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Let φ be a continuous positive function on the open interval (A,∞), and assume that f is a C^2-function on (A,∞) satisfying the differential equation

$$f''(t) = (1+\varphi(t)(f^2(t)-1))f(t).$$

(a) *Given that there exists $a \in (A,\infty)$ such that $f(a) \geq 1$ and $f'(a) \geq 0$, prove that there is a positive constant K such that $f(x) \geq Ke^x$ whenever $x \geq a$.*

(b) *Given instead that there exists $a \in (A,\infty)$ such that $f'(a) < 0$ and $f(x) > 1$ if $x > a$, prove that there exists a positive constant K such that $f(x) \geq Ke^x$ whenever $x \geq a$.*

(c) *Given that f is bounded on (A,∞) and that there exists $\alpha > 0$ such that $\varphi(x) = O(e^{-(1+\alpha)x})$ as $x \to \infty$, prove that $\lim_{x\to\infty} e^x f(x)$ exists and is finite.*

Solution. **(a)** We first claim that $f \geq 1$ and $f' \geq 0$ in $[a,\infty)$. Indeed, assuming the contrary, it follows that f has a local maximum point $x_0 \geq a$ such that $f(x_0) \geq 1$, $f'(x_0) = 0$, and $f''(x_0) \leq 0$. Using now the differential equation satisfied by f we get a contradiction.

In particular, the above claim shows that $f'' \geq f$ in $[a,\infty)$. Set $g := f' - f$. Then $g' + g \geq 0$ in $[a,\infty)$ and the function $h(x) := g(x)e^x$ satisfies $h' \geq 0$ in $[a,\infty)$. We deduce that for all $x \geq a$ we have $g(x) = f'(x) - f(x) \geq g(a)e^{a-x}$. Setting $v(x) := f(x)e^{-x}$ we obtain $v'(x) \geq g(a)e^{a-2x}$ on $[a,\infty)$. By integration on $[a,x]$ we find, for all $x \geq a$,

$$\begin{aligned} f(x) &\geq f(a)e^{x-a} - \frac{g(a)}{2}\,e^{a-x} + \frac{g(a)}{2}\,e^{x-a} \\ &= f(a)e^{x-a} + \frac{f(a)-f'(a)}{2}\,e^{a-x} + \frac{f'(a)-f(a)}{2}\,e^{x-a} \\ &= \frac{f(a)+f'(a)}{2}\,e^{x-a} + \frac{f(a)-f'(a)}{2}\,e^{a-x}. \end{aligned} \tag{1}$$

This shows that there exists a positive constant C such that $f(x) \geq Ce^x$, for any $x \geq a$.

(b) Our hypothesis implies $f'' > f$ in $[a,\infty)$. However, since $f(a) + f'(a)$ is not necessarily positive, estimate (1) does not conclude the proof, as above. For this purpose, using the fact that $f'' > 1$ in (a,∞), we find some $x_0 > a$ such that $f'(x_0) > 0$. Since f is positive in $[x_0,\infty)$, we can repeat the arguments provided in **(a)**, using x_0 instead of a in relation (1). Thus, we find $C > 0$ such that $f(x) \geq Ce^x$, for any $x \geq x_0$. Choosing eventually a smaller positive constant C, we deduce that the same conclusion holds in $[a,\infty)$.

(c) Make the change of variable $e^{-x} = t \in (0, e^{-A})$ and denote $g(t) = f(x)$. Then g satisfies the differential equation

$$g''(t) + \frac{g'(t)}{t} - \frac{g(t)}{t^2} = \frac{\varphi(-\ln t)}{t^2}\,g(t)\left(g^2(t)-1\right), \qquad \text{for all } t \in (0, e^{-A}). \tag{2}$$

We observe that the above equation is equivalent to the first order differential system

$$\begin{cases} g'(t) + \dfrac{g(t)}{t} = h(t), & t \in (0, e^{-A}) \\ h'(t) = \dfrac{\varphi(-\ln t)}{t^2}\, g(t)\left(g^2(t) - 1\right), & t \in (0, e^{-A}). \end{cases} \tag{3}$$

The growth assumption on φ can be written, equivalently, $\varphi(-\ln t) = O\left(t^{1+\alpha}\right)$ as $t \to 0$, where α is a positive number. This implies that the right-hand side member of the second differential equation in (3) is integrable around the origin and, moreover,

$$h(t) = O(1) \qquad \text{as } t \searrow 0. \tag{4}$$

On the other hand, since g is bounded around the origin, the first differential equation in (3) implies

$$g(t) = \frac{1}{t}\int_0^t rh(r)dr, \qquad \text{for all } 0 < t < e^{-A}. \tag{5}$$

Relations (4) and (5) imply that $g(t) = O(t)$ as $t \searrow 0$. Since $tg'(t) + g(t) = th(t)$ we deduce that

$$g'(t) = O(1) \qquad \text{as } t \searrow 0. \tag{6}$$

Let g and g_1 be two arbitrary solutions of (2). Then

$$\left\{t\left[g'(t)g_1(t) - g(t)g_1'(t)\right]\right\}' = \frac{\varphi(-\ln t)}{t}\, g(t)g_1(t)\left(g^2(t) - g_1^2(t)\right), \qquad t \in (0, e^{-A}). \tag{7}$$

Relation (6) and the growth assumption on φ imply that the right-hand side member of (7) is $O(t^{4+\alpha})$ as $t \searrow 0$. So, using again (7),

$$g'(t)g_1(t) - g(t)g_1'(t) = O(t^{4+\alpha}) \qquad \text{as } t \searrow 0. \tag{8}$$

Next, we observe that we can choose g_1 so that $g_1(t) \sim t$ as $t \searrow 0$. Indeed, this follows from the fact that the initial value problem

$$\begin{cases} g''(t) + \dfrac{g'(t)}{t} - \dfrac{g(t)}{t^2} = \dfrac{\varphi(-\ln t)}{t^2}\, g(t)\left(g^2(t) - 1\right) \\ g(0) = 0,\ g'(0) = 1 \end{cases}$$

has a solution defined on some interval $(0, \delta)$. Thus, by (8), $\lim_{t\searrow 0}\left(g(t)/g_1(t)\right)' = 0$. Hence, for any sequence $\{t_n\}_{n\geq 1}$ of positive numbers converging to 0, the sequence $\left\{\dfrac{g(t_n)}{g_1(t_n)}\right\}_{n\geq 1}$ is a Cauchy sequence. Consequently, there exists $\lim_{t\searrow 0} g(t)/g_1(t) = \ell \in \mathbb{R}$. Since $g_1(t) \sim t$ as $t \searrow 0$, we deduce that $\lim_{t\searrow 0} g(t)/t = \ell$ or, equivalently, $\lim_{x\to\infty} e^x f(x) = \ell$.

Remarks. (i) The conclusion stated in **(c)** does not remain true for general potentials φ (like in **(a)** or **(b)**). Indeed, the function $f(x) = x^{-1}$ satisfies the assumption $f(x) \in (-1, 1)$ for all $x \in (1, \infty)$ and is a solution of the differential equation $f'' = [1+\varphi(f^2-1)]f$, provided that $\varphi(x) = (x^2 - 2)(x^2 - 1)^{-1}$. In this case, $\lim_{x\to\infty} e^x f(x)$ exists but is not finite.

(ii) Under the growth assumption on φ imposed in **(c)**, our result shows that an arbitrary solution f of the differential equation $f'' = [1 + \varphi(f^2 - 1)]f$ in (A, ∞) satisfies the following alternative: either

• f is unbounded and, in this case, $f(x)$ tends to $+\infty$ as $x \to +\infty$ (at least like e^x, for general positive potentials φ)

or

• f is bounded and, in this case, $f(x)$ tends to 0 as $x \to +\infty$ (at least like e^{-x}).