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Let Ω be the set of all complex numbers z satisfying $0 < |z| < 1$. Fix a positive integer n and, for all distinct elements $z_1, \cdots, z_n$ in Ω, define the function

$$f(z_1, \cdots, z_n) = \prod_{j=1}^{n} |z_j|^2(1-|z_j|^2) \cdot \prod_{1\le j<k\le n} |z_j|\cdot|z_k|\cdot|z_j-z_k|^2 \cdot \prod_{1\le j<k\le n} |z_j|\cdot|z_k|\cdot\left[|z_j-z_k|^2 + (1-|z_j|^2)(1-|z_k|^2)\right] .$$

(a) *Prove that if $n = 2$ then the maximum of f is attained for a unique configuration (up to a rotation) which consists of two symmetric points with respect to the origin.*

(b) *Prove that if $n = 3$ then the maximal configuration for f is also unique (up to a rotation) and it consists of an equilateral triangle centered at the origin.*

(c)* *Establish similar properties of the maximal configuration for f, provided that $n \ge 4$. Study the asymptotic behavior of the maximal configuration as $n \to \infty$, in the following sense. For any integer $n \ge 4$, let $(z_1, \cdots, z_n) \in \Omega^n$ be an arbitrary configuration that realizes the maximum of f and denote $a_n = \max\{|z_k|;\ 1 \le k \le n\}$. Prove or disprove that $a_n \to 1$ as $n \to \infty$.*

Solution. **(a)** Let z_1, z_2 be two distinct points in Ω. Then

$$\begin{aligned}\log f(z_1,z_2) &= \log|z_1|^2(1-|z_1|^2) + \log|z_2|^2(1-|z_2|^2)\\ &+\log|z_1|\cdot|z_2|(|z_1|^2+|z_2|^2-2|z_1|\cdot|z_2|\cdot\cos\varphi)\\ &+\log|z_1|\cdot|z_2|(1+|z_1|^2|z_2|^2-2|z_1|\cdot|z_2|\cdot\cos\varphi)\,,\end{aligned}$$

where $\varphi \in [0, 2\pi)$ denotes the angle between the vectors $\overrightarrow{Oz_1}$ and $\overrightarrow{Oz_2}$. So, a necessary condition for the maximum of $f(z_1, z_2)$ is $\cos\varphi = -1$, that is, the points z_1, O and z_2 are collinear, with O between z_1 and z_2. Thus, we can suppose that the points z_1 and z_2 lie on the real axis and $-1 < z_2 < 0 < z_1 < 1$. Denote

$$g(z_1,z_2) = \log z_1^2(1-z_1^2) + \log z_2^2(1-z_2^2) + \log z_1(-z_2)(z_1-z_2)^2 + \log z_1(-z_2)(1-z_1z_2)^2 .$$

It remains to prove that the maximum of g is achieved if $z_2 = -z_1$. For this purpose we show that

$$g(z_1,z_2) \le g\left(\frac{z_1-z_2}{2}, \frac{z_2-z_1}{2}\right) . \tag{1}$$

Since the mapping $(0,1) \ni x \longmapsto \log x^2(1-x^2)$ is concave, it follows that

$$\log z_1^2(1-z_1^2) + \log z_2^2(1-z_2^2) \le 2\log\left(\frac{z_1-z_2}{2}\right)^2\left[1-\left(\frac{z_1-z_2}{2}\right)^2\right] . \tag{2}$$

On the other hand, it is obvious that

$$z_1(-z_2)(z_1-z_2)^2 \le \left(\frac{z_1-z_2}{2}\right)^2 (z_1-z_2)^2 \tag{3}$$

and

$$z_1(-z_2)(1-z_1z_2)^2 \le \left(\frac{z_1-z_2}{2}\right)^2 \left[1+\left(\frac{z_1-z_2}{2}\right)^2\right]. \tag{4}$$

Relations (2)–(4) imply (1). This shows that, in order to find the maximum of f, it is enough to maximize the function $\varphi(x) := e^{g(x,-x)} = 4x^{10}(1-x^4)^2$, $x \in (0,1)$. A straightforward computation shows that the maximum of φ is achieved for $x = 5^{1/4}\cdot 3^{-1/2}$, that is,

$$\max f(z_1,z_2) = f(5^{1/4}\cdot 3^{-1/2}, -5^{1/4}\cdot 3^{-1/2}) = 2^6\cdot 3^{-9}\cdot 5^{5/2}.$$

(b) Using the elementary identity

$$3\sum_{j=1}^{3}|z_j|^2 = \left|\sum_{j=1}^{3} z_j\right|^2 + \sum_{1\le j<k\le 3}|z_j-z_k|^2$$

we find

$$3\sum_{j=1}^{3}|z_j|^2 \ge \sum_{1\le j<k\le 3}|z_j-z_k|^2.$$

Put $S=\sum_{j=1}^{3}|z_j|^2$. We try to maximize f keeping S constant. Using the above inequality, we have

$$\begin{aligned}\prod_{1\le j<k\le 3}|z_j|\cdot|z_k|\cdot|z_j-z_k|^2 &\le \left(\frac{\sum_{j=1}^{3}|z_j|^2}{3}\right)^3\left(\frac{\sum_{1\le j<k\le 3}|z_j-z_k|^2}{3}\right)^3\\ &\le \left(\frac{\sum_{j=1}^{3}|z_j|^2}{3}\right)^3\left(\sum_{j=1}^{3}|z_j|^2\right)^3 = \frac{S^6}{3^3}\end{aligned} \tag{5}$$

and

$$\begin{aligned}&\prod_{1\le j<k\le 3}|z_j|\cdot|z_k|\cdot\left[|z_j-z_k|^2+(1-|z_j|^2)(1-|z_k|^2)\right]\\ &\le \left(\frac{\sum_{j=1}^{3}|z_j|^2}{3}\right)^3\left[\frac{\sum_{1\le j<k\le 3}\left(|z_j-z_k|^2+(1-|z_j|^2)(1-|z_k|^2)\right)}{3}\right]^3\\ &\le \frac{S^3}{3^3}\left[\frac{\sum 1-\sum|z_j|^2-\sum|z_k|^2+\sum|z_j|^2\cdot|z_k|^2+\sum|z_j-z_k|^2}{3}\right]^3\\ &\le \frac{S^3}{3^3}\left(\frac{3-2S+\frac{S^2}{3}+3S}{3}\right)^3 = \left(\frac{S^3+3S^2+9S}{3^3}\right)^3.\end{aligned} \tag{6}$$

We have applied above the elementary inequality

$$\sum_{1\le j<k\le 3}|z_j|^2\cdot|z_k|^2 \le \frac{1}{3}\left(\sum_{j=1}^{3}|z_j|^2\right)^2.$$

On the other hand, Jensen's inequality applied to the concave function $(0,1) \ni x \longmapsto \log x(1-x)$ yields

$$\sum_{j=1}^{3} \log |z_j|^2(1-|z_j|^2) \leq 3 \log \frac{S}{3}\left(1-\frac{S}{3}\right).$$

Therefore

$$\prod_{j=1}^{3} |z_j|^2(1-|z_j|^2) \leq \frac{S^3}{3^3} \cdot \left(\frac{3-S}{3}\right)^3. \tag{7}$$

From (5), (6) and (7) we find

$$f(z_1,z_2,z_3) \leq \frac{S^9}{3^6} \cdot \left(\frac{3-S}{3}\right)^3 \cdot \left(\frac{S^3+3S^2+9S}{3^3}\right)^3 = \frac{\left[S^4(27-S^3)\right]^3}{3^{18}}.$$

It follows that the maximum of f is achieved if $S = 3 \cdot 2^{2/3} \cdot 7^{-1/6}$, so $|z_1| = |z_2| = |z_3| = 2^{1/3} \cdot 7^{-1/3}$ and $\max f = 2^8 \cdot 3^6 \cdot 7^{-4}$, with equality when we have equality in (5), (6) and (7), that is, if and only if $z_2 = \varepsilon z_1$, $z_3 = \varepsilon^2 z_1$, where $\varepsilon = \cos(2\pi/3) + i \sin(2\pi/3)$.

Comments. The above problem is related to the minimization of the Ginzburg-Landau energy. In fact, the functional $W(z_1,\cdots,z_n) := -\pi \log f(z_1,\cdots,z_n)$ represents the *renormalized Ginzburg-Landau energy* corresponding to the open set $\Omega = \{z \in \mathbb{C};\ 0 < |z| < 1\}$. This functional has been defined implicitly (see Theorem I.7) for arbitrary domains in the monograph

[BBH] F. Bethuel, H. Brezis, F. Hélein, *Ginzburg-Landau Vortices*, Birkhäuser, Boston, 1994.

As established in Theorem VIII.1 in [BBH], the configuration of *singularities (vortices)* realizes the minimum of W. The renormalized energy is the "finite" part of the Ginzburg-Landau energy, that is, $W(z_1,\cdots,z_n)$ represents what remains in the Ginzburg-Landau energy after the singular "core energy" has been removed. The expression of f shows that the singularities are neither too mutually close nor too close to the boundary and the origin. Our results give an idea about the location of singularities, which have the tendency to be distributed in regular configurations which are called *Abrikosov lattices*. Problems of this type have been studied starting with the pioneering papers by V. Ginzburg and L. Landau in the 1950's. The huge interest for these problems is expressed by the fact that the Nobel Prize in Physics was awarded in 2003 to Alexei Abrikosov, Vitaly Ginzburg and Anthony Leggett "for pioneering contributions to the theory of superconductors and superfluids".

We also point out that the function f can be rewritten as

$$f(z_1,\cdots,z_n) = \prod_{1\leq j<k\leq n} |z_j|\cdot|z_k|\cdot|z_j-z_k|^2 \cdot \prod_{1\leq j\leq k\leq n} |z_j|\cdot|z_k|\cdot|1-z_j\overline{z_k}|^2,$$

where $\overline{z}$ denotes the conjugate of the complex number z.