Applied Mathematics & Optimization (2023) 87:9
https://doi.org/10.1007/s00245-022-09913-9

Low Perturbations and Combined Effects of Critical and Singular Nonlinearities in Kirchhoff Problems

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Accepted: 25 August 2022

Abstract

In this paper, we study three-dimensional Kirchhoff equations with critical growth and singular nonlinearity. We are concerned with the qualitative analysis of solutions to the following nonlocal problem

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=\lambda u^{-\gamma}+u^5, & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $0<\gamma<1$, and a, b, λ are positive constants. By combining variational methods with some delicate decomposition techniques, we obtain the existence of two positive solutions in the case of low perturbations of the singular nonlinearity, namely for small values of the parameter λ.

Keywords Kirchhoff equation · Critical growth · Singular nonlinearity · Multiple positive solutions · Ekeland's variational principle

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Published online: 07 November 2022

Mathematics Subject Classification 35J20 · 35J60 · 58E05 · 58E30

1 Introduction and Main Result

In this article, we consider a class of Kirchhoff-type equations with critical growth and singular nonlinearity including the following important prototype:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = \lambda u^{-\gamma}+u^5, & \text{in } \Omega, \\ u>0, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $0<\gamma<1$, a, b, λ are positive constants.

In recent years, a great attention has been focused on the study of singular elliptic problems like (1.1), see [2, 3, 6, 7, 9, 13–15, 18, 21, 22, 26, 28, 29, 34, 35] and references therein. This type of problems is related with a model proposed by Kirchhoff [20] in 1883 as an extension of the classical d'Alembert's wave equation for free vibrations of elastic strings. More precisely, taking into account the change in length of the string produced by transverse vibrations, Kirchhoff studied the following model

$$\begin{cases} u_{tt}-(a+b\int_\Omega |\nabla u|^2dx)\Delta u = h(x,u), & \text{in } \Omega\times(0,T), \\ u=0, & \text{on } \partial\Omega\times[0,T], \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), \end{cases}$$

where the function u denotes the displacement, the nonlinear term $h(x,u)$ denotes the external force, while the parameter a denotes the initial tension and the parameter b is related to the intrinsic properties of the string (such as Young's modulus). For more details of the physical background of the Kirchhoff equation we refer the reader to [3] and references therein. The driving force for the singular nonlinearity in Eq. (1.1) with $\gamma \in (0,1)$ arises in several physical models such as fluid mechanics, pseudo-plastic flows, chemical heterogeneous catalysts, non-Newtonian fluids, biological pattern formation, as well as in the theory of heat conduction in electrically conducting materials; for more details about these subjects, we refer to [10, 12, 27, 32]. On the other hand, the motivation for the critical nonlinearity arises in some variational problems in geometry and physics where the lack of compactness occurs, such as the Yamabe problem, isoperimetric inequalities, Hardy-Littlewood-Sobolev inequalities, trace inequalities, Plateau problem, H-systems, Yang-Mills-Higgs systems, immersed minimal surfaces problem and so on, see [4, 30, 36]. As for combined effects of singular and critical nonlinearities, there are a lot of works since the seminal paper of Crandall-Rabinowitz-Tartar in [8] with singular nonlinearity, for example, the existence of multiple solutions for Eq. (1.1) without Kirchhoff term was investigated in [16], by using the variational methods and the Nehari method. Our goal in this paper is to employ some novel decompositions to study the existence of multiple solutions for Eq. (1.1) in the Kirchhoff setting.

Eq. (1.1) has a variational structure given by the functional:

$$I(u) = \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 \\ -\frac{1}{6}\int_\Omega |u|^6 dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma} dx \tag{1.2}$$

for $u \in H_0^1(\Omega)$. Due to the singular term $u^{-\gamma}$ $(0 < \gamma < 1)$ contained in Eq. (1.1), the functional I is only continuous in $H_0^1(\Omega)$. A possible way to deal with such problems is to use the critical point theory for nonsmooth functionals, which has been rigorously developed; see [5, 17, 19, 24]. In this paper, we apply another approach, namely the Ekeland variational principle [11], which has extensive applications, in particular it was used to give a short proof of the famous Mountain Pass Lemma [1] even for nonsmooth functionals (see [31] and references therein).

Besides Eq. (1.1), we would like to consider the following more general Kirchhoff-type equation:

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda u^{-\gamma} + u^5, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Clearly, if $M(s) = a + bs$, Eq. (1.3) reduces to Eq. (1.1).

In this paper, we impose the following assumptions on the Kirchhoff function M:

(M_1) $M \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, $M(s) \geq a > 0$, a is a constant, $M(s)$ is increasing in s;
(M_2) $2M(s) \geq sM'(s)$, and $\lim_{s\to+\infty} \frac{M(s)}{s^2} = 0$;
(M_3) $\mathcal{M}(s) - \frac{1}{3}sM(s) \geq \frac{2as}{3}$, and $\frac{1}{s}(\mathcal{M}((s) - \frac{1}{3}sM(s))$ is nondecreasing in s, where $\mathcal{M}(s) = \int_0^s M(t)dt$.

Obviously, the simple example $M(s) = a + bs^\theta$ with $1 \leq \theta < 2$, $a, b > 0$, satisfies the above conditions.

Eq. (1.3) has also a variational structure given by the functional

$$I_\lambda(u) = \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2 dx\right) - \frac{1}{6}\int_\Omega |u|^6 dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma} dx \tag{1.4}$$

for all $u \in H_0^1(\Omega)$.

Notice that 6 is the critical Sobolev exponent for a domain Ω in $\mathbb{R}^3$. In their celebrated work [4], followed by enormous papers, Brézis and Nirenberg studied the following semilinear equation with critical growth

$$\begin{cases} \Delta u + \lambda u + u^{2^*-1} = 0, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $2^* = 2N/(N-2)$ $(2^* = 6$ for $N = 3)$ is the critical Sobolev exponent, $0 < \lambda < \lambda_1$, and λ_1 is the first eigenvalue of the Laplacian operator in Ω. It turns out that there exists a threshold value, only below this threshold value the functional associated with the problem (1.5) satisfies the Palais-Smale condition. Furthermore, the threshold value is related to the energy of solutions for the limit problem

$$\begin{cases} -\Delta u = u^{2^*-1}, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \\ u(x) \to 0, \text{ as } |x| \to \infty, \end{cases} \tag{1.6}$$

which is satisfied by the so-called bubble solutions.

Due to the nonlocal property, the limit problem for the Kirchhoff equation (1.3) with critical growth is a system of coupled equations, satisfied by the weak limit function and the bubbles (see the system $(2b)$ of Lemma 2.5 below). Since M is an abstract function in our paper, it is more difficult than problem (1.1) to determine the threshold value of the energy functional.

In this article, the nonlocal term $M(\int_\Omega |\nabla u|^2 dx)$ causes a serious difficulty in determining the threshold value. To overcome this tricky difficulty, by a concentration-compactness analysis on the Palais-Smale sequence, we decompose the bounded Palais-Smale sequence, and by decomposing the energy functional (see (3.10) below), we find an exact threshold value (see (3.7) below) and prove that the functional I_λ (I_λ is defined in (1.4)) satisfies the Palais-Smale condition under the threshold value. Finally, we estimate the critical value level of the energy functional, it is just the threshold value that we found. To the best of our knowledge, this paper uses for the first time the above decomposition techniques to deal with Kirchhoff-type problems.

Our main result establishes the following multiplicity property in the case of small perturbations of the singular term.

Theorem 1.1 *Assume that* $(M_1) - (M_3)$ *hold, then there exists* $\lambda_* > 0$ *such that for* $0 < \lambda < \lambda_*$, *problem (1.3) has two positive solutions.*

Remark 1.1 (i) In [21], the authors proved the existence of two positive solutions for problem (1.1), one being a local minimizer, the second one being a Mountain-Pass type solution. However, we point out that the Mountain-Pass type solution cannot be obtained. The reason for this is that, they cannot estimate accurately the threshold value of the energy functional I since we found that the accurate threshold value of the energy functional I is not

$$\widetilde{c} := \frac{ab}{4}S^3 + \frac{b^3}{24}S^6 + \frac{aS}{6}\sqrt{b^2S^4 + 4aS} + \frac{b^2S^4}{24}\sqrt{b^2S^4 + 4aS} - D\lambda^{\frac{2}{1+\gamma}},$$

where $D > 0$ is a constant. Indeed, from Lemma 3.3 in [21], we notice the following key inequality

$$B(t_\varepsilon v_\varepsilon) \leq -C_2\varepsilon^{\frac{16(1-\gamma)-(1-\gamma)^2}{64}},$$

where constant C_2 is dependent on the parameter α. Since $\alpha \to 0$, we have $C_2 \to 0$. As a result, it leads to an inability to prove that the critical level value of I is below $\widetilde{c}$. Therefore, the perturbation approach in [21] becomes invalid to obtain the Mountain-Pass type solution. In this sense, Theorem 1.1 is the first contribution to obtain the existence of two positive solutions for problem (1.1). Moreover, in this paper we take a quite different approach from that of [21].

(ii) In the equation (1.3), since the singular term $u^{-\gamma}$ has a low order growth, which makes estimations of the critical value level of energy functional more difficult. Therefore, in this paper, we shall give some new estimates so that problem (1.3) has at least two positive solutions. We believe that our methods can be applied to seek the existence of two positive solutions for the other elliptic problems when the energy functional involves critical and low order growth (below second order), for example:

- Critical and concave-convex nonlinearities:

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda u^{q-1} + u^5, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

  where $1 < q < 2$.
- Nonhomogeneous and critical nonlinearities:

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda f(x) + u^5, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

  where $f \in L^{\frac{6}{5}}(\Omega)$, $f > 0$.

This paper is organized as follows. In Sect. 2, we develop the concentration-compactness analysis and we establish the concrete Palais-Smale condition. In Sect. 3, we demonstrate the threshold value and conclude Theorem 1.1. In Sect. 4, we give two lemmas in the Appendix.

Throughout this paper, we use the same character C to denote several positive constants. Denote $u_+ := \max\{0, u\}$ and $u_- := \min\{0, u\}$.

We refer to the monograph [25] for some of the abstract methods used in this paper.

2 Concentration-Compactness Analysis

In this section we first recall some concepts adapted from the critical point theory for nonsmooth functionals, especially the concept of concrete Palais-Smale sequence (CPS sequence in short). Then we make concentration-compactness analysis on the CPS sequences of the functional I_λ. The results will be used to deduce the system of coupled equations satisfied by the weak limit function of a CPS sequence and the bubbles, and to prove the existence of the local minimizer and the Mountain-Pass type solution in the next section.

Let (X, d) be a complete metric space, $f : X \to \mathbb{R}$ be a continuous functional in X. Denote by $|Df|(u)$ the supermum of δ in $[0, \infty)$ such that there exist $r > 0$ and a continuous map $\sigma : B_r(u) \times [0, r]$ satisfying

$$\begin{cases} f(\sigma(v,t)) \leq f(v) - \delta t, \\ d(\sigma(v,t), v) \leq t \end{cases} \tag{2.1}$$

for $(v, t) \in B_r(u) \times [0, r]$.

A sequence $\{u_n\}$ of X is called Palais-Smale sequence of the functional f, if $|Df|(u_n) \to 0$ as $n \to \infty$ and $f(u_n)$ is bounded. In this paper, however, we use another concept instead, namely the so-called concrete Palais-Smale sequence for our functional I_λ. Since we are looking for positive solutions of the equation (1.5), we consider the functional I_λ as defined on the closed positive cone P of $H_0^1(\Omega)$

$$P = \left\{ u | u \in H_0^1(\Omega), u(x) \geq 0, \text{ a.e. } x \in \Omega \right\}. \tag{2.2}$$

Evidently, P is a complete metric space and I_λ is a continuous functional on P. We first introduce the following definition.

Definition 2.1 Define the concrete weak slope of the functional I_λ at $u \in P$, denoted by $|dI_\lambda|(u)$, by the infimum of $\varepsilon > 0$ such that

$$\lambda \int_\Omega u^{-\gamma}(v-u)dx \leq M\left(\int_\Omega |\nabla u|^2 dx\right) \int_\Omega \nabla u \nabla (v-u)dx - \int_\Omega u^5(v-u)dx + \varepsilon \|v-u\| \tag{2.3}$$

for all $v \in P$. If there is no such a number ε, we understand $|dI_\lambda|(u) = +\infty$. In particular, a sequence $\{u_n\}$ of P is called a concrete Palais-Smale sequence of the functional I_λ, if $|dI_\lambda(u_n)| \to 0$ and $I_\lambda(u_n)$ is bounded. The functional I_λ is said to satisfy the concrete Palais-Smale condition at the level c, if any concrete Palais-Smale sequence $\{u_n\}$ with $I_\lambda(u_n) \to c$ possesses a convergent subsequence.

It turns out that if $|DI_\lambda|(u) < +\infty$, then $u^{-\gamma}\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$ and it holds that

$$\lambda \int_\Omega u^{-\gamma}(v-u)dx \leq M\left(\int_\Omega |\nabla u|^2 dx\right) \int_\Omega \nabla u \nabla (v-u)dx - \int_\Omega u^5(v-u)dx + |DI_\lambda|(u)\|v-u\|$$

for $v \in P$. So in general $|dI_\lambda|(u) \leq |DI_\lambda|(u)$. We have the following lemma.

Lemma 2.2 *If $u \in P$, $|dI_\lambda|(u) = 0$, then u is a weak solution of problem (1.5), that is $u^{-\gamma}\varphi \in L^1(\Omega)$ for all $\varphi \in H_0^1(\Omega)$ and it holds that*

$$M\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla \varphi dx = \int_\Omega u^5 \varphi dx + \lambda \int_\Omega u^{-\gamma}\varphi dx. \tag{2.4}$$

Proof By the definition of $|dI_\lambda|(u) = 0$, we have

$$\lambda \int_\Omega u^{-\gamma}(v-u)dx \le M\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla (v-u)dx - \int_\Omega u^5 (v-u)dx \tag{2.5}$$

for $v \in P$. For $\varphi \in H_0^1(\Omega)$, $s \in \mathbb{R}$, taking $v = (u + s\varphi)_+ \in P$ as test function in (2.5), one has

$$\begin{aligned}
0 \le\ & M\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla ((u+s\varphi)_+ - u)dx \\
& - \int_\Omega u^5((u+s\varphi)_+ - u)dx - \lambda \int_\Omega u^{-\gamma}((u+s\varphi)_+ - u)dx \\
\le\ & s\left[M\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla \varphi dx - \int_\Omega u^5 \varphi dx - \lambda \int_\Omega u^{-\gamma}\varphi dx\right] \\
& - sM\left(\int_\Omega |\nabla u|^2 dx\right)\int_{\{u+s\varphi<0\}} \nabla u \nabla \varphi dx.
\end{aligned}$$

Since $\nabla u = 0$ for a.e. $x \in \Omega$ with $u(x) = 0$ and $\operatorname{meas}\{x \in \Omega | u(x) + s\varphi(x) < 0, u(x) > 0\} \to 0$ as $s \to 0$, we have

$$\int_{\{u+s\varphi<0\}} \nabla u \nabla \varphi dx = \int_{\{u+s\varphi<0, u>0\}} \nabla u \nabla \varphi dx \to 0 \ \text{ as } s \to 0.$$

Therefore

$$0 \le s\left[M\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla \varphi dx - \int_\Omega u^5 \varphi dx - \lambda \int_\Omega u^{-\gamma}\varphi dx\right] + o(s)$$

as $s \to 0$. We obtain

$$M\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla \varphi dx - \int_\Omega u^5 \varphi dx - \lambda \int_\Omega u^{-\gamma}\varphi dx \ge 0.$$

By the arbitrariness of the sign of φ, we obtain

$$M\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla \varphi dx - \int_\Omega u^5 \varphi dx - \lambda \int_\Omega u^{-\gamma}\varphi dx = 0$$

for all $\varphi \in H_0^1(\Omega)$. The proof is thus complete. □

Lemma 2.3 *Any concrete Palais-Smale sequence of I_λ is bounded in $H_0^1(\Omega)$.*

Proof Let $\{u_n\} \subset H_0^1(\Omega)$ be a concrete Palais-Smale sequence of I_λ, namely, $|dI_\lambda|(u_n) \to 0$, $I_\lambda(u_n) \to c$ as $n \to \infty$. By the definition of $|dI_\lambda|(u_n)$, we have

$$\lambda \int_\Omega u_n^{-\gamma}(v-u_n)dx \le M\left(\int_\Omega |\nabla u_n|^2 dx\right)\int_\Omega \nabla u_n \nabla(v-u_n)dx \\ -\int_\Omega u_n^5(v-u_n)dx + |dI_\lambda|(u_n)\|v-u_n\|. \tag{2.6}$$

Taking $v = 2u_n \in P$ in (2.6), we have

$$\lambda \int_\Omega u_n^{1-\gamma}dx \le M\left(\int_\Omega |\nabla u_n|^2 dx\right)\int_\Omega |\nabla u_n|^2 dx \\ -\int_\Omega u_n^6 dx + |dI_\lambda|(u_n)\|u_n\|. \tag{2.7}$$

By (M_3), we obtain

$$\begin{aligned} I_\lambda(u_n) + \frac{1}{6}|dI_\lambda|(u_n)\|u_n\| &\ge \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u_n|^2 dx\right) - \frac{1}{6}M\left(\int_\Omega |\nabla u_n|^2 dx\right) \\ &\quad \times \int_\Omega |\nabla u_n|^2 dx - \left(\frac{1}{1-\gamma} - \frac{1}{6}\right)\lambda \int_\Omega u_n^{1-\gamma}dx \\ &\ge \frac{a}{3}\|u_n\|^2 - C\lambda\|u_n\|^{1-\gamma}, \end{aligned} \tag{2.8}$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ since $0 < 1-\gamma < 1$. Thus, the proof is complete. □

To make the concentration-compactness analysis, we introduce the dilation group D in $\mathbb{R}^3$

$$D = \left\{g_{\sigma,y} | g_{\sigma,y}u(\cdot) = \sigma^{\frac{1}{2}}u(\sigma(\cdot - y)), \quad y \in \mathbb{R}^3, \ \sigma \in \mathbb{R}^+\right\}. \tag{2.9}$$

The dilation g in D is an isometry in both $L^6(\mathbb{R}^3)$ and $\mathfrak{D} = D^{1,2}(\mathbb{R}^3)$, the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|\varphi\|_{\mathfrak{D}} = \left(\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx\right)^{\frac{1}{2}}.$$

Let $\{u_n\} \subset P$ be a concrete Palais-Smale sequence of the functional I_λ. By Lemma 2.3, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By Theorem 3.1 and Corollary 3.2 in [33], $\{u_n\}$ has a profile decomposition

$$u_n = u + \sum_{k\in\Lambda} g_{n,k}U_k + r_n, \tag{2.10}$$

where $u \in H_0^1(\Omega)$, $U_k \in \mathfrak{D}$, $g_{n,k} = g_{\sigma_{n,k},y_{n,k}} \in D$, $\sigma_{n,k} > 0$, $y_{n,k} \in \overline{\Omega}$, $r_n \in \mathfrak{D}$, Λ is an index set, satisfy:

(1) $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $g_{n,k}^{-1} u_n \rightharpoonup U_k$ in $\mathfrak{D}$, as $n \to \infty$, $k \in \Lambda$;
(2) $g_{n,k} \rightharpoonup 0$ in $\mathfrak{D}^*$, $g_{n,k}^{-1} g_{n,l} \rightharpoonup 0$ in $\mathfrak{D}^*$ as $n \to \infty$, $k, l \in \Lambda$, $k \neq l$;
(3) $\|u_n\|_{\mathfrak{D}}^2 = \|u\|_{\mathfrak{D}}^2 + \sum_{k\in\Lambda} \|U_k\|_{\mathfrak{D}}^2 + \|r_n\|_{\mathfrak{D}}^2 + o(1)$, as $n \to \infty$;
(4) $r_n \to 0$ in $L^6(\mathbb{R}^3)$ and

$$\int_\Omega u_n^6 dx = \int_\Omega u^6 dx + \sum_{k\in\Lambda} \int_{\mathbb{R}^3} |U_k|^6 dx + o(1), \text{ as } n \to \infty.$$

Here for a sequence $\{g_n\}$ of D, we say $g_n \rightharpoonup 0$ in $\mathfrak{D}^*$, if for all $v \in \mathfrak{D}$, $g_n v \rightharpoonup 0$ in $\mathfrak{D}$. Moreover since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, we have $\sigma_{n,k} \to \infty$ as $n \to \infty$, $k \in \Lambda$.

We deduce the system of coupled equations satisfied by the weak limit function u and the bubbles U_k, $k \in \Lambda$.

Lemma 2.4 *Let $\{u_n\}$ be a concrete Palais-Smale sequence of the functional I_λ, $A_n \triangleq \int_\Omega |\nabla u_n|^2 dx \to A$ as $n \to \infty$.*

(1) *Assume $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then u satisfies the equation:*

$$M(A) \int_\Omega \nabla u \nabla \varphi dx = \int_\Omega u^5 \varphi dx + \lambda \int_\Omega u^{-\gamma} \varphi dx, \ for\ \varphi \in H_0^1(\Omega). \quad (2.11)$$

(2) *Let $g_n = g_{\sigma_n, y_n} \in D$, $\sigma_n \to \infty$, as $n \to \infty$, $y_n \in \overline{\Omega}$. Assume $\widetilde{u}_n = g_n^{-1} u_n \rightharpoonup U \neq 0$ in $\mathfrak{D}$. Then U satisfies the equation:*

$$M(A) \int_{\mathbb{R}^3} \nabla U \nabla \phi dx = \int_{\mathbb{R}^3} U^5 \phi dx, \ for\ \phi \in \mathfrak{D}. \quad (2.12)$$

Proof (1) By the definition, u_n satisfies the inequality (2.6), namely

$$\lambda \int_\Omega u_n^{-\gamma} (v - u_n) dx \leq M\left(\int_\Omega |\nabla u_n|^2 dx\right) \int_\Omega \nabla u_n \nabla (v - u_n) dx - \int_\Omega u_n^5 (v - u_n) dx + |dI_\lambda|(u_n)\|v - u_n\|$$

for $v \in P$. For $\varphi \in P$, taking $v = u_n + \varphi$ as test function in (2.6) and letting $n \to \infty$, by Fatou's lemma, we obtain

$$\lambda \int_\Omega u^{-\gamma} \varphi dx \leq M(A) \int_\Omega \nabla u \nabla \varphi dx - \int_\Omega u^5 \varphi dx, \text{ for } \varphi \in P. \quad (2.13)$$

Denote $u_n^T = \min\{u_n, T\}$, $T > 0$. Taking $v = u_n - u_n^T \in P$ as test function in (2.6), we have

$$-\lambda \int_{\Omega} u_n^{-\gamma} u_n^T dx \le -M \left(\int_{\Omega} |\nabla u_n|^2 dx \right) \int_{\Omega} \nabla u_n \nabla u_n^T dx + \int_{\Omega} u_n^5 u_n^T dx + |dI_\lambda|(u_n) \|u_n^T\|.$$

Taking the limit $n \to \infty$ first, then $T \to \infty$, we obtain

$$-\lambda \int_{\Omega} u^{1-\gamma} dx \le -M(A) \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} u^6 dx. \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$\lambda \int_{\Omega} u^{-\gamma} (\varphi - u) dx \le M(A) \int_{\Omega} \nabla u \nabla (\varphi - u) dx - \int_{\Omega} u^5 (\varphi - u) dx, \text{ for } \varphi \in P,$$

which implies the equation (2.11), as in the proof of Lemma 2.2.

(2) Denote

$$d_n = \sigma_n \text{dist}(y_n, \partial\Omega).$$

We first assume $d_n \to \infty$. Let $\varphi \ge 0$, $\varphi \in C_0^\infty(\mathbb{R}^3)$. For n large enough, $g_n \varphi \in C_0^\infty(\Omega)$. Taking $v = u_n + g_n \varphi \in P$ as test function in (2.6) and making a variable change

$$y = \sigma_n (x - y_n).$$

Set $\widetilde{u}_n = g_n^{-1} u_n$. In view of

$$g_n^{-1} u_n = \sigma_n^{-\frac{1}{2}} u_n(\sigma_n^{-1} \cdot + x_n)$$

we see that

$$\nabla g_n^{-1} u_n = \sigma_n^{-\frac{1}{2}} \frac{1}{\sigma_n} \nabla u_n.$$

Consequently, let $\sigma_n (x - x_n) = y$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla u_n \nabla (g_n \varphi) dx &= \int_{\Omega_n} \sigma_n \sigma^{\frac{1}{2}} \nabla g_n^{-1} u_n \sigma_n^{\frac{1}{2}} \sigma_n \nabla \varphi \frac{1}{\sigma_n^3} dy \\ &= \int_{\Omega_n} \nabla g_n^{-1} u_n \nabla \varphi dx \\ &= \int_{\Omega_n} \nabla \widetilde{u}_n \nabla \varphi dx, \end{aligned}$$

$$\int_{\mathbb{R}^3} u_n^5 g_n \varphi dx = \int_{\Omega_n} \sigma_n^{\frac{5}{2}} (\sigma_n^{-\frac{1}{2}} u_n (\frac{y}{\sigma_n} + x_n))^5 g_n \varphi(x) \frac{1}{\sigma_n^3} dy$$

$$= \int_{\Omega_n} \sigma_n^{\frac{5}{2}} (\sigma_n^{-\frac{1}{2}} u_n(\frac{y}{\sigma_n} + x_n))^5 \sigma_n^{\frac{1}{2}} \varphi(y) \frac{1}{\sigma_n^3} dy$$
$$= \int_{\Omega_n} \widetilde{u}_n^5 \varphi dx.$$

Then, we get

$$\lambda \sigma_n^{-\frac{1}{2}\gamma - \frac{5}{2}} \int_{\Omega_n} \widetilde{u}_n^{-\gamma} \varphi dx \leq M\left(\int_{\Omega} |\nabla u_n|^2 dx\right) \int_{\Omega_n} \nabla \widetilde{u}_n \nabla \varphi dx - \int_{\Omega_n} \widetilde{u}_n^5 \varphi dx + |dI_\lambda|(\widetilde{u}_n)\|\varphi\|, \tag{2.15}$$

where

$$\Omega_n = \left\{ y | y \in \mathbb{R}^3, x = \sigma_n^{-1} y + y_n \in \Omega \right\}.$$

Taking the limit $n \to \infty$, we have

$$0 \leq M(A) \int_{\mathbb{R}^3} \nabla U \nabla \varphi dx - \int_{\mathbb{R}^3} U^5 \varphi dx, \;\; for\ \varphi \in C_0^\infty(\mathbb{R}^3), \varphi \geq 0.$$

By a density argument, we obtain

$$0 \leq M(A) \int_{\mathbb{R}^3} \nabla U \nabla V dx - \int_{\mathbb{R}^3} U^5 V dx \tag{2.16}$$

for $V \in \mathfrak{D}$, $V \geq 0$.
Taking $\varphi_R \geq 0$, $\varphi_R \in C_0^\infty(\mathbb{R}^3)$ such that

$$\begin{cases} \varphi_R = 1, & \text{for } |x| \leq R, \\ \varphi_R = 0, & \text{for } |x| \geq 2R, \\ |\nabla \varphi_R| \leq \frac{2}{R}. \end{cases}$$

Taking

$$v = g_n(\widetilde{u}_n - (\widetilde{u}_n)^T \varphi_R) = u_n - g_n((\widetilde{u}_n)^T \varphi_R) \in P$$

as test function in (2.6) and making a variable change, we have

$$-\lambda \sigma_n^{-\frac{1}{2}\gamma - \frac{5}{2}} \int_{\Omega_n} \widetilde{u}_n^{-\gamma} (\widetilde{u}_n)^T \varphi_R dx \leq -M\left(\int_{\Omega_n} |\nabla u_n|^2 dx\right) + \int_{\Omega_n} \widetilde{u}_n^5 (\widetilde{u}_n)^T \varphi_R dx + |dI_\lambda|(u_n)\|(\widetilde{u}_n)^T \varphi_R\|. \tag{2.17}$$

Notice that

$$\int_{\Omega_n} \widetilde{u}_n^{-\gamma} (\widetilde{u}_n)^T \varphi_R dx \to \int_{\mathbb{R}^3} U^{-\gamma} U^T \varphi_R dx < +\infty, \text{ as } n \to \infty.$$

Letting $n \to \infty$ in (2.17) first, then $T \to \infty$, $R \to \infty$, we obtain

$$M(A) \int_{\mathbb{R}^3} |\nabla U|^2 dx - \int_{\mathbb{R}^3} U^6 dx \le 0. \tag{2.18}$$

It follows from (2.16) and (2.17) that

$$0 \le M(A) \int_{\mathbb{R}^3} \nabla U \nabla (V-U) dx - \int_{\mathbb{R}^3} U^5 (V-U) dx, \ for \ V \in \mathfrak{D}, V \ge 0,$$

which in turn implies (2.12) in a similar way as we prove Lemma 2.2.

Finally, we assume

$$\widetilde{u}_n = g_n^{-1} u_n \rightharpoonup U \text{ in } \mathfrak{D} \text{ and } d_n = \sigma_n \text{dist}(y_n, \partial\Omega) \to d < +\infty.$$

Without loss of generality we assume $d = 0$. In this case we can prove that U satisfies $U = 0$ in $\mathbb{R}^3 \backslash \mathbb{R}^3_+$ and

$$M(A) \int_{\mathbb{R}^3_+} \nabla U \nabla V dx = \int_{\mathbb{R}^3_+} U^5 V dx, \text{ for } V \in \mathfrak{D}, V \ge 0, V = 0 \text{ in } \mathbb{R}^3 \backslash \mathbb{R}^3_+.$$

By the uniqueness theory for positive solutions of equation (1.6) (see [23]), $U \equiv 0$ in $\mathbb{R}^3$, which is a contradiction. Hence the proof is complete. □

Next we continue to make the concentration-compactness analysis on concrete Palais-Smale sequences.

Lemma 2.5 *Let $\{u_n\}$ be a concrete Palais-Smale sequence of the functional I_λ. Assume the profile decomposition (2.10) holds, namely*

$$u_n = u + \sum_{k \in \Lambda} g_{n,k} U_k + r_n.$$

Then the following conclusions hold:

(1) *The index set Λ is finite, say $\Lambda = \{1, 2, ..., N\}$ (Λ may be empty if $N = 0$).*
(2) *There exist $V_N \in \mathfrak{D}$ and $g_{n,k} \in D$, $k = 1, 2, ..., N$ such that (2a) $U_k = g_{n,k} V_N$, $k = 1, 2..., N$ and the profile decomposition (2.10) reduces to*

$$u_n = u + \sum_{k=1}^{N} g_{n,k} V_N + r_n. \tag{2.19}$$

(2b) u and V_N satisfy the system

$$\begin{cases} M(\int_\Omega |\nabla u|^2dx + N\int_{\mathbb{R}^3} |\nabla V_N|^2dx)\int_\Omega \nabla u\nabla\varphi dx = \int_\Omega (u^5+\lambda u^{-\gamma})\varphi dx, \varphi\in H_0^1(\Omega), \\ M(\int_\Omega |\nabla u|^2dx + N\int_{\mathbb{R}^3} |\nabla V_N|^2dx)\int_{\mathbb{R}^3} \nabla V_N\nabla\phi dx = \int_{\mathbb{R}^3} V_N^5\phi dx, \phi\in\mathfrak{D}. \end{cases}$$

(2c) There holds that

$$\begin{cases} \int_\Omega |\nabla u_n|^2dx = \int_\Omega |\nabla u|^2dx + N\int_{\mathbb{R}^3} |\nabla V_N|^2dx + o(1), \\ \int_\Omega u_n^6dx = \int_\Omega u^6dx + N\int_{\mathbb{R}^3} V_N^6dx + o(1), \ as\ n\to\infty. \end{cases}$$

Proof (1) By Lemma 2.4, we have the system

$$\begin{cases} M(A)\int_\Omega \nabla u\nabla\varphi dx = \int_\Omega u^5\varphi dx + \lambda\int_\Omega u^{-\gamma}\varphi dx, \varphi\in H_0^1(\Omega), \\ M(A)\int_{\mathbb{R}^3} \nabla U_k\nabla\phi dx = \int_{\mathbb{R}^3} U_k^5\phi dx, \phi\in\mathfrak{D},\ k\in\Lambda, \end{cases} \tag{2.20}$$

where $A = \lim_{n\to\infty}\int_\Omega |\nabla u_n|^2dx$. Taking $\phi = U_k$ as test function in the second equation of (2.20), by (M_1) we have

$$a\int_{\mathbb{R}^3} |\nabla U_k|^2dx \le M(A)\int_{\mathbb{R}^3} |\nabla U_k|^2dx = \int_{\mathbb{R}^N} U_k^6dx \le S^{-3}\left(\int_{\mathbb{R}^3} |\nabla U_k|^2dx\right)^3,$$

where S is the Sobolev constant for the embedding $\mathfrak{D}\hookrightarrow L^6(\mathbb{R}^3)$. Hence

$$\int_{\mathbb{R}^3} |\nabla U_k|^2dx \ge a^{\frac{1}{2}}S^{\frac{3}{2}}. \tag{2.21}$$

By the property (3) of the profile decomposition (2.10), Λ is a finite set, say $\Lambda = \{1, 2, ..., N\}$.

(2) By the uniqueness theory of the positive solutions of equation (1.6) (see [23]) and the second equation of (2.20), there exist $V_k\in\mathfrak{D}$, and $g_{n,k}\in D$, $k = 1, 2, ..., N$ such that $U_k = g_{n,k}V_N$ and V_N satisfies

$$M(A)\int_{\mathbb{R}^3} \nabla V_N\nabla\phi dx = \int_{\mathbb{R}^3} V_N^5\phi dx, \phi\in\mathfrak{D},$$

and

$$u_n = u + \sum_{k=1}^N g_{n,k}V_N + r_n,$$

and so (2a) is proved. Since u_n satisfies the inequality (2.6), namely

$$\lambda \int_\Omega u_n^{-\gamma}(v-u_n)dx \le M\left(\int_\Omega |\nabla u_n|^2 dx\right)\int_\Omega \nabla u_n \nabla(v-u_n)dx \\ -\int_\Omega u_n^5(v-u_n)dx + |dI_\lambda|(u_n)\|v-u_n\|$$

for $v \in P$. Taking $v = 2u_n$ and $v = 0$ as test function in the above inequality respectively, it yields that

$$\left| M\left(\int_\Omega |\nabla u_n|^2 dx\right)\int_\Omega |\nabla u_n|^2 dx - \int_\Omega u_n^6 dx - \lambda \int_\Omega u_n^{1-\gamma} dx \right| \\ \le |dI_\lambda|(u_n)\|u_n\| = o(1). \tag{2.22}$$

By (2.20), there holds

$$\begin{cases} M(A)\int_\Omega |\nabla u|^2 dx = \int_\Omega u^6 dx + \lambda \int_\Omega u^{1-\gamma} dx, \\ M(A)\int_{\mathbb{R}^3} |\nabla U_k|^2 dx = \int_{\mathbb{R}^3} U_k^6 dx,\ k \in \Lambda, \end{cases} \tag{2.23}$$

By the property (4) of the profile decomposition (2.10), one has

$$\int_\Omega u_n^6 dx = \int_\Omega u^6 dx + \sum_{k=1}^N \int_{\mathbb{R}^3} U_k^6 dx + o(1). \tag{2.24}$$

Notice that

$$\int_\Omega |\nabla u_n|^2 dx \to A, \quad \int_\Omega u_n^{1-\gamma} dx \to \int_\Omega u^{1-\gamma} dx$$

as $n \to \infty$. It follows from (2.22), (2.23) and (2.24) that

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla u|^2 dx + \sum_{k=1}^N \int_{\mathbb{R}^3} |\nabla U_k|^2 dx + o(1). \tag{2.25}$$

Finally since $g_{n,k} \in D$, $k = 1, 2, ..., N$ are isometry in both $L^6(\mathbb{R}^3)$ and $\mathfrak{D}$, we have

$$\int_{\mathbb{R}^3} |\nabla U_k|^2 dx = \int_{\mathbb{R}^3} |\nabla V_N|^2 dx,$$

and

$$\int_{\mathbb{R}^3} V_k^6 dx = \int_{\mathbb{R}^3} V_N^6 dx,$$

where $k = 1, 2, ..., N$. Hence, from (2.24) and (2.25), we obtain

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla u|^2 dx + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx + o(1),$$

and

$$\int_\Omega u_n^6 dx = \int_\Omega u^6 dx + N \int_{\mathbb{R}^3} V_N^6 dx + o(1)$$

as $n \to \infty$, and hence (2c) is showed. In particular,

$$A = \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla u|^2 dx + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx,$$

and u, V_N satisfy the system (2b). This finishes the proof. □

3 Threshold Value and Multiple Positive Solutions

In this section, we determine the threshold value, below which the functional I_λ satisfies the concrete Palais-Smale condition. Then we apply the Ekeland's variational principle to obtain a concrete Palais-Smale sequence at the Mountain-Pass level, and show that this level is less than the threshold value. Consequently, we can prove the existence of a Mountain-Pass type solution and a local minimizer.

Assume that $\{u_n\}$ is a concrete Palais-Smale sequence of the functional I_λ and the profile decomposition (2.19) holds, namely

$$u_n = u + \sum_{k=1}^{N} g_{n,k} V_N + r_n.$$

By Lemma 2.5, we have

$$\begin{aligned}
&\lim_{n\to\infty} I_\lambda(u_n)\\
&= \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2 dx + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx\right) - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma} dx\\
&\quad -\frac{1}{6}\left(\int_\Omega u^6 dx + N \int_{\mathbb{R}^N} V_N^6 dx\right)\\
&= \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2 dx + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx\right) - \left(\frac{1}{1-\gamma} - \frac{1}{6}\right)\lambda \int_\Omega u^{1-\gamma} dx\\
&\quad -\frac{1}{6}M\left(\int_\Omega |\nabla u|^2 dx + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx\right)\left(\int_\Omega |\nabla u|^2 dx + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx\right).
\end{aligned} \tag{3.1}$$

Here we have used the fact that,

$$M\left(\int_\Omega |\nabla u|^2 dx + N\int_{\mathbb{R}^3} |\nabla V_N|^2 dx\right)\int_\Omega |\nabla u|^2 dx = \int_\Omega u^6 dx + \lambda\int_\Omega u^{1-\gamma} dx,$$

and

$$\begin{aligned} S^{-3}\left(\int_{\mathbb{R}^3} |\nabla V_N|^2 dx\right)^3 &= \int_{\mathbb{R}^3} V_N^6 dx \\ &= M\left(\int_\Omega |\nabla u|^2 dx + N\int_{\mathbb{R}^3} |\nabla V_N|^2 dx\right)\int_{\mathbb{R}^3} |\nabla V_N|^2 dx. \quad (3.2)\end{aligned}$$

Using the following lemma, we can solve the equation (3.2) for $\int_{\mathbb{R}^3} |\nabla V_N|^2 dx$.

Lemma 3.1 *Given $s \geq 0$, the equation $M(s+Nt) = S^{-3}t^2$ has a unique positive solution $t := \mathcal{F}_N(s)$. The function $\mathcal{F}_N$ is continuously differentiable. Moreover, $\mathcal{F}_N(s) \geq \mathcal{F}_1(0) := T$, where T is the unique positive solution of the equation $M(t) = S^{-3}t^2$.*

Proof By (M_2), we deduce that the function

$$g(t,s) = \frac{M(s+Nt)}{t^2} = \frac{M(s+Nt)}{(s+Nt)^2}\frac{(s+Nt)^2}{t^2}$$

is strictly decreasing in t, and

$$\lim_{t\to+\infty} g(t,s) = 0, \quad \lim_{t\to 0^+} g(t,s) = +\infty.$$

Hence there exists a unique $t > 0$, denoted by $\mathcal{F}_N(s)$, satisfies the equation $g(t,s) = S^{-3}$, that is,

$$M(s+Nt) = S^{-3}t^2.$$

Since M is a continuously differentiable function and

$$\frac{\partial}{\partial t} g(t,s) = \frac{1}{t^3}(NtM'(s+Nt) - 2M(s+Nt)) < 0,$$

so the function is $t = \mathcal{F}_N(s)$ by the implicit function theorem. Finally by (M_1), for $t = \mathcal{F}_N(s)$ we have

$$\frac{M(T)}{T^2} = S^{-3} = \frac{M(s+Nt)}{t^2} \geq \frac{M(t)}{t^2},$$

and by (M_3), it yields that $\mathcal{F}_N(s) = t \geq T = \mathcal{F}_1(0)$. This completes the proof. □

Using Lemma 3.1, we solve the equation (3.2) and obtain

$$\int_{\mathbb{R}^3} |\nabla V_N|^2 dx = \mathcal{F}_N\left(\int_\Omega |\nabla u|^2 dx\right) \tag{3.3}$$

and rewrite the formula (3.1) as follows:

$$\begin{aligned} \lim_{n\to\infty} I_\lambda(u_n) = &\frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2 dx + N\int_{\mathbb{R}^3} |\nabla V_N|^2 dx\right) \\ &-\frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma} dx \\ &-\frac{1}{6}\left(\int_\Omega u^6 dx + N\int_{\mathbb{R}^N} V_N^6 dx\right) \\ \triangleq\ & I_N(u), \end{aligned} \tag{3.4}$$

where

$$\int_{\mathbb{R}^N} V_N^6 dx = S^{-3}\left(\int_{\mathbb{R}^3} |\nabla V_N|^2 dx\right)^3 = S^{-3}\mathcal{F}_N^3\left(\int_\Omega |\nabla u|^2 dx\right).$$

Also we rewrite the equation (in Lemma 2.5 (2b)) satisfied by u as follows:

$$\begin{aligned} &M\left(\int_\Omega |\nabla u|^2 dx + N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2 dx\right)\right)\int_\Omega \nabla u \nabla \varphi dx \\ &= \int_\Omega (u^5 + \lambda u^{-\gamma})\varphi dx \end{aligned} \tag{3.5}$$

for $\varphi \in H_0^1(\Omega)$.

Define

$$\Sigma_N = \{u | u \in P,\ u \text{ satisfies the equation (3.5)}\}, \tag{3.6}$$

$$\mu_N = \inf\{I_N(u) | u \in \Sigma_N\}. \tag{3.7}$$

The following lemma gives the lower bound for μ_N.

Lemma 3.2 *There exists a constant C_1 (independent of N) such that*

$$\mu_N \geq ND - C_1\lambda^{\frac{2}{1+\gamma}},$$

where $D = \frac{1}{2}\mathcal{M}(T) - \frac{1}{6}M(T)T$ and T as defined before is the unique solution of the equation $M(t) = S^{-3}t^2$.

Proof Let $u \in \Sigma_N$, then we have

$$M\left(\int_\Omega |\nabla u|^2 dx + N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2 dx\right)\right)\int_\Omega |\nabla u|^2 dx = \int_\Omega u^6 dx + \lambda\int_\Omega u^{1-\gamma} dx.$$

It follows from relations (3.2) and (3.3) that

$$M\left(\int_\Omega |\nabla u|^2dx + N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2dx\right)\right) = S^{-3}\mathcal{F}_N^2\left(\int_\Omega |\nabla u|^2dx\right).$$

Hence, by relation (3.4), we have

$$\begin{aligned} I_N(u) &= \frac{1}{2}\mathcal{M}(A) - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx - \frac{1}{6}\left(\int_\Omega u^6dx + NS^{-3}\mathcal{F}_N^3\left(\int_\Omega |\nabla u|^2dx\right)\right) \\ &= \frac{1}{2}\mathcal{M}(A) - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{6}\right)\int_\Omega u^{1-\gamma}dx - \frac{1}{6}M(A)A, \end{aligned} \tag{3.8}$$

where $A = \int_\Omega |\nabla u|^2dx + N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2dx\right)$.

Let

$$f(s) = \frac{1}{2}\mathcal{M}(s) - \frac{1}{6}M(s)s.$$

By (M_3), it follows that $\frac{f(s)}{s}$ is increasing in s. Hence we have

$$\begin{aligned} f(a+b) &= a\cdot\frac{f(a+b)}{a+b} + b\cdot\frac{f(a+b)}{b} \geq a\cdot\frac{f(a)}{a} + b\cdot\frac{f(a)}{b} \\ &= f(a) + f(b). \end{aligned} \tag{3.9}$$

In view of (3.8) and (3.9), we have

$$\begin{aligned} I_N(u) &\geq \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2dx\right) - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{6}\right)\int_\Omega u^{1-\gamma}dx \\ &\quad -\frac{1}{6}M\left(\int_\Omega |\nabla u|^2dx\right)\int_\Omega |\nabla u|^2dx + \frac{1}{2}\mathcal{M}\left(N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2dx\right)\right) \\ &\quad -\frac{1}{6}M\left(N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2dx\right)\right)\left(N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2dx\right)\right) \\ &\triangleq J_N(u) + G_N(u), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} J_N(u) = \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2dx\right) - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{6}\right)\int_\Omega u^{1-\gamma}dx \\ -\frac{1}{6}M\left(\int_\Omega |\nabla u|^2dx\right)\int_\Omega |\nabla u|^2dx, \end{aligned}$$

and

$$G_N(u) = \frac{1}{2}\mathcal{M}\left(N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2 dx\right)\right)$$
$$-\frac{1}{6}M\left(N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2 dx\right)\right)\left(N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2 dx\right)\right). \quad (3.11)$$

By (M_3), (3.6) and the Sobolev embedding theorem, we obtain

$$J_N(u) \geq \frac{a}{3}\int_\Omega |\nabla u|^2 dx - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{6}\right) S^{-\frac{1-\gamma}{2}} |\Omega|^{\frac{5+\gamma}{6}} \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1-\gamma}{2}}$$
$$\geq -C_1 \lambda^{\frac{2}{1+\gamma}}, \quad (3.12)$$

where the constant $C_1 = C_1(\gamma, a, |\Omega|, S) > 0$. By Lemma 3.1, it follows that

$$\mathcal{F}_N\left(\int_\Omega |\nabla u|^2 dx\right) \geq T.$$

Suppose $N \geq 1$, indeed, if $N < 1$, this means that $N = 0$. Consequently I_λ satisfies the $(PS)_c$ condition. Then

$$N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2 dx\right) \geq NT \geq T.$$

Hence by (M_3) again, one has

$$\frac{\frac{1}{2}\mathcal{M}\left(N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2 dx\right)\right) - \frac{1}{6}M\left(N\mathcal{F}_N(\int_\Omega |\nabla u|^2 dx)\right) N\mathcal{F}_N(\int_\Omega |\nabla u|^2 dx)}{N\mathcal{F}_N(\int_\Omega |\nabla u|^2 dx)}$$
$$\geq \frac{\frac{1}{2}\mathcal{M}(NT) - \frac{1}{6}M(NT)T}{NT}$$
$$\geq \frac{\frac{1}{2}\mathcal{M}(T) - \frac{1}{6}M(T)T}{T}.$$

As a result, by (M_3) again, from (3.11) it follows that

$$G_N(u) \geq \left[\frac{1}{2}\mathcal{M}(T) - \frac{1}{6}M(T)T\right] \frac{N\mathcal{F}_N\left(\int_\Omega |\nabla u|^2 dx\right)}{T}$$
$$\geq N\left[\frac{1}{2}\mathcal{M}(T) - \frac{1}{6}M(T)T\right] \triangleq ND. \quad (3.13)$$

By the definition of (3.7), the estimate for μ_N follows from relations (3.10), (3.11), (3.12) and (3.13)

$$\mu_N = \inf\{I_N(u) | u \in \Sigma_N\} \geq \inf_{u\in\Sigma_N} J_N(u) + \inf_{u\in\Sigma_N} G_N(u) \geq ND - C_1\lambda^{\frac{2}{1+\gamma}}.$$

As claimed. □

Lemma 3.3 *There exists $\Lambda_1 > 0$ such that for $\lambda < \Lambda_1$, μ_1 (where μ_1 is as in (3.7)) is achieved and $\mu_1 < D$.*

Proof Choose $\rho, \Lambda_1 > 0$ such that

$$a\rho^2 - S^{-3}\rho^6 = 0, \quad \frac{a}{6}\rho^2 - \frac{1}{1-\gamma}S^{-\frac{1-\gamma}{2}}|\Omega|^{\frac{5+\gamma}{6}}\Lambda_1\rho^{1-\gamma} = 0. \tag{3.14}$$

Assume $\lambda < \Lambda_1$. If $u \in \Sigma_1$ (Σ_1 as is in (3.6)) and $\|u\| \geq \rho$, then proceeding as the proof of Lemma 3.2, we get

$$\begin{aligned}
I_1(u) &\geq J_1(u) + G_1(u)\\
&\geq \frac{a}{3}\int_\Omega |\nabla u|^2 dx - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{6}\right)S^{-\frac{1-\gamma}{2}}|\Omega|^{\frac{5+\gamma}{6}}\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1-\gamma}{2}} + D\\
&\geq \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1-\gamma}{2}}\left[\frac{a}{3}\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1+\gamma}{2}} - \frac{\lambda}{1-\gamma}S^{-\frac{1-\gamma}{2}}|\Omega|^{\frac{5+\gamma}{6}}\right] + D\\
&\geq \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1-\gamma}{2}}\left[\frac{a}{3}\rho^{1+\gamma} - \frac{\Lambda_1}{1-\gamma}S^{-\frac{1-\gamma}{2}}|\Omega|^{\frac{5+\gamma}{6}}\right] + D\\
&\geq \frac{1}{6}a\rho^{1+\gamma}\rho^{1-\gamma} + D = \frac{1}{6}a\rho^2 + D.
\end{aligned} \tag{3.15}$$

For $u \in P$, $u \in \Sigma_1$ and $\|u\| = \rho$, notice that

$$\begin{aligned}
I_1(u) = &\frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\right) - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx - \frac{1}{6}\int_\Omega u^6 dx\\
&-\frac{1}{6}M\left(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\right)\left(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1(\int_\Omega |\nabla u|^2 dx)\right)\\
&+\frac{1}{6}M\left(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\right)\int_\Omega |\nabla u|^2 dx,
\end{aligned}$$

then according to (3.13) and (M_1), we have

$$\begin{aligned}
I_1(u) \geq &\frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2 dx\right) - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx - \frac{1}{6}\int_\Omega u^6 dx\\
&-\frac{1}{6}M\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla u|^2 dx\\
&+\frac{1}{6}M\left(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\right)\int_\Omega |\nabla u|^2 dx\\
&+\frac{1}{2}\mathcal{M}\left(\mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\right) - \frac{1}{6}M\left(\mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\right)\mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\\
\geq &\frac{a}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx - \frac{1}{6}\int_\Omega u^6 dx + D
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{a}{2}\int_{\Omega}|\nabla u|^2dx-\frac{\Lambda_1}{1-\gamma}S^{-\frac{1-\gamma}{2}}|\Omega|^{\frac{5+\gamma}{6}}\left(\int_{\Omega}|\nabla u|^2dx\right)^{\frac{1-\gamma}{2}}\\
&\quad -\frac{1}{6}S^{-3}\left(\int_{\Omega}|\nabla u|^2dx\right)^3+D\\
&=\frac{a}{2}\rho^2-\frac{\Lambda_1}{1-\gamma}S^{-\frac{1-\gamma}{2}}|\Omega|^{\frac{5+\gamma}{6}}\rho^{1-\gamma}-\frac{1}{6}S^{-3}\rho^6+D\\
&=\frac{1}{6}a\rho^2+D. \qquad (3.16)
\end{aligned}
$$

Define

$$\mu_1^*=\inf\{I_1(u)|u\in B_\rho\}, \qquad (3.17)$$

where $B_\rho=\{u|u\in P,\ \|u\|\leq\rho\}$. Now we claim

(1) $\mu_1^*<D$;
(2) μ_1^* is achieved at an interior point u of B_ρ, which is a solution of the equation (3.5), that is $u\in\Sigma_1$.

Therefore, taking into account the fact that for $u\in\Sigma_1$ with $\|u\|\geq\rho$, it follows from (3.15) and (3.16) that $I_1(u)\geq\frac{1}{6}a\rho^2+D$. By (3.17), we conclude that

$$D-C_1\lambda^{\frac{2}{1+\gamma}}\leq\mu_1\leq\inf\{I_1(u),u\in\Sigma_1,\|u\|\leq\rho\}\leq I_1(u)=\mu_1^*\leq\mu_1.$$

Hence $\mu_1=\mu_1^*<D$ and μ_1 is achieved at $u\in\Sigma_1$ with $\|u\|<\rho$.

We are going to prove the claim via the Ekeland's variational principle. For $\int_\Omega|\nabla u|^2dx$ small enough, we have

$$\mathcal{F}_1\left(\int_{\Omega}|\nabla u|^2dx\right)=\mathcal{F}_1(0)+O\left(\int_{\Omega}|\nabla u|^2dx\right)=T+O\left(\int_{\Omega}|\nabla u|^2dx\right),$$

so that

$$
\begin{aligned}
&\frac{1}{2}\mathcal{M}\left(\int_{\Omega}|\nabla u|^2dx+\mathcal{F}_1\Big(\int_{\Omega}|\nabla u|^2dx\Big)\right)-\frac{1}{6}S^{-3}\mathcal{F}_1^3\left(\int_{\Omega}|\nabla u|^2dx\right)\\
&=\frac{1}{2}\mathcal{M}(T)-\frac{1}{6}S^{-3}T^3+O\left(\int_{\Omega}|\nabla u|^2dx\right)\\
&=\frac{1}{2}\mathcal{M}(T)-\frac{1}{6}M(T)T+O\left(\int_{\Omega}|\nabla u|^2dx\right)\\
&=D+O\left(\int_{\Omega}|\nabla u|^2dx\right),
\end{aligned}
$$

and

$$I_1(u)=\frac{1}{2}\mathcal{M}\left(\int_{\Omega}|\nabla u|^2dx+\mathcal{F}_1\Big(\int_{\Omega}|\nabla u|^2dx\Big)\right)$$

$$
\begin{aligned}
&-\frac{1}{6}\left(\int_\Omega u^6dx+S^{-3}\mathcal{F}_1^3\Big(\int_\Omega |\nabla u|^2dx\Big)\right)-\frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx\\
&\le D+C\int_\Omega |\nabla u|^2dx-\frac{1}{6}\int_\Omega u^6dx-\frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx. \qquad (3.18)
\end{aligned}
$$

Since $0<1-\gamma<2<6$, from relation (3.18) we have $I_1(tu)\le D-Ct^{1-\gamma}$ as $t\to 0^+$, and $\mu_1^*<D$. By Ekeland's variational principle, there exists a sequence $\{u_n\}$ of B_ρ such that

$$
\begin{cases} I_1(u_n)\le \mu_1^*+\frac{1}{n},\\ I_1(u_n)\le I_1(v)+\frac{1}{n}\|v-u_n\|, \quad \text{for } v\in B_\rho. \end{cases} \tag{3.19}
$$

By the estimate (3.16), $I_1(u)\ge \frac{a}{12}\rho^2+D$ near a neighborhood of ∂B_ρ. We can assume that there exists $0<\rho_1<\rho$ such that $\|u_n\|<\rho_1$. For $v\in P$ and sufficiently small $t>0$, $u_{n,t}\triangleq u_n+t(v-u_n)\in B_\rho$. By (3.19) we have

$$
I_1(u_n)\le I_1(u_n+t(v-u_n))+\frac{t}{n}\|v-u_n\|,
$$

that is,

$$
I_1(u_n)\le I_1(u_{n,t})+\frac{t}{n}\|v-u_n\|.
$$

Note that

$$
\begin{aligned}
I_1(u)=&\frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2dx+\mathcal{F}_1\Big(\int_\Omega |\nabla u|^2dx\Big)\right)\\
&-\frac{1}{6}\left(\int_\Omega u^6dx+S^{-3}\mathcal{F}_1^3\Big(\int_\Omega |\nabla u|^2dx\Big)\right)-\frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\frac{\lambda}{1-\gamma}\int_\Omega \frac{u_{n,t}^{1-\gamma}-u_n^{1-\gamma}}{t}dx\\
&\le \frac{1}{2t}\mathcal{M}\left(\int_\Omega |\nabla u_{n,t}|^2dx+\mathcal{F}_1\Big(\int_\Omega |\nabla u_{n,t}|^2dx\Big)\right)\\
&\quad-\frac{1}{2t}\mathcal{M}\left(\int_\Omega |\nabla u_n|^2dx+\mathcal{F}_1\Big(\int_\Omega |\nabla u_n|^2dx\Big)\right)\\
&\quad-\frac{1}{6t}\int_\Omega (u_{n,t}^6-u_n^6)dx+\frac{1}{n}\|v-u_n\|\\
&\quad-\frac{1}{6t}S^{-3}\left[\mathcal{F}_1^3\Big(\int_\Omega |\nabla u_{n,t}|^2dx\Big)-\mathcal{F}_1^3\Big(\int_\Omega |\nabla u_n|^2dx\Big)\right].
\end{aligned}
$$

Let $t \to 0^+$, by the Fatou lemma again, we obtain

$$\begin{aligned}
&\lambda \int_\Omega u_n^{-\gamma}(v-u_n)dx \\
&\le M\Big(\int_\Omega |\nabla u_n|^2dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u_n|^2dx\Big)\Big)\int_\Omega \nabla u_n \nabla(v-u_n)dx \\
&\quad + M\Big(\int_\Omega |\nabla u_n|^2dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u_n|^2dx\Big)\Big)\mathcal{F}_1'\Big(\int_\Omega |\nabla u_n|^2dx\Big)\int_\Omega \nabla u_n \nabla(v-u_n)dx \\
&\quad - S^{-3}\mathcal{F}_1^2\Big(\int_\Omega |\nabla u_{n,t}|^2dx\Big)\mathcal{F}_1'\Big(\int_\Omega |\nabla u_n|^2dx\Big)\int_\Omega \nabla u_n \nabla(v-u_n)dx \\
&\quad - \int_\Omega u_n^5(v-u_n)dx + \frac{1}{n}\|v-u_n\| \\
&= M\Big(\int_\Omega |\nabla u_n|^2dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u_n|^2dx\Big)\Big)\int_\Omega \nabla u_n \nabla(v-u_n)dx \\
&\quad - \int_\Omega u_n^5(v-u_n)dx + \frac{1}{n}\|v-u_n\|.
\end{aligned} \tag{3.20}$$

In the above we have used the fact that

$$M\Big(\int_\Omega |\nabla u_n|^2dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u_n|^2dx\Big)\Big) = S^{-3}\mathcal{F}_1^2\Big(\int_\Omega |\nabla u_{n,t}|^2dx\Big).$$

The inequality (3.20) means that $\{u_n\}$ is a concrete Palais-Smale sequence of the functional I_1, which is defined on the complete metric space P. By the concentration-compactness analysis, as we did for the functional I_1, there exists a profile decomposition for the sequence $\{u_n\}$,

$$u_n = u + \sum_{k\in\Lambda} g_{n,k}U_k + r_n$$

satisfying the properties as in (2.10). In particular, we know:

(1) U_k satisfies the equation

$$M(A+\mathcal{F}_1(A))\int_{\mathbb{R}^3} \nabla U_k \nabla\phi dx = \int_{\mathbb{R}^3} U_k^5 \phi dx, \quad \phi \in \mathfrak{D}, \tag{3.21}$$

where $A = \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2dx$.

(2) The following equalities hold:

$$\int_\Omega |\nabla u_n|^2dx = \int_\Omega |\nabla u|^2dx + \sum_{k\in\Lambda}\int_{\mathbb{R}^3} |\nabla U_k|^2dx + \|r_n\|_{\mathfrak{D}}^2 + o(1), \tag{3.22}$$

and

$$\int_\Omega u_n^6 dx = \int_\Omega u^6 dx + \sum_{k\in\Lambda} \int_{\mathbb{R}^3} U_k^6 dx + o(1). \tag{3.23}$$

By (3.21), it is easy to see that

$$\begin{aligned} a\int_{\mathbb{R}^3} |\nabla U_k|^2 dx &\le M(A+\mathcal{F}_1(A)) \int_{\mathbb{R}^3} |\nabla U_k|^2 dx \\ &= \int_{\mathbb{R}^3} U_k^6 dx \le S^{-3}\left(\int_{\mathbb{R}^3} |\nabla U_k|^2 dx\right)^3, \end{aligned}$$

and so

$$\int_{\mathbb{R}^3} |\nabla U_k|^2 dx \ge a^{\frac{1}{2}} S^{\frac{3}{2}}.$$

If $\Lambda \neq \emptyset$, then by (3.22) and (3.14)

$$\rho_1^2 \ge \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx \ge \int_{\mathbb{R}^3} |\nabla U_k|^2 dx \ge a^{\frac{1}{2}} S^{\frac{3}{2}} = \rho^2.$$

Hence we arrive at a contradiction since $0 < \rho_1 < \rho$. Consequently, $\Lambda = \emptyset$, and so by (3.22) and (3.23), we have

$$\int_\Omega |\nabla u_n|^2 dx \to \int_\Omega |\nabla u|^2 dx, \quad \int_\Omega u_n^6 dx \to \int_\Omega u^6 dx.$$

Taking the limit $n \to \infty$ in (3.20), by Fatou's lemma we obtain

$$\begin{aligned} \lambda \int_\Omega u^{-\gamma}(v-u) dx \le M\left(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\right) \int_\Omega \nabla u \nabla (v-u) dx \\ - \int_\Omega u^5 (v-u) dx \end{aligned} \tag{3.24}$$

for $v \in P$. As in the proof of Lemma 2.2, (3.24) implies that $u^{-\gamma}\varphi \in L^1(\Omega)$ for $\varphi \in H_0^1(\Omega)$ and it holds that

$$\begin{aligned} &M\left(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\right) \int_\Omega \nabla u \nabla \varphi dx \\ &= \int_\Omega u^5 \varphi dx + \lambda \int_\Omega u^{-\gamma} \varphi dx \end{aligned}$$

for $\varphi \in H_0^1(\Omega)$. That is, $u \in \Sigma_1$. On the other hand, by (3.19) it follows that

$$\mu_1^* \le I_1(u) \le \lim_{n\to\infty} I_1(u_n) = \mu_1^*.$$

This concludes the proof. □

Proposition 3.4 *There exists* $\Lambda_2 > 0$ *such that if* $\lambda < \Lambda_2$ *and* $c < \mu_1 = \mu_1(\lambda)$*, then the functional* I_λ *satisfies the concrete Palais-Smale condition at the level* c*.*

Proof By Lemma 3.2, $\mu_N \geq ND - C_1\lambda^{\frac{2}{1+\gamma}}$. By Lemma 3.3, there exits $\Lambda_1 > 0$ such that for $\lambda < \Lambda_1$, $\mu_1 < D$. Choose Λ_0 such that $D = C_1\Lambda_0^{\frac{2}{1+\gamma}}$. Denote $\Lambda_2 = \min\{\Lambda_0, \Lambda_1\}$. For $\lambda < \Lambda_2$, $\mu_1 \leq \mu_N$ for all N.

Now, let $\{u_n\}$ be a concrete Palais-Smale functional I_λ at the level c, $\{u_n\}$ has the profile decomposition (2.19) as follows:

$$u_n = u + \sum_{k=1}^{N} g_{n,k} V_N + r_n.$$

If $N \neq 0$, then

$$c = \lim_{n\to\infty} I_\lambda(u_n) \geq \mu_N \geq \mu_1 > c.$$

We arrive at a contraction. Hence $N = 0$ and $\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla u|^2 dx + o(1)$, which means that $u_n \to u$ in $H_0^1(\Omega)$. As desired. □

Now we are in a position to prove Theorem 1.1, First we prove the existence of a local minimizer. Let ρ be as defined in Lemma 3.3 (see (3.14)). Define

$$c_0^* = \inf_{u\in B_\rho} I_\lambda(u). \tag{3.25}$$

We have the following lemma.

Lemma 3.5 *Let* Λ_2 *be as defined in Proposition 3.4. For* $\lambda < \Lambda_2$*,* I_λ *achieved its local minimum* c_0^* *at an interior point* u_0^* *in* B_ρ*,* u_0^* *is a solution of the equation (1.3).*

Proof Assume $\lambda < \Lambda_2$, $u \in P$, $\|u\| = \rho$. We have

$$\begin{aligned} I_\lambda(u) \geq& \frac{a}{2}\int_\Omega |\nabla u|^2 dx - \frac{\Lambda_1}{1-\gamma} S^{-\frac{1-\gamma}{2}} |\Omega|^{\frac{5+\gamma}{6}} \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1-\gamma}{2}} \\ &- \frac{1}{6} S^{-3}\left(\int_\Omega |\nabla u|^2 dx\right)^3 \\ =& \frac{a}{2}\rho^2 - \frac{\Lambda_1}{1-\gamma} S^{-\frac{1-\gamma}{2}} |\Omega|^{\frac{5+\gamma}{6}} \rho^{1-\gamma} - \frac{1}{6} S^{-3}\rho^6 = \frac{1}{6}a\rho^2. \end{aligned} \tag{3.26}$$

For $u \neq 0$, $I_\lambda(tu) \sim -Ct^{1-\gamma}$ as $t \to 0^+$, hence $c_0^* < 0$ (c_0^* is given in (3.25)). As we did for the functional I_1 in Lemma 3.3, we find a sequence $\{u_n\}$, by the Ekeland's variational principle such that

$$\begin{cases} c_0^* \le I_\lambda(u_n) \le c_0^* + \frac{1}{n}, \\ I_\lambda(u_n) \le I_\lambda(v) + \frac{1}{n}\|v - u_n\|, \quad \text{for } v \in B_\rho. \end{cases} \tag{3.27}$$

Consequently, $\{u_n\}$ satisfies the inequality

$$\lambda \int_\Omega u_n^{-\gamma}(v - u_n)dx \le M\left(\int_\Omega |\nabla u_n|^2 dx\right)\int_\Omega \nabla u_n \nabla(v - u_n)dx - \int_\Omega u_n^5(v - u_n)dx + \frac{1}{n}\|v - u_n\| \tag{3.28}$$

for $v \in P$. Therefore, it follows from (3.27) and (3.28) that $I_\lambda(u_n) \to c_0^*$, $|dI_\lambda|(u_n) \le \frac{1}{n}$ as $n \to \infty$. The sequence $\{u_n\}$ is a concrete Palais-Smale sequence of the functional I_λ at the level $c_0^* < 0$. Assume $\lambda < \Lambda_2$. By Proposition 3.4, I_λ satisfies the concrete Palais-Smale condition at the level c_0^*. Hence $\{u_n\}$ possesses a convergent subsequence, say $u_n \to u_0^*$ in $H_0^1(\Omega)$, $I_\lambda(u_0^*) = c_0^*$, $|dI_\lambda|(u_0^*) = 0$. This implies that u_0^* is a local minimizer of I_λ and satisfies the equation (1.3). Hence the proof is complete. □

We define the Mountain-Pass value

$$c_1^* = \inf_{\sigma \in \Gamma} \sup_{t \in [0,1]} I_\lambda(\sigma(t)), \tag{3.29}$$

where

$$\Gamma = \{\sigma | \sigma \in C([0, 1], P) : \sigma(0) = u, I_\lambda(\sigma(1)) \le 0, \|\sigma(1)\| \ge 100\rho\}, \tag{3.30}$$

and u is the local minimizer of the functional I_1 obtained in Lemma 3.3 and ρ is as defined in (3.14).

By the relation (3.26), we have

$$c_1^* \ge \inf_{u \in \partial B_\rho} I_\lambda(u) \ge \frac{1}{6}a\rho^2.$$

In the following two lemmas we show that $c_1^* < \mu_1$ and there exists a concrete Palais-Smale sequence of I_λ at the level c_1^*. Therefore there exists a Mountain-Pass type solution u_1^* of the equation (1.3) with $I_\lambda(u_1^*) = c_1^*$.

Lemma 3.6 *There exists a concrete Palais-Smale sequence of I_λ at the Mountain-Pass value c_1^*, that is, a sequence $\{u_n\}$ in P such that $I_\lambda(u_n) \to c_1^*$, $|dI_\lambda|(u_n) \to 0$ as $n \to \infty$.*

Proof The proof is an application of the Ekeland's variational principle, and will be given in Appendix. □

Let u be the local minimizer of the functional I_1, obtained in Lemma 3.4, $I_1(u) = \mu_1$ and u satisfies the equation (3.5). Similar to the proof of Lemma 11 in [16], we can deduce that $u \in L^\infty(\Omega)$. By the weak Harnack inequality, we have $u > 0$ in Ω. By regularity theory, $u \in C^2_{\mathrm{loc}}(\Omega)$.

Denote

$$U(x) = \frac{3^{\frac{1}{4}}}{(1+|x|^2)^{\frac{1}{2}}}, \quad U_\varepsilon(x) = \frac{3^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}}{(\varepsilon^2+|x|^2)^{\frac{1}{2}}}, \ x \in \mathbb{R}^3, \ \varepsilon > 0. \tag{3.31}$$

U (and U_ε) satisfies the limit equation

$$\Delta U + U^5 = 0, \quad U > 0 \text{ in } \mathbb{R}^3.$$

Choose $\eta \in C_0^\infty(B_\delta(x_0), [0,1])$ where $B_\delta(x_0) \subset \Omega$ such that $\eta(x) = 1$ near $x = x_0$ and $u(x) \geq m > 0$ for all $x \in B_\delta(x_0)$, where m is a constant. Denote $\varphi_\varepsilon = U_\varepsilon \eta$.

Lemma 3.7 *There holds* $c_1^* \leq \sup_{t\geq 0} I_\lambda(u + t\varphi_\varepsilon) \leq \mu_1 - c\varepsilon^{\frac{1}{2}}$ *for some constant* $c > 0$ *and sufficiently small* $\varepsilon > 0$.

Proof From [4], we know

$$\begin{cases} \displaystyle\int_\Omega |\nabla\varphi_\varepsilon|^2 dx = \int_{\mathbb{R}^3} |\nabla U|^2 dx + O(\varepsilon) = S^{\frac{3}{2}} + O(\varepsilon), \\ \displaystyle\int_\Omega \varphi_\varepsilon^6 dx = \int_{\mathbb{R}^3} U^6 dx + O(\varepsilon^3) = S^{\frac{3}{2}} + O(\varepsilon^3), \end{cases}$$

$$\int_\Omega |\nabla\varphi_\varepsilon| dx \leq C\varepsilon^{\frac{1}{2}},$$

and

$$\int_\Omega \varphi_\varepsilon^q dx = \begin{cases} C\varepsilon^{3-\frac{q}{2}}, & 3 < q < 6, \\ C\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, & q = 3, \\ C\varepsilon^{\frac{q}{2}}, & 0 < q < 3. \end{cases}$$

Let u be the local minimizer of the functional I_1, we have

$$I_\lambda(u + t\varphi_\varepsilon) = \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla(u+t\varphi_\varepsilon)|^2 dx\right) - \frac{1}{6}\int_\Omega (u+t\varphi_\varepsilon)^6 dx \\ - \frac{\lambda}{1-\gamma}\int_\Omega (u+t\varphi_\varepsilon)^{1-\gamma} dx.$$

Since $M(t) = o(t^2)$, $\mathcal{M}(t) = o(t^3)$ as $t \to +\infty$, $I_\lambda(u+t\varphi_\varepsilon) \to -\infty$ as $t \to \infty$. Moreover, $I_\lambda(u) < 0$, we can assume there exist $0 < t_1 < t_2$ such that

$$\sup_{t\geq 0} I_\lambda(u+t\varphi_\varepsilon) = \sup_{t\in[t_1,t_2]} I_\lambda(u+t\varphi_\varepsilon).$$

Note that

$$\begin{aligned}
&\frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla(u+t\varphi_\varepsilon)|^2dx\right)\\
&\quad=\frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2dx+t^2\int_{\mathbb{R}^3}|\nabla U|^2dx+2t\int_\Omega \nabla u\nabla\varphi_\varepsilon dx+O(\varepsilon)\right)\\
&\quad=\frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2dx+t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\\
&\qquad+tM\left(\int_\Omega |\nabla u|^2dx+t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\int_\Omega \nabla u\nabla\varphi_\varepsilon dx+O(\varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{6}\int_\Omega (u+t\varphi_\varepsilon)^6dx\\
&\quad\ge\frac{1}{6}\int_\Omega u^6dx+t\int_\Omega u^5\varphi_\varepsilon dx+t^5\int_\Omega u\varphi_\varepsilon^5dx+\frac{t^6}{6}\int_\Omega \varphi_\varepsilon^6dx\\
&\quad\ge\frac{1}{6}\int_\Omega u^6dx+t\int_\Omega u^5\varphi_\varepsilon dx+Ct^5\int_\Omega \varphi_\varepsilon^5dx+\frac{t^6}{6}\int_{\mathbb{R}^3} U^6dx+O(\varepsilon^3).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\frac{1}{1-\gamma}\int_\Omega (u+t\varphi_\varepsilon)^{1-\gamma}dx\\
&\quad\ge\frac{1}{1-\gamma}\int_\Omega u^{1-\gamma}dx+t\int_\Omega u^{-\gamma}\varphi_\varepsilon dx-Ct^2\int_\Omega u^{-\gamma-1}\varphi_\varepsilon^2dx\\
&\quad\ge\frac{1}{1-\gamma}\int_\Omega u^{1-\gamma}dx+t\int_\Omega u^{-\gamma}\varphi_\varepsilon dx-C\int_\Omega \varphi_\varepsilon^2dx\\
&\quad\ge\frac{1}{1-\gamma}\int_\Omega u^{1-\gamma}dx+t\int_\Omega u^{-\gamma}\varphi_\varepsilon dx-C\varepsilon,
\end{aligned}$$

where $c > 0$ is a constant. In the above, we have used the inequality

$$(1+s)^{1-\gamma}\ge 1+(1-\gamma)s-cs^2 \quad \text{for } s>0.$$

Hence

$$\begin{aligned}
I_\lambda(u+t\varphi_\varepsilon)\le&\frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2dx+t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)-\frac{t^6}{6}\int_{\mathbb{R}^3}U^6dx-\frac{1}{6}\int_\Omega u^6dx\\
&-\frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx+t\Big[M\left(\int_\Omega |\nabla u|^2dx+t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\int_\Omega \nabla u\nabla\varphi_\varepsilon dx\\
&-\int_\Omega u^5\varphi_\varepsilon dx-\lambda\int_\Omega u^{-\gamma}\varphi_\varepsilon dx\Big]-Ct^5\int_\Omega \varphi_\varepsilon^5dx+O(\varepsilon). \qquad (3.32)
\end{aligned}$$

Define

$$g(t) = \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2 dx + t^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right) - \frac{t^6}{6}\int_{\mathbb{R}^3} U^6 dx \\ -\frac{1}{6}\int_\Omega u^6 dx - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma} dx.$$

Then

$$\begin{aligned} g'(t) &= tM\left(\int_\Omega |\nabla u|^2 dx + t^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)\int_{\mathbb{R}^3} |\nabla U|^2 dx - t^5 \int_{\mathbb{R}^3} U^6 dx \\ &= t\int_{\mathbb{R}^3} |\nabla U|^2 dx \Bigg[M\left(\int_\Omega |\nabla u|^2 dx + t^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right) \\ &\quad -S^{-3}\left(t^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)^2\Bigg]. \end{aligned}$$

Let $t_0 > 0$ be the unique positive zero, according to $g'(t_0) = 0$, one has

$$M\left(\int_\Omega |\nabla u|^2 dx + t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right) = S^{-3}\left(t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)^2. \tag{3.33}$$

By the definition of $\mathcal{F}_1$ in Lemma 3.1, we have

$$t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx = \mathcal{F}_1\left(\int_\Omega |\nabla u|^2 dx\right) = \int_{\mathbb{R}^3} |\nabla V_1|^2 dx,$$

and then

$$t_0^6 \int_{\mathbb{R}^3} U^6 dx = S^{-3}\left(t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)^2 = S^{-3}\left(\int_{\mathbb{R}^3} |\nabla V_1|^2 dx\right)^2 = \int_{\mathbb{R}^3} |V_1|^6 dx.$$

Since u satisfies the equation (3.5), combining the above equalities, one has

$$\begin{aligned} 0 &= M\left(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\right)\int_\Omega \nabla u \nabla \varphi_\varepsilon dx \\ &\quad - \int_\Omega u^5 \varphi_\varepsilon dx - \lambda \int_\Omega u^{-\gamma}\varphi_\varepsilon dx \\ &= M\left(\int_\Omega |\nabla u|^2 dx + t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)\int_\Omega \nabla u \nabla \varphi_\varepsilon dx \\ &\quad - \int_\Omega u^5 \varphi_\varepsilon dx - \lambda \int_\Omega u^{-\gamma}\varphi_\varepsilon dx, \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} g(t_0) &= \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2 dx + t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right) - \frac{t_0^6}{6}\int_{\mathbb{R}^3} U^6 dx - \frac{1}{6}\int_\Omega u^6 dx \\ &= \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\nabla V_1|^2 dx\right) - \frac{1}{6}\left(\int_\Omega u^6 dx + \int_{\mathbb{R}^3} |V_1|^6 dx\right) \\ &\quad - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma} dx \\ &= I_1(u) = \mu_1. \end{aligned} \tag{3.35}$$

Moreover, we have

$$\begin{aligned} g''(t) &= M\left(\int_\Omega |\nabla u|^2 dx + t^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)\int_{\mathbb{R}^3} |\nabla U|^2 dx \\ &\quad + 2t^2 M'\left(\int_\Omega |\nabla u|^2 dx + t^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)\left(\int_{\mathbb{R}^3} |\nabla U|^2 dx\right)^2 - 5t^4 \int_{\mathbb{R}^3} U^6 dx. \end{aligned}$$

Note that

$$0 = g'(t_0) = t_0 M\left(\int_\Omega |\nabla u|^2 dx + t^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)\int_{\mathbb{R}^3} |\nabla U|^2 dx - t_0^5 \int_{\mathbb{R}^3} U^6 dx.$$

Hence by (M_1) and (M_2), we have

$$\begin{aligned} g''(t_0) &= -4M\left(\int_\Omega |\nabla u|^2 dx + t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)\int_{\mathbb{R}^3} |\nabla U|^2 dx \\ &\quad + 2t_0^2 M'\left(\int_\Omega |\nabla u|^2 dx + t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)\left(\int_{\mathbb{R}^3} |\nabla U|^2 dx\right)^2 \\ &\le -2t_0^2 M'\left(\int_\Omega |\nabla u|^2 dx + t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)\int_\Omega |\nabla u|^2 dx \int_{\mathbb{R}^3} |\nabla U|^2 dx \\ &< 0 \end{aligned}$$

provided that $M'\left(\int_\Omega |\nabla u|^2 dx + t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right) > 0$. In case

$$M'\left(\int_\Omega |\nabla u|^2 dx + t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right) = 0,$$

then

$$g''(t_0) = -4M\left(\int_\Omega |\nabla u|^2 dx + t_0^2 \int_{\mathbb{R}^3} |\nabla U|^2 dx\right)\int_{\mathbb{R}^3} |\nabla U|^2 dx.$$

Again we obtain $g''(t_0) < 0$. Since t_0 is the unique stationary point of g and $g''(t_0) < 0$, there exists a positive constant C such that

$$g(t) \leq g(t_0) - C(t-t_0)^2, \quad \text{for } t \in [t_1, t_2]. \tag{3.36}$$

Therefore, by (3.32),(3.34), (3.35) and (3.36), we have for $t \in [t_1, t_2]$

$$\begin{aligned}
I_\lambda(u+t\varphi_\varepsilon) \leq\ & \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2dx + t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right) - \frac{t^6}{6}\int_{\mathbb{R}^3}U^6dx - \frac{1}{6}\int_\Omega u^6dx \\
& -\frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx + t\Big[M\Big(\int_\Omega |\nabla u|^2dx + t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\Big)\int_\Omega \nabla u\nabla\varphi_\varepsilon dx \\
& -\int_\Omega u^5\varphi_\varepsilon dx - \lambda\int_\Omega u^{-\gamma}\varphi_\varepsilon dx\Big] - Ct^5\int_\Omega \varphi_\varepsilon^5 dx + O(\varepsilon) \\
\leq\ & \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2dx + t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right) - \frac{t_0^6}{6}\int_{\mathbb{R}^3}U^6dx - \frac{1}{6}\int_\Omega u^6dx \\
& -\frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx - C(t-t_0)^2 - C\int_\Omega \varphi_\varepsilon^5dx + O(\varepsilon) \\
& +t\Big[M\Big(\int_\Omega |\nabla u|^2dx + t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\Big)\int_\Omega \nabla u\nabla\varphi_\varepsilon dx \\
& -\int_\Omega u^5\varphi_\varepsilon dx - \lambda\int_\Omega u^{-\gamma}\varphi_\varepsilon dx\Big] + C|t-t_0|\left|\int_\Omega \nabla u\nabla\varphi_\varepsilon dx\right| \\
\leq\ & \mu_1 - C(t-t_0)^2 + C|t-t_0|\varepsilon^{\frac{1}{2}} - C\varepsilon^{\frac{1}{2}} + O(\varepsilon) \\
\leq\ & \mu_1 - C\varepsilon^{\frac{1}{2}},
\end{aligned}$$

for some $C > 0$. In the above, we have used the following inequality:

$$\begin{aligned}
& t\left|\Big[M\Big(\int_\Omega |\nabla u|^2dx + t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\Big)\right. \\
& \qquad \left.-M\Big(\int_\Omega |\nabla u|^2dx + t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\Big)\Big]\int_\Omega \nabla u\nabla\varphi_\varepsilon dx\right| \\
& \leq t_2M'(v)|t^2-t_0^2|\int_{\mathbb{R}^3}|\nabla U|^2dx\left|\int_\Omega \nabla u\nabla\varphi_\varepsilon dx\right| \\
& \leq t_2M'(v)(t_2+t_0)|t-t_0|\int_{\mathbb{R}^3}|\nabla U|^2dx\left|\int_\Omega \nabla u\nabla\varphi_\varepsilon dx\right| \\
& \leq C|t-t_0|\int_\Omega |\nabla u\nabla\varphi_\varepsilon|dx \\
& \leq C|t-t_0|\varepsilon^{\frac{1}{2}},
\end{aligned}$$

where v is between $\int_\Omega |\nabla u|^2dx + t^2\int_{\mathbb{R}^3}|\nabla U|^2dx$ and $\int_\Omega |\nabla u|^2dx + t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx$. This leads us to the proof. □

Proof of Theorem 1.1 Assume $0 < \lambda < \Lambda := \Lambda_2$. By Lemma 3.5, I_λ has a local minimizer u_0^* in B_ρ with $I_\lambda(u_0^*) = c_0^* < 0$, $|dI_\lambda|(u_0^*) = 0$. By Lemma 3.7, $0 <$

$c_1^* < \mu_1$. By Proposition 3.4, I_λ satisfies the concrete Palais-Smale condition at the level c_1^*. By Lemma 3.6, there exists a concrete Palais-Smale sequence $\{u_n\}$ such that $|dI_\lambda|(u_n) \to 0$, $I_\lambda(u_n) \to c_1^*$ as $n \to \infty$. Up to a subsequence, $u_n \to u_1^*$ in $H_0^1(\Omega)$, and

$$I_\lambda(u_1^*) = \lim_{n\to\infty} I_\lambda(u_n) = c_1^*, \; |dI_\lambda|(u_n) \to 0.$$

By Lemma 2.2, u_0^*, u_1^* satisfy the equation (2.4). By the weak Harnack inequality, we have $u_0^*, u_1^* > 0$ in Ω. By regularity theory, $u_0^*, u_1^* \in C^2_{\mathrm{loc}}(\Omega)$ and they are positive solutions of the equation (1.3). The proof is now complete. □

Acknowledgements The authors would like to express sincere gratitude to the anonymous referees and editors for their valuable comments. Chunyu Lei is supported by Science and Technology Foundation of Guizhou Province (No. KJ[2019]1163). The research of Vicenţiu D. Rădulescu was supported by a grant of the Romanian Ministry of Research, Innovation and Digitization, CNCS/CCCDI–UEFISCDI, project number PCE 137/2021, within PNCDI III. Binlin Zhang was supported by the National Natural Science Foundation of China (No. 11871199), the Heilongjiang Province Postdoctoral Startup Foundation, PR China (LBH-Q18109), and the Cultivation Project of Young and Innovative Talents in Universities of Shandong Province.

4 Appendix

This appendix contains two lemmas. In Lemma A_1, we give the proof of Lemma 3.6, which is an application of the Ekeland's variational principle, and adapted from [31]. In Lemma A_2, we show that the functional I_λ fails to satisfy the concrete Palais-Smale condition at the level μ_1, by finding a sequence $\{u_n\}$ of I_λ such that $I_\lambda(u_n) \to \mu_1$, $|dI_\lambda|(u_n) \to 0$, but $\{u_n\}$ possesses no convergent subsequence in $H_0^1(\Omega)$. Consequently, μ_1 is exactly the threshold value for I_λ, since we have proved in Proposition 3.4 that below μ_1, the functional I_λ satisfies the concrete Palais-Smale condition.

Lemma A_1. *Let c_1^* be the Mountain Pass value as defined in (3.29). Then there exists a concrete Palais-Smale sequence of I_λ at the level c_1^*, that is a sequence $\{u_n\}$ such that $I_\lambda(u_n) \to c_1^*$ and $|dI_\lambda|(u_n) \to 0$ as $n \to \infty$.*

Proof We first recall (3.30) as follows:

$$\Gamma = \{\sigma | \sigma \in C([0,1], P) : \sigma(0) = u, I_\lambda(\sigma(1)) \le 0, \|\sigma(1)\| \ge 100\rho\}.$$

As a closed subset of $C([0,1], P)$, Γ is a complete metric space. For $g \in \Gamma$, define

$$F(g) = \sup_{t\in[0,1]} I_\lambda(g(t)).$$

Then F is continuous in Γ. By relation (3.26), we have

$$F(g) \ge \inf_{u\in\partial B_\rho} I_\lambda(u) \ge \frac{1}{6} a\rho^2.$$

Therefore, $F(g)$ is bounded from below.

Given $\varepsilon > 0$, by Ekeland's variational principle, there exists $g \in \Gamma$ such that

$$\begin{cases} F(g) \le \inf_{h\in\Gamma} F(h) + \varepsilon = c_1^* + \varepsilon, \\ F(g) \le F(h) + \varepsilon\|g - h\|, \quad h \in \Gamma. \end{cases}$$

Denote

$$\widetilde{M}(g) = \left\{ t \in [0,1] | I_\lambda(g(t)) = F(g) = \sup_{s\in[0,1]} I_\lambda(g(s)) \right\}.$$

Then

$$c_1^* \le I_\lambda(g(t)) \le c_1^* + \varepsilon, \quad \text{for } t \in \widetilde{M}(g).$$

We claim that there exists $t_\varepsilon \in \widetilde{M}(g)$ such that $|dI_\lambda|(g(t_\varepsilon)) \le \varepsilon$, which completes the proof. Otherwise, for all $t \in \widetilde{M}(g)$, $|dI_\lambda|(g(t_\varepsilon)) > \varepsilon$. By the definition, for $t \in \widetilde{M}(g)$, there exists $v(t) \in P$ such that

$$\lambda \int_\Omega g^{-\gamma}(t)(v(t) - g(t))dx > M\left(\int_\Omega |\nabla g(t)|^2 dx\right) \int_\Omega \nabla g(t)\nabla(v(t) - g(t))dx \\ - \int_\Omega g^5(t)(v(t) - g(t))dx + \varepsilon\|v(t) - g(t)\|. \quad (4.1)$$

By the Fatou lemma, in a neighborhood $B_{\delta(t)}(t)$ of t in $[0, 1]$, it holds that

$$\lambda \int_\Omega g^{-\gamma}(s)(v(t) - g(s))dx > M\left(\int_\Omega |\nabla g(s)|^2 dx\right) \int_\Omega \nabla g(s)\nabla(v(t) - g(s))dx \\ - \int_\Omega g^5(s)(v(t) - g(s))dx + \varepsilon\|v(t) - g(s)\|$$

for $s \in B_{\delta(t)}(t)$.

We may assume $B_{\delta(t)}(t) \cap \{0, 1\} = \emptyset$, since $I_\lambda(g(0)) < 0$, $I_\lambda(g(1)) < 0$, $\{B_{\delta(t)}(t) | t \in \widetilde{M}(g)\}$ is an open covering of $\widetilde{M}(g)$. There exists a finite covering $B_i = B_{\delta(t_i)}(t_i)$, $i = 1, 2, ..., n$. Let

$$\varphi_0(t) = \text{dist}\left(t, \bigcup_{i=1}^n B_i\right), \ \varphi_i(t) = \text{dist}\left(t, [0, 1]\backslash B_i\right), i = 1, 2, ..., n.$$

$\varphi_0(t) = 0$ for $t \in \widetilde{M}(g)$ and $\varphi_i(0) = \varphi_i(1) = 0$ for $i = 1, 2, ..., n$. Also define

$$\psi_i(t) = \frac{\varphi_i(t)}{\sum_{i=0}^n \varphi_i(t)}, \ t \in [0, 1]; \ \omega(t) = \sum_{i=1}^n \psi_i(t)(v(t_i) - g(t)), \ t \in [0, 1].$$

For $t \in \widetilde{M}(g)$, by (4.1) we have

$$M\left(\int_\Omega |\nabla g(t)|^2 dx\right)\int_\Omega \nabla g(t)\nabla\omega(t)dx - \int_\Omega g^5(t)\omega(t)dx - \lambda\int_\Omega g^{-\gamma}(t)\omega(t)dx$$
$$< -\varepsilon\sum_{i=1}^n \psi_i(t)\|v(t_i) - g(t)\|$$
$$\le -\varepsilon\|\sum_{i=1}^n \psi_i(t)(v(t_i) - g(t))\| = -\varepsilon\|\omega(t)\|.$$

Hence $\omega(t) \neq 0$ for $t \in \widetilde{M}(g)$. There exists $\delta > 0$ such that $\|\omega(t)\| \geq \delta$ for $t \in \widetilde{M}(g)$. Let $\varphi(t) = \min\left\{1, \frac{\delta}{\|\omega(t)\|}\right\}$, $t \in [0, 1]$, then $\varphi \in C([0, 1], \mathbb{R}^+)$. Define

$$\begin{cases} h(t) = \varphi(t)\omega(t), & t \in [0, 1], \\ \|h\| = \delta, \ \|h(t)\| = \delta, & t \in \widetilde{M}(g). \end{cases}$$

Since $h(0) = h(1) = 0$, for τ small enough, $g + \tau h \in \Gamma$, we have

$$F(g) \leq F(g + \tau h) + \varepsilon\|\tau h\| = F(g + \tau h) + \varepsilon\tau\delta. \tag{4.2}$$

Choose $t = t(\tau) \in \widetilde{M}(g + \tau h)$, one has

$$I_\lambda(g(t(\tau)) + \tau h(t(\tau))) \geq I_\lambda(g(s) + \tau h(s)), \quad \text{for } s \in [0, 1].$$

Let $\tau_n \to 0^+$, $t_n = t(\tau_n) \to t_\varepsilon$, we have

$$I_\lambda(g(t_\varepsilon)) \geq I_\lambda(g(s)), \quad \text{for } s \in [0, 1].$$

Thus, $t_\varepsilon \in \widetilde{M}(g)$. It follows from (4.2) that

$$-\varepsilon\delta \leq \frac{1}{\tau_n}[F(g + \tau_n h) - F(g)] \leq \frac{1}{\tau_n}[I_\lambda(g(t_n) + \tau_n h(t_n)) - I_\lambda(g(t_n))]. \tag{4.3}$$

Taking the limit $n \to \infty$ in (4.3), by the Fatou's lemma we obtain

$$-\varepsilon\delta \leq M\left(\int_\Omega |\nabla g(t_\varepsilon)|^2 dx\right)\int_\Omega \nabla g(t_\varepsilon)\nabla h(t_\varepsilon)dx$$
$$- \int_\Omega g^5(t_\varepsilon)h(t_\varepsilon)dx - \lambda\int_\Omega g^{1-\gamma}(t_\varepsilon)h(t_\varepsilon)dx$$
$$\leq \varphi(t_\varepsilon)\left\{M\left(\int_\Omega |\nabla g(t_\varepsilon)|^2 dx\right)\int_\Omega \nabla g(t_\varepsilon)\nabla\omega(t_\varepsilon)dx\right.$$
$$\left.- \int_\Omega g^5(t_\varepsilon)\omega(t_\varepsilon)dx - \lambda\int_\Omega g^{1-\gamma}(t_\varepsilon)\omega(t_\varepsilon)dx\right\}$$

$$< -\varphi(t_\varepsilon)\cdot\varepsilon\|\omega(t_\varepsilon)\| = -\varepsilon\delta,$$

which is a contradiction. Hence, the proof is completed. □

Here we use the notations U, U_ε, η, $\varphi_\varepsilon = \eta U_\varepsilon$ and t_0 as in Lemma 3.7. In particular, U, t_0 satisfy the equation (3.33), and let u be the local minimizer of I_1 obtained in Lemma 3.3.

Lemma A_2. *Let $u_\varepsilon = u + t_0\varphi_\varepsilon$. Then $I_\lambda(u_\varepsilon) \to \mu_1$, $|dI_\lambda|(u_\varepsilon) \to 0$, $u_\varepsilon \rightharpoonup u$, but $\int_\Omega |\nabla u_\varepsilon|^2 dx = \int_\Omega |\nabla u|^2 dx + t_0^2\int_{\mathbb{R}^3}|\nabla U|^2 dx + o(1)$. Hence u_ε ($\varepsilon \to 0$) is a concrete Palais-Smale sequence of I_λ at the level μ_1, but possesses no convergent subsequence in $H_0^1(\Omega)$.*

Proof Using the estimate for the integrals involving φ_ε, we have

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = \int_\Omega |\nabla u|^2 dx + t_0^2\int_{\mathbb{R}^3}|\nabla U|^2 dx + o(1).$$

Hence we deduce as $\varepsilon \to 0$

$$\begin{aligned} I_\lambda(u_\varepsilon) \to \frac{1}{2}\mathcal{M}\left(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\right) - \frac{1}{6}\int_\Omega u^6 dx \\ -\frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx - \frac{1}{6}S^{-3}\mathcal{F}_1^3\left(\int_\Omega |\nabla u|^2 dx\right) = \mu_1. \end{aligned}$$

For $v \in P$, denote

$$\omega_\varepsilon = v - u_\varepsilon.$$

By estimating, we show

$$M\left(\int_\Omega |\nabla u_\varepsilon|^2 dx\right)\int_\Omega \nabla u_\varepsilon \nabla\omega_\varepsilon dx = \int_\Omega u_\varepsilon^5\omega_\varepsilon dx + \lambda\int_\Omega u_\varepsilon^{-\gamma}\omega_\varepsilon dx + o(1)\|\omega_\varepsilon\|,$$

which means

$$|dI_\lambda|(u_\varepsilon) = o(1) \text{ as } \varepsilon \to 0.$$

Note that $u_\varepsilon = u + t\varphi_\varepsilon$ and $\varphi_\varepsilon = \eta U_\varepsilon$. Then we have

$$\begin{aligned} &\Big|M\Big(\int_\Omega |\nabla u_\varepsilon|^2 dx\Big)\int_\Omega \nabla u_\varepsilon\nabla\omega_\varepsilon dx \\ &\quad -M\Big(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\Big)\Big(\int_\Omega \nabla u\nabla\omega_\varepsilon dx + t_0\int_\Omega \nabla U_\varepsilon\nabla\omega_\varepsilon dx\Big)\Big| \\ &\le \Big|M\Big(\int_\Omega |\nabla u_\varepsilon|^2 dx\Big) - M\Big(\int_\Omega |\nabla u|^2 dx + \mathcal{F}_1\Big(\int_\Omega |\nabla u|^2 dx\Big)\Big)\Big|\Big|\int_\Omega \nabla u_\varepsilon\nabla\omega_\varepsilon dx\Big| \end{aligned}$$

$$
\begin{aligned}
&+\Big|M\Big(\int_\Omega |\nabla u|^2dx+\mathcal{F}_1\Big(\int_\Omega |\nabla u|^2dx\Big)\Big)\int_\Omega \nabla(u_\varepsilon-u)\nabla\omega_\varepsilon dx\\
&-M\Big(\int_\Omega |\nabla u|^2dx+\mathcal{F}_1\Big(\int_\Omega |\nabla u|^2dx\Big)\Big)t_0\int_\Omega \nabla U_\varepsilon\nabla\omega_\varepsilon dx\Big|\\
\le\,& o(1)\|\omega_\varepsilon\|+M\Big(\int_\Omega |\nabla u|^2dx+\mathcal{F}_1\Big(\int_\Omega |\nabla u|^2dx\Big)\Big)t_0\\
&\times\left|\int_\Omega[\nabla(\eta U_\varepsilon)-\nabla U_\varepsilon]\nabla\omega_\varepsilon dx\right|\\
\le\,& o(1)\|\omega_\varepsilon\|+C\left(\int_{\{|x|\ge\delta\}}(|\nabla U_\varepsilon|^2+U_\varepsilon^2)dx\right)^{\frac12}\|\omega_\varepsilon\|=o(1)\|\omega_\varepsilon\|. \qquad (4.4)
\end{aligned}
$$

In the above we assume $\eta(x)=1$ for $|x|\le\delta$. Moreover, we also have

$$
\begin{aligned}
&\left|\int_\Omega u_\varepsilon^5\omega_\varepsilon dx-\int_\Omega u^5\omega_\varepsilon dx-t_0^5\int_\Omega U_\varepsilon^5\omega_\varepsilon dx\right|\\
&\le C\int_\Omega u^4\varphi_\varepsilon|\omega_\varepsilon|dx+C\int_\Omega u^3\varphi_\varepsilon^2|\omega_\varepsilon|dx+C\int_\Omega u^2\varphi_\varepsilon^3|\omega_\varepsilon|dx\\
&\quad+C\int_\Omega u\varphi_\varepsilon^4|\omega_\varepsilon|dx+C\int_\Omega |(\eta U_\varepsilon)^5-U_\varepsilon^5||\omega_\varepsilon|dx\\
&\le o(1)\|\omega_\varepsilon\|+C\int_{\{|x|\ge\delta\}}U_\varepsilon^5|\omega_\varepsilon|dx=o(1)\|\omega_\varepsilon\|, \qquad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_\Omega u_\varepsilon^{-\gamma}\omega_\varepsilon dx-\int_\Omega u^{-\gamma}\omega_\varepsilon dx\right|&\le C\int_\Omega u^{-\gamma-1}\varphi_\varepsilon^2|\omega_\varepsilon|dx\\
&\le C\int_\Omega \varphi_\varepsilon^2|\omega_\varepsilon|dx=o(1)\|\omega_\varepsilon\|. \qquad (4.6)
\end{aligned}
$$

Since u, U_ε solve the system:

$$
\begin{cases}
M(A)\int_\Omega \nabla u\nabla\omega_\varepsilon dx-\int_\Omega u^5\omega_\varepsilon dx-\lambda\int_\Omega u^{-\gamma}\omega_\varepsilon dx=0,\\
M(A)t_0\int_{\mathbb{R}^3}\nabla U_\varepsilon\nabla\omega_\varepsilon dx=t_0^5\int_{\mathbb{R}^3}U_\varepsilon^5\omega_\varepsilon dx,
\end{cases}
$$

where $A=\int_\Omega|\nabla u|^2dx+\mathcal{F}_1(\int_\Omega|\nabla u|^2dx)$, the estimates (4.4) and (4.5) follow from (4.6). The proof is thus complete. □

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