



On Some Non-existence Results in a Semilinear Theory of the Dipolar Thermoelastic Bodies

Marin Marin¹ · Vicențiu D. Rădulescu^{2,3} 

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Abstract

We consider a thermoelastic theory in which the equations that govern the evolution are linear with respect to the thermal displacement and nonlinear with regards to gradients of displacements and temperature. Our results refer to the non-existence of solutions for some mixed problems, considered in this context. We also address the instability of solutions of the considered problems. We will treat separately the case where the mechanical effects are neglected, taking into account only the thermal effect. In this case our problem has a nonlinear structure.

Keywords Thermoelasticity without energy dissipation · Dipolar bodies · Semilinear theory · Nonexistence · Instability

Mathematics Subject Classification 74A15 · 35Q74 · 74F05 · 35Q79

1 Introduction

It can be noticed that in the last years great attention has been paid to the phenomenon of heat transmission in elastic solids. Many studies, including those of Green and Naghdi [1–3], have proposed abandoning Fourier’s classical law and considering some theories that allow the propagation of thermal waves with finite speed. In this way, the foundations of a new theory have been laid, namely thermoelasticity without energy

✉ Vicențiu D. Rădulescu
radulescu@inf.ucv.ro

Marin Marin
m.marin@unitbv.ro

¹ Department of Mathematics and Computer Science, Transilvania University of Braşov, 500093 Braşov, Romania

² Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland

³ Department of Mathematics, University of Craiova, Street A.I. Cuza 13, 200585 Craiova, Romania

dissipation. After the start given by the studies of Green and Naghdi, this theory became very consistent by the presence of a large number of papers addressing all aspects of this theory of thermoelasticity without energy dissipation. There are studies that address the existence or non-existence of solutions, the uniqueness, the continuous dependence, stability or instability of solutions. In this respect, we must note the works [4–7]. All these works are dedicated to the linear case only. Our study is an attempt to address the semilinear framework, that is, models driven by equations that govern the theory of elastic dipolar bodies. These models arise in the context of thermoelasticity without energy dissipation and they are linear with respect to the thermal displacement and nonlinear with regards to temperature and gradients of displacements. But it is important to specify that in obtaining our results for the semilinear case we used some suggestions given in nonlinear studies, but in the case of isothermal elasticity. Thus, we have used methods and techniques found in the following papers: [8–13].

In our study we take into consideration an extra aspect, namely that of the dipolar structure, which is part of the modern theories dedicated to the microstructure. We have to say that the first works dedicated to the microstructure are due to Eringen (see [14, 15]). In recent years the number of papers on this subject has increased significantly. All aim to eliminate some shortcomings of classical theory. In this context, unlike the classic case, the equation of energy contains some elastic terms and it is not a parabolic type equation. Another important leap: in all theories dedicated to the microstructure, the heat waves will propagate at a finite speed. As an important component of the microstructure, the dipolar structure has attracted the attention of a large number of researchers and as a consequence, many studies have been published on this topic. In this regard, we must mention, in particular, the works [16–19]. In the studies [20–28] we find some results dedicated to different aspects of the microstructure. The methods and procedures used in the present paper have some similarities to those of classical elasticity. However, the classical results are getting worse in the new context, since the equations governing the behavior of these structures become more complicated.

In our paper we put down the equations that govern the behavior of a dipolar bodies in the context of thermoelasticity without energy dissipation. The mixed problem is completed by some specific initial and boundary conditions.

We highlight that the basic equations are linear with regards to the displacement of the heat and nonlinear with regards to temperature and gradients of displacements. Then we prove a conservation law, which will be useful in obtaining of the main results, in the first place, in obtaining an energy inequality. The result of instability of the solution is obtained for homogeneous boundary conditions. In the last part of the study we neglect the mechanical effects and get a nonlinear problem regarding the conduction of heat. We prove that some solutions cannot exist on a finite time interval.

2 Preliminaries

This paper is dedicated to a body with dipolar structure in the theory of thermoelasticity without energy dissipation. Our material is anisotropic and is located in a regular region D from usual space R^3 and its boundary ∂D is supposed be a sufficient regular surface

to permit the application of the divergence theorem. For the closure $\bar{D}$ of the domain D and we have $\bar{D} = D \cup \partial D$.

A system of Cartesian axes Ox_i , ($i = 1, 2, 3$) for the representation of the points is used. The tensor notation is adopted (letters in boldface). A superposed dot is used to designate the derivative with respect to time while for the differentiation regarding the coordinates x_i we will use the notation “ $\cdot_i$ ”. The Einstein rule the summation over repeated indices is assumed.

Throughout this paper, the following variables are used to characterize the evolution of our dipolar material:

$$u_i(t, x), \varphi_{ij}(t, x), \theta(t, x), \beta(t, x), (t, x) \in [0, t_0) \times D. \quad (1)$$

Here $u = (u_i)$ is the vector of displacement and $\varphi = (\varphi_{ij})$ is the tensor of dipolar displacement.

Also, θ is the temperature, referred to the reference configuration, and β is the displacement of the temperature, defined by

$$\beta(t, x) = \int_0^t \theta(\tau, x) d\tau. \quad (2)$$

With the help of the variables defined in Eq. (1), we define the tensors of the strain, which have the components ε_{ij} , κ_{ij} and χ_{ijk} , by means of the geometric relations:

$$2\varepsilon_{ij} = u_{j,i} + u_{i,j}, \kappa_{ij} = u_{j,i} - \varphi_{ij}, \chi_{ijk} = \varphi_{ij,k}. \quad (3)$$

All our considerations are made within a linear theory. Therefore, it is natural to think that Helmholtz’s free energy is a quadratic form with respect to all its constitutive variables. The Helmholtz’s free energy in the reference configuration will be denoted by Ψ . So, in accordance with the principle of energy conservation of energy, we develop in series the function Ψ and we only keep the terms only until the second order. Because the reference state was assumed be free of loadings, we deduce that the Helmholtz’s free energy per mass can be considered of the following form

$$\begin{aligned} \Psi = & \frac{1}{2} A_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + D_{ijmn} \varepsilon_{ij} \kappa_{mn} + F_{ijmnr} \varepsilon_{ij} \chi_{mnr} + \frac{1}{2} B_{ijmn} \kappa_{ij} \kappa_{mn} \\ & + G_{ijmnr} \kappa_{ij} \chi_{mnr} + \frac{1}{2} C_{ijkmnr} \chi_{ijk} \chi_{mnr} - a_{ij} \varepsilon_{ij} \theta - b_{ij} \kappa_{ij} \theta - c_{ijk} \chi_{ijk} \theta - \frac{1}{2} c \theta^2. \end{aligned} \quad (4)$$

We will use this form of free energy used in the entropy production inequality, where we will deduce the motion equations. Also, from the same inequality the constitutive equations are obtained. These equations express the tensors of stress with the help of the tensors of deformation. We will denote the components of the stress measures by τ_{ij} , σ_{ij} and μ_{ijk} . In this way, the constitutive equations establish a connection between the tensors τ_{ij} , σ_{ij} , μ_{ijk} and the tensors ε_{ij} , κ_{ij} , χ_{ijk} .

We will use a procedure similar to that used by Green and Rivlin in [16], so that taking into account (4) we obtain

$$\begin{aligned}\tau_{ij} &= \frac{\partial \Psi}{\partial \varepsilon_{ij}} = A_{ijmn} \varepsilon_{mn} + D_{mnij} \kappa_{mn} + F_{mnrij} \chi_{mnr} - a_{ij} \theta, \\ \sigma_{ij} &= \frac{\partial \Psi}{\partial \kappa_{ij}} = D_{ijmn} \varepsilon_{mn} + B_{ijmn} \kappa_{mn} + G_{ijkmnr} \chi_{mnr} - b_{ij} \theta, \\ \mu_{ijk} &= \frac{\partial \Psi}{\partial \chi_{ijk}} = F_{ijkmn} \varepsilon_{mn} + G_{mnijk} \kappa_{mn} + C_{ijkmnr} \chi_{mnr} - c_{ijk} \theta, \\ \eta &= -\frac{\partial \Psi}{\partial \theta} = a_{ij} \varepsilon_{ij} + b_{ij} \kappa_{ij} + c_{ijk} \chi_{ijk} + c \theta.\end{aligned}\quad (5)$$

Here we denoted by η the entropy per unit mass.

Also, we can deduce the main equations that govern the thermoelasticity of bodies with a dipolar structure without energy dissipation (see also [19]), namely:

- the motion equations:

$$\begin{aligned}(\tau_{ij} + \eta_{ij})_{,j} + \varrho f_i &= \varrho \ddot{u}_i, \\ \mu_{ijk,i} + \eta_{jk} + \varrho g_{jk} &= I_{kr} \ddot{\varphi}_{jr};\end{aligned}\quad (6)$$

- the equation of energy is given by (see [18]):

$$\varrho T_0 \dot{\eta} = q_{i,i} + \varrho r. \quad (7)$$

The signification of the notions that we introduced in previous equations is as follows: ϱ -the density of mass, which is a constant; I_{ij} -the tensor of microinertia; k -the intrinsic inertia; ε_{ij} , κ_{ij} , χ_{ijk} -the strain tensors; τ_{ij} , η_{ij} , μ_{ijk} -the stress tensors; h_i -the components of the vector for equilibrated stress; f_i -the body forces; g_{jk} -the dipolar charges; A_{ijmn} , B_{ijmn} , ..., a_{ij} -the functions what describe the properties of the material in terms of elasticity. Suppose the following symmetry relations take place:

$$\begin{aligned}A_{ijmn} &= A_{jimn} = A_{mnij}, \quad B_{ijmn} = B_{mnij}, \quad a_{ij} = a_{ji}, \\ C_{ijkmnr} &= C_{mnrijk}, \quad F_{ijkmn} = F_{ijknm}, \quad D_{ijmn} = D_{ijnm}.\end{aligned}\quad (8)$$

Assuming that there are no supply terms and taking into account the constitutive Eq. (5) and the kinematic Eq. (3), the Eqs. (6) and (7) receive the form

$$\begin{aligned}\varrho \ddot{u}_i &= [(C_{ijmn} + G_{ijmn}) u_{n,m} + (G_{mnij} + B_{ijmn}) (u_{n,m} - \varphi_{mn}) + \\ &\quad + (F_{mnrij} + D_{ijkmnr}) \varphi_{nr,m} - (a_{ij} + b_{ij}) \theta]_{,j}, \\ I_{kr} \ddot{\varphi}_{jr} &= [F_{ijkmn} u_{n,m} + D_{mnijk} (u_{n,m} - \varphi_{mn}) + A_{ijkmnr} \varphi_{nr,m} - c_{ijk} \theta]_{,i} + \\ &\quad + G_{jkmn} u_{m,n} + B_{jkmn} (u_{n,m} - \varphi_{mn}) + D_{jkmnr} \varphi_{nr,m} - b_{jk} \theta, \\ k_{ij} (\beta_{,j})_{,i} &= -T_0 [a_{ij} \dot{u}_{i,j} + b_{ij} (\dot{u}_{j,i} - \dot{\varphi}_{ij}) + c_{ijk} \dot{\varphi}_{ij,k} + c \dot{\theta}].\end{aligned}\quad (9)$$

We will complete the mixed problem for dipolar bodies in the context of thermoelasticity without energy dissipation by adding some prescribed initial conditions

$$\begin{aligned} u_i(0, x) &= u_i^0(x), \quad \dot{u}_i(0, x) = u_i^1(x), \\ \varphi_{jk}(0, x) &= \varphi_{jk}^0(x), \quad \dot{\varphi}_{jk}(0, x) = \varphi_{jk}^1(x), \\ \theta(0, x) &= \theta^0(x), \quad \beta(0, x) = \beta^0(x), \quad x \in \bar{D}, \end{aligned} \quad (10)$$

and, also, the given boundary conditions

$$u_i = 0, \quad \varphi_{jk} = 0, \quad \theta = 0 \quad \text{on } \partial D. \quad (11)$$

Clearly, taking into account the definition (2) and the fact that, according to (11), $\theta = 0$ on ∂D , we deduce that $\beta = 0$ on ∂D .

We will denote by $\mathcal{P}$ the mixed initial boundary value problem of the theory of thermoelasticity without energy dissipation for a dipolar material over the set $\Omega_0 = D \times [0, t_0)$ which consists of the system of Eq. (9), for all $(t, x) \in \Omega_0$, the boundary conditions (11) and the initial conditions (10).

Naturally, an ordered array $(u_i, \varphi_{ij}, \theta, \beta)$ can be a solution for the problem $\mathcal{P}$ if it satisfies the system (9) and meet the conditions (10) and (11) of the problem.

Our first result relates to an energy conservation law corresponding to the solutions of the problem $\mathcal{P}$.

For the energy corresponding to a solution to the problem $\mathcal{P}$ we will use the notation

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} \int_D [\rho \dot{u}_i \dot{u}_i + I_{jk} \dot{\varphi}_{jl} \dot{\varphi}_{kl}] dV \\ &+ \frac{1}{2} \int_D [A_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + 2D_{ijmn} \varepsilon_{ij} \kappa_{mn} + 2F_{ijmnr} \varepsilon_{ij} \chi_{mnr} \\ &+ B_{ijmn} \kappa_{ij} \kappa_{mn} + 2G_{ijmnr} \kappa_{ij} \chi_{mnr} + C_{ijkmnr} \chi_{ijk} \chi_{mnr} \\ &- 2a_{ij} \varepsilon_{ij} \theta - 2b_{ij} \kappa_{ij} \theta - 2c_{ijk} \chi_{ijk} \theta - c\theta^2] dV. \end{aligned} \quad (12)$$

If we take into account (4), energy $\mathcal{E}$ receives the simplified form

$$\mathcal{E}(t) = \frac{1}{2} \int_D [\rho \dot{u}_i \dot{u}_i + I_{jk} \dot{\varphi}_{jl} \dot{\varphi}_{kl}] dV + \int_D \Psi dV. \quad (13)$$

Proposition 1 *If the ordered array $(u_i, \varphi_{ij}, \theta, \beta)$ is a solution of the problem $\mathcal{P}$, then the energy $\mathcal{E}$ satisfies the following conservation law:*

$$\mathcal{E}(t) = \mathcal{E}(0). \quad (14)$$

Proof We multiply the Eq. (9)₁ by $\dot{u}_i$, (9)₂ by $\dot{\varphi}_{jk}$ and (9)₃ by $\dot{u}_i$. We sum up the obtained equalities and the equality that result is integrated on the interval $[0, t]$. If we take into account the boundary conditions from (11) and integrate on the domain D , we obtain the desired relation (14). $\square$

In what follows, we will consider that the material is linear with respect to the thermal displacement gradient, that is β_i , and can be nonlinear with respect to the variables $u_{i,j}$, $\varphi_{ij,k}$, θ . In other words, we can write the free energy in

$$\Psi = \omega(u_{i,j}, \varphi_{ij,k}, \theta) + \frac{1}{2} K_{ij} \beta_{,i} \beta_{,j}, \quad (15)$$

where the tensor K_{ij} is positively defined.

As a consequence, the law of energy conservation from (14) becomes

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} \int_D [\varrho \dot{u}_i \dot{u}_i + I_{jkl} \dot{\varphi}_{jl} \dot{\varphi}_{kl} + K_{ij} \beta_{,i} \beta_{,j}] dV \\ &+ \int_D [\omega(u_{i,j}, \varphi_{ij,k}, \theta) + \eta \theta] dV = \mathcal{E}(0). \end{aligned} \quad (16)$$

Based on the considerations made in the papers [10, 12] in the case of classical elasticity, without taking into account the thermal effect, we can replace the Eq. (16) with the following inequality,

$$\mathcal{E}(t) \leq \mathcal{E}(0). \quad (17)$$

Also, if energy $\mathcal{E}$ is associated not with a classical solution of the system (9), but with a weak solution, it is more appropriate to use the inequality (17) than the Eq. (16).

We now aim to achieve a result of instability. For this, we will first consider the following boundary values problem

$$\begin{aligned} (K_{ij} q_{,j})_{,i} &= -\eta (u_{i,j}^0, \varphi_{ij,k}^0, \theta^0), \\ q &= 0, \quad x \in \partial D, \end{aligned} \quad (18)$$

where the initial data $(u_{i,j}^0, \varphi_{ij,k}^0, \theta^0)$ are introduced in (10).

If q is a solution to the problem (18), then we introduce the notation

$$\vartheta = \int_0^t \beta(\tau) d\tau + q.$$

In the following we will consider that energy has the following form:

$$\mathcal{E}_1(t) = \frac{1}{2} \int_D [\varrho u_i u_i + I_{jkl} \varphi_{jl} \varphi_{kl} + K_{ij} \vartheta_{,i} \vartheta_{,j}] dV. \quad (19)$$

Theorem 1 *We consider that a weak solution of the problem $\mathcal{P}$ is the ordered array $(u_i, \varphi_{ij}, \theta, \beta)$. If, in addition, the following condition is met*

$$\int_D \left(-\frac{\partial \omega}{\partial u_{i,j}} u_{i,j} - \frac{\partial \omega}{\partial \varphi_{ij,k}} \varphi_{ij,k} + \frac{\partial \omega}{\partial \theta} \theta \right) dV \geq 0, \quad (20)$$

then the energy $\mathcal{E}_1(t)$ becomes unbounded, that is, the equilibrium state becomes unstable.

Proof By direct calculations, from (19) we are led to the inequality

$$\begin{aligned} \frac{d^2 \mathcal{E}_1}{dt^2} = & \int_D [\varrho u_i u_i + I_{jk} \varphi_{jl} \varphi_{kl} + K_{ij} \beta_{,i} \beta_{,j}] dV \\ & + \int_D \left(-\frac{\partial \omega}{\partial u_{i,j}} u_{i,j} - \frac{\partial \omega}{\partial \varphi_{ij,k}} \varphi_{ij,k} + \frac{\partial \omega}{\partial \theta} \theta \right) dV. \end{aligned} \quad (21)$$

We take into account that $\varrho > 0$ and the tensors I_{ij} and K_{ij} are positively defined. Considering the condition (20) and using Schwarz's inequality, we obtain the inequality

$$\frac{d^2 \mathcal{E}_1^{1/2}}{dt^2} \geq 0, \quad (22)$$

which involves the condition

$$\mathcal{E}_1^{1/2}(t) \geq \frac{t \frac{d\mathcal{E}_1}{dt}(0) + 2\mathcal{E}_1(0)}{\mathcal{E}_1^{1/2}(0)}. \quad (23)$$

It is clear that if $\frac{d\mathcal{E}_1}{dt}(0) > 0$, then, the condition (23) guarantees that energy $\mathcal{E}_1^{1/2}(t)$ becomes unbounded, which ends the demonstration of the theorem. $\square$

Now we will address a non-existence result of the solution of the problem $\mathcal{P}$. To this aim we will construct an auxiliary function $F(t)(t)$, with the help of the function $\mathcal{E}_1$ (from (19)), by

$$F(t) = \mathcal{E}_1(t) + \frac{1}{2} \alpha (t + t_0)^2, \quad \alpha \geq 0. \quad (24)$$

Theorem 2 Suppose that the inequality (22) is satisfied and for $\zeta > 2$ the following condition is checked

$$\int_D \left[-\frac{\partial \omega}{\partial u_{i,j}} u_{i,j} - \frac{\partial \omega}{\partial \varphi_{ij,k}} \varphi_{ij,k} + \frac{\partial \omega}{\partial \theta} \theta + \zeta(\omega + \eta\theta) \right] dV \geq 0. \quad (25)$$

Then, there is no a solution $(u_i, \varphi_{ij}, \theta, \beta)$ of the problem $\mathcal{P}$ which should be defined in finite time.

Proof By direct calculations we obtain

$$\frac{dF(t)}{dt} = \int_D (\varrho u_i \dot{u}_i + I_{jk} \varphi_{jl} \dot{\varphi}_{kl} + K_{ij} \vartheta_{,i} \beta_{,j}) dV + \alpha (t + t_0). \quad (26)$$

By using the energy inequality, from (26) we are led to

$$\begin{aligned} \frac{d^2 F(t)}{dt^2} \geq & \frac{\zeta + 2}{2} \int_D (\varrho \dot{u}_i \dot{u}_i + I_{jkl} \dot{\varphi}_{jl} \dot{\varphi}_{kl} + K_{ij} \beta_{,i} \beta_{,j}) dV \\ & + \int_D \left(-\frac{\partial \omega}{\partial u_{i,j}} u_{i,j} - \frac{\partial \omega}{\partial \varphi_{ij,k}} \varphi_{ij,k} + \frac{\partial \omega}{\partial \theta} \theta + \zeta(\omega + \eta \theta) \right) dV + (\alpha - \zeta \mathcal{E}(0)), \end{aligned} \quad (27)$$

and from here, with the help of Schwarz's inequality, we get

$$\frac{d^2 F(t)}{dt^2} F(t) - \frac{\zeta + 2}{4} \left(\frac{dF(t)}{dt} \right)^2 \geq -\zeta(\alpha + \mathcal{E}(0)) F(t). \quad (28)$$

Taking into account that $\zeta > 2$, we can reverse in inequality (28), so that using the notation $p = (\zeta - 2)/2$, inequality can be reformulated so

$$\frac{d^2 F^{-p}(t)}{dt^2} F(t) \leq 2p(2p + 1) F^{-(p+)}(t)(\alpha + \mathcal{E}(0)). \quad (29)$$

If we give to α a particular value, namely $\alpha = -\mathcal{E}(0)$, then from inequality (28) we obtain

$$\frac{d^2 F^{-p}(t)}{dt^2} \leq 0,$$

which ensures the concavity of the function $F^{-p}(t)$.

Taking into account the definition (25) we deduce that

$$F(t) \geq F(0) \left(1 - pt \frac{dF}{dt}(0) F(0)^{-1} \right)^{-1/p}, \quad t \in [0, t_0]. \quad (30)$$

We choose a t_0 large enough to guarantee the positivity of the derivative dF/dt . As a consequence, the lower bound from the right-hand side of the inequality (30) allows to the function F to grow as much as possible in a finite time. Thus, the proof of the theorem is completed. $\square$

In the last part of our study, we will neglect the mechanical effects. In other words, we consider that the function W which designates the free energy is only depending on temperature θ and of the gradient of the displacement of the heat $\beta_{,i}$, that is,

$$W = \omega(\theta) + \frac{1}{2} K_{ij} \beta_{,i} \beta_{,j}. \quad (31)$$

As such, we approach the problem (nonlinear) of the conduction of heat, consisting of the nonlinear equation

$$\frac{d}{dt} \left(\frac{\partial W}{\partial \theta} \right) + \left(\frac{\partial W}{\partial \beta_{,i}} \right) = 0, \quad (32)$$

the following initial conditions

$$\theta(0, x) = \theta^0(x), \quad \beta(0, x) = \beta^0(x), \quad x \in \bar{D}, \quad (33)$$

and the homogeneous boundary condition

$$\beta = 0 \text{ on } \partial D. \quad (34)$$

In order to determine the conditions in which the mixed problem (32)–(34) admits a solution, it is recommended the works [29,30].

We will show that, under specific conditions, the problem (32)–(34) can not admit a solution in finite time.

To make evaluations on the solutions to the problem (32)–(34) we introduce the function $G(t)$, defined with the help of the free energy $W(t)$ from (31), through

$$G(t) = \int_D \frac{\partial W}{\partial \theta} \beta dV. \quad (35)$$

Let us suppose that the free energy $W(t)$ satisfies the following condition

$$W + \frac{\partial W}{\partial \beta_i} \beta_{,i} \geq 0. \quad (36)$$

Theorem 3 *We consider a solution of problem (32)–(34) for which the initial data (33) is chosen so that the energy $\mathcal{E}$ is negative.*

If the function $G(t)$ attached to this solution satisfies the condition (36), then the solution ceases to exist for $t \rightarrow \infty$.

Proof In fact, we have to show that the solution becomes unbounded if $t \rightarrow \infty$.

If we take into account the condition (34) and use the theorem of divergence, from (32) we are led to the relation:

$$G(t) = \int_0^t \int_D \left(\frac{\partial W}{\partial \theta} \theta + \frac{\partial W}{\partial \beta_i} \beta_{,i} \right) dV d\tau + G(0). \quad (37)$$

Clearly, by direct derivation in (37) we obtain the following identity

$$\frac{dG(t)}{dt} = \int_D \left(\frac{\partial W}{\partial \theta} \theta + \frac{\partial W}{\partial \beta_i} \beta_{,i} \right) dV,$$

and from here, with the help of energy inequality, we are led to the next estimate

$$\frac{dG(t)}{dt} \geq \int_D \left(W + \frac{\partial W}{\partial \beta_i} \beta_{,i} \right) dV - \mathcal{E}(0). \quad (38)$$

Finally, taking into account the hypothesis (36), from (38) we get the inequality

$$G(t) \geq -G(0) - t\mathcal{E}(0),$$

which ends the proof of the theorem. $\square$

3 Conclusions

In the analysis developed in the present paper, we approach a semilinear initial boundary value problem describing thermoelastic dipolar bodies without energy dissipation. In our setting, the basic equations that describe the problem are linear with regard to the thermal displacement and nonlinear with respect to the gradients of displacements and temperature. This approach is original and it cannot be found at present for environments with dipolar structure. Our results refer to the non-existence of solutions for some mixed problems, considered in this context. Another feature of this paper is that we also address the instability of the solutions of the considered problems. We have paid special attention to the case when the mechanical effects are absent. It is noteworthy that if the mechanical effects are neglected, then the theory becomes a nonlinear one. However, we have managed to achieve both a result regarding the non-existence of the solutions and a result concerning instability issues.

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