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NONLINEAR EIGENVALUE PROBLEMS ARISING IN EARTHQUAKE INITIATION

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Abstract. We study a symmetric, nonlinear eigenvalue problem arising in earthquake initiation, and we establish the existence of infinitely many solutions. Under the effect of an arbitrary perturbation, we prove that the number of solutions becomes greater and greater if the perturbation tends to zero with respect to a prescribed topology. Our approach is based on nonsmooth critical-point theories in the sense of De Giorgi and Degiovanni.

1. Introduction

The minimax method has been used intensively in constructing critical points for functionals defined on Hilbert or Banach spaces as solutions of nonlinear partial differential equations or boundary-value problems for inequality problems. In particular, when the problems possess symmetry, one can construct multiple critical points by the minimax method. This is the general Lusternik-Schnirelmann-type theory (see [2, 18, 19, 21, 23, 25]). When an order structure is present, one can also use fixed-point theory, topological degree arguments, or variational methods to construct solutions of differential equations or variational inequalities (see [1, 6, 7, 12, 14]). However, little work has been done for invariant energy functionals under group actions when one expects to obtain multiplicity of critical points.

The main purpose of this paper is to consider a concrete nonlinear eigenvalue variational inequality arising in earthquake initiation and to establish, in the setting of the nonsmooth Lusternik-Schnirelmann theory, the existence of infinitely many solutions. The main novelty in our framework is the

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presence of the convex cone of functions with nonnegative jump across an internal boundary which is composed of a finite number of bounded, connected arcs.

Under some natural assumptions, we prove the existence of infinitely many solutions, as well as further properties of eigensolutions and eigenvalues. Since the associated energy functional is included neither in the theory of monotone operators, nor in their Lipschitz perturbations, we employ the notion of lower subdifferential which is originally due to De Giorgi.

Next, we are concerned with the study of the effect of a small nonsymmetric perturbation, and we prove that the number of solutions of the perturbed problem becomes greater and greater if the perturbation tends to zero with respect to an appropriate topology. Our proof relies on powerful methods from algebraic topology developed in Krasnoselski [18] combined with adequate tools in the sense of the Degiovanni nonsmooth critical-point theory (see [8, 12, 13]).

2. Physical motivation

Consider, as in [3, 5, 10, 16, 27], the anti-plane shearing on a system of finite faults under a slip-dependent friction in a homogeneous linear elastic domain. Let $\Omega \subset \mathbb{R}^2$ be a domain, not necessarily bounded, containing a finite number of cuts. Its boundary $\partial\Omega$ is supposed to be smooth and divided into two disjoint parts: the exterior boundary $\Gamma_d = \partial\bar{\Omega}$ and the internal one Γ composed of N_f bounded, connected arcs Γ_f^i, $i = 1\ldots, N_f$, called cracks or faults. We suppose that the displacement field is 0 in directions Ox and Oy and that u_z does not depend on z. The displacement is therefore denoted simply by $w = w(t,x,y)$. The elastic medium has the shear rigidity G, the density ρ and the shear velocity $c = \sqrt{G/\rho}$. The nonvanishing shear stress components are $\sigma_{zx} = \tau_x^\infty + G\partial_x w$, $\sigma_{zy} = \tau_y^\infty + G\partial_y w$, and $\sigma_{xx} = \sigma_{yy} = -S$ ($S > 0$ is the normal stress on the fault plane). We look for $w : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, a solution of the wave equation,

$$\partial_{tt} w(t) = c^2 \Delta w(t) \quad \text{in} \quad \Omega, \tag{2.1}$$

with the boundary condition

$$w(t) = 0 \ \text{ on } \ \Gamma_d. \tag{2.2}$$

On Γ we denote by $[\ \ ]$ the jump across Γ (i.e., $[w] = w^+ - w^-$) and by $\partial_n = \nabla \cdot n$ the corresponding normal derivative with the unit normal n outwards on the positive side. On the contact zone Γ we have $[\partial_n w] = 0$ and a slip-dependent friction law (introduced in the geophysical context of earthquake modelling) is assumed:

$$G\partial_n w(t) = -\mu(|[w(t)]|)S\text{sign}([\partial_t w(t)]) - q, \quad \text{if} \quad [\partial_t w(t)] \neq 0, \tag{2.3}$$

$$|G\partial_n w(t) + q| \leq \mu(|[w(t)]|)S \quad \text{if} \quad \partial_t [w(t)] = 0, \tag{2.4}$$

where $q = \tau_x^\infty n_x + \tau_y^\infty n_y$. The initial conditions are

$$w(0) = w_0, \quad \partial_t w(0) = w_1 \quad \text{in} \quad \Omega. \tag{2.5}$$

Any solution of the above problem satisfies the following variational problem (VP): find $w : [0, T] \longrightarrow V$ such that

$$\begin{aligned} &\int_\Omega \frac{1}{c^2} \partial_{tt} w(t)(v - \partial_t w(t)) \, dx \; + \int_\Omega \nabla w(t) \cdot \nabla (v - \partial_t w(t)) \, dx \; + \\ &\int_\Gamma \frac{S}{G} \mu(|[w(t)]|)(|[v]| - |[\partial_t w(t)]|) \, d\sigma \; \geq \int_\Gamma \frac{1}{G} q([v] - [\partial_t w(t)]) \, d\sigma, \end{aligned} \tag{2.6}$$

for all $v \in V$, where

$$V = \{v \in H^1(\Omega) : v = 0 \;\text{ on }\; \Gamma_d\}. \tag{2.7}$$

The main difficulty in the study of the above evolution variational inequality is the nonmonotone dependence of μ with respect to the slip $|[w]|$. However, in modelling unstable phenomena, as earthquakes, we have to expect “bad” mathematical properties of the operators involved in the abstract problem. The existence of a solution w of the following regularity,

$$w \in W^{1,\infty}(0, T, V) \cap W^{2,\infty}(0, T, L^2(\Omega)), \tag{2.8}$$

in the two-dimensional case was recently proved by Ionescu et al. [17]. The uniqueness was obtained only in the one-dimensional case.

Since our intention is to study the evolution of the elastic system near an unstable equilibrium position, we shall suppose that $q = \mu(0)S$. We remark that $w \equiv 0$ is an equilibrium solution of (2.6), and w_0 and w_1 may be considered as small perturbations of it.

For simplicity, let us assume in the following that the friction law is homogeneous on the fault plane having the form of a piecewise-linear function (see [24]),

$$\mu(x, u) = \mu_s - \frac{\mu_s - \mu_d}{2D_c} u \;\text{ if }\; u \leq 2D_c, \;\; \mu(x, u) = \mu_d \;\text{ if }\; u > 2D_c, \tag{2.9}$$

where u is the relative slip, μ_s and μ_d ($\mu_s > \mu_d$) are the static and dynamic friction coefficients, and D_c is the critical slip. This piecewise-linear function is a reasonable approximation of the experimental observations reported by [22]. Since the initial perturbation (w_0, w_1) of the equilibrium ($w \equiv 0$) is small we have $[w(t, x))] \leq 2D_c$ for $t \in [0, T_c]$ for all $x \in \Gamma$, where T_c is a critical time for which the slip on the fault reaches the critical value $2D_c$ at least at one point. Hence for a first period $[0,T_c]$, called the *initiation phase*, we deal with a linear function μ.

Our aim is to analyze the evolution of the perturbation during this initial phase. That is why we are interested in the existence of solutions of the type

$$w(t,x)=\sinh(|\lambda|ct)u(x),\quad w(t,x)=\sin(|\lambda|ct)u(x) \tag{2.10}$$

during the initiation phase $t\in[0,T_c]$. If we put the above expression in (2.6) and we have in mind that from (2.9) we have $\mu(s)=\mu_s-(\mu_s-\mu_d)/(2D_c)s$, then we deduce that (u,λ^2) is the solution of the nonlinear eigenvalue problem

$$\left\{\begin{array}{l}\text{find } u\in K \text{ and } \lambda^2\in\mathbb{R} \text{ such that}\\ \displaystyle\int_\Omega \nabla u\cdot\nabla(v-u)dx-\beta\int_\Gamma [u]\,[v-u]d\sigma+\lambda^2\int_\Omega u(v-u)dx\geq 0,\end{array}\right. \tag{2.11}$$

for all $v\in K$, where K is the convex closed cone centered at the origin

$$K=\{v\in V:\ [v]\geq 0 \text{ on } \Gamma\}$$

and $\beta=(\mu_s-\mu_d)S/(2D_cG)>0$. The first type of solution from (2.10) has an exponential growth in time and corresponds to $\lambda^2>0$. The second one has the same amplitude during the initiation phase and corresponds to $\lambda^2<0$.

The nonlinear eigenvalue problem (2.11) can be written as a classical eigenvalue for the Laplace operator with Signorini-type boundary conditions:

$$\text{find } u:\Omega\to\mathbb{R} \text{ and } \lambda^2\in\mathbb{R} \text{ such that}$$

$$\Delta u=\lambda^2 u \text{ in } \Omega,\qquad u=0 \text{ on } \Gamma_d, \tag{2.12}$$

$$[\partial_n u]=0,\ \ [u]\geq 0,\ \ \partial_n u\geq 0,\ \ [u](\partial_n u-\beta[u])=0 \text{ on } \Gamma. \tag{2.13}$$

The linear case, that is, equation (2.12) with the boundary condition

$$[\partial_n u]=0,\ \ \partial_n u-\beta[u]=0 \text{ on } \Gamma, \tag{2.14}$$

was analyzed in [9]. For bounded domains, they proved that the spectrum of (2.12) and (2.14) consists of a decreasing and unbounded sequence of eigenvalues. The greatest one, λ_0^2, which may be positive, is showed to be an increasing function of the friction parameter β. Let us remark that if u is a solution of (2.12) and (2.14) and $[u]\geq 0$ on Γ_f, then u is a solution for (2.12) and (2.13) too. For co-linear faults the first eigenfunction u_0, corresponding to λ_0^2, was found in numerical computations to be positive on Γ_f (see [9, 10]); hence the linear case was sufficient to give a good model for the initiation of instabilities. If the faults are not co-linear, then this condition is not anymore satisfied; that is, the first eigenfunction of the linear problem has no physical significance. Hence, in modeling initiation-of-friction instabilities only the nonlinear eigenvalue problem has to be considered. As was reported in [28], where the case of two parallel faults was analyzed, there exists an important gap between the first eigenvalues of the linear and nonlinear problems.

3. The main results

Let Ω be a smooth, bounded open set in $\mathbb{R}^N$ ($N \geq 2$) as in the preceding section, that is, containing a finite number of cuts. The internal boundary is denoted by Γ and the exterior one by Γ_d. Denote by $\|\cdot\|$ the norm in the space V, as defined in (2.7), and by $\Lambda_0 : L^2(\Omega) \to L^2(\Omega)^*$ and $\Lambda_1 : V \to V^*$ the duality isomorphisms defined by

$$\Lambda_0 u(v) = \int_\Omega uv\, dx, \qquad \text{for any } u, v \in L^2(\Omega)$$

and

$$\Lambda_1 u(v) = \int_\Omega \nabla u \cdot \nabla v\, dx, \qquad \text{for any } u, v \in V.$$

In order to reformulate our problem, consider the Lipschitz map $\gamma = i \circ \eta : V \to L^2(\Gamma)$, where $\eta : V \to H^{1/2}(\Gamma)$ is the trace operator, $\eta(v) = [v]$ on Γ and $i : H^{1/2}(\Gamma) \to L^2(\Gamma)$ is the embedding operator. Then γ is a compact operator.

Thus, problem (2.11) can be written, equivalently,

$$\left\{ \begin{array}{l} \text{find } u \in K \text{ and } \lambda^2 \in \mathbb{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla(v-u)dx + \int_\Gamma j'\left(\gamma(u(x)); \gamma(v(x)) - \gamma(u(x))\right) d\sigma + \\ \displaystyle\lambda^2 \int_\Omega u(v-u)dx \geq 0, \quad \forall v \in K, \end{array} \right. \tag{3.1}$$

where $j : \mathbb{R} \to \mathbb{R}$, $j(t) = -\frac{\beta}{2}t^2$, and $j'(\cdot\,;\,\cdot)$ stands for the Gâteaux directional derivative.

Due to the homogeneity of (3.1), we can reformulate this problem in terms of a constrained inequality problem as follows. For any fixed $r > 0$, set

$$M = \left\{ u \in V : \int_\Omega u^2 dx = r^2 \right\}.$$

Then M is a smooth manifold in the Hilbert space V. We shall study the problem

$$\left\{ \begin{array}{l} \text{find } u \in K \cap M \text{ and } \lambda^2 \in \mathbb{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla(v-u)dx + \int_\Gamma j'\left(\gamma(u(x)); \gamma(v(x)) - \gamma(u(x))\right) d\sigma + \\ \displaystyle\lambda^2 \int_\Omega u(v-u)dx \geq 0, \quad \forall v \in K. \end{array} \right. \tag{3.2}$$

Our multiplicity result is

Theorem 3.1. *Problem* (3.2) *has infinitely many solutions* (u, λ^2), *and the set of eigenvalues* $\{\lambda^2\}$ *is bounded from above and its infimum equals* $-\infty$. *Let* $\lambda_0^2 = \sup\{\lambda^2\}$. *Then there exists* u_0 *such that* (u_0, λ_0^2) *is a solution of* (3.2). *Moreover, the function* $\beta \longmapsto \lambda_0^2(\beta)$ *is convex and the following inequality holds:*

$$\int_\Omega |\nabla v|^2 \, dx + \lambda_0^2(\beta) \int_\Omega v^2 \, dx \geq \beta \int_\Gamma [v]^2 \, d\sigma, \qquad \forall v \in K. \tag{3.3}$$

Next, we study the effect of an arbitrary perturbation in problem (3.1). More precisely, we consider the problem

$$\begin{cases} \text{find } u_\varepsilon \in K \text{ and } \lambda_\varepsilon^2 \in \mathbb{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla(v - u_\varepsilon) dx + \int_\Gamma \left(j' + \varepsilon g'\right)\left(\gamma(u_\varepsilon(x)); \gamma(v(x)) - \gamma(u_\varepsilon(x))\right) d\sigma + \\ \quad \lambda_\varepsilon^2 \displaystyle\int_\Omega u_\varepsilon(v - u_\varepsilon) dx \geq 0, \quad \forall v \in K, \end{cases} \tag{3.4}$$

where $\varepsilon > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with no symmetry hypothesis, but satisfies the growth assumption

$$\begin{array}{ll} \exists\, a > 0,\ \exists\, 2 \leq p \leq \dfrac{2(N-1)}{N-2} \text{ such that } |g(t)| \leq a(1 + |t|^p) & , \text{ if } N \geq 3; \\ \exists\, a > 0,\ \exists\, 2 \leq p < +\infty \text{ such that } |g(t)| \leq a(1 + |t|^p) & , \text{ if } N = 2. \end{array} \tag{3.5}$$

We prove that the number of solutions of problem (3.4) becomes greater and greater if the perturbation "tends" to zero. This is a very natural phenomenon that occurs often in concrete situations. We illustrate it with the following elementary example: consider on the real axis the equation $\sin x = 1/2$. This is a "symmetric" problem (due to the periodicity) with infinitely many solutions. Let us now consider an arbitrary nonsymmetric "small" perturbation of the above equation, say $\sin x = 1/2 + \varepsilon x^2$. This equation has finitely many solutions, for any $\varepsilon \neq 0$. However, the number of solutions of the perturbed equation tends to infinity as the perturbation (that is, $|\varepsilon|$) becomes smaller and smaller.

More precisely, we have

Theorem 3.2. *For every positive integer* n, *there exists* $\varepsilon_n > 0$ *such that problem* (3.4) *has at least* n *distinct solutions* $(u_\varepsilon, \lambda_\varepsilon^2)$ *if* $\varepsilon < \varepsilon_n$. *There exists and is finite* $\lambda_{0\varepsilon}^2 = \sup\{\lambda_\varepsilon^2\}$, *and there exists* $u_{0\varepsilon}$ *such that* $(u_{0\varepsilon}, \lambda_{0\varepsilon}^2)$ *is a solution of* (3.4). *Moreover,* $\lambda_{0\varepsilon}^2$ *converges to* λ_0^2 *as* ε *tends to* 0, *where* λ_0^2 *was defined in Theorem* 3.1.

4. Auxiliary results

Several times in this paper we shall apply the following basic embedding inequality:

Proposition 4.1. (Lemma 5.1 in [15]). *Let $2 \leq \alpha \leq 2(N-1)/(N-2)$ if $N \geq 3$ and $2 \leq \alpha < +\infty$ if $N = 2$. Then for $\beta = [(\alpha-2)N+2]/(2\alpha)$ if $N \geq 3$ or if $N = 2$ and $\alpha = 2$ and for all $(\alpha-1)/\alpha < \beta < 1$ if $N = 2$ and $\alpha > 2$, there exists $C = C(\beta)$ such that*

$$\Big(\int_\Gamma |[u]|^\alpha d\sigma\Big)^{1/\alpha} \leq C\Big(\int_\Omega u^2 dx\Big)^{(1-\beta)/2}\Big(\int_\Omega |\nabla u|^2 dx\Big)^{\beta/2}, \quad \text{for any} \quad u \in V. \tag{4.1}$$

An important role in our arguments in order to locate the solution of (3.2) will be played by the indicator function of M; that is,

$$I_M(u) = \begin{cases} 0 & , \quad \text{if } u \in M \\ +\infty & , \quad \text{if } u \in V \setminus M. \end{cases}$$

Then I_M is lower semicontinuous. However, since the natural energy functional associated to problem (3.2) is neither smooth nor convex, it is necessary to introduce a more general concept of gradient. We shall employ the following notion of lower subdifferential which is due to De Giorgi, Marino, and Tosques [11]. The following definition agrees with the corresponding notions of gradient and critical point in the sense of Fréchet (for C^1 mappings), Clarke (for locally Lipschitz functionals), or in the sense of the convex analysis.

Definition 4.2. Let X be a Banach space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary proper functional. Let $x \in D(f)$. The gradient of f at x is the (possibly empty) set

$$\partial^- f(x) = \left\{\xi \in X^* : \liminf_{y \to x} \frac{f(y) - f(x) - \xi(y-x)}{\|y-x\|} \geq 0\right\}.$$

An element $\xi \in \partial^- f(x)$ is called a lower subdifferential of f at x.

Accordingly, we say that $x \in D(f)$ is a critical (lower stationary) point of f if $0 \in \partial^- f(x)$.

Then $\partial^- f(x)$ is a convex set. If $\partial^- f(x) \neq \emptyset$ we denote by $\text{grad}^- f(x)$ the element of minimal norm of $\partial^- f(x)$; that is,

$$\text{grad}^- f(x) = \min\{\|\xi\|_{X^*};\ \xi \in \partial^- f(x)\}.$$

This notion plays a central role in the statement of our basic compactness condition.

Definition 4.3. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary functional. We say that $(x_n) \subset D(f)$ is a Palais-Smale sequence if

$$\sup_n |f(x_n)| < +\infty \qquad \text{and} \qquad \lim_{n\to\infty} \operatorname{grad}^- f(x_n) = 0.$$

The functional f is said to satisfy the Palais-Smale condition provided that any Palais-Smale sequence is relatively compact.

Remark 4.4. (i) Definition 4.2 implies that if $g : X \to \mathbb{R}$ is Fréchet differentiable and $f : X \to \mathbb{R} \cup \{+\infty\}$ is an arbitrary proper function, then

$$\partial^-(f+g)(x) = \left\{\xi + g'(x) : \xi \in \partial^- f(x)\right\},$$

for any $x \in D(f)$.

(ii) Similary, if $f : X \to \mathbb{R} \cup \{+\infty\}$ is an arbitrary proper functional and $g : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous, then

$$\partial^-(f+g)(x) = \left\{\xi + g'(x) : \xi \in \partial^- f(x)\right\},$$

for any $x \in D(f) \cap D(g)$.

As established in [7],

$$\partial^- I_M(u) = \{\lambda \Lambda_0 u : \lambda \in \mathbb{R}\} \subset L^2(\Omega)^* \subset V^*, \qquad \text{for any } u \in M. \tag{4.2}$$

In the proof of Theorems 3.1 and 3.2 we shall use several auxiliary notions and properties. For the convenience of the reader we recall them in what follows. For further details and proofs we refer to [12, 19, 21, 23, 26].

A topological space X is said to be *contractible* if the identity of X is homotopical to a constant map; that is, there exists $u_0 \in X$ and a continuous map $F : X \times [0,1] \to X$ such that $F(\cdot, 0) = \mathrm{Id}_X$ and $F(\cdot, 1) = u_0$. A subset M of X is said to be *contractible* in X if there exists $u_0 \in X$ and a continuous map $F : M \times [0,1] \to X$ such that $F(\cdot, 0) = \mathrm{Id}_M$ and $F(\cdot, 1) = u_0$. If A is a subset of X, we define the category of A in X as follows:

$\mathrm{Cat}_X(A) = 0$, if $A = \emptyset$.

$\mathrm{Cat}_X(A) = n$, if n is the smallest integer such that A can be covered by n closed sets which are contractible in X.

$\mathrm{Cat}_X(A) = +\infty$, otherwise.

Some basic properties of the notion of category are summarized in

Proposition 4.5. *The following properties hold true:*

(*i*) *If* $A \subset B \subset X$, *then* $\mathrm{Cat}_X(A) \le \mathrm{Cat}_X(B)$.

(*ii*) $\mathrm{Cat}_X(A \cup B) \le \mathrm{Cat}_X(A) + \mathrm{Cat}_X(B)$

(*iii*) *Let* $h : A \times [0,1] \to X$ *be a continuous mapping such that* $h(x, 0) = x$ *for every* $x \in A$. *If* A *is closed and* $B = h(A, 1)$, *then* $\mathrm{Cat}_X(A) \le \mathrm{Cat}_X(B)$.

Let (X, d) be a metric space. Consider $h : X \to \mathbb{R} \cup \{+\infty\}$ an arbitrary functional, and set, as usual, $D(h) := \{u \in X : h(u) < +\infty\}$. We recall the following definitions, which are due essentially to De Giorgi (see, e.g., De Giorgi, Marino, and Tosques [11]).

Definition 4.6. (i) For $u \in D(h)$ and $\rho > 0$, let $h_u(\rho) = \inf\{h(v) : d(v, u) < \rho\}$. Then the number $-D_+h_u(0)$ is called the slope of h at u, where D_+ denotes the right lower derivative.

(ii) Let $I \subset \mathbb{R}$ be an arbitrary nontrivial interval, and consider a curve $U : I \to X$. We say that U is a curve of maximal slope for h if the following properties hold true:

– U is continuous;
– $h \circ U(t) < +\infty$, for any $t \in I$;
– $d(U(t_2), U(t_1)) \le \int_{t_1}^{t_2} \left[D_+h_{U(t)}(0)\right]^2 dt$, for any $t_1, t_2 \in I$, $t_1 < t_2$;
– $h \circ U(t_2) - h \circ U(t_1) \le -\int_{t_1}^{t_2} \left[D_+h_{U(t)}(0)\right]^2 dt$, for any $t_1, t_2 \in I$, $t_1 < t_2$.

In what follows, X denotes a metric space, A is a subset of X and i stands for the inclusion map of A in X.

Definition 4.7. (i) A map $r : X \to A$ is said to be a retraction if it is continuous, surjective, and $r_{|A} = Id$.

(ii) A retraction r is called a strong deformation retraction provided that there exists a homotopy $\zeta : X \times [0, 1] \to X$ of $i \circ r$ and Id_X which satisfies the additional condition $\zeta(x, t) = \zeta(x, 0)$, for any $(x, t) \in A \times [0, 1]$.

(iii) The metric space X is said to be weakly locally contractible, if for every $u \in X$ there exists a neighborhood U of u contractible in X.

For every $a \in \mathbb{R}$, denote $f^a = \{u \in X : f(u) \le a\}$, where $f : X \to \mathbb{R}$ is a continuous function.

Definition 4.8. (i) Let $a, b \in \mathbb{R}$ with $a \le b$. The pair (f^b, f^a) is said to be trivial provided that, for every neighbourhood $[a', a'']$ of a and $[b', b'']$ of b, there exist some closed sets A and B such that $f^{a'} \subseteq A \subseteq f^{a''}$, $f^{b'} \subseteq B \subseteq f^{b''}$, and such that A is a strong deformation retraction of B.

(ii) A real number c is an essential value of f provided that, for every $\varepsilon > 0$, there exist $a, b \in (c - \varepsilon, c + \varepsilon)$ with $a < b$ such that the pair (f^b, f^a) is not trivial.

The following property of essential values is due to Degiovanni and Lancelotti (see [12], Theorem 2.6).

Proposition 4.9. *Let c be an essential value of f. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that every continuous function $g : X \to \mathbb{R}$ with*

$$\sup\{|g(u) - f(u)| : u \in X\} < \delta$$

admits an essential value in $(c-\varepsilon, c+\varepsilon)$.

For every $n \geq 1$, define $\Gamma_n = \{S \subset S_r : S \subset \mathcal{F}, \gamma(S) \geq n\}$, where $\mathcal{F}$ is the class of closed, symmetric subsets of the sphere S_r of radius r in a certain Banach space and $\gamma(S)$ represents the Krasnoselski genus of $S \in \Gamma_n$, that is, the smallest $k \in \mathbb{N} \cup \{+\infty\}$ for which there exists a continuous and odd map from S into $\mathbb{R}^k \setminus \{0\}$.

5. Proof of Theorem 3.1

Define $E = F + G : V \to \mathbb{R} \cup \{+\infty\}$, where

$$F(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx \quad , & \text{if } u \in K \\ +\infty \quad , & \text{if } u \notin K \end{cases}$$

and

$$G(u) = -\frac{\beta}{2}\int_\Gamma [\gamma(u(x))]^2 d\sigma.$$

Then $E + I_M$ is lower semicontinuous.

The following auxiliary result shows that $E + I_M$ is the canonical energy functional associated to problem (3.2).

Proposition 5.1. *If* (u, λ^2) *is a solution of problem* (3.2), *then* $0 \in \partial^-(E + I_M)(u)$. *Conversely, let* u *be a critical point of* $E + I_M$ *and denote* $\lambda^2 = -2E(u)r^{-2}$. *Then* (u, λ^2) *is a solution of problem* (3.2).

Proof. Let (u, λ^2) be a solution of problem (3.2). So, by the definition of the lower subdifferential,

$$-\lambda^2 u \in \partial^- E(u). \tag{5.1}$$

On the other hand,

$$\partial^-(E + I_M)(u) = \partial^- E(u) + \partial^- I_M(u), \qquad \text{for any } u \in K \cap M. \tag{5.2}$$

So, by (4.2) and (5.1), $0 \in \partial^-(E + I_M)(u)$.

Conversely, let $0 \in \partial^-(E + I_M)(u)$. Thus, by (4.2) and (5.2), there exists $\lambda^2 \in \mathbb{R}$ such that (u, λ^2) is a solution of problem (3.2). If we put $v = 0$ in (3.2) then we deduce $\lambda^2 r^2 \leq -2E(u)$, and for $v = 2u$ we get $\lambda^2 r^2 \geq -2E(u)$, that is, $\lambda^2 = -2E(u)r^{-2}$. $\square$

The above result reduces our study to finding the critical points of $E + I_M$. In order to estimate the number of lower stationary points of this functional we shall apply a nonsmooth version of the Lusternik-Schnirelmann theorem. For this purpose we need some preliminary results.

We first observe that a direct argument combined with Proposition 5.1 shows that problem (3.2) has at least one solution. Indeed, the associated energy functional is bounded from below. This follows directly by our basic inequality (4.1) since

$$(E+I_M)(u) \geq \frac{1}{2}\|u\|^2 - |\beta| \cdot \|[u]\|^2_{L^2(\Gamma)} \geq \frac{1}{2}\|u\|^2 - C\|u\| \geq C_0, \qquad (5.3)$$

for any $u \in V$. So, by standard minimization arguments based on the compactness of the embedding $i \circ \eta : V \to L^2(\Gamma)$ we deduce that there exists a global minimum point $u_0 \in K \cap M$ of $E + I_M$. Let $\lambda_0^2 = -2E(u_0)/r^2$. Hence $0 \in \partial^-(E+I_M)(u_0)$ and (u_0, λ_0^2) is a solution of problem (3.2). Since for any eigenvalue λ^2 there exists $u \in K$ such that $\lambda^2 = -2E(u)r^{-2}$ we deduce that $\lambda_0^2 = \sup\{\lambda^2\}$.

The next step in our proof consists in showing that

Proposition 5.2. *The functional $E + I_M$ satisfies the Palais-Smale condition.*

Proof. Let (u_n) be an arbitrary Palais-Smale sequence of $E + I_M$. So, by (5.3), (u_n) is bounded in V. Thus, by the Rellich-Kondratchov theorem (see for instance [4]) and passing eventually to a subsequence,

$$u_n \rightharpoonup u \qquad \text{weakly in } V \qquad (5.4)$$

$$u_n \to u \qquad \text{strongly in } L^2(\Omega) \qquad (5.5)$$

$$u_n \to u \qquad \text{strongly in } L^2(\Gamma). \qquad (5.6)$$

In particular, it follows that $u \in K \cap M$.

Using now the second piece of information contained in the statement of the Palais-Smale condition and applying (4.2), we obtain a sequence (λ_n) of real numbers such that

$$\lim_{n\to\infty} \|E'(u_n) + \lambda_n \Lambda_0 u_n\|_{V^*} = 0. \qquad (5.7)$$

On the other hand, by the compact embeddings $V \subset L^2(\Omega)$ and $V \subset L^2(\Gamma)$ and using (5.4)–(5.6), it follows that

$$E'(u_n) \to E'(u) \quad \text{and} \quad \Lambda_1 u_n \to \Lambda_1 u \qquad \text{in } V^*.$$

So, by (5.7), the sequence (λ_n) is bounded. Hence we can assume that, up to a subsequence, $\lambda_n \to \lambda$ as $n \to \infty$. Therefore $0 \in \partial^-(E + I_M)(u)$.

From (5.4) we get $\|u\| \leq \liminf_{n\to\infty} \|u_n\|$; hence, it follows that for concluding the proof it is enough to show that

$$\|u\| \geq \limsup_{n\to\infty} \|u_n\|. \qquad (5.8)$$

But, since F is convex, $F(u) \geq F(u_n) + F'(u_n)(u-u_n)$. It follows that

$$
\begin{aligned}
E(u) = \quad & F(u) + G(u) \geq \limsup_{n\to\infty} \big(F(u_n) + F'(u_n)(u-u_n) + G(u_n)\big) = \\
& \limsup_{n\to\infty} \big(F(u_n) + F'(u_n)(u-u_n) + G(u_n) + G'(u_n)(u-u_n)\big) = \\
& \limsup_{n\to\infty} \big(F(u_n) + E'(u_n)(u-u_n)\big) + \lim_{n\to\infty} G(u_n).
\end{aligned}
\tag{5.9}
$$

Using now $\lambda_n \to \lambda$ combined with (5.4)–(5.7), relation (5.9) yields

$$E(u) \geq \limsup_{n\to\infty} F(u_n) + G(u).$$

This inequality implies directly our claim (5.8), so the proof is completed. □

Due to the symmetry of our problem (3.2), we can extend our study to the symmetric cone $(-K)$. More precisely, if (u, λ^2) is a solution of (3.2) then $u_0 := -u \in (-K) \cap M$ satisfies

$$
\begin{aligned}
& \int_\Omega \nabla u_0 \cdot \nabla(v-u_0)dx + \int_\Gamma j'\left(\gamma(u_0(x)); \gamma(v(x)) - \gamma(u_0(x))\right) d\sigma + \\
& \lambda^2 \int_\Omega u_0(v-u_0)dx \geq 0, \qquad \text{for all } v \in (-K).
\end{aligned}
$$

This means that we can extend the energy functional associated to problem (3.2) to the symmetric set $\widetilde{K} := K \cup (-K)$. We put, by definition,

$$
\widetilde{E}(u) = \begin{cases} E(u) & , \quad \text{if } u \in K \\ E(-u) & , \quad \text{if } u \in (-K) \\ +\infty & , \quad \text{otherwise.} \end{cases}
$$

We are interested from now on in finding the lower stationary points of the extended energy functional $J := \widetilde{E} + I_M$.

We endow the set $\widetilde{K} \cap M$ with the graph metric of $\widetilde{E}$ defined by

$$d(u,v) = \|u-v\| + |\widetilde{E}(u) - \widetilde{E}(v)|, \qquad \text{for any } u, v \in \widetilde{K} \cap M.$$

Denote by $\mathcal{X}$ the metric space $(\widetilde{K} \cap M, d)$.

We are now in position to state the basic abstract result that we shall apply for concluding the proof of Theorem 3.1. More precisely, we use the following nonsmooth variant of the Lusternik-Schnirelmann theory that we reformulate in terms of our energy functional J.

Theorem 5.3. (Marino and Scolozzi [20]). *Assume that J satisfies the following properties:*

(i) J is bounded from below;

(ii) J satisfies the Palais-Smale condition;

(iii) for any lower stationary point u of J there exists a neighborhood of u in $\mathcal{X}$ which is contractible in $\mathcal{X}$;

(iv) there exists $\Theta : (\widetilde{K} \cap M) \times [0, \infty) \to \widetilde{K} \cap M$ such that $\Theta(\cdot, 0) = \mathrm{Id}$, $\Theta(u, \cdot)$ is a curve of maximal slope for J (with respect to the usual metric in V), and, moreover, the mapping $\Theta : \mathcal{X} \times [0, \infty) \to \mathcal{X}$ is continuous.

Then J has at least $\mathrm{Cat}_{\mathcal{X}}(\widetilde{K} \cap M)$ lower stationary points.

Moreover, if $\mathrm{Cat}_{\mathcal{X}}(\widetilde{K} \cap M) = +\infty$, then J does not have a maximum and

$$\sup\{J(u) : u \in \widetilde{K} \cap M, 0 \in \partial^- J(u)\} = \sup\{J(u) : u \in \widetilde{K} \cap M\}.$$

We have already proved (i) and (ii). Property (iii) is proved in a more general framework in De Giorgi, Marino, and Tosques [11], while (iv) is deduced in Chobanov, Marino, and Scolozzi [7]. So, using Theorem 5.3, it follows that for concluding the proof of Theorem 3.1 it remains to prove

Proposition 5.4. *We have*

$$\mathrm{Cat}_{\mathcal{X}}(\widetilde{K} \cap M) = +\infty. \tag{5.10}$$

Proof. Fix $\psi \in K \setminus \{0\}$ such that $\|\psi\|_{L^2(\Omega)} > r$, and let $(e_n)_{n\geq 1} \subset V$ be an orthonormal basis of $L^2(\Omega)$. Fix arbitrarily an integer $n \geq 1$ and denote

$$M^{(n)} = \Big\{ \sum_{i=1}^{n} \alpha_i e_i;\ \sum_{i=1}^{n} \alpha_i^2 = r^2 \Big\}.$$

As usual, we denote $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$, for any real number a. Define the mapping $\varphi_1 : M^{(n)} \times [0, 1] \to V \setminus \{0\}$ by

$$\varphi_1(u, t) = (1 - t) \left[(u - \psi)^+ - (u + \psi)^-\right] + P_K \left(\min\{\max(u, -\psi), \psi\}\right),$$

where P_K denotes the canonical projection onto K. Then

$$\varphi_1(u, 1) \in K \qquad \text{and} \qquad \|\varphi_1(u, 1)\|_{L^2} \leq \|u\|_{L^2} \leq r.$$

We also define $\varphi_2 : (\widetilde{K} \setminus \{0\}) \times [0, 1) \to \widetilde{K} \setminus \{0\}$ by

$$\varphi_2(u, t) = \min\Big[\max\Big(\frac{1}{1-t}\, u, -\psi\Big), \psi\Big].$$

Fix arbitrarily $u \in \varphi_1(M^{(n)}, 1)$. Then

$$\lim_{t \nearrow 1} \|\varphi_2(u, t)\|_{L^2} = \|\psi\|_{L^2} > r.$$

The compactness of $\varphi_1(M^{(n)}, 1)$ implies that there exists $t_0 \in (0, 1)$ such that

$$\|\varphi_2(u, t)\|_{L^2} > r \qquad \forall t \in [t_0, 1),\ \forall u \in \varphi_1(M^{(n)}, 1).$$

Let P be the canonical projection of V onto the closed ball of radius r in $L^2(\Omega)$ centered at the origin. Define the map $\Phi : M^{(n)} \times [0, 1+t_0] \to V \setminus \{0\}$ by

$$\Phi(u,t) = \begin{cases} \varphi_1(u,t) & , \qquad \text{if } (u,t) \in M^{(n)} \times [0,1] \\ P\left(\varphi_2(\varphi_1(u,1), t-1)\right) & , \qquad \text{if } (u,t) \in M^{(n)} \times [0, 1+t_0]. \end{cases}$$

Then $\Phi(u,0) = 0$ and $\Phi(u, 1+t_0) \in M$. Since $\Phi(\cdot, t)$ is odd and continuous from $L^2(\Omega)$ in the L^2-topology, it follows by Proposition 4.5 that

$$n \leq \mathrm{Cat}_{L^2}(M^{(n)}) \leq \mathrm{Cat}_{L^2}\left(\Phi(M^{(n)}, 1+t_0)\right) \leq \mathrm{Cat}_{H_0^1}\left(\Phi(M^{(n)}, 1+t_0)\right).$$

Since the set $\Phi(M^{(n)}, 1+t_0)$ is compact in V and the topology of $\mathcal{X}$ is stronger than the H_0^1-topology, we obtain

$$n \leq \mathrm{Cat}_{H_0^1}\left(\Phi(M^{(n)}, 1+t_0)\right) \leq \mathrm{Cat}_{\mathcal{X}}\left(\Phi(M^{(n)}, 1+t_0)\right) \leq \mathrm{Cat}_{\mathcal{X}}(\widetilde{K} \cap M).$$

This completes the proof of Proposition 5.4. □

Proof of Theorem 3.1 completed. Up to now, using Theorem 5.3, we have established that problem (3.2) admits infinitely many solutions (u, λ^2). We first observe that the set of eigenvalues is bounded from above. Indeed, if (u, λ^2) is a solution of our problem then choosing $v = 0$ in (3.2) and using (4.1), it follows that

$$\lambda^2 r^2 \leq -2\|u\|^2 + \frac{\beta}{2}\,\|u\|^2_{L^2(\Gamma)} \leq C,$$

where C does not depend on u. It remains to prove that

$$\inf\{\lambda^2 : \lambda^2 \text{ is an eigenvalue of } (3.2)\} = -\infty.$$

For this purpose, it is sufficient to show that $\sup\{J(u) : u \in \widetilde{K} \cap M\} = +\infty$. But this follows directly from (4.1) and

$$\sup_{u \in \widetilde{K} \cap M} \int_\Omega |\nabla u|^2 dx = +\infty.$$

In order to prove the last part of the theorem we remark that $-\lambda_0$, as a function of β, is the upper bound of a family of affine functions

$$-\lambda_0^2(\beta) = \inf_{v \in K \cap M} \frac{1}{r^2}\Big\{ \int_\Omega |\nabla v|^2 \, dx - \beta \int_\Gamma [v]^2 \, d\sigma \Big\}; \tag{5.11}$$

hence, it is a concave function. Thus $\beta \longmapsto \lambda_0^2(\beta)$ is convex and (3.3) yields.

This concludes the proof of Theorem 3.1. □

6. Proof of Theorem 3.2

We shall establish the multiplicity result with respect to a prescribed level of energy. More precisely, let us fix $r > 0$. Consider the manifold

$$N = \Big\{ u \in V : \int_\Gamma [u]^p d\sigma = r^p \Big\},$$

where p is as in (3.5).

We reformulate problem (3.4) as follows:

$$\left\{ \begin{array}{l} \text{find } u_\varepsilon \in K \cap N \text{ and } \lambda_\varepsilon^2 \in \mathbb{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla (v - u_\varepsilon) dx + \int_\Gamma \left(j' + \varepsilon g' \right) \left(\gamma(u_\varepsilon(x)); \gamma(v(x)) - \gamma(u_\varepsilon(x)) \right) d\sigma \\ \displaystyle+\lambda_\varepsilon^2 \int_\Omega u_\varepsilon (v - u_\varepsilon) dx \geq 0, \quad \forall v \in K. \end{array} \right. \tag{6.1}$$

We start with the preliminary result

Lemma 6.1. *There exists a sequence (b_n) of essential values of E such that $b_n \to +\infty$ as $n \to \infty$.*

Proof. For any $n \geq 1$, set $a_n = \inf_{S \in \Gamma_n} \sup_{u \in S} E(u)$, where Γ_n is the family of compact subsets of $K \cap N$ of the form $\varphi(S^{n-1})$, with $\varphi : S^{n-1} \to K \cap N$ continuous and odd. The function E restricted to $K \cap N$ is continuous, even, and bounded from below. So, by Theorem 2.12 in [12], it is sufficient to prove that $a_n \to +\infty$ as $n \to \infty$. But, by Proposition 5.2, the functional E restricted to $K \cap N$ satisfies the Palais-Smale condition. So, taking into account Theorem 3.5 in [8] and Theorem 3.9 in [12], we deduce that the set E^c has finite genus for any $c \in \mathbb{R}$. Using now the definition of the genus combined with the fact that $K \cap N$ is a weakly locally contractible metric space, we deduce that $a_n \to +\infty$. This completes our proof. $\square$

The canonical energy associated to problem (6.1) is the functional J restricted to $K \cap N$, where $J = E + \Phi$ and Φ is defined by

$$\Phi(u) = \varepsilon \int_\Gamma g(\gamma(u(x))) d\sigma.$$

A straightforward computation with the same arguments as in the proof of Proposition 5.1 shows that if u is a lower stationary point of J then there exists $\lambda^2 \in \mathbb{R}$ such that (u, λ^2) is a solution of problem (6.1). By virtue of this result, it is sufficient for concluding the proof of Theorem 3.2 to show that the functional J has at least n distinct critical values, provided that $\varepsilon > 0$ is sufficiently small. We first prove that J is a small perturbation of E. More precisely, we have

Lemma 6.2. *For every $\eta > 0$, there exists $\delta = \delta_\eta > 0$ such that*

$$\sup_{u\in K\cap N} |J(u) - E(u)| \le \eta,$$

provided that $\varepsilon \le \delta$.

Proof. We have

$$|J(u) - E(u)| = |\Phi(u)| \le \varepsilon \int_\Gamma |g(\gamma(u(x)))|\, d\sigma.$$

So, by (3.5) and Proposition 4.1,

$$|J(u) - E(u)| \le \varepsilon\, a \int_\Gamma (1 + [u(x)]^p)\, d\sigma \le C\varepsilon \le \eta,$$

if ε is sufficiently small. $\square$

By Lemma 6.1, there exists a sequence (b_n) of essential values of $E_{|K\cap N}$ such that $b_n \to +\infty$. Without loss of generality we can assume that $b_i < b_j$ if $i < j$. Fix an integer $n \ge 1$ and choose $\varepsilon_0 > 0$ such that $\varepsilon_0 < 1/2 \min_{2\le i\le n}(b_i - b_{i-1})$. Applying now Proposition 4.9, we obtain that for any $1 \le j \le n$, there exists $\eta_j > 0$ such that if $\sup_{K\cap N} |J(u) - E(u)| < \eta_j$ then $J_{|K\cap N}$ has an essential value $c_j \in (b_j - \varepsilon_0, b_j + \varepsilon_0)$. So, by Lemma 6.2 applied for $\eta = \min\{\eta_1, \ldots, \eta_n\}$, there exists $\delta_n > 0$ such that $\sup_{K\cap N} |J(u) - E(u)| < \eta$, provided that $\varepsilon \le \delta_n$. This shows that the energy functional J has at least n distinct essential values $c_1, \ldots, c_n$ in $(b_1 - \varepsilon_0, b_n + \varepsilon_0)$.

The next step consists in showing that $c_1, \ldots, c_n$ are critical values of $J_{|K\cap N}$. Arguing by contradiction, let us suppose that c_j is not a critical value of $J_{|K\cap N}$. We show in what follows that

(A_1) There exists $\bar\delta > 0$ such that $J_{|K\cap N}$ has no critical value in $(c_j - \bar\delta, c_j + \bar\delta)$.

(A_2) For every $a, b \in (c_j - \bar\delta, c_j + \bar\delta)$ with $a < b$, the pair $(J^b_{|K\cap N}, J^a_{|K\cap N})$ is trivial.

Suppose, for the sake of contradiction, that (A_1) is not valid. Then there exists a sequence (d_k) of critical values of $J_{|K\cap N}$ with $d_k \to c_j$ as $k \to \infty$. Since d_k is a critical value, it follows that there exists $u_k \in K \cap N$ such that

$$J(u_k) = d_k \quad \text{and} \quad 0 \in \partial^- J(u_k).$$

Using now the fact that J satisfies the Palais-Smale condition at the level c_j, it follows that, up to a subsequence, (u_k) converges to some $u \in K \cap N$ as $k \to \infty$. So, by the continuity of J and the lower semicontinuity of $\operatorname{grad} J(\,\cdot\,)$, we obtain $J(u) = c_j$ and $0 \in \partial^- J(u)$, which contradicts the initial assumption on c_j.

Let us now prove assertion (A_2). For this purpose we apply the noncritical point theorem (see [8], Theorem 2.15]). So, there exists a continuous map

$\chi : (K \cap N) \times [0,1] \to K \cap N$ such that

$$\begin{aligned} &\chi(u,0) = u, \quad J(\chi(u,t)) \le J(u), \\ &J(u) \le b \Rightarrow J(\chi(u,1)) \le a, \quad J(u) \le a \Rightarrow \chi(u,t) = u. \end{aligned} \tag{6.2}$$

Define the map $\rho : J^b_{|K\cap N} \to J^a_{|K\cap N}$ by $\rho(u) = \chi(u,1)$. From (6.2) we obtain that ρ is well defined and it is a retraction. Set

$$\mathcal{J} : J^b_{|K\cap N} \times [0,1] \to J^b_{|K\cap N}, \qquad \mathcal{J}(u,t) = \chi(u,t).$$

The definition of $\mathcal{J}$ implies that, for every $u \in J^b_{|K\cap N}$,

$$\mathcal{J}(u,0) = u \qquad \text{and} \qquad \mathcal{J}(u,1) = \rho(u) \tag{6.3}$$

and, for any $(u,t) \in J^a_{|K\cap N} \times [0,1]$,

$$\mathcal{J}(u,t) = \mathcal{J}(u,0). \tag{6.4}$$

From (6.3) and (6.4) it follows that $\mathcal{J}$ is $J^a_{|K\cap N}$-homotopic to the identity of $J^a_{|K\cap N}$, that is, $\mathcal{J}$ is a strong deformation retraction, so the pair $(J^b_{|K\cap N}, J^a_{|K\cap N})$ is trivial. Assertions (A_1) and (A_2) and Definition 4.8 (ii) show that c_j is not an essential value of $J_{|K\cap N}$. This contradiction concludes our proof. □

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