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Compactly supported solutions of Schrödinger equations with small perturbation

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We establish the existence of entire compactly supported solutions for a class of Schrödinger equations with competing terms and indefinite potentials. The analysis developed in this paper corresponds to the case of small perturbations of the reaction term.

We study the existence of compactly supported solutions for the following Schrödinger equation

$$-\Delta u+V(x)u=a(x)|u|^{q-1}u+\lambda b(x)g(u),\quad x\in\mathbb{R}^N, \tag{0.1}$$

where $N\geq 3, \lambda>0$, $0<q<1$ and a, b, V are indefinite potentials.

Let S denote the best Sobolev constant, namely $S\|u\|_{2^*}^2\leq\|\nabla u\|_2^2$ for all $u\in H^1(\mathbb{R}^N)$.

We assume that the following hypotheses are fulfilled.

(A) $a\in L^\infty(\mathbb{R}^N)$, $\Omega^+=\{x\in R^N,\ \ a(x)>0\}\neq\emptyset$, $\lim_{|x|\to+\infty}a(x)<0$, and there exist positive numbers R_1 and α such that $a^-(x)\geq\alpha$ for all $|x|\geq R_1$;

(B) $b\in C_c(\mathbb{R}^N,\mathbb{R}_+)$ and $\operatorname{supp}(b)\subset\Omega^+$;

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(G) $g : \mathbb{R} \to \mathbb{R}$ is a continuous and $g(x) \leq g(|x|)$ for all $x \in \mathbb{R}^N$;
(V) $V \in L^\infty(\mathbb{R}^N)$, $\lim_{|x|\to+\infty} V(x) > 0$, $V(x) \geq 0$ for all $x \in \overline{\Omega^+}^c$ and $\|V^-\|_{\frac{N}{2}} < S$.

The main result in this paper establishes that problem (0.1) has solutions with compact support, provided that a suitable perturbation of the second reaction terms is sufficiently small. This perturbation is described below in terms of the real parameter λ in relationship with the small values of the first reaction term with respect to a certain topology.

Theorem 0.1. *Assume that conditions (A), (B), (G) and (V) hold. Moreover, suppose that $N \geq 3$ and $0 < q < 1$. Then there exist positive numbers λ_0 and m such that if $|\lambda| < \lambda_0$ and $\|a^+ + \chi_{B(0,R_1)}\|_{\frac{2^*}{2^*-q-1}} < m$, then problem (0.1) has at least one nonnegative solution with compact support.*

We first study the following auxiliary equation:

$$-\Delta u + V(x)u = a(x)|u|^{q-1}u, \quad x \in \mathbb{R}^N. \tag{0.2}$$

Theorem 0.2. *Let (A) and (V) be satisfied. Assume that $N \geq 3$ and $0 < q < 1$. Then there exists $m > 0$ such that problem (0.2) has at least one nonnegative solution with compact support, provided that $\|a^+ + \chi_{B(0,R_1)}\|_{\frac{2^*}{2^*-q-1}} < m$.*

Let $E := H^1\left(\mathbb{R}^N\right) \cap L^{q+1}\left(\mathbb{R}^N\right)$. Define the following energy functional on E:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx - \frac{1}{q+1}\int_{\mathbb{R}^N} a(x)|u|^{q+1} dx. \tag{0.3}$$

Under assumptions (A) and (V), the functional I is well-defined, of class C^1 on E and any critical point of I is a weak solution of problem (0.2).

1. Proof of Theorems 0.1 and 0.2

1.1. Study of problem (0.2)

In our previous paper [1] we have proved, under the same assumptions of Theorem 0.2, that problem (0.2) has infinitely many solutions. However, we did not establish some qualitative properties of these solutions. This is the main purpose of this paper.

Lemma 1.1. *Let $d \in \mathbb{R}$ and $F \subset E$ be a closed subset. Then I satisfies the $(PS)_{F,d}$ Palais–Smale condition.*

Proof. The proof is identical to that of Lemma 4.3 of [1] and will be omitted. $\square$

Lemma 1.2. *There exists $m > 0$ such that if $\|a^+ + \chi_{B(0,R_1)}\|_{2^*/(2^*-q-1)} < m$, then problem (0.2) has at least one positive solution.*

Proof. We start by showing that there exists $\gamma > 0$ such that

$$I(u) \geq \gamma \ \text{ for all } \ u \in E \ \text{ and } \ \|u\| = 1. \tag{1.4}$$

Let $u \in E$. From conditions (A) and (V), Sobolev's and Young's inequalities, we have

$$I(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)u^2dx + \frac{\min(\alpha,1)}{q+1}\|u\|_{q+1}^{q+1} - \frac{1}{q+1}\int_{\mathbb{R}^N}(a^+ + \chi_{B(0,R_1)})(x)|u|^{q+1}dx \geq$$
$$\left(\frac{1}{2} - \frac{\|V^-\|_{\frac{N}{2}}}{2S}\right)\|\nabla u\|_2^2 + \frac{\min(\alpha,1)}{q+1}\|u\|_{q+1}^{q+1} - C_S\|a^+ + \chi_{B(0,R_1)}\|_{\frac{2^*}{2^*-q-1}}\|\nabla u\|_2^{q+1} \geq$$
$$\left(\frac{1}{4} - \frac{\|V^-\|_{\frac{N}{2}}}{4S}\right)\|\nabla u\|_2^2 + \frac{\min(\alpha,1)}{q+1}\|u\|_{q+1}^{q+1} - \frac{1-q}{2}C_S^{\frac{2}{1-q}}\left[\frac{2(q+1)}{1-\frac{\|V^-\|_{\frac{N}{2}}}{S}}\right]^{\frac{q+1}{1-q}}\|a^+ + \chi_{B(0,R_1)}\|_{\frac{2^*}{2^*-q-1}}^{\frac{2}{1-q}},$$

where C_S is a positive constant. Therefore, using the inequality $(x+y)^2/2 \leq x^2 + y^{q+1}$, for all $x \geq 0$, $0 \leq y \leq 1$, we obtain, for $\|u\| \leq 1$,

$$I(u) \geq c_0\|u\|^2 - \frac{1-q}{2}C_S^{\frac{2}{1-q}}\left[\frac{2(q+1)}{1-\frac{\|V^-\|_{\frac{N}{2}}}{S}}\right]^{\frac{q+1}{1-q}}\|a^+ + \chi_{B(0,R_1)}\|_{\frac{2^*}{2^*-q-1}}^{\frac{2}{1-q}}, \tag{1.5}$$

with $c_0 = \min\left\{(\frac{1}{8} - \frac{\|V^-\|_{\frac{N}{2}}}{8S}), \frac{\min(\alpha,1)}{2(q+1)}\right\}$. Then, by (1.5) and for

$$\|a^+ + \chi_{B(0,R_1)}\|_{\frac{2^*}{2^*-q-1}}^{\frac{2}{1-q}} \leq \frac{c_0}{(1-q)C_S^{\frac{2}{1-q}}\left(\frac{2(q+1)}{1-\frac{\|V^-\|_{N/2}}{S}}\right)^{\frac{q+1}{1-q}}},$$

we deduce that $I(u) \geq c_0/2 = \gamma$. This proves our claim (1.4).

Next, we consider the minimization problem

$$c = \inf_{u \in \overline{B}(0,1)} I(u). \tag{1.6}$$

It is clear that $-\infty < c < 0$. Then, by Lemma 1.1 and [2, Lemma 4], there exists $u_0 \in E$ such that u_0 is a solution of problem (0.2) and $c = I(u_0)$. More precisely, there exists a $(PS)_{\overline{B}(0,\rho),c}$ sequence $(u_n)_n \subset E$ such that $u_n \to u_0$ and $c = \lim_{n\to+\infty} I(u_n)$. Since $c \leq I(|u_n|) \leq I(u_n)$, then $(|u_n|)_n$ is also a minimizing sequence of problem (1.6). Consequently, we can take $u_0 \geq 0$ a.e. in $\mathbb{R}^N$. $\square$

Lemma 1.3. *Let $c(y) \in L^t(B(x,2))$ for some $t > \frac{N}{2}$ with $\|c(y)\|_{L^t(B(x,2))} \leq 1$. Furthermore, assume that hypothesis (B) holds and that $u \geq 0$ satisfies*

$$\int_{B(x,2)} \nabla u \nabla\varphi + cu\varphi \leq \int_{B(x,2)} b\varphi \ \text{ for all } \ \varphi \in H_0^1(B(x,2)) \ \text{ and } \ \varphi \geq 0 \ \text{ in } \ B(x,2).$$

Then $\|u\|_{L^\infty(B(x,1))} \leq C(\|u\|_{L^2(B(x,2))} + \|b\|_{L^t(B(x,2))})$, where $C = C(N,t)$ is a positive constant.

Proof. Invoking condition (B), we deduce that $b \in L^t(B(x,2))$. The rest of the proof is a simple application of Theorem 4.1 in [3, p. 67]. $\square$

Lemma 1.4. *Let (A) and (V) be satisfied. Then any nonnegative weak solution u of Eq. (0.2) is a classical solution and $\lim_{|x|\to+\infty} u(x) = 0$.*

Proof. The regularity of u follows by boostrap arguments; see [4, Appendix B]. Let $R \geq R_1$ such that $V(x) \geq 0$, for all $|x| \geq R$. For $x \in B^c(0, R+3)$, we have $B(x,1) \subset B(x,2) \subset B^c(0,R)$ and $-\Delta u(y) \leq 0$ for all $y \in B(x,2)$. Applying Lemma 1.3 with $c = b = 0$, we obtain $\|u\|_{L^\infty(B(x,1))} \leq c\|u\|_{L^2(B(x,2))}$, for some positive constant c. Hence, since $u \in L^2(\mathbb{R}^\mathbb{N})$, we have $\lim_{|x|\to+\infty} u(x) = 0$. $\square$

Lemma 1.5. *Every nonnegative classical solution of problem (0.2) is compactly supported.*

Proof. From conditions (A) and (V) and Lemma 1.4, there exist positive numbers R and a_0 such that for every $x \in B^c(0,R)$, we infer that

$$u(x) < A, \quad a^-(x) > a_0 \quad \text{and} \quad V(x) \geq 0, \tag{1.7}$$

with $A = \frac{1}{2}[\frac{a_0}{\frac{2}{1-q}(\frac{2}{1-q}+N-2)}]^{\frac{1}{1-q}}$. If we take, for any $y \in B^c(0, R+2)$,

$$W(x) = \left[\frac{a_0}{\frac{2}{1-q}(\frac{2}{1-q}+N-2)}\right]^{\frac{1}{1-q}} |x-y|^{\frac{2}{1-q}},$$

we deduce that

$$-\Delta W(x) = -a_0 W^q(x) \quad \text{in} \quad B(y,1). \tag{1.8}$$

In what follows, we show that $W(x) \geq u(x)$ in $B(y,1)$. Arguing by contradiction, we assume that there exists $x_0 \in B(y,1)$ such that $W(x_0) < u(x_0)$. By Lemma 1.4, we may assume that $W - u$ attains minimal value at x_0. Hence, using (1.7) and (1.8), we get

$$0 \geq -\Delta(W-u)(x_0) = (a^- u^q - a_0 W^q + V u^2)(x_0) \geq (a_0 u^q - a_0 W^q)(x_0) > 0,$$

contradiction. Thus, $0 = W(y) \geq u(y) \geq 0$ for all $y \in B^c(0, R+2)$. $\square$

Proof of Theorem 0.2 concluded. By Lemma 1.2, u_0 is a nonnegative solution of problem (0.2). Applying Lemma 1.5 with $u = u_0$, we conclude that u_0 is a nonegative classical compactly supported solution of problem (0.2). $\square$

1.2. Study of problem (0.1)

We first establish uniform estimates for solutions of problem (0.2). Next, by exploiting the previous section and some ideas coming from [5], we conclude the proof of our main result.

Lemma 1.6. *Let $u \in E$ be an arbitrary classical nonnegative solution of problem (0.2). Then $\|u\|_{L^\infty(\mathbb{R}^N)}$ is attained in $\overline{\Omega^+}$.*

Proof. By Lemma 1.4, we may assume that $\|u\|_{L^\infty(\mathbb{R}^N)}$ is attained at x_1, but is not in $\overline{\Omega^+}$. Let Ω be the connected component of $(\overline{\Omega^+})^c$, which contains x_1. By the strong maximum principle and conditions (A) and (V), we deduce that $u(x) = u(x_1)$ in $\overline{\Omega}$. Taking into account that $\overline{\Omega^+} \cap \overline{\Omega} \neq \emptyset$, we conclude the proof of Lemma 1.6. $\square$

Let O be an open bounded set in $\mathbb{R}^N$ such that $\Omega^+ \cup \{x \in \mathbb{R}^\mathbb{N},\ V(x) \leq 0\} \subset\subset O$.

Lemma 1.7. *For any positive integer $s \geq 2$, we have $\|u\|_{L^{(s+1)N/(N-2)}(O)} \leq C\|u\|_{2^*}$, where $u \in E$ is a nonnegative classical solution of Eq. (0.2) and $C = C(a,q,N,s,V)$.*

Proof. Fix $k \geq 2$. Multiplying Eq. (0.2) by $|u|^{k-1}u$, we obtain

$$k\int_{\mathbb{R}^N} |u|^{k-1}|\nabla u|^2 dx + \int_{\mathbb{R}^N} V(x)|u|^{k+1} dx = \int_{\mathbb{R}^N} a(x)|u|^{k+q} dx,$$

and thus

$$k\int_{\mathbb{R}^N} |u|^{k-1}|\nabla u|^2 dx \leq \int_{\mathbb{R}^N} V^-(x)|u|^{k+1} dx + \int_{\mathbb{R}^N} a^+(x)|u|^{k+q} dx.$$

By the Sobolev inequality, it follows that

$$\frac{4kC_S}{(k+1)^2}\|u^{\frac{k+1}{2}}\|^2_{L^{\frac{2N}{N-2}}(O)} \leq \int_O V^-(x)|u|^{k+1} dx + \int_O a^+(x)|u|^{k+q} dx. \tag{1.9}$$

Now, we estimate the integral on the second part of right-hand side of Eq. (1.9) as

$$\begin{aligned}\int_O a^+(x)|u|^{q+k} dx &= \int_{\{x\in O,\ \ |u(x)|<1\}} a^+(x)|u|^{q+k} dx + \int_{\{x\in O,\ \ |u(x)|\geq 1\}} a^+(x)|u|^{q+k} dx \\ &\leq \int_O a^+(x) dx + \int_O a^+(x)|u|^{k+1} dx.\end{aligned} \tag{1.10}$$

Therefore

$$\frac{4kC_S}{(k+1)^2}\|u\|^{k+1}_{L^{\frac{(k+1)N}{N-2}}(O)} \leq \|V^- + a^+\|_\infty \|u\|^{k+1}_{L^{k+1}(O)} + \|a^+\|_1. \tag{1.11}$$

Fix $s \geq 2$. In (1.11), we start with $k+1 = 2^* = \frac{2N}{N-2}$ and then we follow the sequence $(k+1)\frac{N}{N-2}$, $(k+1)(\frac{N}{N-2})^2$, $\ldots$, until we pass $s+1$. Since O is bounded, the proof is completed. $\square$

Lemma 1.8. *There exists $M > 0$ so that, for every classical nonnegative solution $u \in E$ of problem (0.2), we have $\|u\|_{L^\infty(\mathbb{R}^N)} \leq M$, with $M = M(N, q, \Omega^+, \|a^+\|_{L^\infty(\mathbb{R}^N)})$.*

Proof. Invoking conditions (A) and (V), we infer that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)u^2 dx \leq \int_{\mathbb{R}^N} a^+(x)|u|^{q+1} dx \leq \|a^+\|_{\frac{2^*}{2^*-q-1}} \|u\|^{q+1}_{2^*}.$$

By (V), it follows that $(S - \|V^-\|_{N/2})\|\nabla u\|_2^2 \leq 2\|a^+\|_{\frac{2^*}{2^*-q-1}}\|u\|^{q+1}_{2^*}$. Thus, by Sobolev's inequality and (V), there exists $C > 0$ such that $\|u\|_{2^*} \leq C\|a^+\|^{1/(1-q)}_{2^*/(2^*-q-1)}$. Choosing $s \geq 2$ so that $(s+1)\frac{N}{N-2} \geq \frac{N+1}{2}$, in view of Lemma 1.7, we know that $\|u\|_{L^{N(s+1)/(N-2)}(O)}$ is uniformly bounded. By elliptic estimates (see [6]), we obtain $\|u\|_{W^{2,\frac{N+1}{2}}(\Omega^+)} \leq C\left(\|\Delta u\|_{L^{\frac{N+1}{2}}(O)} + \|u\|_{L^{\frac{N+1}{2}}(O)}\right)$. Therefore, in light of Lemma 1.6, we have shown our desired result. $\square$

1.2.1. Proof of Theorem 0.1

Choose a smooth function $h : \mathbb{R}^N \to \mathbb{R}$ such that $0 \leq h \leq 1$ in $\mathbb{R}^N$, $h(x) = 1$ for $|x| \leq 2M$ and $h(x) = 0$ for $|x| \geq 4M$ (M is given in Lemma 1.8). Then the function

$$\overline{G}(t, u) := h(u(t))G(t, u(t)) = h(u(t))\int_0^{u(t)} b(s)g(s)ds$$

is of class C^1. Hence, by (B) and (G), $\overline{G}(t, u)$ and $\overline{G}_u(t, u)$ are bounded on $\mathbb{R} \times \mathbb{R}^N$.

Next, we define $J_\lambda : E \to \mathbb{R}$ by

$$J_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(x)u^2)dx - \frac{1}{q+1}\int_{\mathbb{R}^N} a(x)|u|^{q+1}dx - \lambda \int_{\mathbb{R}^N} h(u(x))G(x,u(x))dx.$$

A critical point of J_λ is a solution of the problem

$$-\Delta u + V(x)u = a(x)|u|^{q-1}u + \lambda h(u)G_u(x,u) + \lambda h'(u)G(x,u). \tag{1.12}$$

We say that u is an *Ekeland solution* of J_λ if $J'_\lambda(u) = 0$ and $J_\lambda(u) = c$, where c given in (1.6). We say that J_λ has an *Ekeland geometry* if there exist $R > 0$ and v with $\|v\| < R$ such that $J_\lambda(v) < \inf \|u\| = RJ_\lambda(u)$. We also observe that with the same arguments as in the proof of Lemma 1.1, we deduce that J_λ satisfies the Palais–Smale condition.

Lemma 1.9. *There exists $\lambda_0 > 0$ such that J_λ has an Ekeland geometry when $|\lambda| \le |\lambda_0|$.*

Proof. By the boundedness of $\overline{G}(x,u)$, we have $I(u) - C\lambda \le J_\lambda(u) \le I(u) + C\lambda$ for all $u \in E$, where $C > 0$ is independent of λ and u. Thus, for $|\lambda|$ small enough, it follows that $-\infty < c_\lambda = \inf_{u\in\overline{B}(0,1)} J_\lambda(u) < 0$ and $0 < \inf_{u\in\partial B(0,1)} I(u) + C\lambda < \inf_{u\in\partial B(0,1)} J_\lambda(u)$. □

Lemma 1.10. *Let $(\lambda_n)_n \subset \mathbb{R}$ be a sequence converging to zero and u_n be an Ekeland solution of J_{λ_n}. Then, up to a subsequence, (u_n) converges to an Ekeland solution $v \in E$ of I.*

Proof. We recall that $I(u) - C\lambda_n \le J_{\lambda_n}(u) \le I(u) + C\lambda_n$ for all $u \in E$, hence $\inf_{u\in\overline{B}(0,1)} I(u) - C\lambda_n \le \inf_{u\in\overline{B}(0,1)} J_{\lambda_n}(u) \le \inf_{u\in\overline{B}(0,1)} I(u) + C\lambda_n$ for all $u \in E$. Therefore $c_{\lambda_n} \to c$ as $n \to +\infty$. By (G) and since $\overline{G}(x,u)$ and $(\overline{G})_u(x,u)$ are bounded and $\lambda_n \to 0$, we deduce that (u_n) is a (PS) sequence of I. So, by Lemma 1.1, $u_n \to v$ in E. We conclude that $I(v) = c$ and $I'(v) = 0$. □

Lemma 1.11. *Let $(\lambda_n)_n \subset \mathbb{R}$ be a sequence converging to zero and u_n be a nonnegative Ekeland solution of J_{λ_n}. Then $\|u_n\|_\infty = \|u_n\|_{L^\infty(\overline{\Omega^+})}$ for all $n \in \mathbb{N}$. Moreover, up to a subsequence, u_n converges to a limit w in $L^\infty(\overline{\Omega}^+)$, where w is an Ekeland solution of I.*

Proof. For the first part, the proof is identical to that of Lemma 1.6.

For the second part, we apply Lemma 1.10. Thus, $u_n \to w$ in E, where w is an Ekeland solution of I. So, it is sufficient to prove that u_n is bounded in $W^{2,\frac{N+1}{2}}(O)$. We claim that for all $s \ge 2$, there exists $C_1 = C_1(O,s,q,N,b,g,a,V)$ such that $\|u_n\|_{L^{\frac{(s+1)N}{N-2}}(O)} \le C\|u_n\|_{2^*}$ for all $n \in \mathbb{N}$. Fix $k \ge 2$. Multiplying Eq. (1.12) by $|u_n|^{k-1}u_n$, we obtain

$$\begin{aligned} k\int_{\mathbb{R}^N} |u_n|^{k-1}|\nabla u_n|^2 dx + \int_{\mathbb{R}^N} V(x)|u_n|^{k+1}dx = \int_{\mathbb{R}^N} a(x)|u_n|^{k+q}dx \\ + \lambda_n\Big(\int_{\mathbb{R}^N} |u_n|^{k-1}u_n h(u_n)G_u(x,u_n)dx + \int_{\mathbb{R}^N} |u_n|^{k-1}u_n h'(u_n)G(x,u_n)dx\Big). \end{aligned}$$

So, for n large enough, we infer that

$$k\int_{\mathbb{R}^N} |u_n|^{k-1}|\nabla u_n|^2 dx \le \int_{\mathbb{R}^N} V^-(x)|u_n|^{k+1}dx + \int_{\mathbb{R}^N} a^+(x)|u_n|^{k+q}dx + C\int_{\mathbb{R}^N} b(x)|u_n|^{k-1}u_n dx.$$

Using the arguments as in Lemma 1.7 and the above estimation, we deduce our claim. By this claim and $\|u_n\|_\infty = \|u_n\|_{L^\infty(\overline{\Omega^+})}$, the rest of the proof is the same as for Lemma 1.8. □

Lemma 1.12. *There exists $\lambda_0 > 0$ such that any nonnegative Ekeland solution $v \in E$ of J_λ with $|\lambda| \leq \lambda_0$ satisfies $\|v\|_\infty \leq 2M$.*

Proof. By contradiction, there exist $\lambda_n \in \mathbb{R}$ and $u_n \in E$ such that $\lambda_n \to 0$, u_n is a nonnegative Ekeland solution of J_{λ_n} and $\|u_n\|_\infty > 2M$. By Lemmas 1.10 and 1.8, (u_n) converges to an Ekeland solution $w \in E$ of I with $\|w\|_\infty < M$. By Lemma 1.11, $\|u_n\|_\infty < 2M$ for n large, contradiction. □

Choose $\lambda_0 > 0$ that satisfies Lemmas 1.9 and 1.12. By Lemma 1.9, [2, Lemma 4] and since J_λ satisfies the Palais–Smale condition, there exists $u_\lambda \in E$ such that u_λ is a critical point of J_λ and $c_\lambda = J_\lambda(u_\lambda)$ with $|\lambda| < |\lambda_0|$. By Lemma 1.12, $\|u_\lambda\|_\infty < 2M$. Thus, $h'(u_\lambda) = 0$ and $h(u_\lambda) = 1$. By a standard bootstrap argument, u_λ is a classical nonnegative solution of (0.1). It remains to prove that u_λ is compactly supported. Indeed, by (B), (G) and (V), for every $0 < \lambda < \lambda_0$, we have $V(x) \geq 0$ and $b(x) = 0$ for all $|x| \geq R$, with R large enough. From now on the proof is identical to that of Lemma 1.5 and it will be omitted. The proof of Theorem 0.1 is now completed. □

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