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Double phase implicit obstacle problems with convection term and multivalued operator

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This paper is devoted to studying a complicated implicit obstacle problem involving a nonhomogenous differential operator, called double phase operator, a nonlinear convection term (i.e. a reaction term depending on the gradient), and a multivalued term which is described by Clarke's generalized gradient. We develop a general framework to deliver an existence result for the double phase implicit obstacle problem under consideration. Our proof is based on the Kakutani–Ky Fan fixed point theorem together with the theory of nonsmooth analysis and a surjectivity theorem for multivalued mappings generated by the sum of a maximal monotone multivalued operator and a bounded pseudomonotone mapping.

Keywords: Double phase problem; implicit obstacle problem; Clarke's generalized gradient; Kakutani–Ky Fan fixed point theorem; surjectivity theorem; existence of solution.

Mathematics Subject Classification 2020: 35J20, 35J25, 35J60

1. Introduction

Assume that Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, such that its boundary $\partial\Omega$ is Lipschitz-continuous. Let $1 < p < q < N$ and $\mu : \overline{\Omega} \to [0,\infty)$ be a bounded function, $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous with respect to the second variable. In this paper, we consider the following nonlinear partial differential inclusion problem involving implicit obstacle effect, a nonlinear convection term and a multivalued term which is formulated by the generalized Clarke subgradient

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) &\in h\partial j(x,u) + f(x,u,\nabla u) \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ T(u) &\leq U(u), && \end{aligned} \tag{1.1}$$

where $h \in \mathbb{R}$, $T : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R} \cup \{-\infty\}$ and $U : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are given functions, which will be specialized by the appropriate conditions in Sec. 3. Here, $W_0^{1,\mathcal{H}}(\Omega)$ is a closed subspace of the Sobolev–Musielak–Orlicz space $W^{1,\mathcal{H}}(\Omega)$ (see Sec. 2).

Definition 1.1. We say that $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a weak solution of problem (1.1), if there exists $w \in L^{p'}(\Omega)$ such that $w(x) \in \partial j(x, u(x))$ for a.a. $x \in \Omega$ and

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u, \nabla(v-u)\right)_{\mathbb{R}^N} dx \\ &\quad = h\int_\Omega w(x)(v(x)-u(x))dx + \int_\Omega f(x,u,\nabla u)(v(x)-u(x))dx \end{aligned}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$ with $T(v) \leq U(u)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

To highlight the general form of our problem, we list the following particular cases of problem (1.1).

(i) If $h = -1$ and f is independent of u, then problem (1.1) reduces the following one:

$$\begin{aligned} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) + \partial j(x,u) \ni f(x) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \\ T(u) \le U(u), & \end{aligned} \tag{1.2}$$

which has been studied by Zeng *et al.* [59].

(ii) When $j \equiv 0$, then problem (1.1) becomes the following double phase implicit obstacle problem involving convection term:

$$\begin{aligned} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) = f(x,u,\nabla u) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \\ T(u) \le U(u). & \end{aligned}$$

(iii) If $j \equiv 0$, $U \equiv 0$ and $T(u) = \int_\Omega (u(x) - \Phi(x))^+ \, dx$, where $\Phi : \Omega \to \mathbb{R}$ is a given obstacle function, then problem (1.1) becomes the following elliptic obstacle problem:

$$\begin{aligned} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) = f(x,u,\nabla u) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \\ u(x) \le \Phi(x) \quad & \text{on } \Omega, \end{aligned}$$

which has been investigated by Zeng *et al.* [58].

(iv) If $j \equiv 0$ and $U \equiv +\infty$, then problem (1.1) becomes the following double phase problem:

$$\begin{aligned} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) = f(x,u,\nabla u) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

which has been studied by Gasiński and Winkert [31].

Indeed, under the assumption that ∂j satisfies relaxed monotone, Zeng *et al.* [59] applied Kluge fixed point theorem to explore the existence of solutions to problem (1.2). However, an interesting question arises on how to prove the existence of a solution of problem (1.2) without the relaxed monotonicity condition on the subgradient operator ∂j. One of the main aims of the current paper is to give a positive answer to this open question. Besides, the method applied in this paper is completely different from that used in [59]. More precisely, our approach is based on the Kakutani–Ky Fan fixed point theorem.

Note that the double phase operator defined by

$$-\operatorname{div}(|\nabla\omega|^{p-2}\nabla\omega + \mu(x)|\nabla\omega|^{q-2}\nabla\omega), \quad \omega \in W_0^{1,\mathcal{H}}(\Omega) \tag{1.3}$$

is related to the energy functional

$$\omega \mapsto \int_\Omega (|\nabla\omega|^p + \mu(x)|\nabla\omega|^q)dx. \tag{1.4}$$

Functionals of type (1.4) have first been studied by Zhikov [62–64] in order to provide models for strongly anisotropic materials. The main characteristic of the functional defined in (1.4) is the change of ellipticity on the set where the weight function is zero, that is, on the set $\{x \in \Omega : \mu(x) = 0\}$. To be more precise, the energy density of (1.4) exhibits ellipticity in the gradient of order q on the points x where $\mu(x)$ is positive and of order p on the points x where $\mu(x)$ vanishes. After that, Bahrouni *et al.* [5] found that double phase operators (1.3) can be applied exactly to study the transonic flow with nonlinear patterns and stationary waves; Colombo and Mingione [20] proved sharp regularity theorems for minimizers of a class of variational integrals whose integrand switches between two different types of degenerate elliptic phases; Baasandorj *et al.* [4] proved Calderón–Zygmund type estimates for distributional solutions to non-uniformly elliptic equations of generalized double phase type in divergence form. Further results on the topic concerning the theory and applications of double phase operators can be found in the papers of Zhikov *et al.* [65], Baroni *et al.* [7, 8, 10], Baroni *et al.* [9], Cupini *et al.* [22], Migórski and Zeng [46], Colombo and Mingione [19], Marcellini [41, 42], Colasuonno and Squassina [18], Gasiński and Papageorgiou [27], Gasiński and Winkert [30, 31], Liu and Dai [36], Farkas and Winkert [25], Perera and Squassina [55], Biagi *et al.* [11], Carl *et al.* [13], Cencelj *et al.* [16], Papageorgiou *et al.* [50, 52], Zhang and Rădulescu [61], Gasiński and Papageorgiou [28, 29], Rădulescu [56], Bahrouni *et al.* [6].

It is well known that, in a single or a multiphase fluid flow, the convection effect may appear spontaneously because of the combined effects of material heterogeneity and the influence of body forces on a fluid (commonly density and gravity). Whereas, the reaction terms which depend on the gradient of unknown functions can precisely model the convection effect for various fluids flow. Such reactions are usually called convection terms. Whereas, the second interesting phenomena in this paper is the appearance of a nonlinearity on the right-hand side which also depends on the gradient of the solution. The essential difficulty with the gradient dependent term is the nonvariational character of the problem. Based on this motivation, in the past years, several interesting works have been published with convection terms, for instance, Liu *et al.* [39] applied the Leray–Schauder alternative principle, method of sub-supersolution, nonlinear regularity, truncation techniques, and set-valued analysis to examine the existence of positive solutions for a nonlinear Dirichlet problem of p-Laplacian type with combined effects of nonlinear singular and convection terms; via employing a topological approach based on the Leray–Schauder alternative principle together with suitable truncation and comparison techniques, Papageorgiou *et al.* [51] proved the existence of a positive smooth solution to a nonlinear Neumann problem driven by the p-Laplacian, in which the reaction has the competing effects of a singular and a convection term. For more representative references on the topics related to nonlinear elliptic problems with gradient dependence, we refer to the recent works Faraci *et al.* [23], Gasiński and Papageorgiou [29], Faraci and Puglisi [24], Figueiredo and Madeira [26], Araujo and Faria [1], Cen *et al.* [14], Cen *et al.* [15], as well as to the monograph Papageorgiou *et al.* [53].

Moreover, the third and fourth interesting phenomena are implicit obstacle effect and the appearance of a multivalued term which is described by Clarke's generalized gradient, respectively. Originally, the study of obstacle problems is due the pioneering contribution by Stefan [57] in which the temperature distribution in a homogeneous medium undergoing a phase change, typically a body of ice at zero degrees centigrade submerged in water, was studied. However, in many critical situations arising in engineering and economic models, such as Nash equilibrium problems with shared constraints, semipermeability problems with free boundary conditions and transport route optimization with feedback control, the constraint conditions, usually, depend implicitly on the unknown solution. Some interesting and challenging problems have been considered and studied, for example, in order to answer the question of F. Camilli *et al.* mentioned in [12], Gomes and Serra [33] used the methods from the theory of viscosity solutions and weak KAM theory to extend the notion of Aubry set for the systems of weakly coupled Hamilton-Jacobi equations with implicit obstacles that arise in optimal switching problems. On the other hand, the theory of hemivariational inequalities was initially introduced by Panagiotopoulos [47–49] to study the nonsmooth mechanical problems in which the main idea behind hemivariational inequalities is to remove the hypotheses on differentiability and convexity of energy functionals with the help of the generalized subgradient of Clarke. More recently, the problems with implicit obstacle effect and Clarke's generalized gradient have been studied extendly, such as Migórski *et al.* [44] studied an identification inverse problem in a complicated mixed elliptic boundary value problem with p-Laplace operator and an implicit obstacle condition by using Kluge fixed point theorem combined with the theory of nonsmooth analysis and the Minty technique. For more details on this topic, the reader is referred to Alleche and Rădulescu [2], Liu *et al.* [37], Liu *et al.* [38], Aussel *et al.* [3], Migórski *et al.* [45], Liu *et al.* [40], Gwinner [32], Zeng *et al.* [60], and the cited references therein.

The primary objective of this research is to develop a general framework for examining the existence of a (weak) solution of problem (1.1) by applying a surjectivity theorem for multivalued mappings, the theory of nonsmooth analysis and Kakutani–Ky Fan fixed point theorem. In fact, to the best of our knowledge, this is the first work which combines a double phase phenomena along with Clarke's generalized gradient, an implicit obstacle and a nonlinear convection term (i.e. a reaction term depending on the gradient).

The paper is organized as follows. In Sec. 2, we recall the definition of the used function spaces, some embedding results and we state the surjectivity results of Le [34] for multivalued mappings as well as Kakutani–Ky Fan fixed point theorem. In Sec. 3, we provide the full assumptions on the data of problem (1.1), and introduce an auxiliary problem defined in (3.6). Then, we show that the auxiliary problem (3.6) is unique solvability, see Theorem 3.1, and deliver *a priori* estimate result for the solutions of problem (1.1). Moreover, we introduce a solution map L of (3.6) and a multivalued mapping J, and prove that the graph of (L, J) is weakly sequentially closed. Taking these results and Kakutani–Ky Fan fixed point theorem into account we are able to prove our main result which says that the solution set of (1.1) is nonempty and weakly compact in $W_0^{1,\mathcal{H}}(\Omega)$, see Theorem 3.2.

2. Mathematical Prerequisites

Throughout the paper, we assume that Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. For any $1 \leq r \leq \infty$ fixed, we denote by $L^r(\Omega)$ and $L^r(\Omega;\mathbb{R}^N)$ the usual Lebesgue spaces equipped with the norm $\|\cdot\|_p$. Set $W^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ the Sobolev spaces endowed with the norms $\|\cdot\|_{1,r}$ and $\|\cdot\|_{1,r,0}$, respectively. Let $1 < r < +\infty$, we adopt the symbol $r' > 1$ for the conjugate of r, namely, $\frac{1}{r} + \frac{1}{r'} = 1$.

We, first, recall a critical inequality, its detailed proof can be found in [17, Sec. 4].

Lemma 2.1. *Let $r \geq 2$ and $N \in \mathbb{N}$. Then there exists a constant $k(r) > 0$ such that*

$$(|x|^{r-2}x - |y|^{r-2}y, x - y)_{\mathbb{R}^N} \geq k(r)|x - y|^r$$

for all $x, y \in \mathbb{R}^N$.

In what follows, we suppose that the weight function μ enjoys the following conditions (see e.g., [21]).

H(μ): $\mu \in L^\infty(\Omega)$ with $\mu(x) \geq 0$ for a.a. $x \in \Omega$ and $1 < p < q < N$ are chosen such that

$$\frac{q}{p} < 1 + \frac{1}{N}.$$

We, further, consider the modular function $\mathcal{H} : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\mathcal{H}(x, t) = t^p + \mu(x)t^q \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}_+,$$

where $\mathbb{R}_+ := [0, \infty)$. Let us recall the Musielak–Orlicz space $L^{\mathcal{H}}(\Omega)$, which is formulated by

$$L^{\mathcal{H}}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable} \,\middle|\, \rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x, |u|)dx < +\infty \right\}$$

and equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf \left\{ \tau > 0 \,\middle|\, \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1 \right\}.$$

Obviously, the function space $L^{\mathcal{H}}(\Omega)$ is uniformly convex, so it is a reflexive Banach space. Additionally, let us introduce the seminormed function space $L^q_\mu(\Omega)$,

$$L^q_\mu(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable} \,\middle|\, \int_\Omega \mu(x)|u|^q dx < +\infty \right\}$$

endowed with the seminorm

$$\|u\|_{q,\mu} = \left(\int_\Omega \mu(x)|u|^q \, dx \right)^{\frac{1}{q}}.$$

Whereas, Colasuonno and Squassina [18, Propositions 2.15(i), 2.15(iv) and 2.15(v)] point out that the following embeddings:

$$L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_\mu(\Omega)$$

are both continuous. Besides, it is not difficult to verify that the following inequalities are valid:

$$\min\{\|u\|^p_{\mathcal{H}}, \|u\|^q_{\mathcal{H}}\} \le \|u\|^p_p + \|u\|^q_{q,\mu} \le \max\{\|u\|^p_{\mathcal{H}}, \|u\|^q_{\mathcal{H}}\} \tag{2.1}$$

for all $u \in L^{\mathcal{H}}(\Omega)$.

Moreover, we review the Sobolev–Musielak–Orlicz space $W^{1,\mathcal{H}}(\Omega)$ given by

$$W^{1,\mathcal{H}}(\Omega) = \{u \in L^{\mathcal{H}}(\Omega) \mid |\nabla u| \in L^{\mathcal{H}}(\Omega)\}$$

and is equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \||\nabla u|\|_{\mathcal{H}}$.

The Sobolev–Musielak–Orlicz space with zero traces, denoted by $W^{1,\mathcal{H}}_0(\Omega)$, is the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$, namely,

$$W^{1,\mathcal{H}}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{W^{1,\mathcal{H}}(\Omega)}.$$

It follows from $\mu : \overline{\Omega} \to \mathbb{R}_+$ in H(μ) and Colasuonno and Squassina [18, Proposition 2.18] that $W^{1,\mathcal{H}}_0(\Omega)$ endowed the norm

$$\|u\|_{1,\mathcal{H},0} = \|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W^{1,\mathcal{H}}_0(\Omega)$$

becomes a reflexive and separable Banach space. Therefore, $\|\cdot\|_{1,\mathcal{H},0}$ is an equivalent norm on $W^{1,\mathcal{H}}_0(\Omega)$. So, in the sequel, we use $\|\cdot\|_{1,\mathcal{H},0}$ to be the norm of $W^{1,\mathcal{H}}_0(\Omega)$. Using (2.1), it is easy to get that

$$\min\{\|u\|^p_{1,\mathcal{H},0}, \|u\|^q_{1,\mathcal{H},0}\} \le \|\nabla u\|^p_p + \|\nabla u\|^q_{q,\mu} \le \max\{\|u\|^p_{1,\mathcal{H},0}, \|u\|^q_{1,\mathcal{H},0}\} \tag{2.2}$$

for all $u \in W^{1,\mathcal{H}}_0(\Omega)$. Observe that $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$ are uniformly convex, therefore, they are both reflexive.

Colasuonno and Squassina [18, Proposition 2.15] indictes directly that the embedding

$$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega) \tag{2.3}$$

is continuous for all $1 \le r \le p^*$, and is compact for every $1 \le r < p^*$, where p^* stands for the critical exponent to p given by

$$p^* := \frac{Np}{N-p}. \tag{2.4}$$

We, now, pay our attention to recall some critical properties of the eigenvalue problem for the r-Laplacian ($1 < r < \infty$) with homogeneous Dirichlet boundary condition given by

$$\begin{aligned} -\Delta_r u &= \lambda |u|^{r-2}u \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{aligned} \tag{2.5}$$

We denote by σ_r the set of all eigenvalues of (2.5). It is known that the set σ_r has a smallest element $\lambda_{1,r}$ which is positive, isolated, simple and it can be variationally characterized through

$$\lambda_{1,r} = \inf\left\{ \frac{\|\nabla u\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \ne 0 \right\},$$

see Lê [35].

Let us introduce a nonlinear operator $A : W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ given by

$$\langle A(u), v\rangle_{\mathcal{H}} := \int_\Omega (|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) \cdot \nabla v \, dx \tag{2.6}$$

for $u, v \in W_0^{1,\mathcal{H}}(\Omega)$, where $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ stands for the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$. The properties of the operator $A : W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ can be summarized as follows, for more details see Liu and Dai [36].

Proposition 2.1. *The operator A defined by* (2.6) *is bounded, continuous, monotone* (*hence maximal monotone*) *and of type* (S_+).

Given a real Banach $(E, \|\cdot\|_E)$, we say that a function $j : E \to \mathbb{R}$ is locally Lipschitz at $x \in E$, if there is a neighborhood $O(x)$ of x and a constant $L_x > 0$ such that

$$|j(y) - j(z)| \le L_x\|y - z\|_E \quad \text{for all } y, z \in O(x).$$

We denote by $j^0(x; y)$,

$$j^\circ(x; y) := \limsup_{z\to x, \lambda\downarrow 0} \frac{j(z + \lambda y) - j(z)}{\lambda},$$

the generalized directional derivative of j at the point x in the direction y and $\partial j : E \to 2^{E^*}$

$$\partial j(x) := \{\xi \in E^* \mid j^0(x; y) \ge \langle \xi, y\rangle_{E^*\times E} \text{ for all } y \in E\} \quad \text{for all } x \in E$$

the generalized gradient in the sense of Clarke of j.

We next collect some critical properties for the generalized gradient and generalized directional derivative of a locally Lipschitz function as follows, see for example, Migórski *et al.* [43, Proposition 3.23].

Proposition 2.2. *Let $j : E \to \mathbb{R}$ be locally Lipschitz with Lipschitz constant $L_x > 0$ at $x \in E$. Then we have the following*:

(i) *The function $y \mapsto j^0(x; y)$ is positively homogeneous, subadditive, and satisfies*

$$|j^0(x; y)| \leq L_x \|y\|_E \quad \textit{for all } y \in E.$$

(ii) *The function $(x, y) \mapsto j^0(x; y)$ is upper semicontinuous.*
(iii) *For each $x \in E$, $\partial j(x)$ is a nonempty, convex, and weak* compact subset of E^* with $\|\xi\|_{E^*} \leq L_x$ for all $\xi \in \partial j(x)$.*
(iv) *$j^0(x; y) = \max\{\langle \xi, y\rangle_{E^* \times E} \,|\, \xi \in \partial j(x)\}$ for all $y \in E$.*
(v) *The multivalued function $E \ni x \mapsto \partial j(x) \subset E^*$ is upper semicontinuous from E into w^*-E^*.*

Furthermore, let us recall the Kakutani–Ky Fan fixed point theorem for a reflexive Banach space, see e.g., [54, Theorem 2.6.7], which will be applied to prove the main result in the paper concerning the existence of solutions to problem (1.1).

Theorem 2.1. *Let Y be a reflexive Banach space and $D \subseteq Y$ be a nonempty, bounded, closed and convex set. Let $\Lambda : D \to 2^D$ be a multivalued map with nonempty, closed and convex values such that its graph is sequentially closed in $Y_w \times Y_w$ topology. Then Λ has a fixed point.*

Finally, we end this section by recalling the following surjectivity theorem for multivalued mappings in which its detailed proof can be found in Le [34, Theorem 2.2]. Set $B_R(0) := \{u \in X : \|u\|_X < R\}$.

Theorem 2.2. *Let X be a real reflexive Banach space, let $F : D(F) \subset X \to 2^{X^*}$ be a maximal monotone operator, let $G : D(G) = X \to 2^{X^*}$ be a bounded multivalued pseudomonotone operator and let $l \in X^*$. Assume that there exist $u_0 \in X$ and $R \geq \|u_0\|_X$ such that $D(F) \cap B_R(0) \neq \emptyset$ and*

$$\langle \xi + \eta - l, u - u_0\rangle_{X^* \times X} > 0$$

for all $u \in D(F)$ with $\|u\|_X = R$, for all $\xi \in F(u)$ and for all $\eta \in G(u)$. Then the inclusion

$$F(u) + G(u) \ni l$$

has a solution in $D(F)$.

3. Main Results

We, first, impose the following assumptions for the data of problem (1.1).

H(f): $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carethéodory function such that

(i) There exist $a_j, b_j\ge 0$ and a function $\alpha_f\in L^{\frac{q_1}{q_1-1}}(\Omega)_+$ satisfying

$$|f(x,s,\xi)|\le a_j|\xi|^{\frac{p(q_1-1)}{q_1}}+b_j|s|^{q_1-1}+\alpha_f(x)$$

for a.a. $x\in\Omega$, for all $s\in\mathbb{R}$ and for all $\xi\in\mathbb{R}^N$, where $1<q_1<p^*$ with the critical exponent p^* given in (2.4).

(ii) There exist $c_f, d_f\ge 0$, $\theta_1,\theta_2\in[1,p]$ and a function $\beta_f\in L^1(\Omega)_+$ satisfying

$$f(x,s,\xi)s\le c_f|\xi|^{\theta_1}+d_f|s|^{\theta_2}+\beta_f(x)$$

for a.a. $x\in\Omega$, for all $s\in\mathbb{R}$ and for all $\xi\in\mathbb{R}^N$.

(iii) There exist $e_f, h_f\ge 0$ such that

$$(f(x,s,\xi)-f(x,t,\xi))(s-t)\le e_f|s-t|^p,$$
$$|f(x,s,\xi_1)-f(x,s,\xi_2)|\le h_f|\xi_1-\xi_2|^{p-1}$$

for a.a. $x\in\Omega$, for all $s,t\in\mathbb{R}$ and for all $\xi,\xi_1,\xi_2\in\mathbb{R}^N$.

H(j): $j:\Omega\times\mathbb{R}\to\mathbb{R}$ is such that

(i) $x\mapsto j(x,s)$ is measurable on Ω for all $s\in\mathbb{R}$ and there exists a function $l\in L^p(\Omega)$ such that the function $x\mapsto j(x,l(x))$ belongs to $L^1(\Omega)$.

(ii) $s\mapsto j(x,s)$ is locally Lipschitz continuous for a.a. $x\in\Omega$.

(iii) there exist $c_j\ge 0$ and $\gamma_j\in L_+^{\frac{p}{p-1}}(\Omega)$ satisfying

$$|\eta|\le c_j|r|^{p-1}+\gamma_j(x)$$

for a.a. $x\in\Omega$, for all $\eta\in\partial j(x,s)$ and for all $s\in\mathbb{R}$.

H(T): $T:W_0^{1,\mathcal{H}}(\Omega)\to\mathbb{R}\cup\{-\infty\}$ is positively homogeneous and subadditive such that $T\not\equiv-\infty$ and

$$T(u)\le\limsup_{n\to\infty}T(u_n) \tag{3.1}$$

whenever $\{u_n\}\subset W_0^{1,\mathcal{H}}(\Omega)$ is such that $u_n\xrightarrow{w}u$ in $W_0^{1,\mathcal{H}}(\Omega)$ for some $u\in W_0^{1,\mathcal{H}}(\Omega)$.

H(U): $U:W_0^{1,\mathcal{H}}(\Omega)\to(0,+\infty]$ with $U\not\equiv+\infty$ is sequentially weakly continuous, that is, for any sequence $\{u_n\}\subset W_0^{1,\mathcal{H}}(\Omega)$ such that $u_n\xrightarrow{w}u$ for some $u\in W_0^{1,\mathcal{H}}(\Omega)$, we have

$$U(u_n)\to U(u).$$

H(0): $h \in \mathbb{R}$ and the inequalities hold

$$e_f \lambda_{1,p}^{-1} + h_f \lambda_{1,p}^{-\frac{1}{p}} < k(p)$$

and

$$c_f \delta(\theta_1) + d_f \lambda_{1,p}^{-1} \delta(\theta_2) + |h| c_j \lambda_{1,p}^{-1} < 1,$$

where $k(p) > 0$ is given in Lemma 2.1 and $\delta : [1, p] \to \{1, 0\}$ is defined by

$$\delta(\theta) = \begin{cases} 1 & \text{if } \theta = p, \\ 0 & \text{otherwise} \end{cases}$$

and $\lambda_{1,p}$ denotes the first eigenvalue of the Dirichlet eigenvalue problem for the p-Laplacian, see (2.5).

Remark 3.1. Recall that T is positively homogeneous and subadditive, so, we can see that T is also a convex function. Obviously, when $T : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ is lower semicontinuous, then inequality (3.1) holds automatically.

Let $p = 2$. Then, it is not difficult to apply Young inequality to see that the function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$f(x, s, \xi) := \lambda \left(\sum_{i=1}^{N} \eta_i \xi_i + \beta_f(x) + s \right) \text{ for all } (x, s, \xi) \in \Omega \times \mathbb{R}^{N+1},$$

satisfies hypotheses H(f) and H(0), when $\lambda > 0$ is small enough, where $\eta = (\eta_1, \ldots, \eta_N) \in \mathbb{R}^N$ and $\beta_f \in L^2(\Omega)$ are given.

Let us introduce the following multivalued map $K : W_0^{1,\mathcal{H}}(\Omega) \to 2^{W_0^{1,\mathcal{H}}(\Omega)}$ defined by

$$K(u) := \{ v \in W_0^{1,\mathcal{H}}(\Omega) \mid T(v) - U(u) \leq 0 \} \tag{3.2}$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$. Then, we have the following useful properties for function K.

Lemma 3.1. *Let $U : W_0^{1,\mathcal{H}}(\Omega) \to (0, +\infty)$ and $T : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ satisfy hypotheses $H(U)$ and $H(T)$, respectively. Then, we have*

- (i) *for each $u \in W_0^{1,\mathcal{H}}(\Omega)$, the set $K(u)$ is nonempty (more precisely, $0 \in K(u)$ for all $u \in W_0^{1,\mathcal{H}}(\Omega)$), closed and convex in $W_0^{1,\mathcal{H}}(\Omega)$.*
- (ii) *the graph* GrK *of K is sequentially closed in $W_0^{1,\mathcal{H}}(\Omega)_w \times W_0^{1,\mathcal{H}}(\Omega)_w$, i.e. K is sequentially closed from $W_0^{1,\mathcal{H}}(\Omega)$ endowed with the weak topology into the subsets of $W_0^{1,\mathcal{H}}(\Omega)$ with the weak topology.*
- (iii) *if $\{u_n\} \subset W_0^{1,\mathcal{H}}(\Omega)$ and $u \in W_0^{1,\mathcal{H}}(\Omega)$ are such that*

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \text{ as } n \to \infty,$$

then for each $v \in K(u)$, *we are able to find a sequence* $\{v_n\} \subset W_0^{1,\mathcal{H}}(\Omega)$ *such that*

$$v_n \in K(u_n) \quad \textit{and} \quad v_n \to v \textit{ in } W_0^{1,\mathcal{H}}(\Omega) \textit{ as } n \to \infty.$$

Proof. (i) It is a direct consequence of Zeng *et al.* [59, Lemma 3.2].

(ii) Let $\{(u_n, v_n)\} \subset \mathrm{Gr}K$ be such that

$$(u_n, v_n) \xrightarrow{w} (u, v) \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \times W_0^{1,\mathcal{H}}(\Omega) \text{ as } n \to \infty \tag{3.3}$$

for some $(u, v) \in W_0^{1,\mathcal{H}}(\Omega) \times W_0^{1,\mathcal{H}}(\Omega)$. Then, for each $n \in \mathbb{N}$, it has $v_n \in K(u_n)$, that is, $T(v_n) \leq U(u_n)$. Recall that U is weakly continuous on $W_0^{1,\mathcal{H}}(\Omega)$ (see $H(U)$), it yields

$$\lim_{n\to\infty} U(u_n) = U(u). \tag{3.4}$$

On the other side, we apply hypotheses $H(T)$ to get

$$T(v) \leq \limsup_{n\to\infty} T(v_n). \tag{3.5}$$

Taking into account (3.4) and (3.5), we have

$$T(v) \leq \limsup_{n\to\infty} T(v_n) \leq \limsup_{n\to\infty} U(u_n) = U(u).$$

Hence, $v \in K(u)$, namely, $(u, v) \in \mathrm{Gr}K$. So, the graph $\mathrm{Gr}K$ of K is sequentially closed in $W_0^{1,\mathcal{H}}(\Omega)_w \times W_0^{1,\mathcal{H}}(\Omega)_w$.

(iii) Let $\{u_n\} \subset W_0^{1,\mathcal{H}}(\Omega)$ and $u \in W_0^{1,\mathcal{H}}(\Omega)$ be such that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \text{ as } n \to \infty.$$

Let $v \in K(u)$ be arbitrary. Recall that $U(z) > 0$ for all $z \in W_0^{1,\mathcal{H}}(\Omega)$, we consider the sequence $\{v_n\} \subset W_0^{1,\mathcal{H}}(\Omega)$ defined by

$$v_n = \frac{U(u_n)}{U(u)} v.$$

However, the following estimates guarantee that $v_n \in K(u_n)$ for each $n \in \mathbb{N}$,

$$T(v_n) = T\left(\frac{U(u_n)}{U(u)} v\right) = \frac{U(u_n)}{U(u)} T(v) \leq \frac{U(u_n)}{U(u)} U(u) = U(u_n),$$

where we have used the fact that $v \in K(u)$ (i.e. $T(v) \leq U(u)$). Additionally, a simple computation gives

$$\|v_n - v\|_{1,\mathcal{H},0} = \left|\frac{U(u_n)}{U(u)} - 1\right| \|v\|_{1,\mathcal{H},0} = 0.$$

Therefore, it concludes that $v_n \to v$ in $W_0^{1,\mathcal{H}}(\Omega)$ as $n \to \infty$. This completes the proof of the lemma. □

Let $p' > 1$ be the conjugate index of p, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

For any $w \in L^{p'}(\Omega)$ and $z \in W_0^{1,\mathcal{H}}(\Omega)$ fixed, consider the following intermediate problem:

$$
\begin{aligned}
-\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) &= hw(x) + f(x, u, \nabla u) \quad && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega, \\
T(u) &\le U(z).
\end{aligned} \tag{3.6}
$$

The following theorem provides the unique solvability of problem (3.6).

Theorem 3.1. *Assume that $H(\mu)$, $H(f)$, $H(T)$ and $H(0)$ are fulfilled. Then, for each pair of functions $(z, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$, problem* (3.6) *has a unique solution $u \in W_0^{1,\mathcal{H}}(\Omega)$.*

Proof. Uniqueness. For any $(z, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$ fixed, let $u_1, u_2 \in W_0^{1,\mathcal{H}}(\Omega)$ be two solutions of problem (3.6) corresponding to $(z, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$. Then, we have

$$
\begin{aligned}
-\,\mathrm{div}(|\nabla u_i|^{p-2}\nabla u_i + \mu(x)|\nabla u_i|^{q-2}\nabla u_i) &= hw(x) + f(x, u_i, \nabla u_i) \quad && \text{in } \Omega, \\
u_i &= 0 && \text{on } \partial\Omega, \\
T(u_i) &\le U(z)
\end{aligned}
$$

for $i = 1, 2$. A simple calculation gives

$$
\begin{aligned}
&\int_\Omega (|\nabla u_i|^{p-2}\nabla u_i + \mu(x)|\nabla u_i|^{q-2}\nabla u_i, \nabla(v - u_i))_{\mathbb{R}^N}\, dx \\
&= h\int_\Omega w(x)(v(x) - u_i(x))dx + \int_\Omega f(x, u_i, \nabla u_i)(v(x) - u_i(x))\, dx
\end{aligned}
$$

for all $v \in K(z)$ and $i = 1, 2$. Inserting $v = u_2$ and $v = u_1$ into the inequalities above for $i = 1$ and $i = 2$, respectively, we sum up the resulting inequalities to obtain

$$
\begin{aligned}
&\int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 + \mu(x)|\nabla u_1|^{q-2}\nabla u_1, \nabla(u_1 - u_2))_{\mathbb{R}^N}\, dx \\
&\qquad - \int_\Omega (|\nabla u_2|^{p-2}\nabla u_2 + \mu(x)|\nabla u_2|^{q-2}\nabla u_2, \nabla(u_1 - u_2))_{\mathbb{R}^N}\, dx \\
&\quad = \int_\Omega (f(x, u_1, \nabla u_1) - f(x, u_2, \nabla u_2))(u_1(x) - u_2(x))\, dx \\
&\quad = \int_\Omega (f(x, u_1, \nabla u_1) - f(x, u_2, \nabla u_1))(u_1(x) - u_2(x))\, dx \\
&\qquad + \int_\Omega (f(x, u_2, \nabla u_1) - f(x, u_2, \nabla u_2))(u_1(x) - u_2(x))\, dx.
\end{aligned}
$$

It follows from Lemma 2.1, $H(\mu)$ and $H(f)$(iii) that

$$k(p)\|\nabla u_1 - \nabla u_2\|_p^p \le \int_\Omega e_f |u_1(x) - u_2(x)|^p\,dx + \int_\Omega h_f |\nabla u_1(x) - \nabla u_2(x)|^{p-1}|u_1(x) - u_2(x)|\,dx.$$

This combined with Hölder inequality deduces that

$$\begin{aligned} k(p)\|\nabla u_1 - \nabla u_2\|_p^p &\le e_f\|u_1 - u_2\|_p^p + h_f\|\nabla u_1 - \nabla u_2\|_p^{p-1}\|u_1 - u_2\|_p \\ &\le e_f\lambda_{1,p}^{-1}\|\nabla u_1 - \nabla u_2\|_p^p + h_f\lambda_{1,p}^{-\frac{1}{p}}\|\nabla u_1 - \nabla u_2\|_p^p, \end{aligned}$$

namely,

$$(k(p) - e_f\lambda_{1,p}^{-1} - h_f\lambda_{1,p}^{-\frac{1}{p}})\|\nabla u_1 - \nabla u_2\|_p^p \le 0.$$

From the inequality $k(p) - e_f\lambda_{1,p}^{-1} - h_f\lambda_{1,p}^{-\frac{1}{p}} > 0$, we conclude that $u_1 = u_2$. Therefore, problem (3.6) has at most one solution $u \in W_0^{1,\mathcal{H}}(\Omega)$.

Existence. If $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a solution to problem (3.6), then we have $u \in K(z)$ and

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u, \nabla(v-u)\right)_{\mathbb{R}^N}\,dx = h\int_\Omega w(x)(v(x) - u(x))\,dx + \int_\Omega f(x,u,\nabla u)(v(x) - u(x))dx \tag{3.7}$$

for all $v \in K(z)$. Let $N_f : W_0^{1,\mathcal{H}}(\Omega) \subset L^{q_1}(\Omega) \to L^{q_1'}(\Omega)$ be the Nemytskii operator associated to f and i be the embedding operator from $W_0^{1,\mathcal{H}}(\Omega)$ to $L^{q_1}(\Omega)$ with its adjoint operator $i^* : L^{q_1'}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$. Also, let us introduce the function $\mathcal{M} : W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$, $\mathcal{M}u = Au - i^*N_f(u)$ for all $u \in W_0^{1,\mathcal{H}}(\Omega)$, where $A : W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ is defined by (2.6). Let $I_{K(z)} : W_0^{1,\mathcal{H}}(\Omega) \to \overline{\mathbb{R}}$ be the indicator function of the set $K(z)$, thus,

$$I_{K(z)}(u) := \begin{cases} 0 & \text{if } u \in K(z), \\ +\infty & \text{otherwise.} \end{cases}$$

It is not difficult to prove that problem (3.7) is equivalent to the following inclusion problem: find $u \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$\mathcal{M}(u) + \partial_c I_{K(z)}(u) \ni hw \text{ in } W_0^{1,\mathcal{H}}(\Omega)^*, \tag{3.8}$$

where $\partial_c I_{K(z)}$ stands for the convex subdifferential operator of $I_{K(z)}$.

Arguing as in the proof of [31, Theorem 3.2], we conclude that $\mathcal{M}$ is a bounded and pseudomonotone operator. Since $I_{K(z)}$ is a proper convex and l.s.c. function, so, its convex subdifferential $\partial_c I_{K(z)} : K(z) \to 2^{W_0^{1,\mathcal{H}}(\Omega)^*}$ is a maximal monotone operator. However, using the same arguments as in the proof of [58, Theorem 3.3], it finds that there exists $R_0 > 0$ such that

$$\langle \mathcal{M}u + \eta, u \rangle > 0$$

for all $\eta \in \partial_c I_{K(z)}(u)$ and for all $u \in W_0^{1,\mathcal{H}}(\Omega)$ with $\|u\|_{1,\mathcal{H},0} = R_0$.

Therefore, all conditions of Theorem 2.2 are verified. Using this theorem, we could find an element $u \in W_0^{1,\mathcal{H}}(\Omega)$ such that (3.8) holds, i.e. problem (3.6) admits a solution.

Consequently, we conclude that for every $(z, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{q_1}(\Omega)$ problem (3.6) has a unique solution. $\square$

Especially, if $U(z) = +\infty$ for all $z \in W_0^{1,\mathcal{H}}(\Omega)$, then problem (3.6) reduces to a non-obstacle problem. So, we have the following proposition.

Proposition 3.1. *Assume that $H(\mu)$, $H(f)$ and $H(0)$ are fulfilled. Then for each $w \in L^{p'}(\Omega)$, the following problem has a unique solution $u \in W_0^{1,\mathcal{H}}(\Omega)$:*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) = hw(x) + f(x, u, \nabla u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega.$$

Remark 3.2. More particular, if $w = 0$, $p = 2$ and $U(z) = +\infty$ for all $z \in W_0^{1,\mathcal{H}}(\Omega)$, then our result, Theorem 3.1, reduces the one [31, Theorem 3.4] with $k(2) = 1$. Besides, it should be mentioned that in Theorem 3.1 we do not require f to satisfy the following condition:

- there exist $\rho \in L^{r'}(\Omega)$ with $1 < r' < p^*$ and $c_2 > 0$ such that $\xi \mapsto f(x, s, \xi) - \rho(x)$ is linear for a.a. $x \in \Omega$, for all $x \in \mathbb{R}$ and
$$|f(x, s, \xi) - \rho(x)| \le c_2|\xi|$$
for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$,

which is used in [31, Theorem 3.4] for verifying the uniqueness of solution.

Theorem 3.1 allows us to introduce the function $L : W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)$ defined by

$$L(z, w) := u(z, w) \quad \text{for all } (z, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega), \tag{3.9}$$

where $u(z, w) \in W_0^{1,\mathcal{H}}(\Omega)$ is the unique solution of problem (3.6) corresponding to $(z, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$. Moreover, we consider the multivalued mapping $J : W_0^{1,\mathcal{H}}(\Omega) \to 2^{L^{p'}(\Omega)}$ defined by

$$J(u) := \{w \in L^{p'}(\Omega) \mid w(x) \in \partial j(x, u(x)) \text{ for a.a. } x \in \Omega\} \tag{3.10}$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$. Indeed, for each $u \in W_0^{1,\mathcal{H}}(\Omega)$ fixed, it is not difficult to apply hypotheses $H(j)$ for showing that $x \mapsto \partial j(x, u(x))$ is measurable. Employing Yankov–von Neumann–Aumann selection theorem (see e.g., [54, Theorem 2.6.8]), there exists a measurable selection $w : \Omega \to \mathbb{R}$ such that $w(x) \in \partial j(x, u(x))$ for a.a. $x \in \Omega$. However, by virtue of $H(j)$(iv), we can see that $w \in L^{p'}(\Omega)$. Hence, J given in (3.10) is well-defined.

The following lemmas deliver the closedness of J and continuity of L, respectively, which will play the significant role to show the existence of a solution for problem (1.1).

Lemma 3.2. *Suppose that $H(j)$ hold. Then, the graph* GrJ *of J is sequentially closed in $W_0^{1,\mathcal{H}}(\Omega)_w \times L^{p'}(\Omega)_w$.*

Proof. Let sequence $\{(u_n, w_n)\} \subset W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$ be such that

$$(u_n, w_n) \xrightarrow{w} (u, w) \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega) \text{ as } n \to \infty$$

for some $(u, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$. Recall that the embedding from $W_0^{1,\mathcal{H}}(\Omega)$ into $L^p(\Omega)$ is compact, so, it has

$$u_n \to u \quad \text{in } L^p(\Omega) \text{ as } n \to \infty. \tag{3.11}$$

Employing Mazur's theorem, we are able to find a sequence $\{\zeta_n\}$ of convex combinations of $\{w_n\}$ such that

$$\zeta_n \to w \quad \text{in } L^{p'}(\Omega) \text{ as } n \to \infty. \tag{3.12}$$

From (3.11) and (3.12), without loss of generality, we may assume that

$$u_n(x) \to u(x) \quad \text{and} \quad \zeta_n(x) \to w(x) \text{ as } n \to \infty \text{ for a.a. } x \in \Omega. \tag{3.13}$$

The convexity of $\partial j(x, u_n(x))$ implies

$$\zeta_n(x)s \le j^0(x, u_n(x); s) \quad \text{for all } s \in \mathbb{R}.$$

Passing to the upper limit as $n \to \infty$ to the inequality above, we utilize Proposition 2.2(ii) and (3.13) to get

$$w(x)s = \lim_{n\to\infty} \zeta_n(x)s \le \limsup_{n\to\infty} j^0(x, u_n(x); s) \le j^0(x, u(x); s) \quad \text{for all } s \in \mathbb{R}$$

for a.a. $x \in \Omega$. This means that $w(x) \in \partial j(x, u(x))$ for a.a. $x \in \Omega$, i.e. $w \in J(u)$. Therefore, the graph GrJ of J is sequentially closed in $W_0^{1,\mathcal{H}}(\Omega)_w \times L^{p'}(\Omega)_w$. □

Lemma 3.3. *Assume that $H(\mu)$, $H(f)$, $H(T)$, $H(U)$ and $H(0)$ are fulfilled. Then, the function $L : W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)$ is completely continuous.*

Proof. Let $\{(z_n, w_n)\} \subset W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$ be such that

$$(z_n, w_n) \xrightarrow{w} (z, w) \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega) \text{ as } n \to \infty$$

for some $(z, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$. For every $n \in \mathbb{N}$, set $u_n = L(z_n, w_n)$. Then, for each $n \in \mathbb{N}$, we have $u_n \in K(z_n)$ and

$$\begin{aligned}&\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n, \nabla(v - u_n)\right)_{\mathbb{R}^N}\, dx\\&\qquad = h\int_\Omega w_n(x)(v(x) - u_n(x))\, dx + \int_\Omega f(x, u_n, \nabla u_n)(v(x) - u_n(x))dx\end{aligned} \tag{3.14}$$

for all $v \in K(z_n)$.

We assert that $\{u_n\}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Note that $U(z) > 0$ for all $z \in W_0^{1,\mathcal{H}}(\Omega)$ and T is positively homogeneous and subadditive, it has that $0 \in K(z)$ for all $z \in W_0^{1,\mathcal{H}}(\Omega)$. Taking $v = 0$ into (3.14), we have

$$\begin{aligned}&\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n, \nabla u_n\right)_{\mathbb{R}^N}\, dx\\&\qquad = h\int_\Omega w_n(x)u_n(x)\, dx + \int_\Omega f(x, u_n, \nabla u_n)u_n(x)\, dx.\end{aligned} \tag{3.15}$$

Because the embedding from $W_0^{1,\mathcal{H}}(\Omega)$ into $L^{q_1}(\Omega)$ is continuous, it follows from Hölder inequality to find

$$h\int_\Omega w_n(x)u_n(x)\, dx \le |h|\|w_n\|_{p'}\|u_n\|_p \le \widehat{c}_0|h|\|w_n\|_{p'}\|u_n\|_{1,\mathcal{H},0}, \tag{3.16}$$

where $\widehat{c}_0 > 0$ is such that

$$\|u\|_p \le \widehat{c}_0\|u\|_{1,\mathcal{H},0} \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega). \tag{3.17}$$

Applying hypothesis $H(f)$(ii), we have

$$\int_\Omega f(x, u_n, \nabla u_n)u_n(x)\, dx \le \int_\Omega c_j|\nabla u_n(x)|^{\theta_1} + d_j|u_n(x)|^{\theta_2} + \beta_f(x)\, dx.$$

Let us distinguish the following cases:

- $\theta_1 = \theta_2 = p$;
- $\theta_1 < \theta_2 = p$;
- $\theta_2 < \theta_1 = p$;
- $\theta_1 < p$ and $\theta_2 < p$.

Let $\varepsilon > 0$. If $\theta_1 = \theta_2 = p$, then we have

$$\begin{aligned}\int_\Omega f(x, u_n, \nabla u_n)u_n(x)\, dx &\le c_f\|\nabla u_n\|_p^p + d_f\|u_n\|_p^p + \|\beta_f\|_1\\&\le c_f\|\nabla u_n\|_p^p + d_f\lambda_{1,p}^{-1}\|\nabla u_n\|_p^p + \|\beta_f\|_1.\end{aligned} \tag{3.18}$$

When $\theta_1 < \theta_2 = p$, we use Young inequality to find

$$\int_\Omega f(x, u_n, \nabla u_n)u_n(x)\, dx \le c_f \|\nabla u_n\|_{\theta_1}^{\theta_1} + d_f \|u_n\|_p^p + \|\beta_f\|_1$$
$$\le \varepsilon \|\nabla u_n\|_p^p + d_f \lambda_{1,p}^{-1} \|\nabla u_n\|_p^p + \|\beta_f\|_1 + \widehat{c}_1(\varepsilon) \tag{3.19}$$

for some $\widehat{c}_1(\varepsilon) > 0$. Likewise, for $\theta_2 < \theta_1 = p$, it has

$$\int_\Omega f(x, u_n, \nabla u_n)u_n(x)\, dx \le c_f \|\nabla u_n\|_p^p + d_f \|u_n\|_{\theta_2}^{\theta_2} + \|\beta_f\|_1$$
$$\le c_f \|\nabla u_n\|_p^p + \varepsilon \|\nabla u_n\|_p^p + \|\beta_f\|_1 + \widehat{c}_2(\varepsilon) \tag{3.20}$$

for some $\widehat{c}_2(\varepsilon) > 0$. Assume that $\theta_1 < p$ and $\theta_2 < p$, we have

$$\int_\Omega f(x, u_n, \nabla u_n)u_n(x)\, dx \le c_f \|\nabla u_n\|_{\theta_1}^{\theta_1} + d_f \|u_n\|_{\theta_2}^{\theta_2} + \|\beta_f\|_1$$
$$\le 2\varepsilon \|\nabla u_n\|_p^p + \|\beta_f\|_1 + \widehat{c}_1(\varepsilon) + \widehat{c}_2(\varepsilon) \tag{3.21}$$

for some $\widehat{c}_1(\varepsilon), \widehat{c}_2(\varepsilon) > 0$. Taking into account (3.15)–(3.21) and letting $\varepsilon > 0$ small enough, we conclude that

$$\widehat{c}_3(\|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q) \le \widehat{c}_0 |h| \|w_n\|_{p'} \|u_n\|_{1,\mathcal{H},0} + \widehat{c}_4$$

for some $\widehat{c}_3, \widehat{c}_4 > 0$. The latter combined with inequality (2.2) implies that

$$\widehat{c}_3 \|u_n\|_{1,0,\mathcal{H}}^p \le \widehat{c}_0 |h| \|w_n\|_{p'} \|u_n\|_{1,\mathcal{H},0} + \widehat{c}_4$$

for all $n \in \mathbb{N}$. Because of $p > 1$, from the estimate above, it is not difficult to see that $\{u_n\}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \text{ as } n \to \infty \tag{3.22}$$

for some $u \in W_0^{1,\mathcal{H}}(\Omega)$.

Next, we are going to show that u is the unique solution of problem (3.6) associated with $(z, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$, i.e. $u = L(z, w)$. Because the graph GrK of K is sequentially closed in $W_0^{1,\mathcal{H}}(\Omega)_w \times W_0^{1,\mathcal{H}}(\Omega)_w$ (see Lemma 3.1(ii)). Note that $\{(u_n, z_n)\} \subset \mathrm{Gr}K$ and $(u_n, z_n) \xrightarrow{w} (u, z)$ in $W_0^{1,\mathcal{H}}(\Omega) \times W_0^{1,\mathcal{H}}(\Omega)$ as $n \to \infty$, we have $u \in K(z)$. However, Lemma 3.1(iii) allows us to find a sequence $\{y_n\} \subset W_0^{1,\mathcal{H}}(\Omega)$ such that

$$y_n \to u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \text{ as } n \to \infty. \tag{3.23}$$

Inserting $v = y_n$ into (3.14) and passing to the upper limit as $n \to \infty$ to the resulting inequality, we have

$$\limsup_{n\to\infty} \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n, \nabla(u_n - y_n)\right)_{\mathbb{R}^N} dx$$
$$\le \limsup_{n\to\infty} h \int_\Omega w_n(x)(u_n(x) - y_n(x))dx$$
$$+ \limsup_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)(u_n(x) - y_n(x))dx.$$

Recall that the embedding from $W_0^{1,\mathcal{H}}(\Omega)$ into $L^p(\Omega)$ is compact and the sequences $\{w_n\}$ and $\{f(\cdot, u_n, \nabla u_n)\}$ are both bounded in $L^{p'}(\Omega)$ and $L^{q_1'}(\Omega)$ (see hypothesis $H(f)$(i)), respectively, it finds

$$\limsup_{n\to\infty} \int_\Omega w_n(x)(u_n(x) - y_n(x))\, dx \\ + \limsup_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)(u_n(x) - y_n(x))\, dx = 0.$$

Hence, one has

$$\begin{aligned}
&\limsup_{n\to\infty} \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n, \nabla(u_n - u)\right)_{\mathbb{R}^N} dx \\
&\quad \le \limsup_{n\to\infty} \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n, \nabla(u_n - u)\right)_{\mathbb{R}^N} dx \\
&\qquad + \liminf_{n\to\infty} \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n, \nabla(u - y_n)\right)_{\mathbb{R}^N} dx \\
&\quad \le \limsup_{n\to\infty} \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n, \nabla(u_n - y_n)\right)_{\mathbb{R}^N} dx \\
&\quad \le 0.
\end{aligned}$$

The latter together with Proposition 2.1 (i.e. A is of type (S_+)) implies that

$$u_n \to u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \text{ as } n \to \infty.$$

For any $v \in K(z)$ fixed, it follows from Lemma 3.1(iii) that there exists a sequence $\{v_n\} \subset W_0^{1,\mathcal{H}}(\Omega)$ satisfying

$$v_n \in K(z_n) \quad \text{and} \quad v_n \to v \text{ in } W_0^{1,\mathcal{H}}(\Omega) \text{ as } n \to \infty.$$

Putting $v = v_n$ into (3.14), and passing to the limit as $n \to \infty$ for the resulting inequality, we use Lebesgue dominated convergence theorem to obtain

$$\begin{aligned}
&\int_\Omega (|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u, \nabla(v - u))_{\mathbb{R}^N}\, dx \\
&\quad = \lim_{n\to\infty} \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n, \nabla(v_n - u_n))_{\mathbb{R}^N}\, dx \\
&\quad = \lim_{n\to\infty} h \int_\Omega w_n(x)(v_n(x) - u_n(x))\, dx \\
&\qquad + \lim_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)(v_n(x) - u_n(x))\, dx \\
&\quad = h \int_\Omega w(x)(v(x) - u(x))\, dx + \int_\Omega f(x, u, \nabla u)(v(x) - u(x))dx.
\end{aligned}$$

Since $v \in K(z)$ is arbitrary, so, we conclude that u is the unique solution of problem (3.6) corresponding to (z, w), namely, $u = L(z, w)$. Notice that every convergent subsequence of $\{u_n\}$ converges to the same limit u, so, we conclude that the whole sequence $\{u_n\}$ converges strongly to u in $W_0^{1,\mathcal{H}}(\Omega)$. This means that $L : W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)$ is completely continuous. □

Furthermore, we deliver *a priori* estimate for the solutions of problem (1.1).

Lemma 3.4. *Suppose that* $H(\mu)$, $H(f)$, $H(T)$, $H(U)$, $H(j)$ *and* $H(0)$ *are satisfied. If, the solution set of problem* (1.1) *is nonempty, then there exists a constant* $M > 0$ *such that*

$$\|u\|_{1,\mathcal{H},0} \leq M.$$

Proof. Let $u \in W_0^{1,\mathcal{H}}(\Omega)$ be a solution of problem (1.1). Then, there exists a function $\zeta \in L^{p'}(\Omega)$ with $\zeta(z) \in \partial j(x, u(x))$ for a.a. $x \in \Omega$ such that

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u, \nabla(v-u)\right)_{\mathbb{R}^N} dx$$
$$= \int_\Omega h\zeta(x)(v(x)-u(x))\,dx + \int_\Omega f(x,u,\nabla u)(v(x)-u(x))dx \quad (3.24)$$

for all $v \in K(u)$. Owing to $0 \in K(u)$, we take $v = 0$ for the inequality above to get

$$\int_\Omega (|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u, \nabla u)_{\mathbb{R}^N}\, dx$$
$$= \int_\Omega h\zeta(x)u(x)dx + \int_\Omega f(x,u,\nabla u)u(x)dx.$$

Employing hypothesis $H(j)$(iii) deduces

$$\begin{aligned}\int_\Omega h\zeta(x)u(x)\,dx &\leq |h| \int_\Omega |\zeta(x)u(x)|\,dx \\ &\leq |h| \int_\Omega c_j|u(x)|^{\theta_3} + \gamma_j(x)|u(x)|\,dx \\ &\leq |h|c_j\|u\|_{\theta_3}^{\theta_3} + |h|\|\gamma_j\|_{\theta_3'}\|u\|_{\theta_3}.\end{aligned}$$

Then, we have

$$\begin{aligned}\int_\Omega h\zeta(x)u(x)\,dx &\leq |h|c_j\|u\|_p^p + |h|\|\gamma_j\|_{p'}\|u\|_p \\ &\leq |h|c_j\lambda_{1,p}^{-1}\|\nabla u\|_p^p + |h|\|\gamma_j\|_{p'}\lambda_{1,p}^{\frac{-1}{p}}\|\nabla u\|_p. \end{aligned} \quad (3.25)$$

For the term $\int_\Omega f(x,u,\nabla u)u(x)\,dx$, we could carry out the same arguments as in the proof of Lemma 3.3 for obtaining similar estimates as (3.18)–(3.21). Because of $c_f\delta(\theta_1) + d_f\lambda_{1,p}^{-1}\delta(\theta_2) + |h|c_j\lambda_{1,p}^{-1} < 1$, it is not difficult to find a constant $M > 0$ such that $\|u\|_{1,\mathcal{H},0} \leq M$. □

Under the analysis above, we are now in a position to provide the main result of the paper concerning the existence of a solution to problem (1.1) as follows.

Theorem 3.2. *Assume that $H(\mu)$, $H(f)$, $H(T)$, $H(U)$, $H(j)$ and $H(0)$ are satisfied. Then, the solution set of problem* (1.1) *is nonempty and weakly compact in* $W_0^{1,\mathcal{H}}(\Omega)$.

Proof. For any $u \in W_0^{1,\mathcal{H}}(\Omega)$ and $w \in J(u)$, from H(j)(iii), we have

$$\int_\Omega w(x)u(x)dx \le |h|c_j\|u\|_p^p + |h|\|\gamma_j\|_{p'}\|u\|_p. \tag{3.26}$$

We assert that there exists a constant $C_0 > 0$ such that

$$\|L(B_{W_0^{1,\mathcal{H}}(\Omega)}(0, C_0), J(\iota(B_{W_0^{1,\mathcal{H}}(\Omega)}(0, C_0))))\|_{1,\mathcal{H},0} \le C_0, \tag{3.27}$$

where $\iota : W_0^{1,\mathcal{H}}(\Omega) \to L^{p'}(\Omega)$ is the embedding operator from $W_0^{1,\mathcal{H}}(\Omega)$ to $L^{p'}(\Omega)$ and $B_{W_0^{1,\mathcal{H}}(\Omega)}(0, C_0) := \{u \in W_0^{1,\mathcal{H}}(\Omega) \mid \|u\|_{1,\mathcal{H},0} \le C_0\}$. Arguing by contradiction, for each $n \in \mathbb{N}$, there are functions $u_n = L(z_n, w_n)$ such that $w_n \in J(z_n)$ and

$$\|z_n\|_{1,\mathcal{H},0} \le n \quad \text{and} \quad \|u_n\|_{1,\mathcal{H},0} > n.$$

Let $\varepsilon > 0$. From (3.26) and hypothesis $H(f)$(ii), we can find a constant $C_1 > 0$ such that

$$\begin{aligned} 0 &\ge \|u_n\|_{1,\mathcal{H},0}^p - \int_\Omega w(x)u(x)\,dx - \int_\Omega f(x, u_n, \nabla u_n)u_n\,dx \\ &\ge \begin{cases} (1 - c_f - d_f\lambda_{1,p}^{-1})\|\nabla u_n\|_p^p + \|\nabla u_n\|_{\mu,q}^q & \text{if } \theta_1 = \theta_2 = p, \\ (1 - \varepsilon - d_f\lambda_{1,p}^{-1})\|\nabla u_n\|_p^p + \|\nabla u_n\|_{\mu,q}^q & \text{if } \theta_1 < \theta_2 = p, \\ (1 - c_f - \varepsilon)\|\nabla u_n\|_p^p + \|\nabla u_n\|_{\mu,q}^q & \text{if } \theta_2 < \theta_1 = p, \\ (1 - 2\varepsilon)\|\nabla u_n\|_p^p + \|\nabla u_n\|_{\mu,q}^q & \text{if } \max\{\theta_1, \theta_2\} < p, \end{cases} \\ &\quad - \int_\Omega w(x)u(x)\,dx - C_1. \end{aligned}$$

Therefore, by (3.26), it yields

$$0 \ge (1 - c_f\delta(\theta_1) - d_f\lambda_{1,p}^{-1}\delta(\theta_2) - |h|c_j\lambda_{1,p}^{-1}\delta)\|u_n\|_{1,\mathcal{H},0}^p - C_1.$$

Passing to the limit as $n \to \infty$ for the inequality above, it gives a contradiction. So, we conclude that there exists a constant $C_0 > 0$ such that (3.27) holds.

Let us introduce a multivalued mapping $\Pi : W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega) \to 2^{W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)}$ defined by

$$\Pi(z, w) = (L(z, w), J(z)) \quad \text{for all } (z, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega).$$

We are going to apply Kakutani–Ky Fan fixed point theorem, Theorem 2.1, for examining the existence of a fixed point of Π. It follows from hypothesis H(j)(iii) that there is a constant $C_2 > 0$ such that $\|J(B_{W_0^{1,\mathcal{H}}(\Omega)}(0, C_0))\|_{p'} \le C_2$. Set

$$D := \{(z, w) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega) \,|\, \|z\|_{1,\mathcal{H},0} \le C_0 \text{ and } \|w\|_{p'} \le C_2\}.$$

By (3.27), it is easy to verify that Π maps D into itself.

We next shall show that the graph of Π is sequentially closed in $(W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega))_w \times (W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega))_w$. Let sequences $\{(z_n, w_n)\} \subset W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$ and $\{(u_n, v_n)\} \subset W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$ be such that $v_n \in J(z_n)$, $u_n = L(z_n, w_n)$ and

$$(z_n, w_n) \xrightarrow{w} (z, w) \text{ in } W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega) \text{ as } n \to \infty,$$

$$(u_n, v_n) \xrightarrow{w} (u, v) \text{ in } W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega) \text{ as } n \to \infty$$

for some $(z, w), (u, v) \in W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$. Employing Lemma 3.3, we conclude that

$$u_n = L(z_n, w_n) \to L(z, w) = u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \text{ as } n \to \infty.$$

It follows from Lemma 3.2 that the graph of $u \mapsto J(u)$ is sequentially closed in $W_0^{1,\mathcal{H}}(\Omega)_w \times L^{p'}(\Omega)_w$. Therefore, we conclude that $\Pi : W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega) \to 2^{W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)}$ has nonempty, closed and convex values, and its graph is sequentially closed in $(W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega))_w \times (W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega))_w$.

All conditions of Theorem 2.1 are verified. Applying this theorem, we conclude that Π has at least a fixed point, say (z^*, w^*). Hence, it has $(z^*, w^*) \in \Pi(z^*, w^*)$, i.e. $z^* = L(z^*, w^*)$ and $w^* \in J(z^*)$. From the definition of L and J, we can see that $z^* \in W_0^{1,\mathcal{H}}(\Omega)$ is a solution to problem (1.1).

The boundedness of solution set of problem (1.1) is a direct consequence of Lemma 3.4. We will illustrate the weak closedness of the solution set of problem (1.1). Let $\{u_n\}$ be a sequence of solutions to problem (1.1) such that $u_n \xrightarrow{w} u$ in $W_0^{1,\mathcal{H}}(\Omega)$ as $n \to \infty$. So, there exists a sequence $\{w_n\} \subset L^{p'}(\Omega)$ such that $(u_n, w_n) \in \Pi(u_n, w_n)$. But, the growth condition $H(j)$(iii) guarantees that $\{w_n\}$ is bounded in $L^{p'}(\Omega)$. Passing to a subsequence if necessary, we may suppose that $w_n \xrightarrow{w} w$ in $L^{p'}(\Omega)$. Recall that Π is weakly sequentially closed, $(u_n, w_n) \in \Pi(u_n, w_n)$ and $(u_n, w_n) \xrightarrow{w} (u, w)$ in $W_0^{1,\mathcal{H}}(\Omega) \times L^{p'}(\Omega)$, so, we have $(u, w) \in \Pi(u, w)$. This means u is a solution to problem (1.1). Consequently, the solution set of problem (1.1) is weakly compact. □

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