proposition 14: *If hypotheses* H_1(i), (ii), (iii) *and* $\widehat{H}$ *hold, then* $\lambda^* \in \mathcal{L}$ *and so* $\mathcal{L} = (0, \lambda^*]$.

Proof. Let $\{\lambda_n\}_{n \geqslant 1} \subseteq \mathcal{L}$ such that $\lambda_n \uparrow \lambda^*$ and let $u_n \in S(\lambda_n) \subseteq \operatorname{int} C_+$. From the proof of Proposition 8 and Corollary 9, we know that we can assume that $\{u_n\}_{n \geqslant 1}$ is increasing (that is, $u_n \leqslant u_{n+1}$ for all $n \geqslant 1$) and

$$\hat{\varphi}_{\lambda_n}(u_n) < 0 \text{ for all } n \geqslant 1. \tag{53}$$

We have

$$\begin{aligned}\langle A(u_n), h \rangle = \int_\Omega f(z, u_n) h dz + \lambda_n \int_{\partial\Omega} \beta(z) u_n^{q-1} h d\sigma& \\ \text{for all } h \in W^{1,p}(\Omega), \text{ all } n \geqslant 1.&\end{aligned} \tag{54}$$

Using (53) and (54) and reasoning as in the proof of Proposition 4, we can show that $\{u_n\}_{n \geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded. So by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_* \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{55}$$

In (54) we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (55). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_* \rangle = 0, \tag{56}$$
$$\Rightarrow u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 3).}$$

Evidently u_* (recall $\{u_n\}_{n\geqslant 1} \subseteq \operatorname{int} C_+$) is increasing.

If in (54) we pass to the limit as $n \to \infty$ and use (56), then

$$\langle A(u_*),h\rangle = \int_\Omega f(z,u_*)hdz + \lambda_* \int_{\partial\Omega} \beta(z)u_*^{q-1}hdz \text{ for all } h \in W^{1,p}(\Omega),$$
$$\Rightarrow u_* \in S(\lambda_*) \subseteq \operatorname{int} C_+ \text{ (see the proof of Proposition 7) and so } \lambda_* \in \mathcal{L}.$$

Therefore by virtue of Proposition 8 and 13, we have $\mathcal{L} = (0, \lambda^*]$. ∎

Next, we show that for every $\lambda \in \mathcal{L}$ problem (P_λ) admits a minimal positive solution.

PROPOSITION 15: *If hypotheses H_1(i), (ii), (iii) and $\widehat{H}$ hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem (P_λ) admits a smallest positive solution $\underline{u}_\lambda \in \operatorname{int} C_+$.*

Proof. From Filippakis, Kristaly and Papageorgiou [6], we have that $S(\lambda)$ is downward directed, that is, if $u_1, u_2 \in S(\lambda)$, then we can find $u \in S(\lambda)$ such that $u \leqslant u_1, u \leqslant u_2$. Since we are looking for the smallest positive solution of $S(\lambda)$, without any loss of generality, we may assume that

$$||u||_\infty \geqslant c_{18} \text{ for some } c_{18} > 0 \text{ all } u \in S(\lambda). \tag{57}$$

From Hu and Papageorgiou [14, p. 178], we know that there exist

$$\{u_n\}_{n\geqslant 1} \subseteq S(\lambda)$$

such that

$$\inf S(\lambda) = \inf_{n\geqslant 1} u_n \text{ and } \tilde{u}_\lambda \leqslant u_n \text{ for all } n \geqslant 1 \text{ (see Proposition 11).} \tag{58}$$

We have

$$\langle A(u_n), h\rangle = \int_\Omega f(z,u_n)hdz + \lambda \int_{\partial\Omega} \beta(z)u_n^{q-1}hd\sigma \text{ for all } h \in W^{1,p}(\Omega). \tag{59}$$

In (59) we choose $h = u_n \in W^{1,p}(\Omega)$. Then using (57) and hypotheses $\widehat{H}$, we have

$$||Du_n||_p^p \leqslant c_{19}(1 + ||u_n||^q) \text{ for some } c_{19} > 0, \text{ all } n \geqslant 1. \tag{60}$$

From (57) and (60) it follows that $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded. Hence, we may assume that

$$u_n \xrightarrow{w} \underline{u}_\lambda \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to \underline{u}_\lambda \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{61}$$

If in (59) we choose $h = u_n - \underline{u}_\lambda \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (61), then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(u_n), u_n - \underline{u}_\lambda \rangle = 0, \\ &\Rightarrow u_n \to \underline{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 3) and } \tilde{u}_\lambda \leqslant \underline{u}_\lambda \text{ (see (58)).} \end{aligned} \tag{62}$$

So if in (59) we pass to the limit as (59), then

$$\begin{aligned} \langle A(\underline{u}_\lambda), h\rangle = \int_\Omega f(z, \underline{u}_\lambda) h dz + \lambda \int_{\partial\Omega} \beta(z) \underline{u}_\lambda^{q-1} h d\sigma \text{ for all } h \in W^{1,p}(\Omega), \\ \Rightarrow \underline{u}_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+ \text{ (see (62)) and } \underline{u}_\lambda = \inf S(\lambda). \end{aligned}$$

∎

We examine the map $\lambda \longmapsto \underline{u}_\lambda$.

Proposition 16: *If hypotheses* H_1(i), (ii), (iii) *and* $\widehat{H}$ *hold, then the map* $\lambda \to \underline{u}_\lambda$ *from* $\mathcal{L} = (0, \lambda^*]$ *into* $C^1(\overline{\Omega})$ *is*

- *increasing (that is, if* $\nu < \lambda$*, then* $\underline{u}_\nu \leqslant \underline{u}_\lambda$*),*
- *left continuous.*

Proof. Let $\lambda, \nu \in [0, \lambda^*]$ with $\nu < \lambda$. From Corollary 9 we know that there exist $u_\nu \in S(\nu)$ such that

$$\begin{aligned} u_\nu \leqslant &\underline{u}_\lambda, \\ &\Rightarrow \underline{u}_\nu \leqslant \underline{u}_\lambda, \\ &\Rightarrow \lambda \longmapsto \underline{u}_\lambda \text{ is increasing.} \end{aligned}$$

Next, let $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathcal{L}$ such that $\lambda_n \uparrow \lambda \in \mathcal{L}$. We can find $u_n \in S(\lambda_n)$ $n \geqslant 1$ such that

$$\{u_n\}_{n\geqslant 1} \subseteq \operatorname{int} C_+ \text{ is increasing and } \hat{\varphi}_{\lambda_n}(u_n) < 0 \text{ for all } n \geqslant 1 \tag{63}$$

(see the proof of Proposition 7). We have

$$\begin{aligned} \langle A(u_n), h\rangle = \int_\Omega f(z, u_n) h dz + \lambda_n \int_{\partial\Omega} \beta(z) u_n^{q-1} h d\sigma \\ \text{for all } h \in W^{1,p}(\Omega), \text{ all } n \geqslant 1. \end{aligned} \tag{64}$$

Using (63) and (64) and reasoning as in the proof of Proposition 4, we have

$$\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

From Lieberman [16], we know that there exist $\alpha \in (0,1)$ and $c_{20} > 0$ such that

$$\underline{u}_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad ||\underline{u}_n||_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_{20} \text{ for all } n \geqslant 1.$$

Because of the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and since

$$\{\underline{u}_n\}_{n\geqslant} \subseteq \operatorname{int} C_+$$

is increasing (we have already established that $\lambda \to \underline{u}_\lambda$ is increasing), we have

$$\underline{u}_n \to \tilde{u} \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{65}$$

By passing to the limit as $n \to \infty$ in (64), we see that $\tilde{u} \in S(\lambda)$. Suppose $\tilde{u} \neq \underline{u}_\lambda$. Then we can find $z_0 \in \overline{\Omega}$ such that

$$\begin{aligned} &\underline{u}_\lambda(z_0) < \tilde{u}(z_0) \\ &\quad \Rightarrow \underline{u}_\lambda(z_0) < \underline{u}_n(z_0) \text{ for all } n \geqslant n_0 \text{ (see (65))}, \end{aligned}$$

which contradicts the monotonicity of $\lambda \to \underline{u}_\lambda$ established in the first part of the proof. This proves the left continuity of $\lambda \to \underline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C^1(\overline{\Omega})$. ∎

In the semilinear case ($p = 2$) and $\lambda \in (0, \lambda^*)$, we can prove a multiplicity result for the positive solutions of (P_λ).

Proposition 17: *If $p = 2$, hypotheses H_1 and $\widehat{H}$ hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has at least two positive solutions,*

$$u_n, \hat{u} \in \operatorname{int} C_+ \quad \textit{and} \quad u_0 \neq \hat{u}.$$

Proof. Let $\nu < \lambda < \mu < \lambda^*$. From Proposition 8 we know that $\nu, \lambda, \mu \in \mathcal{L}$. Also, from Proposition 10, we know that we can find $u_\nu \in S(\nu) \subseteq \operatorname{int} C_+$, $u_0 \in S(\lambda) \subseteq \operatorname{int} C_+$ and $u_\mu \in S(\mu) \subseteq \operatorname{int} C_+$ such that

$$u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[u_\nu, u_\mu]. \tag{66}$$

From the proof of Proposition 8, we know that if we consider the truncated functional $\hat{\psi}_\mu \in C^1(H^1(\Omega))$ (see (27), (28) with u_λ replaced by u_μ and ν by λ), then

$$u_0 \text{ is a minimizer of } \hat{\psi}_\mu. \tag{67}$$

We consider the following Carathéodory functions

$$g(z,x)=\begin{cases} f(z,u_\nu(z))+u_\nu(z) & \text{if } x\leqslant u_\nu(z)\\ f(z,x)+x & \text{if } u_\nu(z)<x\end{cases} \quad \text{for all } (z,x)\in\Omega\times\mathbb{R}, \tag{68}$$

$$w_\lambda(z,x)=\begin{cases} \lambda\beta(z)u_\nu(z)^{q-1} & \text{if } x\leqslant u_\nu(z)\\ \lambda\beta(z)x^{q-1} & \text{if } u_\nu(z)<x\end{cases} \quad \text{for all } (z,x)\in\partial\Omega\times\mathbb{R}. \tag{69}$$

We set

$$G(z,x)=\int_0^x g(z,s)ds \quad \text{and} \quad W_\lambda(z,s)=\int_0^x w_\lambda(z,s)ds$$

and introduce the C^1-functional $\hat{\sigma}_\lambda : H^1(\Omega)\to\mathbb{R}$ defined by

$$\hat{\sigma}_\lambda(u)=\frac{1}{2}||Du||_2^2+\frac{1}{2}||u||_2^2-\int_\Omega G(z,u(z))dz-\int_{\partial\Omega}W_\lambda(z,u(z))d\sigma \quad \text{for all } u\in H^1(\Omega).$$

From (68) and (69), we have

$$\hat{\sigma}_\lambda=\hat{\varphi}_\lambda+\hat{\xi}_\lambda^* \text{ with } \hat{\xi}_\lambda^*\in\mathbb{R} \text{ for all } u\geqslant u_\nu. \tag{70}$$

From (70) it follows that

(71) • $\hat{\sigma}_\lambda$ satisfies the C-condition (see Proposition 4);

(72) • for all $u\in \operatorname{int} C_+$, $\hat{\sigma}_\lambda(tu)\to-\infty$ as $t\to+\infty$(see Proposition 6).

Moreover, note that

$$\begin{aligned} &\hat{\sigma}_\lambda|_{[u_\nu,u_\mu]}=\hat{\varphi}_\lambda|_{[u_\nu,u_\mu]} \text{ (see (68), (69) and the proof of Proposition 8),}\\ &\Rightarrow u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \hat{\sigma}_\lambda\\ &\qquad \text{(see (68), (69) and the proof of Proposition 8),}\\ &\Rightarrow u_0 \text{ is a local } H^1(\Omega)\text{-minimizer of } \hat{\sigma}_\lambda \text{ (see Proposition 2).}\end{aligned} \tag{73}$$

Let

$$[u_\nu)=\{u\in H^1(\Omega): u_\nu(z)\leqslant u(z) \text{ for almost all } z\in\Omega\}.$$

We can easily check that

$$K_{\hat{\sigma}_\lambda}\subseteq[u_\nu). \tag{74}$$

We may assume that $K_{\hat{\sigma}_\lambda}$ is finite, or otherwise we already have infinitely many distinct positive solutions (see (68), (69), (74)). Then because of (73), we can find $\rho \in [0,1)$ small such that

$$\hat{\sigma}_\lambda(u_0) < \inf[\hat{\sigma}_\lambda(u) : ||u - u_0|| = \rho] = \hat{m}_\lambda \tag{75}$$

(see Aizicovici, Papageorgiou and Staicu [1] (proof of Proposition 29)). Then (71), (72) and (75) permit the use of Theorem 1 (the mountain pass theorem). So we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\hat{\sigma}_\lambda} \quad \text{and} \quad \hat{m}_\lambda \leqslant \hat{\sigma}_\lambda(\hat{u}). \tag{76}$$

From (68), (69), (74) and (76), we see that

$$\hat{u} \in S(\lambda) \subseteq \operatorname{int} C_+,$$

while from (75) and (76) it follows that $\hat{u} \neq u_0$. ∎

We can summarize our investigation of the positive solutions for problem (P_λ) with two theorems. The first concerns the nonlinear equation $(1 < p < \infty)$.

Theorem 18: *If hypotheses H_1(i), (ii), (iii) and $\widehat{H}$ hold, then there exists $\lambda^* \in (0, +\infty)$ such that for every $\lambda \in (0, \lambda^*]$, problem (P_λ) has a positive solution, in fact it has a smallest positive solution $\underline{u}_\lambda \in \operatorname{int} C_+$ and the map $\lambda \to \underline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C^1(\overline{\Omega})$ is increasing and left continuous. For $\lambda > \lambda^*$ problem (P_λ) has no positive solution.*

For the semilinear equation $(p \equiv 2)$, we have a bifurcation-type result.

Theorem 19: *If $p = 2$ and hypotheses H_1 and $\widehat{H}$ hold, then there exists $\lambda^* \in (0, +\infty)$ such that*

(a) *for every $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions,*

$$u_0, \hat{u} \in \operatorname{int} C_+, \ u_0 \neq \hat{u};$$

(b) *for $\lambda = \lambda^*$ problem (P_{λ^*}) has at least one positive solution, $u_* \in \operatorname{int} C_+$;*
(c) *for $\lambda > \lambda^*$ problem (P_λ) has no positive solution;*
(d) *for every $\lambda \in \mathcal{L} = (0, \lambda^*]$ problem (P_λ) has a smallest positive solution, $\underline{u}_\lambda \in \operatorname{int} C_+$, and the map $\lambda \to \underline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C^1(\overline{\Omega})$ is increasing and left-continuous.*

4. Nodal Solutions

In this section we produce nodal (sign changing) solutions, by imposing bilateral conditions on $f(z,\cdot)$.

The new conditions on the reaction $f(z,x)$ are the following:

H_2: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that for almost all $z\in\Omega$, $f(z,0)=0$ and

(i) $|f(z,x)|\leqslant a(z)(1+|x|^{r-1})$ for almost all $z\in\Omega$, all $z\in\mathbb{R}$, with $z\in L^\infty(\Omega)_+$, $p<r<p^*$;

(ii) if
$$F(z,x)=\int_0^x f(z,s)ds$$
then there exist $\eta>p$ and $M>0$ such that
$$0<\eta F(z,x)\leqslant f(z.x)x \text{ for almost all } z\in\Omega,\ \text{all } |x|\geqslant M,$$
$$\operatorname{essinf}_\Omega F(\cdot,\pm M)>0;$$

(iii) $\lim_{x\to 0}\frac{f(z,x)}{|x|^{p-2}x}=0$ uniformly for almost all $z\in\Omega$ and for all $\tau>0$,
$$f(z,x)x\geqslant\mu_\tau>0$$
for almost all $z\in\Omega$, all $|x|\geqslant\tau$;

(iv) if $p=2$, then for every $\rho>0$ there exists $\xi_\rho>0$ such that for almost all $z\in\Omega$, the function
$$x\longmapsto f(z,x)+\xi_\rho x$$
is increasing on $[-\rho,\rho]$.

Arguing as in Section 3, this time on the negative semi-axis $(-\infty,0]$, we can produce a critical parameter value $\hat{\lambda}^*>0$ such that for all $\lambda\in(0,\hat{\lambda}^*]$ problem (P_λ) has a maximal negative solution $\bar{v}_\lambda\in-\operatorname{int}C_+$. So for $\lambda\in(0,\lambda^*]$ with $\lambda_0^*=\min\{\lambda^*,\hat{\lambda}^*\}$, problem (P_λ) admits extremal constant sign solutions
$$\underline{u}_\lambda\in\operatorname{int}C_+\quad\text{and}\quad\bar{v}_\lambda\in-\operatorname{int}C_+.$$
Using them, we can generate a nodal solution.

Proposition 20: *If hypotheses* H_2(i), (ii), (iii) *hold and* $\lambda\in(0,\lambda_0^*]$, *then problem* (P_λ) *admits a nodal solution* $y_\lambda\in[\bar{v}_\lambda,\underline{u}_\lambda]\cap C^1(\overline{\Omega})$.

Proof. Let $\underline{u}_\lambda \in \operatorname{int} C_+$ and $\bar{v}_\lambda \in -\operatorname{int} C_+$ be the two extremal constant sign solutions. We introduce the following Carathéodory functions:

$$j(z,x) = \begin{cases} f(z,\bar{v}_\lambda(z)) + |\bar{v}_\lambda(z)|^{p-2}\bar{v}_\lambda(z) & \text{if } x < \bar{v}_\lambda(z) \\ f(z,x) + |x|^{p-2}x & \text{if } \bar{v}_\lambda(z) \leqslant x \leqslant \underline{u}_\lambda(z) \\ f(z,\underline{u}_\lambda(z)) + \underline{u}_\lambda(z)^{p-1} & \text{if } \underline{u}_\lambda(z) < x \end{cases} \tag{77}$$
$$\text{for all } (z,x) \in \Omega \times \mathbb{R},$$

$$\tau_\lambda(z,x) = \begin{cases} \lambda\beta(z)|\bar{v}_\lambda(z)|^{q-2}\bar{v}_\lambda(z) & \text{if } x < \bar{v}_\lambda(z) \\ \lambda\beta(z)|x|^{q-2}x & \text{if } \bar{v}_\lambda(z) \leqslant x \leqslant \underline{u}_\lambda(z) \\ \lambda\beta(z)\underline{u}_\lambda(z)^{q-1} & \text{if } \underline{u}_\lambda(z) < x \end{cases} \tag{78}$$
$$\text{for all } (z,x) \in \partial\Omega \times \mathbb{R}.$$

We set

$$J(z,x) = \int_0^x j(z,s)ds \quad \text{and} \quad T_\lambda(z,x) = \int_0^x \tau_\lambda(z,s)ds$$

and consider the C^1-functional $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p - \int_\Omega J(z,u(z))dz - \int_{\partial\Omega} T_\lambda(z,u(z))d\sigma$$
$$\text{for all } u \in W^{1,p}(\Omega).$$

Also, we consider the positive and negative truncations of $j(z,\cdot)$, $\tau_\lambda(z,\cdot)$, namely the Carathéodory functions

$$j_\pm(z,x) = j(z,\pm x^\pm) \quad \text{and} \quad \tau_\lambda^\pm(z,x) = \tau_\lambda(z,\pm x^\pm).$$

We set

$$J_\pm(z,x) = \int_0^x j_\pm(z,s)ds \quad \text{and} \quad T_\lambda^\pm(z,x) = \int_0^x \tau_\lambda^\pm(z,s)$$

and introduce the C^1-functionals $\psi_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda^\pm(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p - \int_\Omega J_\pm(z,u(z))dz - \int_{\partial\Omega} T_\lambda^\pm(z,u(z))d\sigma$$
$$\text{for all } u \in W^{1,p}(\Omega).$$

As in the proof of Proposition 8, we can see that

$$K_{\psi_\lambda} \subseteq [\bar{v}_\lambda, \underline{u}_\lambda], \quad K_{\psi_\lambda^+} \subseteq [0, \underline{u}_\lambda], \quad K_{\psi_\lambda^-} \subseteq [\bar{v}_\lambda, 0] \quad \text{(see (77) and (78))}.$$

The extremality of $\bar{v}_\lambda$ and $\underline{u}_\lambda$, implies that

$$K_{\psi_\lambda} \subseteq [\bar{v}_\lambda, \underline{u}_\lambda], \quad K_{\psi_\lambda^+} = \{0, \underline{u}_\lambda\}, \quad K_{\psi_\lambda} = \{0, \bar{v}_\lambda\}. \tag{79}$$

Claim 1: *$\bar{v}_\lambda \in -\operatorname{int} C_+$ and $\underline{u}_\lambda \in \operatorname{int} C_+$ are both local minimizers of ψ_λ.*

Note that (77) and (78) imply that ψ_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So we can find $\tilde{\underline{u}}_\lambda \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda^+(\tilde{\underline{u}}_\lambda) = \inf[\psi_\lambda^+(u) : u \in W^{1,p}(\Omega)].$$

As in the proof of Proposition 8 and since $q < p$, we show that

$$\begin{aligned} \psi_\lambda^+(\tilde{\underline{u}}_\lambda) &< 0 = \psi_\lambda^+(0), \\ &\Rightarrow \tilde{\underline{u}}_\lambda \neq 0. \end{aligned}$$

Since $\tilde{\underline{u}}_\lambda \in K_{\psi_\lambda^+}$, from (79) we have $\tilde{\underline{u}}_\lambda = \underline{u}_\lambda \in \operatorname{int} C_+$. But

$$\begin{aligned} &\psi_\lambda|_{C_+} = \psi_\lambda^+|_{C_+}, \\ &\qquad \Rightarrow \underline{u}_\lambda \in \operatorname{int} C_+ \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \psi_\lambda, \\ &\qquad \Rightarrow \underline{u}_\lambda \in \operatorname{int} C_+ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \psi_\lambda \text{ (see Proposition 2)}. \end{aligned}$$

Similarly for $\bar{v}_\lambda \in -\operatorname{int} C_+$, using this time the functional ψ_λ^-. This proves Claim 1.

Without any loss of generality, we may assume that

$$\psi_\lambda(\bar{v}_\lambda) \leqslant \psi_\lambda(\underline{u}_\lambda)$$

(the reasoning is similar, if the opposite inequality holds). By virtue of Claim 1, we can find $\rho \in (0,1)$ small such that

$$\psi_\lambda(\bar{v}_\lambda) \leqslant \psi_\lambda(\underline{u}_\lambda) < \inf[\psi_\lambda(u) : ||u - \underline{u}_\lambda|| = \rho] = m_\lambda, \quad ||\bar{v}_\lambda - \underline{u}_\lambda|| > \rho \tag{80}$$

(see [1]). Since ψ_λ is coercive (see (77) and (78)), we have

$$\varphi_\lambda \text{ satisfies the } C\text{-condition.} \tag{81}$$

Then (80) and (81) permit the use of Theorem 1 (the mountain pass theorem). So we can find $y_\lambda \in W^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\psi_\lambda} \quad \text{and} \quad m_\lambda \leqslant \psi_\lambda(y_\lambda). \tag{82}$$

From (79), (80) and (82) it follows that

$$y_\lambda \notin \{\bar{v}_\lambda, \underline{u}_\lambda\} \text{ and } y_\lambda \text{ is a solution of } (P_\lambda).$$

Note that y_λ is a critical point of mountain pass type for ψ_λ . If

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}||Du||_p^p - \int_\Omega F(z,u(z))dz - \frac{1}{q}\int_{\partial\Omega}\beta(z)|u(z)|^q d\sigma$$

for all $u \in W^{1,p}(\Omega)$, then

$$\psi_\lambda|_{[\bar{v}_\lambda,\underline{u}_\lambda]} = \hat{\varphi}_\lambda|_{[\bar{v}_\lambda,\underline{u}_\lambda]} \text{ (see (77), (78)).}$$

Since $\bar{v}_\lambda \in -\operatorname{int} C_+$, $\underline{u}_\lambda \in \operatorname{int} C_+$ and $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, we see that y_λ is a critical point of mountain pass type for $\hat{\varphi}_\lambda$ too (see Gasinski and Papageorgiou [11, p. 645]). On the other hand, since $q<p$, $u=0$ cannot be a critical point mountain pass type for $\hat{\varphi}_\lambda$. Therefore $y_\lambda \neq 0$ and, since $y_\lambda \in [\bar{v}_\lambda, \underline{u}_\lambda]$ (see (79)), the extremality of $\bar{v}_\lambda$ and $\underline{u}_\lambda$ implies that y_λ is nodal. Finally, the nonlinear regularity theory of Lieberman [16] implies $y_\lambda \in C^1(\overline{\Omega})$. ∎

So we can state two multiplicity theorems for problem (P_λ). First the nonlinear case $(1 < p < \infty)$.

Theorem 21: *If hypotheses H_2(i), (ii), (iii) and $\widehat{H}$ hold, then there exists $\lambda_0^* > 0$ such that for all $\lambda \in (0, \lambda_0^*]$ problem (P_λ) has at least three nontrivial solutions:*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+, \quad y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \textit{ nodal.}$$

In the semilinear case $(p = 2)$, we can improve this multiplicity result.

Theorem 22: *If $p = 2$ and hypotheses H_1 and $\widehat{H}$ hold, then there exists $\lambda_0^* > 0$ such that for all $\lambda \in (0, \lambda_0^*)$ problem (P_λ) has at least five nontrivial solutions:*

$$u_0, \hat{u} \in \operatorname{int} C_+, \quad u_0 \leqslant \hat{u}, \quad u_0 \neq \hat{u},$$
$$u_0, \hat{v} \in -\operatorname{int} C_+, \quad \hat{v} \leqslant v_0, \quad v_0 \neq \hat{v},$$

and

$$y_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0] \quad \textit{nodal.}$$

Remark 4: An interesting open problem is whether we can extend the work of this paper to equations driven by the more general nonlinear, nonhomogeneous differential operators used by Papageorgiou and Rădulescu [20]. Such an extension will require new methods and techniques.

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