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Constant sign and nodal solutions for resonant double phase problems

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Abstract. We consider a double phase Dirichlet problem with a reaction which asymptotically as $x \to \pm\infty$ can be resonant with respect to the principle eigenvalue $\hat{\lambda}_1 > 0$ of the Dirichlet weighted p-Laplacian. Using variational tools, together with truncation and comparison techniques and critical groups, we show that the problem has at least three bounded solutions which are ordered and we provide sign information for all of them (positive, negative and nodal).

Resonoivan kaksivaiheongelman vakio- ja vaihtuvamerkkiset ratkaisut

Tiivistelmä. Tarkastelemme kaksivaiheista Dirichlet'n ongelmaa, jonka reaktiotermi voi resonoida painotetun Dirichlet'n p-Laplacen operaattorin päämominaisarvon $\hat{\lambda}_1 > 0$ kanssa asymptoottisesti, kun $x \to \pm\infty$. Käyttämällä variaatiomenetelmiä yhdessä katkaisu- ja vertailutekniikoiden sekä kriittisten ryhmien kanssa osoitamme, että ongelmalla on ainakin kolme rajallista ratkaisua, joilla on keskinäinen suuruusjärjestys ja määrätyt etumerkkiominaisuudet (positiivinen, negatiivinen ja vaihtuvamerkkinen).

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We study the following double phase Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta_p^a u - \Delta_q u = f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ 1 < q < p < N. \end{array} \right\} \tag{1}$$

For $a \in L^\infty(\Omega)\backslash\{0\}$ with $a(z) \geq 0$ for a.a. $z \in \Omega$ and $1 < r < \infty$, by Δ_r^a we denote the weighted p-Laplace differential operator with weight $a(\cdot)$ defined by

$$\Delta_r^a u = \operatorname{div}(a(z)|Du|^{r-2}Du).$$

If $a \equiv 1$, then we have the standard r-Laplace differential operator. Equation (1) is driven by the sum of two such operators with distinct exponents. So, the differential operator in (1) is not homogeneous. This operator is related to the so-called "double phase" integral functional

$$u \to \int_\Omega [a(z)|Du|^p + |Du|^q] \, \mathrm{d}z.$$

The density of this functional is the integrand

$$\eta(z,t) = a(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \geqslant 0.$$

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We do not assume that the weight $a(\cdot)$ is bounded away from zero (that is, we do not require that $0 < \operatorname{ess\,inf}_\Omega a$). So, the density integrand $\eta(z,\cdot)$ exhibits unbalanced growth

$$t^q \leqslant \eta(z,t) \leqslant \hat{c}\,(1+t^p) \ \text{ for a.a. } z \in \Omega, \ \text{ all } t \geqslant 0, \ \text{ some } \hat{c} > 0.$$

Such functionals were first examined by Marcellini [13, 14] and Zhikov [24, 25] in the context of problems of the calculus of variations (including the Lavrentiev gap phenomenon) and of nonlinear elasticity theory.

Recently, the interest for these problems was revived and there have been efforts to develop a regularity theory. We refer to the works of Marcellini [15], Mingione–Rădulescu [16], Ragusa–Tachikawa [23] and the references therein. So far we have only local regularity results. A global regularity theory (that is, regularity up to the boundary), remains elusive. This removes from consideration many powerful tools which are available when dealing with balanced growth problems. So, the task of proving multiplicity theorems with sign information for all the solutions for double phase problems, is much more difficult.

In the past, most multiplicity results for double phase equations, assumed that the reaction is $(p-1)$-superlinear. We mention the works of Deregowska–Gasinski–Papageorgiou [2], Gasinski–Papageorgiou [7], Gasinski–Winkert [8], Liu–Dai [12], Papageorgiou–Vetro–Vetro [21], Papageorgiou–Zhang [22]. Recently, Papageorgiou–Rădulescu–Zhang [20] and Papageorgiou–Pudelko–Rădulescu [17], developed the spectral properties of the weighted p-Lapiacian Δ_p^a and proved multiplicity theorems for resonant problems. They prove the existence of two solutions but do not provide sign information for them.

Here using a combination of variational tools, with truncation and comparison techniques and critical groups, under resonance conditions on the reaction, we prove a multiplicity theorem producing three nontrivial bounded solutions, two of constant sign (positive and negative) and the third nodal (sign changing). It appears that our result here is the first multiplicity result for double phase equations with sign information for all the solutions.

2. Mathematical background and hypotheses

A first consequence of the unbalanced growth of $\eta\,(z,\cdot)$ is that the standard Sobolev spaces do not provide an adequate framework to deal with problem (1). We need to use generalized Orlicz spaces. A comprehensive presentation of the theory of these spaces can be found in the book of Harjulehto and Hästo [10].

We introduce the conditions on the exponents and on the weight function.

Recall that $C^{0,1}(\bar{\Omega})$ is the space of all Lipschitz continuous functions $u\colon \bar{\Omega} \to \mathbb{R}$.

Also, by $\mathcal{A}_p$ we denote the class of all p-Muckenhoupt weights (see Cruz Uribe–Fiorenza [1, p. 142] and Harjulehto–Hästo [10, p. 114]).

$\mathbf{H_0}$: $a \in C^{0,1}(\bar{\Omega}) \cap \mathcal{A}_p$, $a(z) > 0$ for all $z \in \Omega$, $1 < q < p < N$, $2 \leqslant p$, $\frac{p}{q} < 1 + \frac{1}{N}$.

Remark 2.1. The hypothesis that $a \in C^{0,1}(\bar{\Omega})$, implies that the Poincaré inequality holds in the corresponding Orlicz–Sobolev space. The hypothesis that $a \in \mathcal{A}_p$ permits the use of the spectral analysis for Δ_p^a which was done in [17].

The last inequality in H_0 is common in double phase Dirichlet problems and it says that the two exponents p, q can not be far apart. Also, it implies that $p < q^* = \frac{Nq}{N-q}$ and this in turn leads to compact embeddings of some relevant spaces.

Recall that $\eta(z,t)$ is the density of the double phase integral functional, that is,

$$\eta(z,t) = a(z)t^p + t^q \quad \text{for a.a. } z \in \Omega, \text{ all } t \geqslant 0.$$

Note that for all $z \in \Omega$, $\eta(z,\cdot)$ is uniformly convex.

Let $L^0(\Omega)$ be the space of all measurable functions $u\colon \Omega \to \mathbb{R}$. As usual we identify two such functions which differ only on a Lebesgue-null set.

The generalized Orlicz–Lebesgue space $L^\eta(\Omega)$ is defined by

$$L^\eta(\Omega) = \left\{ u \in L^0(\Omega)\colon \rho_\eta(u) = \int_\Omega \eta(z,|u|)\,\mathrm{d}z < \infty \right\}.$$

The function $\rho_\eta(\cdot)$ is the "modular function" corresponding to the density η. We equip $L^\eta(\Omega)$ with the so-called "Luxemburg norm" $\|\cdot\|_\eta$ defined by

$$\|u\|_\eta = \inf\left\{\lambda > 0\colon \rho_\eta\left(\frac{u}{\lambda}\right) \leqslant 1\right\} \text{ for all } u \in L^\eta(\Omega).$$

With this norm $L^\eta(\Omega)$ becomes a separable Banach space which is uniformly convex, thus reflexive. Using $L^\eta(\Omega)$, we can define the corresponding generalized Orlicz–Sobolev space $W^{1,\eta}(\Omega)$ by

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega)\colon |Du| \in L^\eta(\Omega)\}.$$

We equip this space with the norm $\|\cdot\|_{1,\eta}$ defined by

$$\|u\|_{1,\eta} = \|u\|_\eta + \|Du\|_\eta \quad \text{for all } u \in W^{1,\eta}(\Omega),$$

where $\|Du\|_\eta = \|\,|Du|\,\|_\eta$. Also we set

$$W_0^{1,\eta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\eta}}.$$

As we already mentioned, since $a \in C^{0,1}(\bar{\Omega})$, on $W_0^{1,\eta}(\Omega)$ the Poincare inequality holds, namely we can find $c = c(\Omega) > 0$ such that

$$\|u\|_\eta \leqslant c\|Du\|_\eta \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Therefore on $W_0^{1,\eta}(\Omega)$, we can use the equivalent norm

$$\|u\| = \|Du\|_\eta \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Both $W^{1,\eta}(\Omega)$ and $W_0^{1,\eta}(\Omega)$ are separable Banach spaces, which are uniformly convex (thus reflexive). We have some useful embeddings between these spaces.

Proposition 1. *The following results hold:*

(a) *let $s \in [1,q]$. Then $L^\eta(\Omega) \hookrightarrow L^s(\Omega), W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,s}(\Omega)$ continuously.*
(b) *$W_0^{1,\eta}(\Omega) \hookrightarrow L^s(\Omega)$ continuously for all $s \in [1,q^*]$.*
(c) *$W_0^{1,\eta}(\Omega) \hookrightarrow L^s(\Omega)$ compactly for all $s \in [1,q^*)$.*

There is a close relation between the norm $\|\cdot\|$ and the modular function $\rho_\eta(\cdot)$.

Proposition 2. *The following results hold:*

(a) $\|u\| = \lambda \Leftrightarrow \rho_\eta\left(\frac{Du}{\lambda}\right) = 1$.
(b) $\|u\| < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow \rho_\eta(Du) < 1$ (*resp.* $= 1, > 1$).
(c) $\|u\| < 1 \Rightarrow \|u\|^p \leqslant \rho_\eta(Du) \leqslant \|u\|^q$.
(d) $\|u\| > 1 \Rightarrow \|u\|^q \leqslant \rho_\eta(Du) \leqslant \|u\|^p$.
(e) $\|u\| \to 0$ (*resp.*, $\|u\| \to \infty$) $\Leftrightarrow \rho_\eta(Du) \to 0$ (*resp.* $\rho_\eta(Du) \to \infty$).

We introduce the operator $V\colon W_0^{1,\eta}(\Omega) \to W_0^{1,\eta}(\Omega)^*$ defined by

$$\langle V(u), h\rangle = \int_\Omega \left[a(z)|Du|^{p-2} + |Du|^{q-2}\right](Du, Dh)_{\mathbb{R}^N}\,\mathrm{d}z \quad \text{for all } u, h \in W_0^{1,\eta}(\Omega).$$

This operator has the following properties (see Liu–Dai [12]).

Proposition 3. *The operator $V\colon W_0^{1,\eta}(\Omega) \to W_0^{1,\eta}(\Omega)^*$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$, that is, "if $u_n \xrightarrow{w} u$ in $W_0^{1,\eta}(\Omega)$ and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,\eta}(\Omega)$".*

Let $\eta_0(z,t) = a(z)t^p,\ z \in \Omega,\ t \geqslant 0$. For this integrand we introduce the generalized Orlicz spaces $L^{\eta_0}(\Omega)$ and $W_0^{1,\eta_0}(\Omega)$. We equip $L^{\eta_0}(\Omega)$ with the Luxemburg norm

$$\|u\|_{\eta_0} = \inf\left\{\lambda > 0\colon \rho_{\eta_0}\left(\frac{u}{\lambda}\right) \le 1\right\}$$

and $W_0^{1,\eta_0}(\Omega)$ with the norm

$$\|u\|_{1,\eta_0} = \|u\|_{\eta_0} + \|Du\|_{\eta_0}.$$

These are separable reflexive (in fact uniformly convex) Banach spaces. From Papageorgiou–Rădulescu–Zhang [20] (Lemma 2.1), we know that

$$W_0^{1,\eta_0}(\Omega) \hookrightarrow L^{\eta_0}(\Omega) \text{ compactly.} \tag{2}$$

We consider the following nonlinear eigenvalue problem

$$\left\{\begin{array}{l} -\Delta_p^a u(z) = \hat{\lambda}\alpha(z)|u(z)|^{p-2}u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{array}\right\}. \tag{3}$$

Using (2), we can show that the eigenvalue problem (3) has a smallest eigenvalue $\hat{\lambda}_1 > 0$, which has the following variational characterization

$$\hat{\lambda}_1 = \inf\left\{\frac{\rho_{\eta_0}(Du)}{\rho_{\eta_0}(u)}\colon u \in W_0^{1,\eta_0}(\Omega),\ u \neq 0\right\}, \tag{4}$$

where $\rho_{\eta_0}(u) = \int_\Omega \eta_0(z, |u|)\,\mathrm{d}z$.

This eigenvalue is simple (that is, if $\hat{u}, \hat{v}$, are eigenfunctions corresponding to $\hat{\lambda}_1 > 0$, then $\hat{u} = \vartheta\hat{v}$ for some $\vartheta \in \mathbb{R}\backslash\{0\}$), isolated.

The infimum in (4) is realized on the corresponding one-dimensional eigenspace, the elements of which have fixed sign. By $\hat{u}_1$ we denote the corresponding positive $L^{\eta_0}(\Omega)$-normalized eigenfunction (that is, $\|\hat{u}_1\|_{\eta_0} = 1$). We have

$$\hat{u}_1 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$$

and for every $K \subseteq \Omega$ compact, we have

$$0 < c_K \leqslant \hat{u}_1(z) \quad \text{for a.a. } z \in \Omega.$$

In the sequel for every $u \in L^0(\Omega)$ with this property, we write $0 \prec u$. We mention that all higher eigenvalues of (3), have nodal eigenfunctions. For details, see [17].

If $u \in L^0(\Omega)$, then $u^+ = \max\{u, 0\}, u^- = \max\{-u, 0\}$. We have $u = u^+ - u^-, |u| = u^+ + u^-$ and if $u \in W_0^{1,\eta}(\Omega)$, then $u^\pm \in W_0^{1,\eta}(\Omega)$. If $h_1, h_2 \in L^0(\Omega)$, then

$$[h_1, h_2] = \left\{u \in W_0^{1,\eta}(\Omega)\colon h_1(z) \leqslant u(z) \leqslant h_2(z) \text{ for a.a. } z \in \Omega\right\}.$$

Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets:

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi),$$
$$\varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

We say that $\varphi(\cdot)$ satisfies the "C-condition", if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

$$\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R} \text{ is bounded},$$
$$(1 + \|u_n\|_X)\,\varphi'(u_n) \to 0 \ \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence.

This is a compactness condition on $\varphi(\cdot)$, which compensates for the fact that the ambient space X need not be locally compact (being in general infinite dimensional).

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}-relative singular homology group with integer coefficients. Let $u \in K_\varphi$ be isolated and set $c = \varphi(u)$. Then the critical groups of $\varphi(\cdot)$ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap \mathcal{U}, \varphi^c \cap \mathcal{U} \backslash \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with $\mathcal{U}$ being an open neighborhood of u such that $K_\varphi \cap \varphi^c \cap \mathcal{U} = \{u\}$. The excision property of singular homology implies that this definition is independent of the choice of the isolating neighborhood $\mathcal{U}$.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and that $-\infty < \inf \varphi(K_\varphi)$. The critical groups of $\varphi(\cdot)$ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see [19, p. 386]) implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We introduce the following series in $t \in \mathbb{R}$.

$$M(t, u) = \sum_{k\in\mathbb{N}_0} \operatorname{rank} C_k(\varphi, u) t^k \quad \text{for all } u \in K_\varphi,$$
$$P(t, u) = \sum_{k\in\mathbb{N}_0} \operatorname{rank} C_k(\varphi, \infty) t^k.$$

The "Morse relation" says that

$$\sum_{u\in K_\varphi} M(t, u) = P(t, \infty) + (1 + t)Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{5}$$

with $Q(t) = \sum_{k\in\mathbb{N}_0} \beta_k t^k$ a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients (see [19]).

We will use critical groups to overcome the difficulties we encounter due to the lack of a global regularity theory.

To do this, We will need the notion of $L^\infty(\Omega)$-locally Lipschitz integrand. We say that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is an $L^\infty(\Omega)$-locally Lipschitz integrand, if

- for all $x \in \mathbb{R}$, $z \to g(z, x)$ is measurable;
- for a.a. $z \in \Omega$ and all compact $K \subseteq \mathbb{R}$, there exists $g_K \in L^\infty(\Omega)$ such that

$$|g(z, x) - g(z, y)| \leqslant g_K(z)|x - y| \quad \text{for a.a. } z \in \Omega \text{ all } x, y \in K.$$

The hypotheses on the reaction $f(z, x)$ are the following:

$\mathbf{H_1}$: $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^∞-locally Lipschitz integrand such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,x)x \geqslant 0$ for all $x \in \mathbb{R}$ and

(i) $|f(z,x)| \leqslant \hat{a}(z)\,(1 + |x|^{p-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\hat{a} \in L^\infty(\Omega)$;

(ii) $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$, then $\lim_{x\to\pm\infty} \frac{pF(z,x)}{a(z)|x|^p} \leq \hat{\lambda}_1$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\beta_0 > 0$ such that

$$-\beta_0 \leqslant f(z,x)x - pF(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R};$$

(iv) there exist $\delta > 0$ and $\tau \in (1,q)$ such that

$$c_0|x|^\tau \leq f(z,x)x \leq \tau F(z,x)$$

for a.a. $z \in \Omega$, all $|x| \leq \delta$, some $c_0 > 0$.

Remark 2.2. Hypothesis H_1 (ii) implies that we also have

$$\limsup_{x\to\pm\infty} \frac{f(z,x)}{a(z)|x|^{p-2}x} \leqslant \hat{\lambda}_1 \quad \text{unformly for a.a. } z \in \Omega$$

So, our hypotheses cover the resonant case. Hypothesis H_1 (iv) implies the presence of a local concave term near zero.

Let $\varphi\colon W_0^{1,\eta}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega F(z,u)\,\mathrm{d}z \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Evidently $\varphi \in C^1(W_0^{1,\eta}(\Omega))$.

Also in order to produce solutions of constant sign, we consider the positive and negative truncations of $\varphi(\cdot)$, namely the C^1-functionals $\varphi_\pm\colon W_0^{1,\eta}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\pm(u) = \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega F(z,\pm u^\pm)\,\mathrm{d}z \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

3. Solutions of constant sign

In this section we produce two bounded constant sign solutions (positive and negative) using the direct method of the calculus of variations.

Proposition 4. *If hypotheses H_0, H_1 hold, then the functionals $\varphi_\pm, \varphi$ are coercive.*

Proof. We do the proof for $\varphi_+(\cdot)$, the proofs for $\varphi_-(\cdot), \varphi(\cdot)$ being similar.

We argue by contradiction. So, suppose we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ such that

$$\|u_n\| \to \infty \text{ as } n \to \infty \text{ and } \varphi_+(u_n) \leqslant c_1 \text{ for some } c_1 > 0, \text{ all } n \in \mathbb{N}. \tag{6}$$

From the inequality in (6), we see that if $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ is bounded, then so is $\{u_n^-\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ and then $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ is bounded, contradicting (6). Therefore we may assume that

$$\|u_n^+\| \to \infty \quad \text{as } n \to \infty. \tag{7}$$

First suppose that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta_0}(\Omega)$ is bounded (recall that $W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,\eta_0}(\Omega)$ continuously). Hypotheses H_1 (i), (ii) imply that we can find $c_2 > 0$ such that

$$F(z,x) \leqslant \frac{1}{p}\big(\hat{\lambda}_1+1\big)a(z)|x|^p + c_2 \quad \text{for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R} \tag{8}$$

from (6), we have

$$\begin{aligned}
&\frac{1}{p}\rho_\eta(Du_n^+) + \frac{1}{q}\left\|Du_n^+\right\|_q^q \leqslant c_1 + \int_\Omega F\left(z,u_n^+\right)\mathrm{d}z,\\
\Rightarrow\ &\frac{1}{p}\left[\rho_{\eta_0}\left(Du_n^+\right) - \hat{\lambda}_1\rho_{\eta_0}\left(u_n^+\right)\right] + \frac{1}{q}\left\|Du_n^+\right\|_q^q \leqslant c_3 + \frac{1}{p}\rho_{\eta_0}\left(u_n^+\right)\\
&\text{for some } c_3>0,\ \text{all } n\in\mathbb{N} \text{ (see [8])},\\
\Rightarrow\ &\frac{1}{q}\left\|Du_n^+\right\|_q^q \leqslant c_4 \text{ for some } c_4>0,\ \text{all } n\in\mathbb{N}\\
&\text{(see (4) and recall that we have assumed that } \left\{u_n^+\right\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta_0}(\Omega) \text{ is bounded)},\\
\Rightarrow\ &\left\{u_n^+\right\}_{n\in\mathbb{N}} \subseteq W_0^{1,q}(\Omega) \text{ is bounded},\\
\Rightarrow\ &\left\{u_n^+\right\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega) \text{ is bounded}.
\end{aligned}$$

But this contradicts (7). Therefore we may assume that

$$\left\|u_n^+\right\|_{1,\eta_0} \to \infty \quad \text{as } n\to\infty. \tag{9}$$

Let $y_n = \frac{u_n^+}{\left\|u_n^+\right\|_{1,\eta_0}}$ for all $n\in\mathbb{N}$. We have

$$\|y_n\|_{1,\eta_0} = 1,\ y_0 \geqslant 0 \quad \text{for all } n\in\mathbb{N}.$$

Recall that $W_0^{1,\eta_0}(\Omega) \hookrightarrow L^{\eta_0}(\Omega)$ compactly (see [20]). Since $W_0^{1,\eta_0}(\Omega)$ is a separable, reflexive Banach space, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,\eta_0}(\Omega),\quad y_n\to y \text{ in } L^{\eta_0}(\Omega) \quad \text{as } n\to\infty. \tag{10}$$

From (6) we have

$$\begin{aligned}
&\frac{1}{p}\rho_{\eta_0}\left(Du_n^+\right) + \frac{1}{q}\|Du_n^+\|^q - \int_n F\left(z,u_n^+\right)\mathrm{d}z \leqslant c_1 \quad \text{for all } n\in\mathbb{N},\\
\Rightarrow\ &\frac{1}{p}\rho_{\eta_0}\left(Dy_n\right) + \frac{1}{q\left\|u_n^+\right\|_{1,\eta_0}^{p-q}}\left\|Dy_n\right\|_q^q - \int_\Omega \frac{F\left(z,u_n^+\right)}{\left\|u_n^+\right\|_{1,\eta_0}^p} \leqslant \frac{c_1}{\left\|u_n^+\right\|_{1,\eta_0}^p} \quad \text{for all } n\in\mathbb{N}.
\end{aligned} \tag{11}$$

Claim. $-\beta_0 \leqslant \hat{\lambda}_1 a(z)v^p - pF(z,v)$ for a.a. $z\in\Omega$, all $v\geqslant 0$.

For $x>0$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{F(z,x)}{x^p}\right) &= \frac{f(z,x)x^p - px^{p-1}F(z,x)}{x^2p} = \frac{f(z,x)x - pF(z,x)}{x^{p+1}}\\
&\geqslant -\frac{\beta_0}{x^{p+1}} \quad \text{for a.a. } z\in\Omega \text{ (see hypothesis } H_1 \text{ (iii))}.
\end{aligned}$$

Integrating this inequality, we obtain

$$\frac{F(z,x)}{x^p} - \frac{F(z,v)}{v^p} \geqslant \frac{\beta_0}{p}\left[\frac{1}{x^p} - \frac{1}{v^p}\right] \quad \text{for a.a. } z\in\Omega,\ \text{all } x\geqslant v>0.$$

Passing to the limit as $x \to +\infty$ and using hypothesis H_1 (ii), we obtain

$$\begin{aligned} &\frac{\hat{\lambda}_1 a(z)}{p} - \frac{F(z,v)}{v^p} \geqslant -\frac{\beta_0}{pv^p}, \\ \Rightarrow\ &\hat{\lambda}_1 \alpha(z) v^p - pF(z,v) \geqslant -\beta_0 \ \text{ for a.a. } z \in \Omega, \text{ all } v \geqslant 0. \end{aligned}$$

This proves the Claim.

The above Claim implies that

$$-F(z,x) \geqslant -\frac{1}{p}\beta_0 - \frac{1}{p}\hat{\lambda}_1 \alpha(z) x^p \ \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{12}$$

We use (12) in (11) and have

$$\frac{1}{p}\left(\rho_{\eta_0}(Dy_n) - \hat{\lambda}_1 \rho_{\eta_0}(y_n)\right) \leqslant \varepsilon_n \text{ for all } n \in \mathbb{N}, \text{ with } \varepsilon_n \to 0^+ \text{ as } n \to \infty.$$

If we pass to the limit as $n \to \infty$ and use (10), we obtain

$$\rho_{\eta_0}(Dy) \leqslant \hat{\lambda}_1 \rho_{\eta_0}(y).$$

Note that the modular function $\rho_{\eta_0}(\cdot)$ is continuous, convex, thus weakly lower semicontinuous on $W_0^{1,\eta_0}(\Omega)$. From (4) if follows that

$$\begin{aligned} &\rho_{\eta_0}(Dy) = \hat{\lambda}_1 \rho_{\eta_0}(y), \\ \Rightarrow\ &y = \hat{u}_1 \succ 0 \ \text{ or } \ y = 0 \quad \text{(recall that } y \geqslant 0). \end{aligned}$$

If $y = 0$, then

$$\begin{aligned} &\rho_{\eta_0}(Dy_n), \rho_{\eta_0}(y_n) \to 0, \\ \Rightarrow\ &y_n \to 0 \text{ in } W_0^{1,\eta_0}(\Omega) \ \text{ as } n \to \infty, \end{aligned}$$

which contradicts the fact that $\|y_n\|_{1,\eta_0} = 1$ for all $n \in \mathbb{N}$,

If $y = \hat{u}_1$, then since $\hat{u}_1 \succ 0$, we infer that

$$u_n^+(z) \to +\infty \ \text{ for a.a. } z \in \Omega \text{ as } n \to \infty. \tag{13}$$

From (6) we have

$$\begin{aligned} &\rho_\eta\left(Du_n^+\right) + \frac{p}{q}\left\|Du_n^+\right\|_q^q - \int_\Omega pF\left(z, u_n^+\right) \mathrm{d}z \leqslant pc_1 \ \text{ for all } n \in \mathbb{N}, \\ \Rightarrow\ &\int_\Omega \left[\hat{\lambda}_1 a(z)\left(u_n^+\right)^p - pF\left(z, u_n^+\right)\right] \mathrm{d}z + \frac{p}{q}\left\|Du_n^+\right\|_q^q \leqslant pc_1 \quad \text{(see the Claim)}. \end{aligned}$$

If $\hat{\lambda}_1(q) > 0$ denotes the principal eigenvalue of $\left(-\Delta_q, W_0^{1,q}(\Omega)\right)$ and since $\hat{\lambda}_1(q)\|v\|_q^q \leqslant \|Dv\|_q^q$ for all $v \in W_0^{1,q}(\Omega)$ (see Gasinski–Papageorgiou [5]), we have

$$\frac{p}{q}\hat{\lambda}_1(q) \int_\Omega \left(u_n^+\right)^q \ \mathrm{d}z \leqslant c_5 \text{ for all } n \in \mathbb{N}, \text{ some } c_5 > 0 \quad \text{(see the Claim)}. \tag{14}$$

Using (13), (14) and Fatou's lemma, we reach a contradiction.

Therefore $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta_0}(\Omega)$ is bounded, which we have seen earlier that it implies that $\{u_n^+\} \subseteq W^{1,q}(\Omega)$ is bounded, hence $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ is bounded, contradicting (7). We conclude that $\varphi_+(\cdot)$ coercive.

Similarly we show that $\varphi_-(\cdot)$ and $\varphi(\cdot)$ are coercive. □

Now we can produce two constant sign solutions.

Proposition 5. *If hypotheses H_0, H_1 hold, then problem* (1) *has at least two constant sign solutions*

$$u_0, v_0 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega),$$
$$v_0 \prec 0 \prec u_0.$$

Proof. By Proposition 4 , $\varphi_+(\cdot)$ is coercive. Also using Proposition 1 and the sequential weak lower semicontinuity of the modular and norm functions, we see that $\varphi_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_0 \in W_0^{1,\eta}(\Omega)$ such that

$$\varphi_+ (u_0) = \inf \left\{\varphi_+(u)\colon u \in W_0^{1,\eta}(\Omega)\right\}. \tag{15}$$

Let $u \in C_0^1(\bar{\Omega})$ with $u(z) > 0$ for all $z \in \Omega$. We an find $t \in (0, 1)$ small such that

$$0 \leqslant tu(z) \leqslant \delta \quad \text{for all } z \in \bar{\Omega}, \tag{16}$$

with $\delta > 0$ as postulated by hypothesis H_1 (iv). We have

$$\varphi_t(tu) \leqslant \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \frac{c_0}{\tau}t^\tau\|u\|_\tau^\tau \quad \text{(see hypothesis } H_1\text{(iv))}.$$

Since $1 < \tau < q < p$, choosing $t \in (0, 1)$ even smaller if necessary, we obtain

$$\begin{aligned} &\varphi_+(tu) < 0,\\ \Rightarrow\ &\varphi_+(u_0) < 0 = \varphi_+(0) \quad \text{(see(15))},\\ \Rightarrow\ &u_0 \neq 0. \end{aligned}$$

From (15) we have

$$\begin{aligned} &\left\langle\varphi_+' (u_0), h\right\rangle = 0 \quad \text{for all } h \in W_0^{1,\eta}(\Omega),\\ \Rightarrow\ &\langle V (u_0), h\rangle = \int_\Omega f(z, u_0^+)h \,\mathrm{d}z \quad \text{for all } h \in W_0^{1,\eta}(\Omega). \end{aligned}$$

Choosing $h = -u_0^- \in W_0^{1,\eta}(\Omega)$, we obtain

$$\begin{aligned} &\rho_\eta\left(Du_0^-\right) = 0,\\ \Rightarrow\ &u_0 \geqslant 0, u_0 \neq 0 \quad \text{(see Propesition 2)}. \end{aligned}$$

Therefore $u_0 \in W_0^{1,\eta}(\Omega)$ is a positive solution. From Gasinski–Winkert [8] (Theorem 3.1), we have $u_0 \in W_0^{1,\eta}(\Omega)\cap L^\infty(\Omega)$, In addition Proposition 2.4 of Papageorgiou–Vetro–Vetro [21] implies that $0 \prec u_0$.

Similarly, working with $\varphi_-(\cdot)$, we generate a negative solution $v_0 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ with $v_0 \prec 0$. □

In fact, we can have extremal constant sign solutions that is, a smallest positive solution and a biggest negative solution. We will need these extremal solutions in order to produce a nodal one. To this end, motivated by hypothesis H_1 (iv), we consider the following auxiliary double phase problem

$$\left\{\begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = c_0|u(z)|^{\tau-2}u(z) \ \text{ in } \Omega,\\ u|_{\partial\Omega} = 0, \quad 1 < \tau < q < p. \end{array}\right\} \tag{Au}$$

Proposition 6. *If hypotheses H_0 hold, then problem* (Au) *has a unique positive solution $\bar{u} \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$, $0 \prec \bar{u}$ and since problem* (Au) *is odd $\bar{v} = -\bar{u} \prec 0$ is the unique solution of* (Au).

Proof. Let $\sigma\colon W_0^{1,\eta}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\sigma(u) = \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \frac{c_0}{\tau}\|u^+\|_\tau^\tau \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Evidently, $\sigma(\cdot)$ is coercive (since $\tau < q < p$) and sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W_0^{1,\eta}(\Omega)$ such that

$$\sigma(\bar{u}) = \inf\left\{\sigma(u)\colon u \in W_0^{1,\eta}(\Omega)\right\} \tag{17}$$

If $u \in W_0^{1,\eta}(\Omega)\backslash\{0\}, u(z) \geqslant 0$ for a.a. $z \in \Omega$ and $t \in (0,1)$, then

$$\sigma(tu) \leqslant \frac{t^q}{q}p_\eta(Du) - \frac{c_0t^\tau}{\tau}\|u\|_\tau^\tau.$$

Since $\tau < q$, by choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned}
&\sigma(tu) < 0,\\
\Rightarrow\ &\sigma(\bar{u}) < 0 = \sigma(0) \quad (\text{see (17)}),\\
\Rightarrow\ &\bar{u} \neq 0.
\end{aligned}$$

From (17) we have

$$\begin{aligned}
&\langle\sigma'(\bar{u}), h\rangle = 0 \quad \text{for all } h \in W_0^{1,\eta}(\Omega),\\
\Rightarrow\ &\langle V(\bar{u}), h\rangle = c_0\int_\Omega \left(\bar{u}^+\right)^{\tau-1} h \,\mathrm{d}z \quad \text{for all } h \in W_0^{1,\eta}(\Omega).
\end{aligned}$$

Using $h = -\bar{u}^- \in W_0^{1,\eta}(\Omega)$, we obtain

$$\begin{aligned}
&\rho_\eta(D\bar{u}^-) = 0,\\
\Rightarrow\ &\bar{u} \geqslant 0,\ \bar{u} \neq 0.
\end{aligned}$$

So, $\bar{u}$ is a positive solution of (Au) and as before, we have $\bar{u} \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$, $0 \prec \bar{u}$.

Suppose $\bar{v}$ is another positive solution of (Au). Again we have $\bar{v} \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$, $0 \prec \bar{v}$. Then we introduce the integral functional $j\colon L^1(\Omega) \to \bar{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}p_{\eta_0}\left(Du^{1/q}\right) + \frac{1}{q}\|Du^{1/q}\|_q^q & \text{if } u \geqslant 0,\ u^{1/q} \in W_0^{1,\eta}(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

We set $\operatorname{dom} j = \{u \in L^1(\Omega)\colon j(u) < \infty\}$ (the effective domain of $j(\cdot)$). As in Papageorgiou–Rădulescu [18], using Lemma 1 of Diaz–Saa [3], we have that $j(\cdot)$ is convex.

For $\varepsilon > 0$, we set

$$\bar{u}_\varepsilon = \bar{u} + \varepsilon \quad \text{and} \quad \bar{v}_\varepsilon = \bar{v} + \varepsilon.$$

Then $\bar{u}_\varepsilon, \bar{v}_\varepsilon \in \operatorname{int} L^\infty(\Omega)_+$ with $L^\infty(\Omega)_+ = \{y \in L^\infty(\Omega) : y(z) \geqslant 0 \text{ for a.a. } z \in \Omega\}$.

So, using Proposition 4.1.22 of Papageorgiou–Rădulescu–Repovs [19, p. 274], we have that

$$\frac{\bar{u}_\varepsilon}{\bar{v}_\varepsilon} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\bar{v}_\varepsilon}{\bar{u}_\varepsilon} \in L^\infty(\Omega). \tag{18}$$

Let $h = \bar{u}_\varepsilon^q - \bar{v}_\varepsilon^q \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$. On account of (18) for $t \in (0,1)$ small we have

$$\bar{u}_\varepsilon + th \in \operatorname{dom} j \quad \text{and} \quad \bar{v}_\varepsilon + th \in \operatorname{dom} j.$$

So, exploiting the convexity of $j(\cdot)$, we can compute the directional derivatives of $j(\cdot)$ at $\bar{u}_\varepsilon$ and at $\bar{v}_\varepsilon$ in the direction h. A direct computation gives

$$j'(\bar{u}_\varepsilon^q)(h) = \frac{1}{q}\int_\Omega \frac{-\Delta_p^a \bar{u} - \Delta_q \bar{u}}{\bar{u}_\varepsilon^{q-1}} h \,\mathrm{d}z = \frac{c_0}{q}\int_\Omega \frac{\bar{u}^{\tau-1}}{\bar{u}_\varepsilon^{q-1}} h \,\mathrm{d}z,$$
$$j'(\bar{v}_\varepsilon^q)(h) = \frac{1}{q}\int_\Omega \frac{-\Delta_p^a \bar{v} - \Delta_q \bar{v}}{\bar{v}_\varepsilon^{q-1}} h \,\mathrm{d}z = \frac{c_0}{q}\int_\Omega \frac{\bar{v}^{\tau-1}}{\bar{v}_\varepsilon^{q-1}} h \,\mathrm{d}z.$$

The convexity of the integral functional $j(\cdot)$ implies the monotonicity of the directional derivative $j'(\cdot)$. So, we have

$$0 \leqslant \frac{c_0}{q}\int_\Omega \left(\frac{\bar{u}^{\tau-1}}{\bar{u}_\varepsilon^{q-1}} - \frac{\bar{v}^{\tau-1}}{\bar{v}_\varepsilon^{q-1}}\right)\left(\bar{u}_\varepsilon^q - \bar{v}_\varepsilon^q\right) \mathrm{d}z.$$

For $\varepsilon \in (0,1]$, note that

$$\left(\frac{\bar{u}^{\tau-1}}{\bar{u}_\varepsilon^{q-1}} - \frac{\bar{v}^{\tau-1}}{\bar{v}_\varepsilon^{q-1}}\right)\left(\bar{u}_\varepsilon^q - \bar{v}_\varepsilon^q\right) \leqslant \bar{u}_1^\tau + \bar{v}_1^\tau \in L^\infty(\Omega).$$

So, by Fatou's lemma, we have

$$\begin{aligned} 0 &\leqslant \frac{c_0}{q}\limsup_{\varepsilon\to 0^+}\int_\Omega \left(\frac{\bar{u}^{\tau-1}}{\bar{u}_\varepsilon^{q-1}} - \frac{\bar{v}^{\tau-1}}{\bar{v}_\varepsilon^{q-1}}\right)\left(\bar{u}_\varepsilon^q - \bar{v}_\varepsilon^q\right) \mathrm{d}z \\ &\leqslant \frac{c_0}{q}\int_n \left(\frac{1}{\bar{u}^{q-\tau}} - \frac{1}{\bar{v}^{q-\tau}}\right)\left(\bar{u}^q - \bar{v}^q\right) \mathrm{d}z \leqslant 0, \\ &\Rightarrow \bar{u} = \bar{v}. \end{aligned}$$

This proves the uniqueness of the positive solution of (Au). Since the equation is odd, $\bar{v} = -\bar{u} \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$, $\bar{v} \prec 0$ is the unique negative solution of (Au). $\square$

We introduce the following two sets

$$S_+ = \{\text{positive solutions of (1)}\},$$
$$S_- = \{\text{negative solutions of (1)}\}.$$

From Proposition 5 and its proof, we have

$$\phi \neq S_+ \subseteq W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega), 0 \prec u \quad \text{for all } u \in S_+,$$
$$\phi \neq S_- \subseteq W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega), v \prec 0 \quad \text{for all } u \in S_-.$$

The set S_+ is downward directed (that is, if $u_1 u_2 \in S_+$, then there is $u \in S_+$ such that $u \leq u_1, u \leq u_2$), while S_- is upward directed (that is, if $v_1, v_2 \in S_-$, then there is $v \in S_-$. such that $v_1 \leqslant v, v_2 \leqslant v$, see Filippakis–Papageorgiou [4]). We prove the existence of extremal elements in these two sets.

Proposition 7. *If hypotheses H_0, H_1 hold, then there exist $u_* \in S_+$ and $v_* \in S_-$ such that*

$$u_* \leqslant u \quad \textit{for all } u \in S_+,$$
$$v \leqslant v_* \quad \textit{for all } v \in S_-.$$

Proof. As we already mentioned, S_+ is downward directed. So, using Theorem 5.109 of Hu–Papageorgiou [11, p. 308], we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\langle V(u_n), h\rangle = \int_\Omega f(z,u_n)\,h\,\mathrm{d}z \quad \text{for all } h \in W_0^{1,\eta}(\Omega), \text{ all } n\in\mathbb{N}, \tag{19}$$

$$0 \le u_n \le u_1. \tag{20}$$

In (19) we use the test function $h = u_n \in W_0^{1,\eta}(\Omega)$. From (20) and hypothesis H_1 (i), we infer that

$$\begin{aligned} &\rho_\eta(Du_n) \leqslant c_6 \quad \text{for some } c_6>0, \text{ all } n\in\mathbb{N},\\ \Rightarrow\ &\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega) \text{ is bounded.}\end{aligned}$$

We may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W_0^{1,\eta}(\Omega),\ u_n \to u_* \text{ in } L^p(\Omega) \quad \text{as } n\to\infty \quad \text{(see Proposition 1).} \tag{21}$$

In (19) we use $h = u_n - u_* \in W_0^{1,\eta}(\Omega)$, pass to the limit as $n\to\infty$ and use (21). We obtain

$$\begin{aligned} &\lim_{n\to\infty} \langle V(u_n), u_n - u_*\rangle = 0,\\ \Rightarrow\ & u_n \to u_* \text{ in } W_0^{1,\eta}(\Omega) \text{ as } n\to\infty \quad \text{(see Proposition 3).}\end{aligned} \tag{22}$$

From hypothesis H_1 (i) and (20), we have

$$0 \leqslant f(z,u_n(z)) \leqslant \hat{a}(z)\left(1+u_1(z)^{p-1}\right) = \xi(z) \in L^\infty(\Omega) \quad \text{for a.a. } z\in\Omega, \text{ all } n\in\mathbb{N}$$

Hence via Moser's iteration process (see Guedda–Veron [9, Proposition 1.3]), we have

$$\|u_n\| \leqslant O(\|u_n\|). \tag{23}$$

So, if $u_* = 0$, from (22) and (23), we have

$$u_n \to 0 \text{ in } L^\infty(\Omega) \quad \text{as } n\to\infty.$$

Therefore, we can find $n_0\in\mathbb{N}$ such that

$$0 \leqslant u_n(z) \leqslant \delta \quad \text{for a.a. } z\in\Omega, \text{ all } n \geqslant n_0 \tag{24}$$

(here $\delta>0$ is as in hypothesis H_1 (iv)). We fix $n\geqslant n_0$ and introduce the Carathéodory function $k_+(z,x)$ defined by

$$k_+(z,x) = \begin{cases} c_0(x^+)^{\tau-1} & \text{if } x \leqslant u_n(z),\\ c_0 u_n(z)^{\tau-1} & \text{if } u_n(z) < x.\end{cases} \tag{25}$$

We set $K_+(z,x) = \int_0^x k_+(z,s)\,\mathrm{d}s$ and consider the C^1-functional $\psi_+\colon W_0^{1,\eta}(\Omega)\mapsto\mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega K_+(z,u)\,\mathrm{d}z \quad \text{for all } u\in W_0^{1,\eta}(\Omega).$$

It is clear from (25) that $\psi_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $\tilde{u}\in W_0^{1,\eta}(\Omega)$ such that

$$\psi_+(\tilde{u}) = \inf\{\psi_+(u)\colon u\in W_0^{1,\eta}(\Omega)\}. \tag{26}$$

Let $u \in C_0^1(\bar{\Omega})$ with $u(z) > 0$ for all $z \in \Omega$. For $t > 0$, we have

$$\begin{aligned}
\psi_+(tu) &= \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \int_\Omega K_+(z,tu)\,\mathrm{d}z \\
&= \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \int_{\{tu\leqslant u_n\}} \frac{c_0 t^\tau}{\tau} u^\tau\,\mathrm{d}z \\
&\quad - \int_{\{u_n<tu\}} \left(\frac{c_0 u_n^\tau}{\tau} + c_0 u_n^{\tau-1}(tu-u_n)\right)\mathrm{d}z \\
&\leqslant \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \frac{c_0 t^\tau}{\tau}\int_{\{tu\leqslant u_n\}} u^\tau\,\mathrm{d}z \\
&= \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \frac{c_0 t^\tau}{\tau}\int_\Omega u^\tau\,\mathrm{d}z + \frac{c_0 t^\tau}{\tau}\int_{\{u_n<tu\}} u^\tau\,\mathrm{d}z
\end{aligned}$$

Note that if $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$, then since $0 \prec u_n$, we have $|\{u_n < tu\}|_N \to 0$ as $t \to 0^+$. Then

$$\begin{aligned}
&\frac{\psi_+(tu)}{t^\tau} \leqslant \frac{t^{p-\tau}}{\tau}\rho_{\eta_0}(Du) + \frac{t^{q-\tau}}{q}\|Du\|_q^q - \frac{c_0}{\tau}\int_\Omega u^\tau\,\mathrm{d}z + \frac{c_0}{\tau}\int_{\{u_n<tu\}} u^\tau\,\mathrm{d}z \\
&\Rightarrow \limsup_{t\to 0^+} \frac{\psi_+(tu)}{t^\tau} < 0, \\
&\Rightarrow \psi_+(tu) < 0 \quad \text{for } t \in (0,1) \quad \text{small}, \\
&\Rightarrow \psi_+(\tilde{u}) < 0 = \psi_+(0) \quad \text{(see (26))} \\
&\Rightarrow \tilde{u} \neq 0.
\end{aligned}$$

From (26) we have

$$\begin{aligned}
&\left\langle \psi_+'(\tilde{u}), h\right\rangle = 0 \quad \text{for all } h \in W_0^{1,\eta}(\Omega), \\
&\Rightarrow \langle V(\tilde{u}), h\rangle = \int_\Omega k_+(z,\tilde{u})h\,\mathrm{d}z \quad \text{for all } h \in W_0^{1,\eta}(\Omega).
\end{aligned} \tag{27}$$

In (27) first we choose the test function $h = -\tilde{u}^- \in W_0^{1,\eta}(\Omega)$ and obtain

$$\begin{aligned}
&\rho_\eta\left(D\tilde{u}^-\right) = 0, \\
&\Rightarrow \tilde{u} \geqslant 0.
\end{aligned}$$

Also, in (27), we choose $h = (\tilde{u} - u_n)^+ \in W_0^{1,\eta}(\Omega)$. Then

$$\begin{aligned}
&\left\langle V(\tilde{u}), (\tilde{u}-u_n)^+\right\rangle \\
&= \int_\Omega c_0 u_n^{\tau-1}(\tilde{u}-u_n)^+\,\mathrm{d}z \quad \text{(see (25))} \\
&\leqslant \int_\Omega f(z,u_n(z))(\tilde{u}-u_n)^+\,\mathrm{d}z \quad \text{(see (24) and hypothesis } H_1 \text{ (iv))} \\
&= \langle V(u_n), (\tilde{u}-u_n)\rangle \quad (\text{since } u_n \in S_+) \\
\Rightarrow \quad &\tilde{u} \leqslant u_n \quad \text{(see Proposition 3)}.
\end{aligned}$$

So, we have proved that

$$\tilde{u} \in [0, u_n], \quad \tilde{u} \neq 0. \tag{28}$$

From (28), (25) and (27), we infer that if is a positive solution of (Au). Using Preposition 6, we infer that

$$\begin{aligned} \tilde{u} &= \bar{u}, \\ \Rightarrow \bar{u} &\leqslant u_n \quad \text{for all } n \geqslant n_0, \end{aligned}$$

which contradicts the fact that $u_n \to 0$ in $W_0^{1,\eta}(\Omega)$ as $n \to \infty$ (see (22) and recall that we have assumed $u_* = 0$). Therefore $u_* \neq 0$ and from (19) and (22), in the limit as $n \to \infty$, we obtain

$$\begin{aligned} \langle V(u_*), h\rangle &= \int_\Omega f(z, u_*)\, h \, \mathrm{d}z \quad \text{for all } h \in W_0^{1,\eta}(\Omega), \\ u &\leqslant u_*. \end{aligned}$$

Therefore $u_* \in S_+, u_* = \inf S_+$.

Similarly, using the fact that S_- is upward directed and $\bar{v} = -\bar{u}$, we produce $v_* \in S_-$ such that $v_* = \sup S_-$. □

Consider the order interval

$$[v_*, u_*] = \{u \in W_0^{1,\eta}(\Omega)\colon v_*(z) \leqslant u(z) \leqslant u_*(z) \text{ for a.a. } z \in \Omega\}.$$

If we can find a nontrivial solution of (1) in this order interval which is distinct from v_*, u_*, then such a solution is necessarily nodal. So, our goal is to produce such a solution. This is done in the next section.

4. Nodal solution

In this section we produce a nodal solution following the strategy outlined at the end of the previous section. To focus on the order interval $[v_*, u_*]$, we introduce the following truncation of the reaction $f(z,\cdot)$

$$l(z,x) = \begin{cases} f(z, v_*(z)) & \text{if } x < v_*(z), \\ f(z,x) & \text{if } v_*(z) \leqslant x \leqslant u_*(z), \\ f(z, u_*(z)) & \text{if } u_*(z) < x. \end{cases} \tag{29}$$

This is a Carathéodory function. We also introduce the positive and negative truncations of $f(z,\cdot)$, namely the Carathéodory functions

$$l_\pm(z,x) = l(z, \pm x^\pm). \tag{30}$$

We set

$$L(z,x) = \int_0^x l(z,s)\,\mathrm{d}s, \quad L_\pm(z,x) = \int_0^x l_\pm(z,s)\,\mathrm{d}s$$

and consider the C^1-functionals $\gamma, \gamma_\pm \colon W_0^{1,\eta}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \gamma(u) &= \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega L(z,u)\,\mathrm{d}z, \\ \gamma_\pm(u) &= \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega L_\pm(z,u)\,\mathrm{d}z \end{aligned}$$

for all $u \in W_0^{1,\eta}(\Omega)$.

From (29),(30) and the extremality of u_* and v_*, we have:

Proposition 8. *If hypotheses H_0, H_1 hold, then*

$$K_\gamma \subseteq [v_*, u_*], \quad K_{\gamma_+} = \{0, u_*\}, \quad K_{\gamma_-} = \{0, v_*\}.$$

Also hypothesis H_1 (iv) and Proposition 3.7 of Papageorgiou–Rădulescu [18], imply the following result.

Proposition 9. *If hypotheses H_0, H_1 hold, then*

$$C_k(\varphi, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Using this proposition and the C^1-continuity property of critical groups (see Theorem 5.126 of Gasinski–Papageorgiou [6, p. 836]), we can compute the critical groups $C_k(\gamma, 0)$ for all $k \in \mathbb{N}_0$.

Proposition 10. *If hypotheses H_0, H_1 hold, then*

$$C_k(\gamma, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Proof. For all $u \in W_0^{1,\eta}(\Omega)$, we have

$$\begin{aligned}
|\gamma(u) - \varphi(u)| &\leqslant \int_\Omega |L(z,u) - F(z,u)| \,\mathrm{d}z \\
&= \int_{\{u \leqslant v_*\}} |F(z, v_*) + (u - v_*) f(z, v_*) - F(z,u)| \,\mathrm{d}z \\
&\quad + \int_{\{u_* \leqslant u\}} |F(z, u_*) + (u - u_*) f(z, u_*) - F(z,u)| \,\mathrm{d}z \quad \text{(see (29))} \\
&\leqslant \int_{\{u < v_*\}} |F(z,u) - F(z, v_*)| \,\mathrm{d}z + \int_{\{u < v_*\}} |u - v_*| \, |f(z, v_*)| \,\mathrm{d}z \\
&\quad + \int_{\{u_* < u\}} |F(z,u) - F(z, u_*)| \,\mathrm{d}z + \int_{\{u_* < u\}} (u - u_*) \, f(z, u_*) \,\mathrm{d}z \\
&\leqslant c_7 \left[\int_{\{u < v_*\}} |u| \,\mathrm{d}z + \int_{\{u_* < u\}} u \,\mathrm{d}z \right] \quad \text{for some } c_7 > 0 \\
&\leqslant c_8 \|u\| \quad \text{for some } c_8 > 0.
\end{aligned} \tag{31}$$

Also for all $u, h \in W_0^{1,\eta}(\Omega)$, we have

$$\begin{aligned}
&|\langle \gamma'(u) - \varphi'(u), h \rangle| \\
&\leqslant \int_{\{u < v_*\}} |\gamma(z,u) - f(z,u)| \, |h| \,\mathrm{d}z + \int_{\{u_* < u\}} |f(z,u) - \gamma(z,u)| \, |h| \,\mathrm{d}z \\
&\leqslant \int_{\{u < v_*\}} |f(z, v_*) - f(z,u)| \, |h| \,\mathrm{d}z + \int_{\{u_* < u\}} |f(z,u) - f(z, u_*)| \, |h| \,\mathrm{d}z \\
&\leqslant c_9 \left[\int_{\{u < v_*\}} |u - v_*| \, |h| \,\mathrm{d}z + \int_{\{u_* < u\}} |u - u_*| \, |h| \,\mathrm{d}z \right] \quad \text{for some } c_9 > 0 \\
&\leqslant c_{10} \|u\| \|h\| \quad \text{for some } c_{10} > 0,
\end{aligned} \tag{32}$$

$$\Rightarrow \quad \|\gamma'(u) - \varphi'(u)\|_* \leqslant c_{10} \|u\|.$$

From (31) and (32), we see that given $\varepsilon > 0$, we can find $\hat{\delta} \in (0,1)$ such that

$$\|\gamma - \varphi\|_{C^1(\bar{B}_{\hat{\delta}})} \leq \varepsilon \quad \text{with } \bar{B}_{\hat{\delta}} = \left\{ u \in W_0^{1,\eta}(\Omega) \colon \|u\| \leqslant \hat{\delta} \right\}. \tag{33}$$

From Proposition 4 we know that $\varphi(\cdot)$ is coercive.

Also from (29) it is clear that $\gamma(\cdot)$ is coercive.

Therefore by [19, Proposition 5.1.15, p. 369], both functionals φ and γ satisfy the C-condition. Then the C^1-continuity property of critical groups (see Theorem 5.126

of Gasinski–Papageorgiou [6, p. 836]) implies that

$$\begin{aligned}&C_k(\gamma,0)=C_k(\varphi,0)\text{ for all }k\in\mathbb{N}_0,\\ \Rightarrow\ &C_k(\gamma,0)=0\ \text{ for all }k\in\mathbb{N}_0\text{ (see Preposition 9).}\end{aligned}$$

The proof is now complete. □

We know that $u_*,v_*\in K_\gamma$ (see (29)) and we assume that K_γ is finite or otherwise on account of Proposition 8, we already have a whole sequence of distinct bounded nodal solutions and so we are done.

Proposition 11. *If hypotheses H_0,H_1 hold, then $C_k(\gamma,u_*)=C_k(\gamma_+,u_*)$ and $C_k(\gamma,v_*)\ =C_k(\gamma_-,v_*)$ for all $k\in\mathbb{N}_0$.*

Proof. From (29) and (30) we see that $L(z,u_*)=L_+(z,u_*)$ for every $u\in W_0^{1,\eta}(\Omega)$ we have

$$\begin{aligned}|\gamma(u)-\gamma_+(u)|&\leqslant\int_\Omega|L(z,u)-L_+(z,u)|\,\mathrm{d}z\\ &\leqslant\int_\Omega|L(z,u)-L(z,u_*)|\,\mathrm{d}z+\int_\Omega|L_+(z,u_*)-L_+(z,u)|\,\mathrm{d}z.\end{aligned}\tag{34}$$

We estimate the two integrals in the right-hand side of (34). We have

$$\begin{aligned}&\int_\Omega|L(z,u)-L\left(z,u_*\right)|\,\mathrm{d}z\\ &=\int_{\{u<v_*\}}|F(z,v_*)+(u-v_*)f(z,v_*)-F(z,u_*)|\,\mathrm{d}z\\ &\quad+\int_{\{v_*\leqslant u\leqslant u_*\}}|F(z,u)-F(z,u_*)|\,\mathrm{d}z\\ &\quad+\int_{\{u_*<u\}}(u-u_*)\,f(z,u_*)\,\mathrm{d}z\quad\text{(see (29)).}\end{aligned}\tag{35}$$

By I_1, we denote the first integral in the right-hand side of (35). Then

$$\begin{aligned}I_1&\leqslant\int_{\{u<v_*\}}|F(z,v_*)-F(z,u_*)|\,\mathrm{d}z+\int_{\{u<v_*\}}(v_*-u)\,|f(z,v_*)|\,\mathrm{d}z\\ &\leqslant\int_{\{u<v_*\}}g_K(z)\,(u_*-v_*)\,\mathrm{d}z+\int_{\{u<v_*\}}(u_*-u)\,|f(z,v_*)|\,\mathrm{d}z\\ &\quad\text{with }K=[-\rho,\rho],\rho=\max\{\|u_*\|,\|v_*\|\}\\ &\leqslant\int_{\{u<v_*\}}g_K(z)\,(u_*-u)\,\mathrm{d}z+\int_{\{u<v_*\}}(u_*-u)\,|f\,(z,v_*)|\,\mathrm{d}z\\ &\leqslant c_{11}\|u-u_*\|\quad\text{for some }c_{11}>0.\end{aligned}\tag{36}$$

By I_2, we denote the second integral in the right-hand side of (35).
Evidently, $F(z,\cdot)$ is L^∞-locally Lipschitz and so

$$I_2\leqslant c_{12}\|u-u_*\|\quad\text{for some }c_{12}>0.\tag{37}$$

Finally, let I_3 denote the integral in the right hand side of (35). Since $f(\cdot,u_*(\cdot))\in L^\infty(\Omega)$ (see hypothesis H_1 (i)), we have

$$I_3\leqslant c_{13}\|u-u_*\|\quad\text{for some }c_{13}>0.\tag{38}$$

Using (36), (37) and (38), we have

$$I_1 + I_2 + I_3 \leqslant c_{14} \|u - u_*\| \quad \text{for some } c_{14} > 0.$$

So, given $\varepsilon > 0$, we can find $\rho_0 > 0$ such that for all $u \in \bar{B}_\rho(u_*)$; $\rho \in (0, \rho_0]$ we have

$$\int_\Omega |L(z,u) - L(z,u_*)| \, \mathrm{d}z \leqslant \frac{\varepsilon}{4} \quad \text{(see (35))}. \tag{39}$$

Next, we estimate the second integral in the right hand side of (34). We have

$$\begin{aligned}
&\int_\Omega |L_+(z,u_*) - L_+(z,u)| \, \mathrm{d}z \\
&= \int_{\{u<0\}} L_+(z,u_*) \, \mathrm{d}z + \int_{\{0\leqslant u \leqslant u_*\}} |F(z,u) - F(z,u_*)| \, \mathrm{d}z \\
&\quad + \int_{\{u_*<u\}} (u - u_*) f(z,u_*) \, \mathrm{d}z \quad \text{see (29), (30)} \\
&\leqslant \int_{\{u<0\}} F(z,u_*) \, \mathrm{d}z + c_{15} \|u - u_*\| \quad \text{for some } c_{15} > 0.
\end{aligned}$$

For $u \in \bar{B}_\rho(u_*)$, we have $|\{u<0\}|_N \to 0$ as $\rho \to 0^+$ (recall that $0 \prec u_*$). Therefore for $\rho \in (0,1)$ small, we have

$$\int_\Omega |L_+(z,u) - L_+(z,u_*)| \, \mathrm{d}z \leqslant \frac{\varepsilon}{4} \quad \text{for all } u \in \bar{B}_\rho(u_*). \tag{40}$$

We return to (34) and use (39), (40). We obtain that for $\rho \in (0,1)$ small

$$|\gamma(u) - \gamma_+(u)| \leqslant \frac{\varepsilon}{2} \quad \text{for all } u \in \bar{B}_\rho(u_*). \tag{41}$$

We estimate the corresponding derivatives. So, for all $u, h \in W_0^{1,\eta}(\Omega)$, we have

$$\begin{aligned}
&\left|\langle \gamma'(u) - \gamma'_+(u), h\rangle\right| \leqslant \int_\Omega |l(z,u) - l_+(z,u)| \, |h| \, \mathrm{d}z \\
&\leqslant \int_\Omega |l(z,u) - l(z,u_*)| \, |h| \, \mathrm{d}z + \int_\Omega |l_+(z,u_*) - l_+(z,u)| \, |h| \, \mathrm{d}z,
\end{aligned} \tag{42}$$

since $l(z,u_*) = l_+(z,u_*)$, see (28),(29).

We have

$$\begin{aligned}
&\int_\Omega |l(z,u) - l(z,u_*)| \, |h| \, \mathrm{d}z \\
&= \int_{\{u<v_*\}} |f(z,v_*) - f(z,u_*)| \, |h| \, \mathrm{d}z + \int_{\{v_*<u\leqslant u_*\}} |f(z,u) - f(z,u_*)| \, |h| \, \mathrm{d}z \\
&\leqslant \int_\Omega c_{16} |u - u_*| \, |h| \, \mathrm{d}z \quad \text{for some } c_{16} > 0
\end{aligned} \tag{43}$$

(note that $0 \leqslant u_* - v_* \leqslant u_* - u$ on $\{u < v_*\}$).

From hypotheses H_0, we have

$$2 \leqslant p < q^*$$

and so $(q^*)' < 2 < q^*$ (recall that if $s \in (1,2)$, then $s' \in (2,\infty)$ satisfies $\frac{1}{s} + \frac{1}{s'} = 1$). From Proposition 1, we have

$$u - u_* \in L^{(q^*)'}(\Omega), \quad h \in L^{q^*}(\Omega).$$

So, from (43) and Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega |l(z,u)-l(z,u_*)|\,|h|\,\mathrm{d}z &\leqslant c_{17}\,\|u-u_*\|_{(q^*)'}\,\|h\|_{q^*} \quad \text{for some } c_{17}>0\\ &\leqslant c_{18}\,\|u-u_*\|\,\|h\| \quad \text{for some } c_{18}>0\end{aligned}$$

(see Proposition 1).

Therefore given $\varepsilon>0$, for $\rho>0$ small we have

$$\int_\Omega |l(z,u)-l(z,u_*)|\,|h|\,\mathrm{d}z \leqslant \frac{\varepsilon}{4}\|h\| \quad \text{for all } u\in \bar{B}_\rho\,(u_*)\,. \tag{44}$$

Also we have

$$\begin{aligned}&\int_\Omega |l_+(z,u_*)-l_+(z,u)|\,|h|\,\mathrm{d}z\\ &=\int_{\{u<v_*\}} f(z,u_*)|h|\,\mathrm{d}z+\int_{\{v_*\leqslant u\leqslant u_*\}} |f(z,u_*)-f(z,u)|\,|h|\,\mathrm{d}z\\ &\leqslant c_{19}\left[\int_{\{u<v_*\}}|h|\,\mathrm{d}z+\int_{\{v_*\leqslant u\leqslant u_*\}}|u-u_*|\,|h|\,\mathrm{d}z\right] \quad \text{for some } c_{19}>0\\ &\leqslant c_{20}\left[|\{u<v_*\}|_N+\|u-u_*\|_{(q^*)'}\right]\|h\|_{q^*} \quad \text{for some } c_{20}>0\\ &\qquad \text{(as before using Hölder's inequality)}\\ &\leqslant c_{21}\big[\,|\{u<v_*\}|_N+\|u-u_*\|\,\big]\,\|h\| \quad \text{for some } c_{21}>0.\end{aligned}$$

If $u\in\bar{B}_\rho\,(u_*)$, then $|\{u<v_*\}|_N\to 0$ as $\rho\to 0^+$. Therefore, for $\rho\in(0,1)$ small, we have

$$\int_\Omega |l_+(z,u)-l_+(z,u_*)|\,|h|\,\mathrm{d}z \leqslant \frac{\varepsilon}{4}\|h\|. \tag{45}$$

We return to (42) and use (44), (45) and obtain

$$\begin{aligned}&\big|\big\langle \gamma'(u)-\gamma_+'(u),h\big\rangle\big| \leqslant \frac{\varepsilon}{2}\|h\|,\\ \Rightarrow\ &\big\|\gamma'(u)-\gamma_+'(u)\big\|_* \leqslant \frac{\varepsilon}{2} \quad \text{for all } u\in\bar{B}_\rho\,(u_*)\,.\end{aligned} \tag{46}$$

From (41) and (46) it follows that for $\rho\in(0,1)$ small

$$\|\gamma-\gamma_+\|_{C^1(\bar{B}_\rho(u_k))}\leqslant\varepsilon.$$

The functionals γ,γ_+ are coercive and so they satisfy the C-condition. Using the C^1-continuity property of critical groups, we have

$$C_k\,(\gamma,u_*)=C_k\,(\gamma_+,u_*) \quad \text{for all } k\in\mathbb{N}_0.$$

Similarly we show that

$$C_k\,(\gamma,v_*)=C_k\,(\gamma,v_*) \quad \text{for all } k\in\mathbb{N}_0.$$

The proof is now complete. □

Now we are ready to produce a nodal solution.

Proposition 12. *If hypotheses H_0,H_1 hold, then problem* (1) *has a nodal solution*

$$\hat{y}\in[v_*,u_*]\,.$$

Proof. We know that $\gamma_+(\cdot)$ is coercive (see (29), (30)). Also it is sequentially weakly lower semicontinuous.

So, by the Weierstrass–Tonelli theorem we can find $\tilde{u}_* \in W_0^{1,\eta}(\Omega)$ such that

$$\gamma_+(\tilde{u}_*) = \inf\left\{\gamma_+(u)\colon u \in W_0^{1,\eta}(\Omega)\right\} \tag{47}$$

Let $u \in C_0^1(\bar{\Omega})\backslash\{0\}$, $u(z) \geqslant 0$ for all $z \in \bar{\Omega}$. For $t \in (0,1)$ small we have $0 \leqslant tu(z) \leqslant \delta$ for all $z \in \bar{\Omega}$ (with $\delta > 0$ as postulated by hypothesis H_1 (iv)). We have

$$\begin{aligned}
\gamma_+(tu) &= \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \int_\Omega L_+(z,tu)\,\mathrm{d}z \\
&= \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \int_{\{tu\leqslant u_*\}} F(z,tu)\,\mathrm{d}z - \int_{\{u_*<tu\}} L_+(z,tu)\,\mathrm{d}z \\
&\leqslant \frac{t^q}{q}\rho_\eta(Du) - \frac{c_0 t^\tau}{\tau}\int_{\{tu\leqslant u_*\}} u^\tau\,\mathrm{d}z \\
&\quad \text{(since } L_+ \geqslant 0 \text{ and using hypothesis } H_1 \text{ (iv))} \\
&= \frac{t^q}{q}\rho_\eta(Du) - \frac{c_0 t^\tau}{\tau}\|u\|_\tau^\tau + \frac{c_0 t^\tau}{\tau}\int_{\{u_*<tu\}} u^\tau\,\mathrm{d}z.
\end{aligned}$$

Since $|\{u_* < tu\}|_N \to 0$ as $t \to 0^+$ (recall that $0 \prec u_*$) and $1 < \tau < q$, we see that for $t \in (0,1)$ small

$$\begin{aligned}
&\gamma_+(tu) < 0, \\
\Rightarrow\ &\gamma_+(\tilde{u}_*) < 0 = \gamma_+(0) \quad \text{(see (47))}, \\
\Rightarrow\ &\tilde{u}_* \neq 0.
\end{aligned} \tag{48}$$

From (47) and Proposition 8, we have

$$\begin{aligned}
&\tilde{u}_* \in \{0, u_*\}, \\
\Rightarrow\ &\tilde{u}_* = u_* \quad \text{(see (42))}, \\
\Rightarrow\ &C_k(\gamma_+, u_*) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \\
\Rightarrow\ &C_k(\gamma, u_*) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see Proposition 11)}.
\end{aligned} \tag{49}$$

Similarly working with $\gamma_-(\cdot)$, we show that

$$C_k\left(\gamma, v_*\right) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{50}$$

From Proposition 10, we have

$$C_k(\gamma, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{51}$$

The functional $\gamma(\cdot)$ is coercive. So, [19, Proposition 6.2.24] implies that

$$C_k(\gamma, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{52}$$

Suppose that $K_\gamma = \{0, u_*, v_*\}$. From (49), (50), (51), (52) and using the Morse relation with $t = -1$ (see (5)), we have

$$2(-1)^0 = (-1)^0,$$

a contradiction. So, there exists

$$\begin{aligned}
&\hat{y} \in K_\gamma \backslash \left\{0, u_*, v_*\right\}, \\
\Rightarrow\ &\hat{y} \in [v_*, u_*] \quad \text{(see Pronosition 8)}
\end{aligned}$$

and so $\hat{y}$ is a nodal solution of (1). ∎

Therefore we can state the following multiplicity theorem for problem (1). We produce three nontrivial bounded solutions, all with sign information and ordered.

Theorem 4.1. *If hypotheses H_0, H_1 hold, then problem* (1) *has at least three nontrivial solutions*

$$u_0 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega),\ 0 \prec u_0,$$
$$v_0 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega),\ v_0 \prec 0,$$
$$y_0 \in [v_0, u_0]\ \textit{nodal}.$$

Remark 4.1. Our multiplicity result here extends the corresponding results in [17, 20], where the authors produce two solutions with no sign information.

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