PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**

with the collaboration of Paul T. Bateman, Mario Benedicty, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Dennis Eichhorn, Tamás Erdélyi, Kevin Ford, Zachary Franco, Christian Friesen, Ira M. Gessel, Jerrold R. Griggs, Jerrold Grossman, Kiran S. Kedlaya, Andre Kündgen, Frederick W. Luttman, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, John Henry Steelman, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before October 31, 2003. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11019. *Proposed by Bernardo Recamán Santos, Universidad Sergio Arboleda, Bogotá, Colombia.*
(a) Find an integer N so that there is a block B of N consecutive integers that can be arranged cyclically so that adjacent pairs have a nontrivial common divisor.
(b)* Show that this can be done for all sufficiently large N.

11020. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* For positive real numbers x and y, evaluate

$$\int_{t=0}^{\pi/2} \log\left[\frac{\sqrt{x^2+y^2\sin^2 t}+y\sin t}{\sqrt{y^2+x^2\sin^2 t}+x\sin t}\right] dt.$$

11021. *Proposed by Mason Foudroyant Possner, Stanford University, Stanford, CA.* Find all solutions in positive integers (x, y, z) to the equations

$$xy \bmod z = yz \bmod x = zx \bmod y = 2.$$

Here $a \bmod b$ denotes $a - b\lfloor a/b \rfloor$, the remainder when a is divided by b.

11022. *Proposed by Razvan Satnoianu, City University, London, U. K.* Let T_1 and T_2 be triangles such that, for $i \in \{1, 2\}$, triangle T_i has circumradius R_i, inradius r_i, and side-lengths a_i, b_i, and c_i. Show that

$$8R_1R_2 + 4r_1r_2 \geq a_1a_2 + b_1b_2 + c_1c_2 \geq 36r_1r_2$$

and determine when equality holds.

11023. *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Dong Province, China.* Find all pairs (x, y) of integers such that

$$x^2 + 3xy + 4006(x + y) + 2003^2 = 0.$$

11024. *Proposed by Vicenţiu Rădulescu, University of Craiova, Romania.* Consider a continuous function $g\colon (0,\infty) \to (0,\infty)$ such that for some $\alpha > 0$

$$\lim_{x\to\infty} \frac{g(x)}{x^{1+\alpha}} = \infty.$$

Let $f : \mathbb{R} \to (0,\infty)$ be a twice differentiable function for which there exists $a > 0$ and $x_0 \in \mathbb{R}$ such that for all $x \geq x_0$

$$f''(x) + f'(x) > ag(f(x)).$$

Prove that $\lim_{x\to\infty} f(x)$ exists and is finite, and evaluate the limit.

11025. *Proposed by Floor van Lamoen, Goes, the Netherlands, and Darij Grinberg, Karlsruhe, Germany.* Let four lines a, b, c, and d be given, no two of which are parallel. Let T_a be the triangle bounded by b, c, and d, and let O_a be its circumcenter. Let T_b, T_c, and T_d as well as O_b, O_c, and O_d be defined similarly. Let H_a be the orthocenter of triangle $O_b O_c O_d$, and again define H_b, H_c, and H_d similarly. Show:
(a) The four circumcircles of T_a, T_b, T_c, and T_d intersect at a single point (call it M).
(b) $O_a O_b O_c O_d$ is a concyclic quadrilateral (call it Q_O), and M lies on its circumcircle.
(c) Each H_x lies on line x for $x \in \{a, b, c, d\}$.
(d) $H_a H_b H_c H_d$ is a quadrilateral congruent to the Q_O (call it Q_H), in fact, congruent via a half-turn about an appropriate point. (Hence Q_H is also concyclic.)
(e) The perpendicular bisectors of the Euler segments of the four triangles T_a, T_b, T_c, and T_d are concurrent (say at E).
(f) The point E is the circumcenter of Q_H.

A *concyclic quadrilateral* is a quadrilateral inscribed in a circle. The *Euler segment* of a triangle is the segment from the orthocenter to the circumcenter.

11001. *Proposed by Rick Mabry, LSUS, Shreveport, LA (corrected).* Let a be a sequence of real numbers.
(a) Given that $\sum_1^\infty a_n$ converges and that p is an odd integer greater than one, must $\sum_1^\infty a_n^p$ converge?
(b) Again given that $\sum a_n$ converges, must there exist a positive integer P such that $\sum a_n^p$ converges whenever p is an odd integer greater than P?
(c) Given that all a_n are positive and that $\sum (-1)^n a_n$ converges, must there be a positive integer P such that $\sum (-1)^n a_n^p$ converges whenever p is an odd integer greater than P?

Editorial comment. The editors introduced the errors in (a) and (c) during typesetting.

SOLUTIONS

Divisibility Conditions Holding as Many Times as You Want

10898 [2001, 770]. *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou City, China.* Let m be a positive integer. Prove that there exist infinitely many integer vectors (a, b, c, d) such that (1) $a > c > 0$, (2) a and c are relatively prime, and (3) there are exactly 2^m values of $n \in \mathbb{N}$ with the property that $an + b$ divides $cn + d$.

Solution by Gerry Myerson, Macquarie University, New South Wales, Australia. We prove much more: for each nonnegative integer k there are infinitely many pairs of integers b and d such that there are exactly k positive integers n for which $2n+b$ divides $n+d$. In fact, we can take $b=-1$ in all cases except $k=0$ and $k=1$.

For $k=0$, take any b and d such that $2+b>1+d>0$. Now $2n+b>n+d>0$ for $n\in\mathbb{N}$, and divisibility never occurs.

For $k=1$, take any b and d such that $2+b=1+d>0$. Now $2n+b>n+d>0$ for all $n>1$, so $2n+b$ divides $n+d$ if and only if $n=1$.

For $k\geq 2$, let p be an odd prime, let $d=(p^{k-1}-1)/2$, and suppose that $2n-1$ divides $n+d$. Thus $n+(p^{k-1}-1)/2=(2n-1)q$ for some integer q, so $n=\frac{p^{k-1}+(2q-1)}{2(2q-1)}$. We conclude that n is an integer if and only if $2q-1$ divides p^{k-1}, and so the number of positive integers for which the divisibility condition holds is the number of divisors of p^{k-1}. This is k, as desired. There are infinitely many choices for d because there are infinitely many odd primes.

Also solved by S. Amghibech, M. R. Avidon, R. Chapman (U. K.), T. Liu, O. P. Lossers (Netherlands), R. Martin, GCHQ Problem Solving Group (U. K.), LSU Problem Solving Group, NSA Problems Group, C. P. Rupert, Szeged Problem Group "Fejéntaláltuka" (Hungary), and the proposer.

Best Polynomial Approximations

10903 [2001, 871]. *Proposed by Richard Bagby, New Mexico State University, Las Cruces, NM.* Let μ be an absolutely continuous measure on an interval $I\subset\mathbb{R}$ with strictly positive density function, and assume that every polynomial is μ-integrable. Let $f\colon\ I\to\mathbb{R}$ be continuous and in $L^2(\mu)$, and let p be the real polynomial of degree at most n that best approximates f in the norm of $L^2(\mu)$. Prove that there are at least $n+1$ points in I at which $p=f$.

Solution by O. P. Lossers, Eindhover University of Technology, Eindhoven, The Netherlands. If every polynomial is integrable, then so is the square of every polynomial, so every polynomial belongs to $L^2(\mu)$. Write $(\cdot\,,\cdot)$ for the usual inner product in $L^2(\mu)$. Since p is the best approximation to f among polynomials of degree at most n, if q is any polynomial of degree at most n, then $(f-p-\lambda q,\ f-p-\lambda q)$ is minimal when $\lambda=0$. That is, the quadratic $(f-p,\ f-p)+2(f-p,q)\lambda+(q,q)\lambda^2$ is minimal when $\lambda=0$. Thus $(f-p,q)=0$ for every real polynomial q of degree at most n.

Let m be the number of points at which $f=p$. If $m<n+1$, then $f-p$ changes sign at most m times. Let q be a polynomial of degree $\leq m$ that changes sign at exactly the points where $f-p$ changes sign. We conclude that $(f-p)q$ is either nonnegative or nonpositive. From the continuity of $(f-p)q$ it follows that there is an interval where $(f-p)q$ differs from zero. As the density function is strictly positive, this implies that $(f-p,q)$ is not zero. Hence the degree of q is larger than n, which is a contradiction.

Solved also by S. Amghibech (France), R. Chapman (U. K.), J. W. Hagood, J. H. Lindsey, R. Martin, R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

A Higher Mean Value Theorem

10935 [2002, 392]. *Proposed by Proposed by Zhibing Chen, Shenzhen University, Shenzhen, China.* Suppose that f and g are continuous on the interval $[a,b]$, that the nth derivatives of f and g exist on (a,b), and that $g^{(n)}(x)\neq 0$ for all $x\in(a,b)$. Let $x_0,\dots,x_n$ be points in $[a,b]$ such that $a=x_0<\cdots<x_n=b$. Let D_f be the determinant of the $n+1$ by $n+1$ matrix whose kth row is $x_0^{k-1},\dots,x_n^{k-1}$ for $1\leq k\leq n$, and whose $(n+1)$st row is $f(x_0),\dots,f(x_n)$. Let D_g denote the determinant when the last row is changed to $g(x_0),\dots,g(x_n)$.

Prove that there is a point $c\in(a,b)$ such that $D_f/D_g=f^{(n)}(c)/g^{(n)}(c)$.

Solution by Shusen Ding, Seattle University, Seattle, WA. For fixed $x_0, \ldots, x_{n-1}$, let

$$D_f(x) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_0 & x_1 & \cdots & x_{n-1} & x \\ x_0^2 & x_1^2 & \cdots & x_{n-1}^2 & x^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} & x^{n-1} \\ f(x_0) & f(x_1) & \cdots & f(x_{n-1}) & f(x) \end{vmatrix},$$

$$D_g(x) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_0 & x_1 & \cdots & x_{n-1} & x \\ x_0^2 & x_1^2 & \cdots & x_{n-1}^2 & x^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} & x^{n-1} \\ g(x_0) & g(x_1) & \cdots & g(x_{n-1}) & g(x) \end{vmatrix},$$

so that $D_f(x_n) = D_f$ and $D_g(x_n) = D_g$. Let $F(x) = D_g \cdot D_f(x) - D_f \cdot D_g(x)$. Now $F(x)$ is continuous on $[a, b]$, its nth derivative exists on (a, b), and $F(x_0) = F(x_1) = \cdots = F(x_n) = 0$.

Applying Rolle's theorem n times, we see that there are n points in (a, b) where F' vanishes, $n-1$ points where F'' vanishes, and so forth. Accordingly, there is a point c where $F^{(n)}(c) = D_g \cdot D_f^{(n)}(c) - D_f \cdot D_g^{(n)}(c) = 0$. To compute $D_f^{(n)}(x)$, differentiate the entries in the last column n times. All but the last one become 0. Thus $D_f^{(n)}(x) = f^{(n)}(x)V$ and $D_g^{(n)}(x) = g^{(n)}(x)V$, where V is the Vandermonde determinant whose kth row is $x_0^{k-1}, \cdots, x_{n-1}^{k-1}$ for $1 \le k \le n$. Since $V \neq 0$, it follows that $D_g f^{(n)}(c) = D_f g^{(n)}(c)$. Note that $D_g \neq 0$, since otherwise we would have $D_g(x_0) = \cdots = D_g(x_n) = 0$. Hence arguing as before would give a point c_1 with $D_g^{(n)}(c_1) = g^{(n)}(c_1)V = 0$, contradicting the assumption that $g^{(n)}(x) \neq 0$. Thus $D_f/D_g = f^{(n)}(c)/g^{(n)}(c)$.

Solved also by S. Amghibech (France), H. M. Bui (U. K.), W. Chau, R. Chapman (U. K.), O. P. Lossers (Netherlands), E. Neuman and Z. Pales (Hungary), M. A. Prasad (India), A. L. Yandl, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Convex n-gons

10936 [2002, 392]. *Proposed by Andor Lucács and Szilard András, Babes–Bolyai University, Cluj–Napoca, Romania.* Prove that there exists a set S of $(n-2)$ points in the interior of a convex n-gon such that for any three vertices of the n-gon, the interior of the triangle determined by the three vertices contains exactly one element of S.

Solution by Nathan D. Cahill, Eastman Kodak Company, Rochester, NY. Label the vertices of the n-gon $0, 1, \ldots, n-1$ sequentially. If a, b, c are three distinct vertices, let $[a, b, c]$ denote the triangle whose vertices are a, b, c, and let $[a, b]$ denote the segment whose endpoints are a and b. The interior of the triangle $[a, b, c]$ is not empty, because the polygon is convex.

Let m be such that $0 < m < n-1$. The vertices $m-1, m, m+1, n-1, 0$ appear in this order in the n-gon. Let U_m be the polygon determined by them. The cases $m+1 = n-1$ and $0 = m-1$ are not excluded, but note that $m-1 \neq m \neq m+1$ and $n-1 \neq 0$. Segment $[m+1, m-1]$ is a diagonal of U_n (or a side, if $m+1 = n-1$ and $0 = m-1$). Segments $[n-1, m]$ and $[0, m]$ are distinct and intersect $[m+1, m-1]$ in two distinct points, which together with m determine a triangle P_m (the intersection of $[0, m, n-1]$ and $[m-1, m, m+1]$) that has interior points. Select one of them, and call it s_m. Let $S = \{s_1, \dots, s_{n-2}\}$. We show that S meets the stated requirements.

Consider an arbitrary triangle $[i, j, k]$ with distinct vertices i, j, k. Assume, without loss of generality, that $0 \le i \le j \le k \le n-1$.

We first prove that there exists at least one point in S that is in $[i, j, k]$. The vertices $0, i, j-1, j, j+1, k, n-1$ appear in this order in the n-gon. No element of $\{j+1 = k, ..., i = j-1\}$ is excluded. Since

$$P_j = [0, j, n-1] \cap [j-1, j, j+1] \subseteq [i, j, k],$$

we conclude that $s_j \in [i, j, k]$.

Now we prove that none of the other points in S is in $[i, j, k]$. If p is such that $0 < p \le i$ or $k \le p < n-1$, then $[0, p, n-1]$ and $[i, j, k]$ are nondegenerate and have no interior points in common, because their interior points lie on opposite sides of the diagonal $[i, k]$. Since s_p is interior to $[0, p, n-1]$, we conclude that $s_p \notin [i, j, k]$. If q is such that $i < q < j$ or $j < q < k$, then $[q-1, q, q+1]$ and $[i, j, k]$ have no interior points in common, because their interior points lie on opposite sides of the diagonal $[i, j]$ or $[j, k]$, respectively. Since s_q is interior to $[q-1, q, q+1]$, we conclude that $s_q \notin [i, j, k]$. These values of p and q exhaust all indices $1, \dots, n-2$ except for j. We have already seen that $s_j \in [i, j, k]$.

Therefore, $s_m \in [i, j, k]$ for $m = j$ and $s_m \notin [i, j, k]$ for $m \ne j$.

Solved also by S. T. Ahearn, M. R. Avidon, D. Beckwith, D. R. Berman, J. Bertorelli & M. Brozinsky & N. J. Moh, M. Brozinsky, E. W. Casey, O. P. Lossers (The Netherlands), R. B. Maddox, R. Martin, L. A. Ray, H. Skala, P. D. Straffin, S. Wagon, L. Wenstrom, J. T. Ward, L. Zhou, GCHQ Problem Solving Group (U. K.) (3 solutions), and the proposer.

The Complex Geometric Mean

10940 [2002, 393]. *Proposed by Yves Nievergelt, Eastern Washington University, Cheney, WA.* Prove that for each real number r with $0 < r < 1$ and any complex numbers $z_1, \cdots, z_k$ in the closed disk of radius r about the origin, there is a complex number z_0 in that disk such that $\prod_{j=1}^{k}(1+z_j) = (1+z_0)^k$.

Solution by Achava Nakhash, Overton, NV. The disk in the complex plane with center at $z = 1$ and radius one can be described in polar coordinates by $r \le 2\cos\theta$, where $-\pi/2 < \theta < \pi/2$. The cosine function is always positive on this interval. Let $1 + z_j = r_j e^{i\theta_j}$ for $1 \le j \le k$. Then

$$\prod_{j=1}^{k}(1+z_j) = (r_1 \cdots r_k)e^{i(\theta_1+\cdots+\theta_k)}.$$

The claim follows if we can establish that the condition $r_j \le 2\cos\theta_j$ for $1 \le j \le k$ implies that

$$(r_1 \cdots r_k)^{1/k} \le 2\cos\left(\frac{\theta_1 + \cdots + \theta_k}{k}\right),$$

since then $z_0 = -1 + (r_1 \cdots r_k)^{1/k} e^{i(\theta_1+\cdots+\theta_k)/k}$ has the desired property. Now

$$(r_1 \cdots r_k)^{1/k} \le (2\cos\theta_1 \cdots 2\cos\theta_k)^{1/k} = 2(\cos\theta_1 \cdots \cos\theta_k)^{1/k},$$

so it suffices to show that

$$(\cos\theta_1 \cdots \cos\theta_k)^{1/k} \le \cos\left(\frac{\theta_1 + \cdots + \theta_k}{k}\right). \tag{1}$$

We will need the fact that the function $f(x) = \ln(\cos x))$ is concave on the interval $-\pi/2 < x < \pi/2$. To see this, use $\ln(\cos x) < 0$ and $\cos(x) > 0$ to get

$$f''(x) = \frac{-\ln(\cos x)\cos x - \dfrac{\sin^2 x}{\ln(\cos x)}}{(\ln(\cos x))^2} > 0$$

for $-\pi/2 < x < \pi/2$. Jensen's inequality then gives

$$\frac{\sum_{j=1}^k \ln(\cos\theta_j)}{k} \le \ln\left(\cos\left(\frac{\sum_{j=1}^k \theta_j}{k}\right)\right).$$

Exponentiating both sides yields exactly (1).

Solved also by P. G. Kirmser, P. W. Lindstrom, O. P. Lossers (Netherlands), M. Reid, A. Stadler (Switzerland), F. Szeged (Hungary), Li Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Decay of a Markov Chain

10941 [2002, 393]. *Proposed by Jeffrey C. Lagarias, AT&T Labs–Research, Florham Park, NJ.* Let N be an integer greater than 1, and consider the polynomial p_N defined by

$$p_N(x) = \left(x + \frac{1}{N}\right)^N - \left(1 + \frac{1}{N}\right)^N x^{N-1},$$

which has a zero θ_1 at 1.
(**a**) Show that p_N has a positive real zero θ_2 that is simple and satisfies

$$1 - \frac{2}{N} < \theta_2 < 1 - \frac{1}{N}.$$

(**b**) Show that the remaining zeros of p_N have absolute value less than θ_2.

*Solution for (**a**) by Thomas Hermann, SDRC, Milford, OH.* Expanding, one has

$$p_N(x) = x^N - \left(\left(1 + \frac{1}{N}\right)^N - 1\right) x^{N-1} + \sum_{k=0}^{N-2} \binom{N}{k} x^k N^{k-N}.$$

As there are two sign changes, by Descartes's rule of signs there are at most two positive zeros (counting multiplicity). Since 1 is trivially a zero, it remains to show only that there is a sign change in the interval $[1 - 2/N, 1 - 1/N]$.

We first write

$$\begin{aligned} p_N\left(1 - \frac{1}{N}\right) &= 1 - \left(1 + \frac{1}{N}\right)^N \left(1 - \frac{1}{N}\right)^{N-1} \\ &= \frac{N}{N-1}\left(\left(1 - \frac{1}{N}\right) - \left(1 - \frac{1}{N^2}\right)^N\right). \end{aligned}$$

Thus $p_N(1 - 1/N) < 0$ if and only if

$$0 > \log\left(1 - \frac{1}{N}\right) - N\log\left(1 - \frac{1}{N^2}\right) = -\sum_{k=1}^{\infty} \frac{1}{kN^k}\left(1 - \frac{1}{N^{k-1}}\right),$$

which is true because $1 - 1/N^{k-1} \ge 0$.

We next write

$$p_N\left(1-\frac{2}{N}\right)=\left(1-\frac{1}{N}\right)^N-\left(1+\frac{1}{N}\right)^N\left(1-\frac{2}{N}\right)^{N-1}.$$

Then $p_N(1-2/N)>0$ if and only if

$$\begin{aligned}0&<N\log\left(1-\frac{1}{N}\right)-N\log\left(1+\frac{1}{N}\right)-(N-1)\log\left(1-\frac{2}{N}\right)\\&=-n\sum_{k=1}^{\infty}\frac{1}{kn^k}-n\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{kn^k}+(n-1)\sum_{k=1}^{\infty}\frac{2^k}{kn^k}\\&=\sum_{k=1}^{\infty}\frac{1}{kn^k}((n-1)2^k-n(1+(-1))^k),\end{aligned}$$

which holds because $(n-1)2^k\geq 2n$ for $k,n\geq 2$.

*Solution for (**b**) by O.P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands.* From the facts (established in the solution to the first part) that p_N has exactly two positive zeros 1 and θ_2, both simple, it follows that $p_N(x)<0$ for $\theta_2<x<1$. Hence for $\theta_2<|z|<1$,

$$\left(1+\frac{1}{N}\right)^N|z|^{N-1}>\left(|z|+\frac{1}{N}\right)^N\geq\left|\left(z+\frac{1}{N}\right)^N\right|.$$

By Rouché's theorem, $p_N(z)$ thus has the same number of zeros in the disc $|z|\leq\theta_2$ as $(1+1/N)^N z^{N-1}$, namely, $N-1$. Moreover, if $|z|=\theta_2$ but $z\neq\theta_2$, then

$$\left|\left(z+\frac{1}{N}\right)^N\right|<\left(\theta_2+\frac{1}{N}\right)^N=\left(1+\frac{1}{N}\right)^N|z|^{N-1},$$

so $p_N(z)\neq 0$. Consequently, the remaining zeros of p_N have absolute value less than θ_2.

Editorial comment. It seems to have been unclear that in part (b) one was expected to prove the inequality for complex zeros as well as real zeros. The proposer notes that $p_N(x)$ is the characteristic polynomial of a Markov chain studied by Haga and Robins (On Kruskal's principle, in *Organic Mathematics*, Canadian Math. Soc. Conf. Proc. no. 20, American Mathematical Society, Providence, 1997, pp. 407-412), in which context the size of θ_2 governs the rate of convergence to the steady state.

Part (**a**) solved also by S. Amghibech (France), M.R. Avidon, O. Furdui, R.C. Rhoades, L. Zhou, and GCHQ Problem Solving Group (United Kingdom). Parts (**a**) and (**b**) solved also by P. Bracken (Canada), R. Chapman (United Kingdom), A. Stadler (Switzerland), A. Stenger, and the proposer.

An Inequality for Triangles

10950 [2002, 569]. *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (corrected).* Consider a triangle T with sides a, b, and c and vertices A, B, and C, and a point P in the plane of T. Let a', b', and c' be the distances from P to A, B, and C respectively. Call P *good* if $a'/a=b'/b=c'/c$.

Show that T has a good point P if and only if T is not obtuse. Show further that the ratio of similarity a'/a is always at least $1/\sqrt{3}$, with equality if and only if T is equilateral.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. If a triangle has sides a, b, and c and if a good point with positive similarity ratio t exists, then (see e. g. L. M. Blumenthal, *Theory and Applications of Distance Geometry,* 1953, Clarendon Press, Oxford)

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & (tc)^2 \\ 1 & a^2 & 0 & c^2 & (tb)^2 \\ 1 & b^2 & c^2 & 0 & (ta)^2 \\ 1 & (tc)^2 & (tb)^2 & (ta)^2 & 0 \end{bmatrix} = 0 \qquad (1)$$

(This is a so-called Cayley–Menger determinant.) Equation (1) is equivalent to the equation

$$\begin{aligned} &- a^2b^2c^2 + (-a^6 + b^4c^2 + b^2c^4 - b^6 + a^4c^2 + a^2c^4 - c^6 + a^4b^2 + a^2b^4)t^2 \\ &+ (-a^6 + b^4c^2 + b^2c^4 - b^6 + a^4c^2 + a^2c^4 - c^6 + a^4b^2 + a^2b^4 - 3a^2b^2c^2)t^4 = 0. \end{aligned}$$

Using the law of cosines, this can be simplified to

$$8 \cos\alpha \cos\beta \cos\gamma \cdot (t^4 + t^2) - (t^2 - 1)^2 = 0. \qquad (2)$$

For an obtuse triangle, $-1 < \cos\alpha \cos\beta \cos\gamma < 0$ and the left-hand side of (2) is negative for all t, so there is no good point.

For a right triangle, $\cos\alpha \cos\beta \cos\gamma = 0$, and $t = 1$ is the only solution of (2). Indeed, in a rectangle $ABCD$, the point D is a good point for triangle ABC.

If we set $K = 8 \cos\alpha \cos\beta \cos\gamma$, then equation (2) can be rewritten as

$$(1 - K)t^4 - (2 + K)t^2 + 1 = 0. \qquad (3)$$

For an acute triangle, $0 < K \le 1$ (O. Bottema et al., *Geometric Inequalities,* Wolters–Noordhoff, Groningen, 1969, p. 25). In the extreme case $K = 1$, we have an equilateral triangle with its center as the unique good point with $t = 1/\sqrt{3}$. In the remaining cases $0 < K < 1$, there are two good points, the product of the similarity ratios equals $1/\sqrt{1-K} > 1$. The left hand side of (3) is negative for $t^2 = 1$ and equals $\frac{4}{9}(1 - K) > 0$ for $t^2 = 1/3$. This shows that the smallest similarity ratio is larger than $1/\sqrt{3}$.

Solved also by M. Bataille (France), S. Klamkin (Canada), R. Chapman (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.