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Inequality problems with Nonlocally Lipschitz Energy Functional: Existence Results and Applications to Nonsmooth Mechanics

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We give an existence result for a double eigenvalue problem in Hemivariational Inequalities whose energetic functional is not locally Lipschitz. It is used a finite dimensional approach based on Kakutani's fixed point theorem.

Keywords: Eigenvalue problem; Generalized gradient; Hemivariational inequality; Linear elasticity

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

The concept of hemivariational inequality has been introduced by Panagiotopoulos as a natural extension of the variational inequalities to the case of nonconvex functionals. This extension is strongly motivated by many problems arising in Mechanics, Engineering or Economics. For a comprehensive overview on this subject we refer to the monographs [9,10].

In this article we deal with a new type of hemivariational inequalities called "double eigenvalue problems" which has been introduced by Motreanu and Panagiotopoulos in an article where there are considered three different approaches: minimization, minimax methods and (sub)critical theory on the sphere (see [7]). Other results on this type of hemivariational inequalities can be found in [1] (multiplicity results) and [2] (a perturbation result).

Let V be a Hilbert space and let $\Omega \subset \mathbf{R}^m$ be an open bounded subset of $\mathbb{R}^m$, $m \geq 1$, with $\partial\Omega$ sufficiently smooth. We shall suppose that V is compactly embedded into $L^p(\Omega; \mathbb{R}^N)$, $N \geq 1$, for some $p \in (1, +\infty)$. In particular, the continuity of this embedding implies the existence of a constant $C_p(\Omega) > 0$ such that

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$$\|u\|_{L^p} \le C_p(\Omega) \cdot \|u\|_V, \quad \text{for all } u \in V, \tag{$*$}$$

where by $\|\cdot\|_{L^p}$ and $\|\cdot\|_V$ we have denoted the norms in $L^p(\Omega; \mathbb{R}^N)$ and V respectively. Throughout the article the symbols $V^*, (\cdot,\cdot)_V$, and $\langle \cdot,\cdot \rangle$ will denote the dual space of V, the inner product on V and the duality pairing over $V^* \times V$, respectively. We suppose that $V \cap L^\infty(\Omega; \mathbb{R}^N)$ is dense in V. Let $a_1, a_2 : V \times V \to \mathbb{R}$ be two bilinear and continuous forms on V which are coercive in the sense that there exist two real valued functions $c_1, c_2 : \mathbb{R}_+ \to \mathbf{R}_+$, with $\lim_{r\to\infty} c_i(r) = +\infty$, such that for all $v \in V$

$$a_i(v, v) \ge c_i(\|v\|_V) \cdot \|v\|_V, \quad i = 1, 2.$$

We denote by $A_1, A_2 : V \to V$ the operators associated to the forms considered above, defined by

$$\langle A_i u, v\rangle = a_i(u, v), \quad i = 1, 2.$$

The operators A_1 and A_2 are linear, continuous and coercive in the sense that for each $i = 1, 2$ we have

$$(A_i u, u)_V \ge c_i(\|u\|_V) \cdot \|u\|_V, \quad \text{for all } u \in V.$$

In addition we shall suppose that the operators A_1 and A_2 are weakly continuous, i.e., if $u_n \rightharpoonup u$, weakly in V then $A_i u_n \rightharpoonup A_i u$, also weakly in V, for each $i = 1, 2$. Let us now consider two bounded selfadjoint linear and weakly continuous operators $B_1, B_2 : V \to V$. Let $j : \Omega \times \mathbf{R}^N \to \mathbb{R}$ be a Carathéodory function which is locally Lipschitz in the second variable for a.e. $x \in \Omega$. Thus, we can define the directional derivative

$$j^0(x; \xi, \eta) = \limsup_{[h,\lambda]\to[0,0^+]} \frac{j(x, \xi + h + \lambda\eta) - j(x, \xi + h)}{\lambda}, \quad \text{for } \xi, \eta \in \mathbb{R}^N,$$

and the generalized gradient of Clarke [5]

$$\partial j(x; \xi) = \{\eta \in \mathbf{R}^N : \eta \cdot \gamma \le j^0(x, \xi, \gamma),\ \forall \gamma \in \mathbf{R}^N\},$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$. Here, the symbol " $\cdot$ " means the inner product on $\mathbf{R}^N$.

In order to ensure the integrability of $j(\cdot, u(\cdot))$ and $j^0(\cdot; u(\cdot), v(\cdot))$ for any $u, v \in V \cap L^\infty(\Omega; \mathbb{R}^N)$ we admit the existence of a function $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ fulfilling the conditions

(β_1) $\beta(\cdot, r) \in L^1(\Omega)$, for each $r \ge 0$;
(β_2) if $r_1 \le r_2$ then $\beta(x, r_1) \le \beta(x, r_2)$, for almost all $x \in \Omega$, and such that

$$|j(x, \xi) - j(x, \eta)| \le \beta(x, r) \cdot |\xi - \eta|, \quad \forall\, \xi, \eta \in B(O, r),\ \ r \ge 0, \tag{1}$$

where $B(O, r) = \{\xi \in \mathbb{R}^N : |\xi| \le r\}$, "$|\cdot|$" denoting the norm in $\mathbb{R}^N$.

Concerning the conditions above, it is important to point out that in the homogenous case (when j is not depending explicitely on $x \in \Omega$) they are negligible (see also [9], p. 146).

Let $1 \le s < p$ and let $k : \Omega \to \mathbb{R}_+$ and $\alpha : \Omega \to \mathbb{R}_+$ be two functions satisfying the assumptions:

$$k(\cdot) \in L^q(\Omega), \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1, \tag{2}$$

$$\alpha(\cdot, r) \in L^t(\Omega), \text{ for each } r > 0, \text{ where } t = \frac{p}{p - s} \tag{3}$$

and

$$\text{if } 0 < r_1 \le r_2 \text{ then } \alpha(x, r_1) \le \alpha(x, r_2), \text{ for almost all } x \in \Omega. \tag{4}$$

We shall impose the following directional growth conditions:

$$j^0(x, \xi, -\xi) \le k(x) \cdot |\xi|, \text{ for all } \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega; \tag{5}$$

$$\begin{aligned} &j^0(x, \xi, \eta - \xi) \le \alpha(x, r)(1 + |\xi|^s), \text{ for all } \xi, \eta \in \mathbb{R}^N, \\ &\quad \text{with } \eta \in B(O, r), r > 0, \text{ and a.e. } x \in \Omega. \end{aligned} \tag{6}$$

Remarks

1. We must pay attention to the fact that the growth conditions (5) and (6) do not ensure the finite integrability of $j(\cdot, u(\cdot))$ and $j^0(\cdot; u(\cdot), v(\cdot))$ in Ω for any $u, v \in V$. We can remark, also, that they do not guarantee that the functional $J : V \to \mathbb{R}$ given by

$$J(v) = \int_\Omega j(x, v(x))dx,$$

is locally Lipschitz on V. In fact, (5) and (6) do not allow us to conclude even that the effective domain of J coincides with the whole space V.

2. Notice that we do not impose any coerciveness assumption on the operators B_i $(i = 1, 2)$, as done in [7], Section 4, for the case of a double eigenvalue problem on a sphere. We suppose however that these operators satisfy the additional hypothesis of weak continuity.

Let us consider two nonlinear monotone and demicontinuous operators $C_1, C_2 : V \to V$. We are ready to consider the following double eigenvalue problem:

(P) Find $u_1, u_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\begin{aligned} &a_1(u_1, v_1) + a_2(u_2, v_2) + (C_1(u_1), v_1)_V + (C_2(u_2), v_2)_V \\ &\quad + \int_\Omega j^0(x; (u_1 - u_2)(x), (v_1 - v_2)(x))dx \ge \lambda_1(B_1u_1, v_1)_V + \lambda_2(B_2u_2, v_2)_V, \quad \forall v_1, v_2 \in V. \end{aligned}$$

From Remark 1 we derive that in order to find a solution for the Problem (P) we cannot follow the classical technique of Clarke [5] and for this reason, the Problem

(P) is a nonstandard one. First of all we have to point out what we shall mean by solution of the problem considered above.

Definition 1 We say that an element $(u_1, u_2, \lambda_1, \lambda_2) \in V \times V \times \mathbb{R} \times \mathbb{R}$ is a solution of (P) if there exists $\chi \in L^1(\Omega; \mathbb{R}^N) \cap V$ such that

$$a_1(u_1, v_1) + a_2(u_2, v_2) + (C_1(u_1), v_1)_V + (C_2(u_2), v_2)_V + \int_\Omega \chi(x) \cdot (v_1 - v_2)(x)dx \\ = \lambda_1(B_1u_1, v_1)_V + \lambda_2(B_2u_2, v_2)_V, \quad \forall v_1, v_2 \in V \cap L^\infty(\Omega; \mathbb{R}^N) \tag{7}$$

and

$$\chi(x) \in \partial j(x; (u_1 - u_2(x)), \quad \text{for a.e. } x \in \Omega. \tag{8}$$

The aim of this article is to prove the following existence result concerning the double eigenvalue Problem (P).

THEOREM 1 *We assume that the hypotheses considered in this section are fulfilled. Then the double eigenvalue Problem* (P) *has at least one solution.*

The difficulties mentioned in the Remark 1 will be surmounted by employing the Galerkin approximation method combined with the finite intersection property. For the treatment of finite dimensional problem we shall use Kakutani's fixed point theorem for multivalued mappings. This technique has been introduced by Naniewicz and Panagiotopoulos (see [9]).

2. A FINITE DIMENSIONAL APPROACH

Let Λ be the family of all finite dimensional subspaces F of $V \cap L^\infty(\Omega; \mathbf{R}^N)$, ordered by inclusion. For any $F \in \Lambda$ we formulate the following finite dimensional problem

(P_F) Find $u_{1F}, u_{2F} \in F, \lambda_1, \lambda_2 \in \mathbb{R}$ and $\chi_F \in L^1(\Omega; \mathbb{R}^N)$ such that

$$a_1(u_{1F}, v_1) + a_2(u_{2F}, v_2) + (C_1(u_{1F}), v_1)_V + (C_2(u_{2F}), v_2)_V \\ + \int_\Omega \chi_F(x) \cdot (v_1 - v_2)(x)dx = \lambda_1(B_1u_{1F}, v_1)_V + \lambda_2(B_2u_{2F}, v_2)_V, \quad \forall v_1, v_2 \in F \tag{9}$$

and

$$\chi_F(x) \in \partial j(x; (u_{1F} - u_{2F})(x)), \quad \text{for a.e. } x \in \Omega. \tag{10}$$

Let $\Gamma_F : F \to 2^{L^1(\Omega; \mathbf{R}^N)}$ defined by

$$\Gamma_F(v_F) = \left\{ \Psi \in L^1(\Omega; \mathbb{R}^N) : \int_\Omega \Psi w dx \le \int_\Omega j^0(x; v_F(x), w(x))dx, \ \forall w \in L^\infty(\Omega; \mathbb{R}^N) \right\}.$$

It is immediately that if $\Psi \in \Gamma_F(v_F)$ then we have $\Psi(x) \in \partial j(x; v_F(x))$, for a.e. $x \in \Omega$. Let $v_F \in F$ for some $F \in \Lambda$. It is proved in [8] (see Lemma 3.1) that $\Gamma(v_F)$ is a nonempty convex and weakly compact subset of $L^1(\Omega;\mathbb{R}^N)$. For $F\in\Lambda$, we shall denote by $i_F: F\to V$ and by $i_F^*: V^*\to F^*$ the inclusion and the dual projection mappings respectively. Throughout, by $\langle\cdot,\cdot\rangle_F$ we mean the duality pairing over $F^*\times F$. Let us define $\gamma_F: L^1(\Omega;\mathbf{R}^N)\to F^*$, by

$$\langle\gamma_F\Psi, v\rangle_F = \int \Psi\cdot v dx,\quad \forall v\in F.$$

We consider the map $T_F: F\to 2^{F^*}$ given by

$$T_F(v_F) = \gamma_F\Gamma_F(v_F).$$

The main properties of T_F are pointed out by the following result which has been established in [8].

LEMMA 1 *For each $v_F\in F$, $T_F(v_F)$ is a nonempty bounded closed convex subset of F^*. Moreover, T_F is upper semicontinuous as a map from F into 2^{F^*}.*

We are now prepared to formulate the existence result for the finite dimensional Problem (P_F).

THEOREM 2 *Suppose that the hypotheses made in Section 1 are fulfilled. Then, for each $F\in\Lambda$, there exist $u_{1F}, u_{2F}\in F$, $\lambda_1,\lambda_2\in\mathbb{R}$ and $\chi_F\in L^1(\Omega;\mathbb{R}^N)$ which solve the Problem (P_F). Moreover, there exists a positive constant M, independent by F such that*

$$\|u_{1F}\|_V + \|u_{2F}\|_V \le M. \tag{11}$$

Proof In what follows we shall be able to find a solution of the Problem (P_F) by restraining the searching area for λ_i, $i\in\{1,2\}$ on the class of all those numbers $\lambda_1,\lambda_2\in\mathbb{R}$ which satisfy the relation

$$\delta := \inf_{w_1,\,w_2\in V\cap L^\infty(\Omega;\mathbf{R}^N)} \frac{\sum_{i=1}^2[(C_i(w_i),w_i)_V - \lambda_i\|B_i\|\|w_i\|_V^2]}{\|u_1\|+\|u_2\|} > -\infty. \tag{12}$$

Define $A_{1F} = i_F^*A_1 i_F, A_{2F} = i_F^*A_2 i_F,$ and let $\overline{G}: V\times V\to V$ be the map given by

$$\overline{G}(v_1,v_2) = v_1 - v_2.$$

Fix $F\in\Lambda$. We denote by G the map $\overline{G}$ restricted to $F\times F$. Let us consider the multi-valued mapping $\Delta: F\times F\to 2^{F^*\times F^*}$, defined by

$$\begin{aligned}\Delta(u_1,u_2) = \big(&A_{1F}u_1 + (C_1(u_1),u_1)_V - \lambda_1(B_1u_1,\cdot),\\ &A_{2F}u_2 + (C_2(u_2),\cdot)_V - \lambda_2(B_2u_2,\cdot)_V\big) + (G^*\circ T_F\circ G)(u_1,u_2),\end{aligned}$$

where by $(G^*\circ T_F\circ G)(u_1,u_2)$ we mean the set

$$\{G^*(f)\colon f\in T_F(u_1-u_2)\}\subset F^*\times F^*.$$

The first step is to prove the upper semicontinuity of $G^* \circ T_F \circ G$. For this aim, let us consider $u_n^1 \to u_1$, $u_n^2 \to u_2$, strongly in F and $\Psi_n \in G^*(T_F(u_n^1 - u_n^2))$ converging strongly to $\Psi \in F^* \times F^*$. It must be proved that $\Psi \in G^*(T_F(u^1 - u^2))$. First we observe that G fulfills the set of conditions which permits to apply the Theorem II. 19 from [3]. From there we draw the conclusion that $\Re(G^*) = \{G^*\theta: \theta \in F^*\}$ is closed. This implies that $\Psi \in \Re(G^*)$ (we have used the fact that $\Psi_n \in \Re(G^*)$, $\forall n \geq 1$ and $\Psi_n \to \Psi$ in $F^* \times F^*$). Thus we obtain the existence of a $\xi^* \in F^*$ such that $\Psi_n = G^*(\gamma_F \chi_n)$. We have

$$\langle G^*(\gamma_F \chi_n, (v, w)\rangle_{F\times F} \to \langle \Psi, (v, w)\rangle_{F\times F}, \quad \text{for all } v, w \in F,$$

which implies that $\langle \gamma_F \chi_n, v - w\rangle_F$ tends to $\langle \xi^*, v - w\rangle_F$, $\forall v, w \in F$ and thus, due to the fact that $\dim F < +\infty$ we get the strong convergence of $\gamma_F \chi_n$ to ξ^* in F^*. Since T_F is upper semicontinuous (see Lemma 1), we obtain that there exists $\chi \in \Gamma_F(u_1 - u_2)$ such that $\xi^* = \gamma_F \chi$. Thus, $\Psi = G^*(\gamma_F \chi)$, which means that $\Psi \in (G^* \circ T_F)(u_1 - u_2)$. This ends the proof of the upper semicontinuity of $G^* \circ T_F \circ G$.

On the other side, the weak continuity of A_1 and A_2 implies the continuity of A_{1F} and A_{2F} from F into F^*. The hypotheses on B_i and C_i $(i = 1, 2)$ and the above considerations lead us to the upper semicontinuity of Δ from $F \times F$ to $2^{F^* \times F^*}$. By using again Lemma 1 and the hypotheses made on B_i, C_i and A_i, we can simply derive that for each $(u_1, u_2) \in F \times F$, $\Delta(u_1, u_2)$ is a nonempty, bounded, closed and convex subset of $F^* \times F^*$. Moreover, from the coercivity of a_1 and a_2 and from the definition of T_F we have

$$\begin{aligned}\langle \Delta(u_1, u_2), (u_1, u_2)\rangle_{F\times F} \geq{} & c_1(\|u_1\|_V)\|u_1\|_V + c_2(\|u_2\|_V)\|u_2\|_V + (C_1(u_1), u_1)_V \\ & + (C_2(u_2), u_2)_V - \lambda_1\|B_1\| \cdot \|u_1\|_V^2 - \lambda_2\|B_2\| \cdot \|u_2\|_V^2 \\ & + \int_\Omega \Psi(u_1 - u_2)dx,\end{aligned}$$

where $\Psi \in \Gamma_F(u_1 - u_2)$. By $(*)$ and (5) we obtain

$$\begin{aligned}\langle \Delta(u_1, u_2), (u_1, u_2)\rangle_{F\times F} \geq{} & c_1(\|u_1\|_V)\|u_1\|_V + c_2(\|u_2\|_V)\|u_2\|_V + (C_1(u_1), u_1)_V \\ & + (C_2(u_2), u_2)_V - \lambda_1\|B_1\| \cdot \|u_1\|_V^2 - \lambda_2\|B_2\| \cdot \|u_2\|_V^2 \\ & - \int_\Omega j^0(x; (u_1 - u_2)(x), -(u_1 - u_2)(x))dx \\ \geq{} & c_1(\|u_1\|_V)\|u_1\|_V + c_2(\|u_2\|_V)\|u_2\|_V + (C_1(u_1), u_1)_V \\ & + (C_2(u_2), u_2)_V - \lambda_1\|B_1\| \cdot \|u_1\|_V^2 - \lambda_2\|B_2\| \cdot \|u_2\|_V^2 \\ & - C_p(\Omega)\|k\|_{L^q}(\|u_1\|_V + \|u_2\|_V).\end{aligned}$$

Taking into account the relation (12) we easily obtain the coercivity of Δ. Thus, Δ fulfills the conditions which allow us to apply Kakutani's fixed point theorem (see [4], Proposition 10, p. 270). Thus $\Re(\Delta) = F^* \times F^*$, which implies the existence of $u_{1F}, u_{2F} \in F$ such that $0 \in \Delta(u_{1F}, u_{2F})$. From the definition of Δ we have that there exists $\chi_F \in L^1(\Omega; \mathbb{R}^N)$ such that (9) and (10) hold. In order to prove the final part of

Theorem 2 we use the estimates:

$$\begin{aligned}\lambda_1\|B_1\|\|u_{1F}\|_V^2+\lambda_2\|B_2\|\|u_{2F}\|_V^2 &\geq \lambda_1(B_1u_{1F},u_{1F})_V+\lambda_2(B_2u_{2F},u_{2F})_V\\ &= a_1(u_{1F},u_{1F})+a_2(u_{2F},u_{2F})+(C_1(u_{1F}),u_{1F})_V\\ &\quad+(C_2(u_{2F}),u_{2F})_V+\int_\Omega \chi_F(u_{1F}-u_{2F})dx\\ &\geq c_1(\|u_{1F}\|_V)\|u_{1F}\|_V+c_2(\|u_{2F}\|_V)\|u_{2F}\|_V\\ &\quad+(C_1(u_{1F}),u_{1F})_V+(C_2(u_{2F}),u_{2F})_V\\ &\quad-\int_\Omega j^0(x;(u_{1F}-u_{2F})(x),\,-(u_{1F}-u_{2F})(x))dx.\end{aligned}$$

Taking into account the relations (5) and (12) we get

$$\frac{c_1(\|u_{1F}\|_V)\|u_{1F}\|_V+c_2(\|u_{2F}\|_V)\|u_{2F}\|_V}{\|u_{1F}\|_V+\|u_{2F}\|_V}\leq C_p(\Omega)\|k\|_{L^q}-\delta,$$

which by the properties of c_1 and c_2 implies the existence of a positive constant M such that (11) holds.

LEMMA 2 *For every* $F\in\Lambda$, *let* $u_{1F},u_{2F}\in F$, $\lambda_1,\lambda_2\in\mathbb{R}$ *and* $\chi_F\in L^1(\Omega;\mathbb{R}^N)$ *which solve the Problem* (P_F). *Then the set* $\{\chi_F\colon F\in\Lambda\}$ *is weakly precompact in* $L^1(\Omega;\mathbb{R}^N)$.

Proof The proof is based on the well-known Dunford–Petis theorem. We have to prove that for each $\epsilon>0$, a $\delta_\epsilon>0$ may be determined such that, for any $\omega\subset\Omega$ with $\mathrm{meas}(\omega)<\delta_\epsilon$,

$$\int_\omega|\chi_F|dx<\epsilon,\ \ F\in\Lambda.$$

Fix $r>0$ and let $\eta\in\mathbb{R}^N$ be such that $|\eta|\leq r$. From $\chi_F\in\partial j(x;(u_{1F}-u_{2F})(x))$, for a.e. $x\in\Omega$ we derive that

$$\chi_F\cdot(\eta-(u_{1F}-u_{2F})(x))\leq j^0(x;(u_{1F}-u_{2F})(x),\eta-(u_{1F}-u_{2F})(x)).$$

Taking into account the relation (6) it follows that

$$\chi_F(x)\cdot\eta\leq\chi_F(x)\cdot(u_{1F}-u_{2F})(x)+\alpha(x,r)(1+|u_{1F}(x)-u_{2F}(x)|^s),\quad \text{for a.e. } x\in\Omega. \tag{13}$$

Let us denote by $\chi_{Fi}(x)$, $i=1,2,\ldots,N$, the components of $\chi_F(x)$ and set

$$\eta(x)=\frac{r}{\sqrt{N}}\left(\mathrm{sgn}\chi_{F1}(x),\ldots,\mathrm{sgn}\chi_{Fn}(x)\right).$$

We can easily verify that $|\eta(x)|\leq r$ a.e. $x\in\Omega$ and that

$$\chi_F(x)\cdot\eta(x)\geq\frac{r}{\sqrt{N}}\cdot|\chi_F(x)|.$$

From (13) we obtain

$$\frac{r}{\sqrt{N}} \cdot |\chi_F(x)| \leq \chi_F(x) \cdot (u_{1F} - u_{2F})(x) + \alpha(x,r)(1 + |u_{1F}(x) - u_{2F}(x)|^s).$$

Integrating over $\omega \subset \Omega$ the above inequality yields

$$\int_\omega |\chi_F(x)| dx \leq \frac{\sqrt{N}}{r} \int_\omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx + \frac{\sqrt{N}}{r} \|\alpha(\cdot,r)\|_{L^{q'}(\omega)} \cdot \mathrm{meas}(\omega)^{s/p} + \frac{\sqrt{N}}{r} \|\alpha(\cdot,r)\|_{L^{q'}(\omega)} \cdot \|u_{1F} - u_{2F}\|^S_{L^p(\omega)}.$$

Thus, from (*) and (11) we obtain

$$\begin{aligned}\int_\omega |\chi_F(x)| dx &\leq \frac{\sqrt{N}}{r} \int_\omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx + \frac{\sqrt{N}}{r} \|\alpha(\cdot,r)\|_{L^{q'}(\Omega)} \cdot \mathrm{meas}(\omega)^{s/p} \\ &\quad + \frac{\sqrt{N}}{r} \|\alpha(\cdot,r)\|_{L^{q'}(\omega)} \cdot (C_p(\Omega))^S \cdot \|u_{1F} - u_{2F}\|^S_V \\ &\leq \frac{\sqrt{N}}{r} \int_\omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx + \frac{\sqrt{N}}{r} \|\alpha(\cdot,r)\|_{L^{q'}(\Omega)} \cdot \mathrm{meas}(\omega)^{s/p} \\ &\quad + \frac{\sqrt{N}}{r} \|\alpha(\cdot,r)\|_{L^{q'}(\omega)} \cdot (C_p(\Omega)^S \cdot M^S. \end{aligned} \tag{14}$$

We shall continue by observing that (5) implies

$$\chi_F(x) \cdot (u_{1F}(x) - u_{2F}(x)) + k(x) \cdot (1 + |u_{1F}(x) - u_{2F}(x)|) \geq 0, \quad \text{for a.e. } x \in \Omega.$$

Thus we have

$$\begin{aligned}&\int_\omega (\chi_F(x) \cdot (u_{1F} - u_{2F})(x) + k(x)(1 + |u_{1F}(x) - u_{2F}(x)|)) dx \\ &\quad \leq \int_\Omega (\chi_F(x) \cdot (u_{1F} - u_{2F})(x) + k(x)(1 + |u_{1F}(x) - u_{2F}(x)|)) dx\end{aligned}$$

and we derive that

$$\begin{aligned}\int_\omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx &\leq \int_\Omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx + \|k\|_{L^q(\Omega)} \cdot C_p(\Omega) \cdot \|u_{1F} - u_{2F}\|_V \\ &\quad + \|k\|_{L^q(\Omega)} \cdot \mathrm{meas}(\Omega)^{1/p} \\ &\leq \int_\Omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx + \|k\|_{L^q(\Omega)} \cdot \mathrm{meas}(\Omega)^{1/p} \\ &\quad + \|k\|_{L^q(\Omega)} \cdot C_p(\Omega) \cdot M.\end{aligned}$$

We have

$$\int_\Omega \chi_F(u_{1F} - u_{2F})dx = -(A_1u_{1F}, u_{1F})_V - (A_2u_{2F}, u_{2F})_V - (C_1(u_{1F}), u_{1F})_V$$
$$- (C_2(u_{2F}), u_{2F})_V + \lambda_1(B_1u_{1F}, u_{1F})_V + \lambda_2(B_2u_{2F}, u_{2F})_V.$$

Taking into account that C_i are monotone operators and that A_i, being weakly continuous maps bounded sets into bounded sets, the relation

$$\int_\Omega \chi_F(u_{1F} - u_{2F})dx \le \sum_{i=1}^{2} \left\{ \|A_i\| \|u_{iF}\|_V^2 + \lambda_i \|B_i\| \|u_{iF}\|_V^2 - (C_i(u_{iF}), u_{iF})_V \right\},$$

imply that there exists a positive constant $\tilde{C}$ such that

$$\int_\Omega \chi_F(u_{1F} - u_{2F})dx \le \tilde{C}. \tag{15}$$

Now, from (14) and (15) we obtain

$$\int_\omega |\chi_F(x)|dx \le \frac{\sqrt{N}}{r} \cdot C + \frac{\sqrt{N}}{r} \cdot \|\alpha(\cdot, r)\|_{L^{q'}(\Omega)} \cdot \text{meas}(\omega)^{s/p}$$
$$+ \frac{\sqrt{N}}{r} \cdot \|\alpha(\cdot, r)\|_{L^{q'}(\omega)} \cdot (C_p(\Omega))^S \cdot M^S, \tag{16}$$

where we have denoted

$$C := \tilde{C} + \|k\|_{L^q(\Omega)} \cdot \text{meas}(\Omega)^{1/p} + \|k\|_{L^q(\Omega)} \cdot C_p(\Omega) \cdot M.$$

Let $\epsilon > 0$. We choose $r > 0$ such that $\left(\sqrt{N}/r\right) \cdot C < \epsilon/2$. Since $\alpha(\cdot, r) \in L^{q'}(\Omega)$ we can determine $\delta_\epsilon > 0$ small enough such that if $\text{meas}(\omega) < \delta_\epsilon$, we have

$$\frac{\sqrt{N}}{r} \|\alpha(\cdot, r)\|_{L^{q'}(\Omega)} \cdot \text{meas}(\omega)^{s/p} + \frac{\sqrt{N}}{r} \|\alpha(\cdot, r)\|_{L^{q'}(\omega)} \cdot (C_p(\Omega))^S \cdot M^S < \frac{\epsilon}{2}.$$

By the relation (16) it follows that

$$\int_\omega |\chi_F(x)|dx \le \epsilon,$$

for any $\omega \subset \Omega$ with $\text{meas}(\omega) < \delta_\epsilon$. This means that the weak precompactness of $\{\chi_F \colon F \in \Lambda\}$ in $L^1(\Omega; \mathbf{R}^N)$ is established.

3. PROOF OF THEOREM 1

We are ready to prove Theorem 1, which is our main existence result. We shall follow a procedure introduced by Naniewicz and Panagiotopoulos (see, for example [9]). For every $F \in \Lambda$ let

$$W_F = \bigcup_{\substack{F' \in \Lambda \\ F' \supset F}} \{(u_{1F'}, u_{2F'}, \chi_{F'})\} \subset V \times V \times L^1(\Omega; \mathbb{R}^N),$$

with $(u_{1F'}, u_{2F'}, \chi_{F'})$ being a solution of $(P_{F'})$. Moreover, let

$$Z = \bigcup_{F \in \Lambda} \{\chi_F\} \subset L^1(\Omega; \mathbb{R}^N).$$

Denoting by weakcl(W_F) the weak closure of W_F in $V \times V \times L^1(\Omega; \mathbb{R}^N)$ and by weakcl(Z) the weak closure of Z in $L^1(\Omega; \mathbb{R}^N)$ we obtain, taking into account the relation (12)

$$\text{weakcl}(W_F) \subset B_V(O, M) \times B_V(O, M) \times \text{weakcl}(Z), \qquad \text{for every } F \in \Lambda.$$

Since V is reflexive it follows that $B_V(O, M)$ is weakly compact in V. Using Lemma 2 we get that the family $\{\text{weakcl}(W_F)\colon F \in \Lambda\}$ is contained in a weekly compact set of $V \times V \times L^1(\Omega; \mathbb{R}^N)$. It follows that this family has the finite intersection property and we may infer that

$$\bigcap_{F \in \Lambda} \text{weakcl}(W_F) \neq \emptyset.$$

We choose (u_1, u_2, χ) belonging to the nonempty set above. In what follows we shall prove that this is the searched solution for the Problem (P).

Let $v_1, v_2 \in L^\infty(\Omega; \mathbb{R}^N)$ and let F be an element of Λ such that $(v_1, v_2) \in F \times F$. We note that such an F exists, for example we can take $F = \text{span}\{v_1, v_2\}$. Since $(u_1, u_2, \chi) \in \bigcap_{F \in \Lambda} \text{weakcl}(W_F)$ it follows that there exists a sequence $\{(u_{1F_n}, u_{2F_n}, \chi_{F_n})\}$ in W_F, simply denoted by (u_{1n}, u_{2n}, χ_n) converging weakly to (u_1, u_2, χ) in $V \times V \times L^1(\Omega; \mathbb{R}^N)$. We have $u_{in} \rightharpoonup u_i$, weakly in V $(i = 1, 2)$ and $\chi_n \rightharpoonup \chi$, weakly in $L^1(\Omega; \mathbb{R}^N)$. Since (u_{1n}, u_{2n}, χ_n) is a solution of (P_F) we get

$$\begin{aligned} &\langle A_1 u_{1n}, v_1 \rangle_V + \langle A_2 u_{2n}, v_2 \rangle_V + (C_1(u_{1n}), v_1)_V + (C_2(u_{2n}), v_2)_V + \int_\Omega \chi_n (v_1 - v_2) dx \\ &\quad = \lambda_1 (B_1 u_{1n}, v_1)_V + \lambda_2 (B_2 u_{2n}, v_2)_V. \end{aligned}$$

The hypotheses on A_i, B_i, C_i $(i = 1, 2)$ and the convergences above imply the equality

$$\sum_{i=1}^{2} \left\{ \langle A_i u_i, v_i \rangle_V + (C_i(u_i), v_i)_V - \lambda_i (B_i u_i, v_i)_V \right\} + \int_\Omega \chi (v_1 - v_2) dx = 0,$$

which is satisfied for any $v_1, v_2 \in V \cap L^\infty(\Omega; \mathbb{R}^N)$. By the density of $V \cap L^\infty(\Omega; \mathbf{R}^N)$ in V we draw the conclusion that the relation (7) is valid for any $v_1, v_2 \in V$.

In what follows we shall prove the relation (8). Due to the compact embedding $V \subset L^p(\Omega; \mathbb{R}^N)$ it results from the weak convergences $u_{in} \rightharpoonup u_i$ in V that we have

$$u_{in} \to u_i \text{ strongly in } L^p(\Omega; \mathbf{R}^N), \text{ for each } i = 1, 2\,.$$

So, by passing eventually to a subsequence we have

$$u_{in} \to u_i \text{ a.e. in } \Omega.$$

From the Egoroff theorem we obtain that for any $\epsilon > 0$ a subset $\omega \subset \Omega$ with $\text{meas}(\omega) < \epsilon$ can be determined such that for each $i \in \{1, 2\}$

$$u_{in} \to u_i \text{ uniformly on } \Omega \backslash \omega,$$

with $u_i \in L^\infty(\Omega \backslash \omega; \mathbb{R}^N)$ for every $i \in \{1, 2\}$. Let $v \in L^\infty(\Omega \backslash \omega; \mathbb{R}^N)$ be arbitrarily chosen. The Fatou's lemma now implies that for any $\mu > 0$ there exists $\delta_\mu > 0$ and a positive integer N_μ such that

$$\int_{\Omega \backslash \omega} \frac{j(x; (u_{1n} - u_{2n})(x) - \theta + \lambda v(x)) - j(x; (u_{1n} - u_{2n})(x) - \theta)}{\lambda} dx \\ \leq \int_{\Omega \backslash \omega} j^0(x; (u_1 - u_2)(x), v(x)) dx + \mu, \tag{17}$$

for every $n \geq N_\mu$, $|\theta| < \delta_\mu$ and $\lambda \in (0, \delta_\mu)$. Taking into account that $\chi_n \in \partial j(x; (u_{1n} - u_{2n})(x))$ for a.e. $x \in \Omega$ we have

$$\int_{\Omega \backslash \omega} \chi_n(x) \cdot v(x) dx \leq \int_{\Omega \backslash \omega} j^0(x; (u_{1n} - u_{2n})(x), v(x)) dx. \tag{18}$$

Passing to the limit as $\lambda \to 0$ in (17) and employing the relation (18) it follows that

$$\int_{\Omega \backslash \omega} \chi_n(x) \cdot v(x) dx \leq \int_{\Omega \backslash \omega} j^0(x; (u_1 - u_2)(x), v(x)) dx + \mu.$$

From the relation above and the weak convergence of χ_n to χ in $L^1(\Omega; \mathbb{R}^N)$ we derive that

$$\int_{\Omega \backslash \omega} \chi(x) \cdot v(x) dx \leq \int_{\Omega \backslash \omega} j^0(x; (u_1 - u_2)(x), v(x)) dx + \mu.$$

Since $\mu > 0$ was chosen arbitrarily,

$$\int_{\Omega\backslash\omega} \chi(x) \cdot v(x)dx \leq \int_{\Omega\backslash\omega} j^0(x; (u_1 - u_2)(x), v(x))dx, \quad \forall v \in L^\infty(\Omega\backslash\omega; \mathbf{R}^N).$$

The last inequality implies that

$$\chi(x) \in \partial j(x; (u_1 - u_2)(x)), \quad \text{for a.e. } x \in \Omega\backslash\omega,$$

where $\text{meas}(\omega) < \epsilon$. Since $\epsilon > 0$ was chosen arbitrarily we have that

$$\chi(x) \in \partial j(x; (u_1 - u_2)(x)), \quad \text{for a.e. } x \in \Omega,$$

which means that the relation (8) holds. The proof of Theorem 1 is now complete.

4. APPLICATION: THE MULTIPLE LOADING BUCKLING

We consider two elastic beams (linear elasticity) of length l measured along the axis Ox of the coordinate system yOx, and with the same cross-section. The beams, numbered here by $i = 1, 2,$ are simply supported at their ends $x = 0$ and $x = l$. On the interval (l_1, l_2), $l_1 < l_2 < l$, they are connected with an adhesive material of negligible thickness. The displacements of the ith beam are denoted by $x \to u_i(x)$, $i = 1, 2,$ and the behavior of the adhesive material is described by a nonmonotone possibly multivalued law between $-f(x)$ and $[u(x)]$, where $x \to f(x)$ denotes the reaction force per unit length vertical to the Ox axis, due to the adhesive material (cf. [9] p. 110 and [10] p. 87) and $[u] = u_1 - u_2$ is the relative deflection of the two beams. Recall that u_i is referred to the middle line of the beam i (the dotted lines in Fig. 1) and that each beam has constant thickness which remains the same after the deformation. The adhesive material can sustain a small tensile force before rupture (debonding). In Fig. 1 a rupture of zig-zag brittle type is depicted in the $(-f, u)$ diagramm. The beams are assumed to have the same moduli of elasticity E and let I be the moment of inertia of them. The sandwich beam is subjected to the compressive forces P_1 and P_2 and we want to determine the buckling loading of it. This problem is yet open problem in Engineering. From the large deflection theory of beams we may write the following relations which describe the behavior of the ith beam:

$$u_i''''(x) + \frac{1}{a_i^2}\, u_i''(x) = f_i(x) \quad \text{on } (0, l)\,; \tag{19}$$

$$u_i(0) = u_i(l) = 0\,, \quad u_i''(0) = u_i''(l) = 0 \quad i = 1, 2\,. \tag{20}$$

Here $a_i^2 := IE/P_i$. We assume that the $(-f, [u])$ graph results from a nonlocally Lipschitz function $j : \mathbb{R} \to \overline{\mathbb{R}}$ such that

$$-f(x) \in \partial j([u(x)])\,, \quad \forall x \in (l_1, l_2)\,, \tag{21}$$

where ∂ denotes the generalized gradient of Clarke. We set

$$V := H^2(\Omega) \cap H_0^1(\Omega) \qquad \Omega = (0, l). \tag{22}$$

It is a Hilbert space with the inner product (see [6], p. 216, Lemma 4.2) $a(u, v) := \int_0^l u''(x)v''(x)dx$.

Let $L : V \to V^\star$ be the linear operator defined by

$$\langle Lu, v\rangle := \int_0^l u'(x)v'(x)dx\,, \quad \forall u, v \in V. \tag{23}$$

We observe easily that L is bounded, weak continuous and satisfies

$$\langle Lu, v\rangle = \langle Lv, u\rangle\,, \quad \text{for all } u, v \in V.$$

The superpotential law (21) implies that

$$j^0([u(x)]; y) \geq -f(x)y\,, \quad \forall x \in (l_1, l_2)\,, \forall y \in \mathbb{R}. \tag{24}$$

Multiplying (19) by $v_i(x) - u_i(x)$, integrating over $(0, l)$ and adding the resulting relations for $i = 1, 2$, implies by taking into account the boundary condition (20), the hemivariational inequality

$$u = \{u_1, u_2\} \in V \times V,$$

$$\sum_{i=1}^{2} \int_0^l u_i''(x)[v_i''(x) - u_i''(x)]dx - \sum_{i=1}^{2} \frac{1}{a_i^2} \int_0^l u_i'(x)[v_i'(x) - u_i'(x)]dx$$
$$+ \int_{l_1}^{l_2} j^0([u(x)]; [v(x)] - [u(x)])dx \geq 0\,, \quad \forall v = \{v_1, v_2\} \in V \times V. \tag{25}$$

Thus buckling of the beam occurs if $\lambda_i := (1/a_i^2)$ $(i = 1, 2)$ is an eigenvalue for the following hemivariational inequality

$$\sum_{i=1}^{2} a_i(u_i, v_i - u_i) - \sum_{i=1}^{2} \lambda_i \langle u_i, v_i - u_i\rangle + \int_{l_1}^{l_2} j^0([u(x)]; [v(x)] - [u(x)])dx \geq 0\,, \tag{26}$$

for all $v = \{v_1, v_2\} \in V \times V$. According to the Theorem 1 the present problem admits at least one solution $\{u_1, u_2, \lambda_1, \lambda_2\}$, provided that j fulfills the growth assumption given in Section 1, i.e., (1), (5) and (6).

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References

[1] M.F. Bocea, D. Motreanu and P.D. Panagiotopoulos (2000). Multiple solutions for a double eigenvalue hemivariational inequality on a spherelike type manifold. *Nonlinear Analysis, T.M.A.*, **42**, 737–749.

[2] M.F. Bocea, P.D. Panagiotopoulos and V.D. Rădulescu (1999). A perturbation result for a double eigenvalue hemivariational inequality with constraints and applications. *J. Global Optimization*, **14**, 137–156.
[3] H. Brezis (1992). *Analyse fonctionnelle. Théorie et Applications*. Masson, Paris.
[4] F. Browder and P. Hess (1972). Nonlinear mappings of monotone type in Banach spaces. *J. Funct. Anal.*, **11**, 251–294.
[5] F.H. Clarke (1983). *Optimization and Nonsmooth Analysis*. Willey, New York.
[6] G. Duvaut and J.-L. Lions (1972). *Les Inéquations en Mécanique et en Physique*. Dunod, Paris.
[7] D. Motreanu and P.D. Panagiotopoulos (1997). Double eigenvalue problems for hemivariational inequalities. *Arch. Rat. Mech. Analysis*, **140**, 225–251.
[8] Z. Naniewicz (1995). Hemivariational inequalities with functionals which are not locally Lipschitz. *Nonlinear Analysis, T.M.A.*, **25**(12), 1307–1320.
[9] Z. Naniewicz and P.D. Panagiotopoulos (1995). *Mathematical Theory of Hemivariational Inequalities and Applications*. Marcel Dekker, New York.
[10] P.D. Panagiotopoulos (1993). Hemivariational Inequalities. *Applications in Mechanics and Engineering*. Springer-Verlag, Berlin–Heidelberg.