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Research Article

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Sensitivity analysis for optimal control problems governed by nonlinear evolution inclusions

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Abstract: We consider a nonlinear optimal control problem governed by a nonlinear evolution inclusion and depending on a parameter λ. First we examine the dynamics of the problem and establish the nonemptiness of the solution set and produce continuous selections of the solution multifunction $\xi \mapsto S(\xi)$ (ξ being the initial condition). These results are proved in a very general framework and are of independent interest as results about evolution inclusions. Then we use them to study the sensitivity properties of the optimal control problem. We show that we have Hadamard well-posedness (continuity of the value function), and we establish the continuity properties of the optimal multifunction. Finally, we present an application on a nonlinear parabolic distributed parameter system.

Keywords: Evolution triple, evolution inclusion, PG and G convergence, compact embedding, Hadamard well-posedness

MSC 2010: Primary 49J24, 49K20; secondary 34G20

1 Introduction

One of the important problems in optimal control theory is the study of the variations of the set of optimal state-control pairs and of the value of the problem when we perturb the dynamics, the cost functional and the initial condition of the problem. Such a sensitivity analysis (also known in the literature as "variational analysis") is important because it gives information about the tolerances which are permitted in the specification of the mathematical models, it suggests ways to solve parametric problems and it can also be useful in the computational analysis of the problem. For infinite dimensional systems (distributed parameter systems), such investigations were conducted for linear [8, 13, 24, 31], semilinear [30, 38] and nonlinear systems [23, 32, 33]. We also mention the books of Buttazzo [7], Dontchev and Zolezzi [17], Ito and Kunisch [25], and Sokolowski and Zolezio [39] (the latter for shape optimization problems). In this paper we conduct such an analysis for a very general class of systems driven by nonmonotone evolution inclusions.

So, let $T = [0, b]$ be the time interval and (X, H, X^*) an evolution triple of spaces (see Section 2). We assume that $X \hookrightarrow H$ compactly. The space of controls is modelled by a separable reflexive Banach space Y, and E is a compact metric space that corresponds to the parameter space. As we have already mentioned, we

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consider systems monitored by evolution inclusions. These inclusions represent a way to model systems with deterministic uncertainties, see the books of Aubin and Frankowska [2], Fattorini [18] and Roubicek [37].

The problem under consideration is the following:

$$\begin{cases} J(x,u,\lambda) = \int_0^b L(t,x(t),\lambda)\,dt + \int_0^b H(t,u(t),\lambda)\,dt + \hat{\psi}(\xi,x(b),\lambda) \to \inf = m(\xi,\lambda), \\ -x'(t) \in A_\lambda(t,x(t)) + F(t,x(t),\lambda) + G(t,u(t),\lambda) \quad \text{for almost all } t \in T, \qquad x(0)=\xi, \\ u(t) \in U(t,\lambda) \quad \text{for almost all } t \in T,\ \lambda \in E. \end{cases} \tag{1.1}$$

In this problem,

$$A_\lambda : T \times X \to 2^{X^*} \quad \text{for every } \lambda > 0, \qquad F : T \times H \times E \to 2^H \setminus \{\emptyset\}, \qquad G : T \times Y \times E \to 2^H \setminus \{\emptyset\},$$

and the precise conditions on them will be given in Section 4. For every initial state $\xi \in H$ and every parameter $\lambda \in E$, we denote the set of admissible state-control pairs (that is, pairs (x,u) which satisfy the dynamics and the constraints of problem (1.1)) by $Q(\xi,\lambda)$. We investigate the dependence of $Q(\xi,\lambda)$ on the two variables $(\xi,\lambda) \in H \times E$. Also, $\Sigma(\xi,\lambda)$ denotes the set of optimal state-control pairs (that is, $(x^*,u^*) \in \Sigma(\xi,\lambda)$ such that $J(x^*,u^*,\xi,\lambda) = m(\xi,\lambda)$). So, $\Sigma(\xi,\lambda) \subseteq Q(\xi,\lambda)$. We establish the nonemptiness of the set $\Sigma(\xi,\lambda)$ and examine the continuity properties of the value function $(\xi,\lambda) \mapsto m(\xi,\lambda)$ and of the multifunction $(\xi,\lambda) \mapsto \Sigma(\xi,\lambda)$.

The nonemptiness and other continuity and structural properties of the set $Q(\xi,\lambda)$ are consequences of general results about evolution inclusions, which we prove in Section 3 and which are of independent interest. The class of evolution inclusions considered in Section 3 is more general than the classes studied by Chen, Wang and Zu [11], Denkowski, Migorski and Papageorgiou [14], Liu [28], and Papageorgiou and Kyritsi [34].

In the next section, for the convenience of the reader, we review the main mathematical tools which we will need in this paper.

2 Mathematical background

Suppose that V and Z are Banach spaces and assume that V is embedded continuously and densely into Z (denoted by $V \hookrightarrow Z$). Then it is easy to check that:
- Z^* is embedded continuously into V^*,
- if V is reflexive, then $Z^* \hookrightarrow V^*$.

Having this observation in mind, we can introduce the notion of evolution triple of spaces, which is central in the class of evolution equations considered here.

Definition 2.1. A triple (X,H,X^*) of spaces is said to be an "evolution triple" (or "Gelfand triple" or "spaces in normal position") if the following hold:
(a) X is a separable reflexive Banach space and X^* is its topological dual.
(b) H is a separable Hilbert space identified with its dual $H^* = H$ (pivot space).
(c) $X \hookrightarrow H$.

According to the remark made in the beginning of this section, we also have $H^* = H \hookrightarrow X^*$. In this paper we also assume that the embedding of X into H is compact. Hence, by Schauder's theorem (see, for example, [20, Theorem 3.1.22]), so is the embedding of $H^* = H$ into X^*. In what follows, by $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$) we denote the norm of the space X (resp. H, X^*). By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X) and by $(\cdot,\cdot)$ the inner product of the Hilbert space H. We know that

$$\langle\cdot,\cdot\rangle|_{H\times X} = (\cdot,\cdot).$$

Also, let $\beta > 0$ be such that

$$|\cdot| \le \beta\|\cdot\|. \tag{2.1}$$

We introduce the following space which has a central role in the study of the evolution inclusions. So, let $1 < p < \infty$ and set

$$W_p(0,b) = \{x \in L^p(T,X) : x' \in L^{p'}(T,X^*)\} \quad \Big(\frac{1}{p} + \frac{1}{p'} = 1\Big).$$

In this definition the derivative of x is understood in the sense of vectorial distributions (weak derivative). In fact, if we view x as an X^*-valued function, then $x(\cdot)$ is absolutely continuous, hence strongly differentiable almost everywhere. Therefore,

$$W_p(0,b) \subseteq AC^{1,p'}(T,X^*) = W^{1,p'}((0,b),X^*).$$

The space $W_p(0,b)$, equipped with the norm

$$\|x\|_{W_p} = \|x\|_{L^p(T,X)} + \|x'\|_{L^{p'}(T,X^*)} \quad \text{for all } x \in W_p(0,b),$$

becomes a separable reflexive Banach space. We know that

$$W_p(0,b) \hookrightarrow C(T,H), \tag{2.2}$$
$$W_p(0,b) \hookrightarrow L^p(T,H) \text{ compactly}. \tag{2.3}$$

The following integration by parts formula is very helpful.

Proposition 2.2. *If $x, y \in W_p(0,b)$, then $t \mapsto (x(t),y(t))$ is absolutely continuous and*

$$\frac{d}{dt}(x(t),y(t)) = \langle x'(t),y(t)\rangle + \langle x(t),y'(t)\rangle \quad \textit{for almost all } t \in T.$$

We know that for all $1 \le p < \infty$,

$$L^p(T,X)^* = L^{p'}(T,X^*)$$

with $p' = +\infty$ if $p = 1$ (see [20, Theorem 2.2.9]).

Now, let (Ω,Σ) be a measurable space and V a separable Banach space. We introduce the following hyperspaces:

$$P_{f(c)}(V) = \{C \subseteq V : C \text{ is nonempty, closed, (convex)}\},$$
$$P_{(w)k(c)}(V) = \{C \subseteq V : C \text{ is nonempty, (weakly-)compact, (convex)}\}.$$

Given a multifunction $F\colon \Omega \to 2^V \setminus \{\emptyset\}$, the "graph" of F is the set

$$\operatorname{Gr} F = \{(\omega,v) \in \Omega \times V : v \in F(\omega)\}.$$

We say that $F(\cdot)$ is "graph measurable" if $\operatorname{Gr} F \in \Sigma \times B(V)$ with $B(V)$ being the Borel σ-field of V. If $\mu(\cdot)$ is a σ-finite measure on Σ and $F\colon \Omega \to 2^V \setminus \{\emptyset\}$ is graph measurable, then the Yankov–von Neumann–Aumann selection theorem (see [22, Theorem 2.14, p. 158]) implies that $F(\cdot)$ admits a measurable selection, that is, there exists a Σ-measurable function $f\colon \Omega \to V$ such that $f(\omega) \in F(\omega)$ μ-almost everywhere. In fact, there is a whole sequence $\{f_n\}_{n\ge1}$ of such measurable selections such that $F(\omega) \subseteq \overline{\{f_n(\omega)\}}$ μ-almost everywhere (see [22, Proposition 2.17, p. 159]). Moreover, the above results are valid if V is only a Souslin space. Recall that a Souslin space need not be metrizable (see [21, p. 232]).

A multifunction $F\colon \Omega \to P_f(V)$ is said to be "measurable" if for all $y \in V$, the function

$$\omega \mapsto d(y,F(\omega)) = \inf_{v \in F(\omega)} \|y - v\|_V$$

is Σ-measurable. A multifunction $F\colon \Omega \to P_f(V)$ which is measurable is also graph measurable. The converse is true if (Ω,Σ) admits a complete σ-finite measure μ. If (Ω,Σ,μ) is a σ-finite measure space and $F\colon \Omega \to 2^V \setminus \{\emptyset\}$ is a multifunction, then for $1 \le p \le \infty$ we introduce the set

$$S_F^p = \{f \in L^p(\Omega,Y) : f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}.$$

Evidently, $S_F^p \neq \emptyset$ if and only if $\omega \mapsto \inf_{v \in F(\omega)} \|v\|_V$ belongs to $L^p(\Omega)$. Moreover, the set S_F^p is "decomposable", that is, if $(A, f_1, f_2) \in \Sigma \times S_F^p \times S_F^p$, then

$$\chi_A f_1 + \chi_{\Omega \setminus A} f_2 \in S_F^p.$$

Here, for $C \in \Sigma$, by χ_C we denote the characteristic function of the set $C \in \Sigma$.

For every $D \subseteq \Sigma$, $D \neq \emptyset$, we define

$$|D| = \sup_{v \in D} \|v\|_V \qquad \text{and} \qquad \sigma(v^*, D) = \sup_{v \in D} \langle v^*, v \rangle_V \quad \text{for all } v^* \in V^*.$$

Here, $\langle \cdot, \cdot \rangle_V$ denotes the duality brackets of the pair (V^*, V). The function $\sigma(\cdot, D) \colon V^* \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is known as the "support function" of D.

Let Z, W be Hausdorff topological spaces. We say that a multifunction $G : Z \to 2^W \setminus \{\emptyset\}$ is "upper semicontinuous" (USC for short), respectively, "lower semicontinuous" (LSC for short), if for all $U \subseteq W$ open, the set

$$G^+(U) = \{z \in Z : G(z) \subseteq U\}, \quad \text{respectively,} \quad G^-(U) = \{z \in Z : G(z) \cap U \neq \emptyset\}$$

is open in Z. If $G(\cdot)$ is both USC and LSC, then we say that $G(\cdot)$ is continuous. On a Hausdorff topological space (W, τ) (τ being the Hausdorff topology), we can define a new topology τ_{seq} whose closed sets are the sequentially τ-closed sets. Then topological properties with respect to this topology have the prefix "sequential". Note that $\tau \subseteq \tau_{\text{seq}}$ and the two are equal if τ is first countable (see [7, p. 9] and [21, p. 808]). We say that $G \colon Z \to 2^W \setminus \{\emptyset\}$ is "closed" if the graph $\operatorname{Gr} G \subseteq Z \times W$ is closed.

For any Banach space V, on $P_f(V)$ we can define a generalized metric, known as the "Hausdorff metric", by setting

$$h(E, M) = \max\Big[\sup_{e \in E} d(e, M), \sup_{m \in M} d(m, E)\Big].$$

Recall that $(P_f(V), h)$ is a complete metric space (see [22, p. 6]). If Z is a Hausdorff topological space, a multifunction $G \colon Z \to P_f(V)$ is said to be "h-continuous", if it is continuous from Z into $(P_f(V), h)$.

Also, if $E, M \subseteq V$ are nonempty, bounded, closed and convex subsets, then (Hörmander's formula)

$$h(E, M) = \sup_{v^* \in V^*, \|v^*\|_{V^*} \leq 1} |\sigma(v^*, E) - \sigma(v^*, M)|.$$

Let (W, τ) be a Hausdorff topological space with topology τ and let $\{E_n\}_{n \geq 1} \subseteq 2^W \setminus \{\emptyset\}$. We define

$$K_{\text{seq}}(\tau) - \liminf_{n \to \infty} E_n = \Big\{y \in W : y = \tau - \lim_{n \to \infty} y_n, y_n \in E_n \text{ for all } n \in \mathbb{N}\Big\},$$

$$K_{\text{seq}}(\tau) - \limsup_{n \to \infty} E_n = \Big\{y \in W : y = \tau - \lim_{n \to \infty} y_{n_k}, y_{n_k} \in E_{n_k}, n_1 < n_2 < \cdots < n_k < \cdots\Big\}.$$

Sometimes we drop the K_{seq}-symbol and simply write $\tau - \limsup_{n \to \infty} E_n$ and $\tau - \liminf_{n \to \infty} E_n$.

Returning to the setting of an evolution triple, we consider a sequence of multivalued maps

$$a_n, a \colon L^p(T, X) \to 2^{L^{p'}(T, X^*)} \setminus \{\emptyset\} \quad (n \in \mathbb{N})$$

such that for every $h^* \in L^{p'}(T, X^*)$ the inclusions

$$y' + a_n(y) \ni h^* \quad (n \in \mathbb{N}) \qquad \text{and} \qquad y' + a(y) \ni h$$

have unique solutions $e_n(h^*), e(h^*) \in W_p(0, b)$.

We say that $\frac{d}{dt} + a_n$ "PG-converges" to $\frac{d}{dt} + a$ (denoted by $\frac{d}{dt} + a_n \xrightarrow{\text{PG}} \frac{d}{dt} + a$ as $n \to \infty$) if for every $h^* \in L^{p'}(T, X^*)$, we have

$$e_n(h^*) \xrightarrow{w} e(h^*) \quad \text{in } W_p(0, b).$$

In what follows, by X_w (respectively, H_w, X_w^*) we denote the space X (respectively, H, X^*) furnished with the weak topology. Also, by $|\cdot|_1$ we denote the Lebesgue measure on $\mathbb{R}$ and by $((\cdot, \cdot))$ we denote the duality brackets for the pair $(L^{p'}(T, X^*), L^p(T, X))$. So, we have

$$((h^*, f)) = \int_0^b \langle h^*(t), f(t) \rangle \, dt \quad \text{for all } h^* \in L^{p'}(T, X^*) \text{ and all } f \in L^p(T, X).$$

Next, let us recall some useful facts from the theory of nonlinear operators of monotone type. So, let V be a reflexive Banach space, $L\colon D(L) \subseteq V \to V^*$ a linear maximal monotone operator and $a\colon V \to 2^{V^*}$. We say that $a(\cdot)$ is "L-pseudomonotone" if the following conditions hold:
(a) For every $v \in V$, $a(v) \in P_{wkc}(V^*)$.
(b) $a(\cdot)$ is bounded (that is, maps bounded sets to bounded sets).
(c) If $\{v_n\}_{n\geq 1} \subseteq D(L)$, $v_n \xrightarrow{w} v \in D(L)$ in V, $L(v_n) \xrightarrow{w} L(v)$ in V^*, $v_n^* \in a(v_n)$ for all $n \in \mathbb{N}$, $v_n^* \xrightarrow{w} v^*$ in X^* and $\limsup_{n\to\infty} \langle v_n^*, v_n - v\rangle_V \leq 0$, then $v^* \in a(v)$ and $\langle v_n^*, v_n\rangle_V \to \langle v^*, v\rangle_V$.

Such maps have nice surjectivity properties.

The next result is due to Papageorgiou, Papalini and Renzacci [35], and it extends an earlier single-valued result of Lions [27, Theorem 1.2, p. 319].

Proposition 2.3. *Assume that V is a reflexive Banach space which is strictly convex, $L\colon D(L) \subseteq V \to V^*$ is a linear maximal monotone operator and $A\colon V \to 2^{V^*}$ is L-pseudomonotone and strongly coercive, that is,*

$$\frac{\inf_{v^* \in A(v)} \langle v^*, v\rangle_V}{\|v\|_V} \to +\infty \quad \text{as } \|v\|_V \to +\infty.$$

Then $R(L + V) = V^*$ (that is, $L + V$ is surjective).

In the next section we obtain some results about a general class of evolution inclusions, which will help us study problem (1.1) (see Section 4).

3 Nonlinear evolution inclusions

Let $T = [0, b]$ and let (X, H, X^*) be an evolution triple with $X \hookrightarrow H$ compactly (see Definition 2.1). In this section we deal with the following evolution inclusion:

$$-x'(t) \in A(t, x(t)) + E(t, x(t)) \quad \text{for almost all } t \in T, \qquad x(0) = \xi. \tag{3.1}$$

The hypotheses on the data of (3.1) are as follows.

(HA1) $A\colon T \times X \to 2^{X^*}$ is a map such that the following hold:
  (i) $t \mapsto A(t, x)$ is graph measurable for all $x \in X$.
  (ii) $\operatorname{Gr} A(t, \cdot)$ is sequentially closed in $X_w \times X_w^*$ and $x \mapsto A(t, x)$ is pseudomonotone for almost all $t \in T$.
  (iii) For almost all $t \in T$, all $x \in X$ and all $h^* \in A(t, x)$, we have

$$\|h\|_* \leq a_1(t) + c_1\|x\|^{p-1}$$

  with $2 \leq p$, $a_1 \in L^{p'}(T)$ and $c_1 > 0$.
  (iv) For almost all $t \in T$, all $x \in X$ and all $h^* \in A(t, x)$, we have

$$\langle h^*, x\rangle \geq c_2\|x\|^p - a_2(t)$$

  with $c_2 > 0$, $a_2 \in L^1(T)$.

Remark 3.1. If $A(\cdot, \cdot)$ is single-valued, then in hypothesis (HA1) (ii) we can drop the condition on the graph of $\operatorname{Gr} A(t, \cdot)$ and only assume that $x \mapsto A(t, x)$ is pseudomonotone for almost all $t \in T$. The same applies if $A(t, \cdot)$ is maximal monotone for almost all $t \in T$. An example of where the condition on the graph of $A(t, \cdot)$ is satisfied is the following (for simplicity we drop the t-dependence):

$$A(x) = -\operatorname{div} \partial\varphi(Dx) - \operatorname{div} \xi(Dx),$$

where $\varphi\colon L^p(\Omega, \mathbb{R}^N) \to \mathbb{R}$ is continuous and convex, and $\xi\colon L^p(\Omega, \mathbb{R}^N) \to \mathbb{R}$ is continuous and satisfies $|\xi(y)| \leq \hat{c}(1 + |y|^{\tau-1})$ for all $y \in \mathbb{R}^N$, $\hat{c} > 0$ and $1 \leq \tau < p$. Then recalling that $W^{1,p}(\Omega) \hookrightarrow W^{1,\tau}(\Omega)$ compactly (see Zeidler [41, p. 1026]), we easily see that $\operatorname{Gr} A$ is sequentially closed in $W^{1,p}(\Omega)_w \times W^{1,p}(\Omega)_w^*$.

$(\mathrm{H}F1)$ $F: T \times H \to P_{f_c}(H)$ is a multifunction such that the following hold:
(i) $t \mapsto F(t, x)$ is graph measurable for all $x \in H$.
(ii) $\mathrm{Gr}\, F(t, \cdot)$ is sequentially closed in $H \times H_w$ for almost all $t \in T$.
(iii) For almost all $t \in T$, all $x \in H$ and all $h \in F(t, x)$, we have

$$|h| \le a_3(t) + c_3|x|$$

with $a_3 \in L^2(T)$, $c_3 > 0$ and if $p = 2$, then $\beta^2 c_3 < c_2$ (see (2.1)).

By a solution of problem (3.1) we understand a function $x \in W_p(0, b)$ such that

$$-x'(t) = h^*(t) + f(t) \quad \text{for almost all } t \in T,$$

where $h^* \in L^{p'}(T, X^*)$ and $f \in L^2(T, H)$ are such that

$$h^*(t) \in A(t, x(t)) \quad \text{and} \quad f(t) \in F(t, x(t)) \quad \text{for almost all } t \in T.$$

By $S(\xi)$ we denote the set of solutions of problem (3.1). In the sequel we investigate the structure of $S(\xi)$.
Consider the multivalued map $a: L^p(T, X) \to 2^{L^{p'}(T,X^*)}$ defined by

$$a(x) = \{h^* \in L^{p'}(T, X^*) : h^*(t) \in A(t, x(t)) \text{ for almost all } t \in T\} \quad \text{for all } x \in L^p(T, X). \tag{3.2}$$

Note that $a(\cdot)$ has values in $P_{wkc}(L^{p'}(T, X^*))$ (see hypotheses $(\mathrm{H}A1)$ (i) and (iii), and use the Yankov–von Neumann–Aumann selection theorem, see [22, Theorem 2.14, p. 158]).

Lemma 3.2. *If hypotheses* $(\mathrm{H}A1)$ *hold,* $x_n \xrightarrow{w} x$ *in* $W_p(0, b)$, $x_n(t) \xrightarrow{w} x(t)$ *in* X *for almost all* $t \in T$, $h_n^* \xrightarrow{w} h^*$ *in* $L^{p'}(T, X^*)$ *and* $h_n^* \in a(x_n)$ *for all* $n \in \mathbb{N}$, *then* $h^* \in a(x)$.

Proof. Let $v \in X$ and consider the function $x \mapsto \sigma(v, A(t, x))$ (see Section 2). We will show that it is sequentially upper semicontinuous. To this end, we need to show that given $\lambda \in \mathbb{R}$, the superlevel set

$$E_\lambda = \{x \in X : \lambda \le \sigma(v, A(t, x))\}$$

is sequentially closed in X_w. So, we consider a sequence $\{x_n\}_{n\ge1} \subseteq E_\lambda$ and assume that

$$x_n \xrightarrow{w} x \quad \text{in } X.$$

Let $h_n^* \in A(t, x_n)$ $(n \in \mathbb{N})$ be such that

$$\langle h_n^*, v\rangle = \sigma(v, A(t, x_n)) \quad \text{for all } n \in \mathbb{N} \tag{3.3}$$

(recall $A(t, x_n) \in P_{wkc}(X^*)$). Evidently, $\{h_n^*\}_{n\ge1} \subseteq X^*$ is bounded (see hypothesis $(\mathrm{H}A1)$ (iii)) and so, by passing to a subsequence if necessary, we may assume that $h_n^* \xrightarrow{w} h^*$ in X^*. Therefore, by hypothesis $(\mathrm{H}A1)$ (ii),

$$h^* \in A(t, x) \tag{3.4}$$

Then, from (3.3) and (3.4), we have

$$\lambda \le \langle h^*, v\rangle \le \sigma(v, A(t, x)),$$

and thus $x \in E_\lambda$. This proves the upper semicontinuity of the map $x \mapsto \sigma(v, A(t, x))$.

Now let $v \in L^p(T, X)$. Then we have (see [22, Theorem 3.24, p. 183])

$$((h_n^*, v)) \le \sigma(v, a(x_n)) = \int_0^b \sigma(v(t), A(t, x_n(t)))\, dt \quad \text{for all } n \in \mathbb{N},$$

and by Fatou's lemma,

$$((h^*, v)) \le \limsup_{n\to\infty} \sigma(v, a(x_n)) \le \int_0^b \limsup_{n\to\infty} \sigma(v(t), A(t, x_n(t)))\, dt.$$

By the first part of the proof and since, by hypothesis, $x_n(t) \xrightarrow{w} x(t)$ in X,

$$((h^*, v)) \le \int_0^b \sigma\big(v(t), A(t, x(t))\big)\, dt = \sigma(v, a(x)).$$

Thus, $h^* \in a(x)$. □

Lemma 3.3. *If hypotheses* (HA1) *hold, then the multivalued map* $a\colon L^p(T, X) \to 2^{L^{p'}(T,X^*)}$*, defined by* (3.2)*, is L-pseudomonotone.*

Proof. Suppose $x_n \xrightarrow{w} x$ in $W_p(0, b)$, $h_n^* \in a(x_n)$ for all $n \in \mathbb{N}$, $h_n^* \xrightarrow{w} h^*$ in $L^{p'}(T, X^*)$ and

$$\limsup_{n\to\infty} ((h_n^*, x_n - x)) \le 0. \tag{3.5}$$

From (2.2), we infer that

$$x_n(t) \xrightarrow{w} x(t) \quad \text{in } H \text{ for all } t \in T \text{ as } n \to \infty. \tag{3.6}$$

We set $\vartheta_n(t) = \langle h_n^*(t), x_n(t) - x(t)\rangle$. Let N be the Lebesgue-null set in $T = [0, b]$ outside of which hypotheses (HA1) (ii), (iii) and (iv) hold. Using hypotheses (HA1) (iii) and (iv), we have

$$\vartheta_n(t) \ge c_2\|x_n(t)\|^2 - a_2(t) - \big(a_1(t) + c_1\|x_n(t)\|^{p-1}\big)\|x(t)\| \quad \text{for all } t \in T \setminus N. \tag{3.7}$$

We introduce the Lebesgue measurable set $D \subseteq T$ defined by

$$D = \Big\{t \in T : \liminf_{n\to\infty} \vartheta_n(t) < 0\Big\}.$$

Suppose that $|D|_1 > 0$. If $t \in D \cap (T \setminus N)$, then from (3.7) we see that $\{x_n(t)\}_{n\ge1} \subseteq X$ is bounded. Then, from (3.6), it follows that

$$x_n(t) \xrightarrow{w} x(t) \quad \text{in } X \text{ for all } t \in D \cap (T \setminus N). \tag{3.8}$$

We fix $t \in D \cap (T \setminus N)$ and choose a subsequence $\{n_k\}$ of $\{n\}$ (in general this subsequence depends on t) such that

$$\lim_{k\to\infty} \vartheta_{n_k}(t) = \liminf_{n\to\infty} \vartheta_n(t).$$

By hypothesis (HA1) (ii), $A(t, \cdot)$ is pseudomonotone, and since $t \in D$, we infer that

$$\lim_{k\to\infty} \langle h_{n_k}^*(t), x_{n_k}(t) - x(t)\rangle = 0,$$

a contradiction. So, $|D|_1 = 0$ and we have

$$0 \le \liminf_{n\to\infty} \vartheta_n(t) \quad \text{for almost all } t \in T. \tag{3.9}$$

Invoking the extended Fatou's lemma (see [15, Theorem 2.2.33]), we have

$$\begin{aligned}
0 &\le \int_0^b \liminf_{n\to\infty} \vartheta_n(t)\, dt && \text{(see (3.9))}\\
&\le \liminf_{n\to\infty} \int_0^b \vartheta_n(t)\, dt\\
&\le \limsup_{n\to\infty} \int_0^b \vartheta_n(t)\, dt\\
&= \limsup_{n\to\infty} \int_0^b \langle h_n^*(t), x_n(t) - x(t)\rangle dt\\
&= \limsup_{n\to\infty} ((h_n^*, x_n - x)) \le 0 && \text{(see (3.5))}.
\end{aligned}$$

Therefore,

$$\int_0^b \vartheta_n(t)\,dt \to 0. \tag{3.10}$$

We write

$$|\vartheta_n(t)| = \vartheta_n^+(t) + \vartheta_n^-(t) = \vartheta_n(t) + 2\vartheta_n^-(t). \tag{3.11}$$

Note that, from (3.9),

$$\vartheta_n^-(t) \to 0 \quad \text{for almost all } t \in T. \tag{3.12}$$

Moreover, from (3.7) we have

$$\vartheta_n(t) \geq \eta_n(t) \quad \text{for almost all } t \in T \text{ and all } n \in \mathbb{N}$$

with $\{\eta_n\}_{n\geq 1} \subseteq L^1(T)$ uniformly integrable. Then

$$\vartheta_n^-(t) \leq \eta_n^-(t) \quad \text{for almost all } t \in T \text{ and all } n \in \mathbb{N}$$

with $\{\eta_n^-\}_{n\geq 1} \subseteq L^1(T)$ uniformly integrable. Using (3.12) and invoking Vitali's theorem we infer that $\vartheta_n^- \to 0$ in $L^1(T)$. Hence, from (3.10) and (3.11),

$$\vartheta_n \to 0 \quad \text{in } L^1(T). \tag{3.13}$$

Then, from (3.13) and the fact that $h_n^* \xrightarrow{w} h^*$ in $L^{p'}(T, X^*))$, we have

$$|((h_n^*, x_n)) - ((h^*, x))| \leq |((h_n^*, x_n - x))| + |((h_n^* - h^*, x))| \to 0,$$

which implies $((h_n^*, x_n)) \to ((h^*, x))$. In addition, from (3.8) and Lemma 3.2, we have that $h^* \in a(x)$. This proves the L-pseudomonotonicity of $a(\cdot)$. ☐

Remark 3.4. From the above proof it is clear why in the case of a single-valued map $A(t, x)$, in hypothesis (HA1) (ii) we can drop the condition on the graph of $A(t, \cdot)$ and only assume that $x \mapsto A(t, x)$ is pseudomonotone for almost all $t \in T$. Indeed, in this case, from (3.13) we have (at least for a subsequence) that

$$\vartheta_n(t) \to 0 \quad \text{for almost all } t \in T,$$

which, since $A(t, \cdot)$ is pseudomonotone, implies

$$A(t, x_n(t)) \xrightarrow{w} A(t, x(t)) \quad \text{for almost all } t \in T \text{ in } X^*.$$

In the multivalued case, there is no canonical way to identify the pointwise limit of the sequence $\{h_n^*(t)\}_{n\geq 1} \subseteq X^*$. If for almost all $t \in T$, $A(t, \cdot)$ is maximal monotone, then again, we do not need the graph hypothesis on $A(t, \cdot)$. In this case $a(\cdot)$ is also maximal monotone and then the lemma is a consequence of (3.5) and [4, Lemma 1.3]. It is worth mentioning that a similar strengthening of the topology in the range space was used by Defranceschi [12], while studying G-convergence of multivalued operators.

Without loss of generality, invoking the Troyanski renorming theorem (see [21, Remark 2.115]), we may assume that both X and X^* are locally uniformly convex, hence $L^p(T, X)$ and $L^{p'}(T, X^*)$ are strictly convex.

We are now ready for the first result concerning the solution set $S(\xi)$.

Theorem 3.5. *If hypotheses* (HA1), (HF1) *hold and $\xi \in H$, then the solution set $S(\xi)$ is nonempty, weakly compact in $W_p(0, b)$ and compact in $C(T, H)$.*

Proof. First suppose that $\xi \in X$. We define

$$A_1(t, x) = A(t, x + \xi) \quad \text{and} \quad F_1(t, x) = F(t, x + \xi).$$

Evidently, $A_1(t, x)$ and $F_1(t, x)$ have the same measurability, continuity and growth properties as the multivalued maps $A(t, x)$ and $F(t, x)$. So, we may equivalently consider the following Cauchy problem:

$$-x'(t) \in A_1(t, x(t)) + F_1(t, x(t)) \quad \text{for almost all } t \in T, \qquad x(0) = 0. \tag{3.14}$$

Note that if $x \in W_p(0,b)$ is a solution of (3.14), then $\hat{x} = x - \xi$ is a solution of (3.1) (when $\xi \in X$, that is, the initial condition is regular). Consider the linear densely defined operator $L\colon D(L) \subseteq L^p(T,X) \to L^{p'}(T,X^*)$ defined by

$$L(x) = x' \quad \text{for all } x \in W_p^0(0,b) = \{y \in W_p(0,b) : y(0) = 0\}$$

(the evaluation $y(0) = 0$ makes sense by virtue of (2.2)).

Consider the multivalued maps $a_1, G_1\colon L^p(T,X) \to 2^{L^{p'}(T,X^*)} \setminus \{\emptyset\}$ defined by

$$a_1(x) = \{h^* \in L^{p'}(T,X^*) : h^*(t) \in A_1(t,x(t)) \text{ for almost all } t \in T\},$$
$$G_1(x) = \{f \in L^{p'}(T,X^*) : f(t) \in F_1(t,x(t)) \text{ for almost all } t \in T\}.$$

We set $K(x) = a_1(x) + G_1(x)$ for all $x \in L^p(T,X)$. Then

$$K\colon L^p(T,X) \to 2^{L^{p'}(T,X^*)} \setminus \{\emptyset\}.$$

Claim 1. *K is L-pseudomonotone.*

Clearly, K has values in $P_{wkc}(L^{p'}(T,X^*))$ and it is bounded (see hypotheses (HA1) (iii), (HF1) (iii)).

Next, we consider a sequence $\{x_n\}_{n\ge1} \subseteq D(L)$ such that

$$\begin{cases} x_n \xrightarrow{w} x \in D(L) \quad \text{in } L^p(T,X), \qquad L(x_n) \to L(x) \quad \text{in } L^{p'}(T,X^*), \\ k_n^* \in K(x_n), \qquad k_n^* \xrightarrow{w} k^* \quad \text{in } L^{p'}(T,X^*) \qquad \text{and} \qquad \limsup\limits_{n\to\infty}((k_n^*, x_n - x)) \le 0. \end{cases} \tag{3.15}$$

Then we have

$$k_n^* = h_n^* + f_n \quad \text{with} \quad h_n^* \in a_1(x_n), \quad f_n \in G_1(x_n) \quad \text{for all } n \in \mathbb{N}.$$

Hypotheses (HA1) (iii) and (HF1) (iii) imply that

$$\{h_n^*\}_{n\ge1} \subseteq L^{p'}(T,X^*) \quad \text{and} \quad \{f_n\}_{n\ge1} \subseteq L^{p'}(T,H) \text{ are bounded.}$$

So, we may assume (at least for a subsequence) that

$$h_n^* \xrightarrow{w} h^* \quad \text{in } L^{p'}(T,X^*) \qquad \text{and} \qquad f_n \xrightarrow{w} f \quad \text{in } L^{p'}(T,H).$$

By (3.15) we have $x_n \xrightarrow{w} x$ in $W_p(0,b)$, and due to (2.3),

$$x_n \to x \quad \text{in } L^p(T,H). \tag{3.16}$$

Hence,

$$((f_n, x_n - x)) = \int_0^b (f_n(t), x_n(t) - x(t))\,dt \to 0,$$

and in view of (3.15),

$$\limsup_{n\to\infty}((h_n^*, x_n - x)) \le 0.$$

Therefore, from Lemma 3.3,

$$h^* \in a_1(x) \quad \text{and} \quad ((h_n^*, x_n)) \to ((h^*, x)).$$

Recall that

$$f_n(t) \in F(t, x_n(t)) \quad \text{for almost all } t \in T \text{ and all } n \in \mathbb{N}. \tag{3.17}$$

By (3.15), (3.16), (3.17) and [34, Proposition 6.6.33], we have

$$f(t) \in \overline{\operatorname{conv}}\, w - \limsup_{n\to\infty} F(t, x_n(t)) \subseteq F(t, x(t)) \quad \text{for almost all } t \in T$$

(see hypothesis (HF1) (ii)), and thus $f \in G_1(x)$.

Since $((f_n, x_n - x)) = \int_0^b (f_n(t), x_n(t) - x(t))dt \to 0$, we conclude that K is L-pseudomonotone. This proves Claim 1.

Claim 2. *K is coercive.*

Let $x \in L^p(T, X)$ and $k^* \in K(x)$. Then

$$k^* = h^* + f \quad \text{with} \quad h^* \in a_1(x),\ f \in G_1(x).$$

We have (see hypothesis (HA1) (iv))

$$((k^*, x)) = ((h^*, x)) + \int_0^b (f(t), x(t))\,dt \geq c_2\|x\|^p_{L^p(T,X)} - \|a_2\|_1 - \int_0^b |f(t)||x(t)|\,dt. \tag{3.18}$$

Note that from hypothesis (HF1) (iii) and using Young's inequality with $\epsilon > 0$, we have

$$\int_0^b |f(t)||x(t)|\,dt \leq \int_0^b \big(a_3(t)|x(t)| + c_3|x(t)|^2\big)\,dt \leq \int_0^b \big(c(\epsilon)a_3(t)^2 + (c_3 + \epsilon)|x(t)|^2\big)\,dt \tag{3.19}$$

Returning to (3.18) and using (3.19), we see that

$$((k^*, x)) \geq c_2\|x\|^p_{L^p(T,X)} - c_4\|x\|^2_{L^p(T,X)} - c_5 \quad \text{for } c_4, c_5 > 0 \tag{3.20}$$

(recall that $2 \leq p$ and in case $p = 2$, choose $\epsilon > 0$ small so that $c_4 < c_2$, see hypothesis (HF1) (iii)). From (3.20) it follows that K is coercive. This proves Claim 2.

Now Claims 1 and 2 permit the use of Proposition 2.3 to find $x \in W_p(0, b)$ solving problem (3.1) when $\xi \in X$.

Next, we remove the restriction $\xi \in X$. So, suppose $\xi \in H$. We can find $\{\xi_n\}_{n\geq 1} \subseteq X$ such that $\xi_n \to \xi$ in H (recall that X is dense in H). From the first part of the proof, we know that we can find $x_n \in S(\xi_n) \subseteq W_p(0, b)$ for all $n \in \mathbb{N}$. We have

$$\begin{cases} -x_n'(t) \in A(t, x_n(t)) + F(t, x_n(t)) & \text{for almost all } t \in T, \\ x_n(0) = \xi_n, \quad n \in \mathbb{N}. \end{cases}$$

It follows that

$$-x_n' = h_n^* + f_n \quad \text{with} \quad h_n^*(t) \in A(t, x_n(t)), \quad f_n(t) \in F(t, x_n(t)) \tag{3.21}$$

for almost all $t \in T$ and all $n \in \mathbb{N}$.

We have

$$((x_n', x_n)) + ((h_n^*, x_n)) \leq \int_0^b |f_n(t)||x_n(t)|\,dt,$$

and thus

$$\frac{1}{2}|x_n(b)|^2 + c_2\|x_n\|^p_{L^p(T,X)} \leq c_6 + c_7\|x_n\|^2_{L^p(T,X)} \quad \text{for } c_6, c_7 > 0 \tag{3.22}$$

(see hypotheses (HA1) (iv), (HF1) (iii) and if $p = 2$,then, as before, we have $c_7 < c_2$).

From (3.21), (3.22) and hypotheses (HA1) (iii), (HF1) (iii), it follows that $\{x_n\}_{n\geq 1} \subseteq W_p(0, b)$ is bounded. So, we may assume that as $n \to \infty$ (see (2.3)),

$$x_n \xrightarrow{w} x \quad \text{in } W_p(0, b) \qquad \text{and} \qquad x_n \to x \quad \text{in } L^p(T, H). \tag{3.23}$$

By (3.21), for all $n \in \mathbb{N}$, we have

$$((x_n', x_n - x)) + ((h_n^*, x_n - x)) = -((f_n, x_n - x)) = -\int_0^b (f_n(t), x_n(t) - x(t))\,dt. \tag{3.24}$$

By Proposition 2.2 we know that

$$((x_n' - x', x_n - x)) = \frac{1}{2}|x_n(b) - x(b)|^2 - \frac{1}{2}|\xi_n - \xi|^2.$$

Hence,

$$((x_n', x - x_n)) \le \frac{1}{2}|\xi_n - \xi|^2 + ((x', x - x_n)).$$

By (3.23) and since $\xi_n \to \xi$ in H,

$$\limsup_{n\to\infty}((x_n', x - x_n)) \le 0. \tag{3.25}$$

Hypothesis (HF1) (iii) implies that $\{f_n\}_{n\ge1} \subseteq L^2(T, H)$ is bounded. Hence, from (3.23),

$$\int_0^b (f_n(t), x(t) - x_n(t))\, dt \to 0,$$

and so, by (3.25),

$$\limsup_{n\to\infty}[((x_n', x - x_n)) + ((f_n, x - x_n))] \le 0.$$

Therefore, from (3.24),

$$\limsup_{n\to\infty}((h_n^*, x_n - x)) \le 0. \tag{3.26}$$

By hypothesis (HA1) (iii) we see that $\{h_n^*\}_{n\ge1} \subseteq L^{p'}(T, X^*)$ is bounded. So, we may assume that

$$h_n^* \xrightarrow{w} h^* \quad \text{in } L^{p'}(T, X^*) \text{ as } n \to \infty. \tag{3.27}$$

From (3.23), (3.26) and (3.27), we see that we can use Lemma 3.3 and infer that

$$h^*(t) \in A(t, x(t)) \quad \text{for almost all } t \in T. \tag{3.28}$$

As we have already mentioned $\{f_n\}_{n\ge1} \subseteq L^2(T, H)$ is bounded and so we may assume that

$$f_n \xrightarrow{w} f \quad \text{in } L^2(T, H). \tag{3.29}$$

Using [34, Proposition 6.6.33], we have (see hypothesis (HF1) (ii))

$$f(t) \in \overline{\mathrm{conv}}\, w - \limsup_{n\to\infty} F(t, x_n(t)) \subseteq F(t, x(t)) \quad \text{for almost all } t \in T. \tag{3.30}$$

In (3.21), we pass to the limit as $n \to \infty$ and use (3.23), (3.27) and (3.29) to obtain

$$-x' = h^* + f \quad \text{with} \quad h^* \in a(x) \text{ (see (3.28))}, \quad f \in G(x) \text{ (see (3.30))}, \quad x(0) = \xi.$$

Hence, $x \in S(\xi)$. So, we have proved that when $\xi \in H$, the solution set $S(\xi)$ is a nonempty subset of $W_p(0, b)$.

Next, we will prove the compactness of $S(\xi)$ in $W_p(0, b)_w$ and in $C(T, H)$. Let $x \in S(\xi)$. For every $t \in T$ we have

$$\int_0^t \langle x'(s), x(s)\rangle\, ds + \int_0^t \langle h^*(s), x(s)\rangle\, ds \le \int_0^t |f(s)||x(s)|\, ds \quad \text{with } h^* \in a(x),$$

which implies

$$\frac{1}{2}|x(t)|^2 \le \frac{1}{2}c_8^2 + c_9 \int_0^t |x(s)|^2\, ds \quad \text{for } c_8, c_9 > 0,$$

and hence, by Gronwall's inequality,

$$|x(t)| \le M \quad \text{for some } M > 0, \text{ all } t \in T \text{ and all } x \in S(\xi). \tag{3.31}$$

Then let $r_M\colon H \to H$ be the M-radial retraction defined by

$$r_M(x) = \begin{cases} x & \text{if } |x| \le M, \\ M\dfrac{x}{|x|} & \text{if } |x| > M. \end{cases}$$

Because of the *a priori* bound (3.31), we can replace $F(t,x)$ by

$$\hat{F}(t,x) = F(t, r_M(x)).$$

Note that for all $x \in H$, $t \mapsto \hat{F}(t,x)$ is graph measurable (hence also measurable, see Section 2) and for almost all $t \in T$, $x \mapsto \hat{F}(t,x)$ has a graph which is sequentially closed in $H \times H_w$. Moreover, we see that

$$|\hat{F}(t,x)| \le a_4(t) \quad \text{for almost all } t \in T \text{ and all } x \in H \text{ with } a_4 \in L^2(T).$$

We introduce the set

$$C = \{f \in L^2(T,H) : |f(t)| \le a_4(t) \text{ for almost all } t \in T\}.$$

We consider the following Cauchy problem:

$$\begin{cases} -x'(t) \in A(t,x(t)) + f(t) & \text{for almost all } t \in T = [0,b], \\ x(0) = \xi \in H, \quad f \in C. \end{cases} \tag{3.32}$$

Let $H \colon C \to 2^{C(T,H)}$ be the map (in general, multivalued) that assigns to each $f \in C$ the set of solutions of problem (3.32). It is a consequence of Proposition 2.3 and Lemma 3.3 that $H(\cdot)$ has nonempty values.

Claim 3. *$H(C) \subseteq C(T,H)$ is compact.*

Let $\{x_n\}_{n\ge1} \subseteq H(C)$. Then

$$-x_n' = h_n^* + f_n \quad \text{with} \quad h_n^* \in a(x_n), \quad f_n \in C \quad \text{for all } n \in \mathbb{N}. \tag{3.33}$$

Hence, for all $t \in T$, we have

$$\frac{1}{2}|x_n(t)|^2 \le \frac{1}{2}c_{10}^2 + \int_0^t a_4(s)|x_n(s)|\,ds \quad \text{for some } c_{10} > 0 \text{ and all } n \in \mathbb{N},$$

which implies (see [6, Lemme A.5])

$$|x_n(t)| \le M_1 \quad \text{for } M_1 > 0, \text{ all } t \in T \text{ and all } n \in \mathbb{N}. \tag{3.34}$$

Also, using hypothesis (HA1) (iv), we have (see (3.34))

$$c_2\|x_n\|^p_{L^p(T,X)} \le c_{11} + \int_0^b a_4(t)|x_n(t)|\,dt \le c_{12} \quad \text{for all } n \in \mathbb{N}. \tag{3.35}$$

From (3.33) and (3.35) it follows that $\{x_n\}_{n\ge1} \subseteq W_p(0,b)$ is bounded. So, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_p(0,b), \qquad h_n^* \xrightarrow{w} h^* \quad \text{in } L^{p'}(T,X^*), \qquad f_n \xrightarrow{w} f \quad \text{in } L^2(T,H). \tag{3.36}$$

Passing to the limit as $n \to \infty$ in (3.33) and using (3.36), we obtain $-x' = h^* + f$. Also, from (3.33) we have

$$((h_n^*, x_n - x)) + ((x_n', x_n - x)) = -\int_0^b (f_n(t), x_n(t) - x(t))\,dt. \tag{3.37}$$

Note that, by (3.36) and (2.3),

$$\int_0^b (f_n(t), x_n(t) - x(t))\,dt \to 0. \tag{3.38}$$

Also using Proposition 2.2, we have (recall that $x_n(0) = x(0) = \xi$ for all $n \in \mathbb{N}$)

$$((x_n' - x', x_n - x)) = \frac{1}{2}|x_n(b) - x(b)|^2 \ge 0 \quad \text{for all } n \in \mathbb{N},$$

and so

$$((x', x_n - x)) \le ((x'_n, x_n - x)) \quad \text{for all } n \in \mathbb{N}. \tag{3.39}$$

From (3.36), it follows that

$$((x', x_n - x)) \to 0. \tag{3.40}$$

Returning to (3.37), passing to the limit as $n \to \infty$ and using (3.38), (3.39) and (3.40), we obtain

$$\limsup_{n\to\infty} ((h_n^*, x_n - x)) \le 0,$$

and thus $h^* \in a(x)$ (see Lemma 3.3 and (3.36)). Therefore, $H(C)$ is w-compact in $W_p(0, b)$. From the proof of Lemma 3.3 (see (3.13)), we know that

$$\int_0^b |\langle h_n^*(t), x_n(t) - x(t)\rangle| \, dt \to 0 \quad \text{as } n \to \infty. \tag{3.41}$$

In a similar fashion, we also have

$$\int_0^b |\langle h^*(t), x_n(t) - x(t)\rangle| \, dt \to 0 \quad \text{as } n \to \infty. \tag{3.42}$$

Also, by (2.3), (3.34), (3.36) and Vitali's theorem, we have

$$\int_0^b |(f_n(t) - f(t), x_n(t) - x(t))| \, dt \to 0 \quad \text{as } n \to \infty. \tag{3.43}$$

For every $t \in T$ and every $n \in \mathbb{N}$, using Proposition 2.2, we have

$$\frac{1}{2}|x_n(t) - x(t)|^2 \le \int_0^b |\langle h_n^*(t) - h^*(t), x_n(t) - x(t)\rangle| \, dt \int_0^b |(f_n(t) - f(t), x_n(t) - x(t))| \, dt,$$

and so, from (3.41), (3.42) and (3.43), $\|x_n - x\|_{C(T,H)} \to 0$. Thus, $H(C) \subseteq C(T, H)$ is compact.

However, from the previous parts of the proof it is clear that $S(\xi) \subseteq H(C)$ is weakly closed in $W_p(0, b)$ and closed in $C(T, H)$. Therefore, we conclude that $S(\xi)$ is weakly compact in $W_p(0, b)$ and compact in $C(T, H)$. □

Next, we want to produce a continuous selection of the multifunction $\xi \mapsto S(\xi)$ (we refer to [36] for more details about continuous selections of multivalued mappings). Note that $S(\cdot)$ is in general not convex-valued, and so the Michael selection theorem (see [22, Theorem 4.6, p. 92]) cannot be used. To produce a continuous selection of the solution multifunction $\xi \mapsto S(\xi)$, we need to strengthen the conditions on the multimap $A(t, \cdot)$ in order to guarantee that certain Cauchy problems admit a unique solution.

The new hypotheses on the map $A(t, x)$ are as follows.

(HA2) $A\colon T \times X \to 2^{X^*} \setminus \{\emptyset\}$ is a multivalued map such that the following hold:

(i) $t \mapsto A(t, x)$ is graph measurable for every $x \in X$.

(ii) $x \mapsto A(t, x)$ is maximal monotone for almost all $t \in T$.

(iii) For almost all $t \in T$, all $x \in X$ and all $h^* \in A(t, x)$, we have

$$\|h^*\|_* \le a_1(t) + c_1\|x\|^{p-1}$$

with $a_1 \in L^{p'}(T)$, $c_1 > 0$, $2 \le p$.

(iv) For almost all $t \in T$, all $x \in X$ and $h^* \in A(t, x)$, we have

$$\langle h^*, x\rangle \ge c_2\|x\|^p - a_2(t)$$

with $c_2 > 0$, $a_2 \in L^1(T)_+$.

Remark 3.6. As we have already mentioned in an earlier remark, since now $A(t,\cdot)$ is maximal monotone for almost all $t \in T$, we do not need the condition on the graph of $A(t,\cdot)$ (see hypothesis (HA1) (ii) and [4, Lemma 1.3]).

Also, we strengthen the condition on the multifunction $F(t,\cdot)$.

(HF2) $F\colon T \times H \to P_{f_c}(H)$ is a multifunction such that the following hold:
 (i) $t \mapsto F(t,x)$ is graph measurable for every $x \in H$.
 (ii) For almost all $t \in T$ and all $x, y \in H$, we have

$$h(F(t,x), F(t,y)) \le k(t)|x-y| \quad \text{with } k \in L^1(T)_+.$$

 (iii) For almost all $t \in T$, all $x \in H$ and all $h \in F(t,x)$, we have

$$|h| \le a_3(t) + c_3|x|$$

with $a_3 \in L^2(T)_+$, $c_3 > 0$, and if $p = 2$, then $\beta^2 c_3 < c_2$ (see (2.1)).

Remark 3.7. Hypothesis (HF2) (ii) is stronger than condition (HF1) (ii). Indeed, suppose that (HF2) (ii) holds and we have

$$x_n \to x, \qquad h_n \xrightarrow{w} h \quad \text{in } H \qquad \text{and} \qquad h_n \in F(t,x_n) \quad \text{for all } n \in \mathbb{N}. \tag{3.44}$$

By the definition of the Hausdorff metric (see Section 2), we have

$$d(h_n, F(t,x)) \le d(h_n, F(t,x_n)) + h(F(t,x_n), F(t,x)) = h(F(t,x_n), F(t,x)),$$

and therefore (see (3.44) and hypothesis (HF2) (ii))

$$d(h_n, F(t,x)) \to 0 \quad \text{as } n \to \infty.$$

The function $y \mapsto d(y, F(t,x))$ is continuous and convex, hence weakly lower semicontinuous. Therefore, by (3.44) we have

$$d(h, F(t,x)) \le \liminf_{n\to\infty} d(h_n, F(t,x)) = 0,$$

which implies $h \in F(t,x)$. This proves that condition (HF1) (ii) holds.

So, we can use Theorem 3.5 and establish that given any $\xi \in H$, the solution set $S(\xi)$ is nonempty, weakly compact in $W_p(0,b)$ and compact in $C(T,H)$. The next result extends an earlier result of Cellina and Ornelas [9] for differential inclusions in $\mathbb{R}^N$ with $A \equiv 0$.

Proposition 3.8. *If hypotheses* (HA2),(HF2) *hold, then there exists a continuous map* $\vartheta\colon H \to C(T,H)$ *such that*

$$\vartheta(\xi) \in S(\xi) \quad \textit{for all } \xi \in H.$$

Proof. Consider the following auxiliary Cauchy problem:

$$-x'(t) \in A(t, x(t)) \quad \text{for almost all } t \in T = [0,b], \qquad x(0) = \xi.$$

This problem has a unique solution $x_0(\xi) \in W_p(0,b)$ (see Proposition 2.3 and use the monotonicity of $A(t,\cdot)$ and Proposition 2.2 to check the uniqueness of this solution).

If $\xi_1, \xi_2 \in H$, then

$$-x_0'(\xi_1) = h_1^* \quad \text{and} \quad -x_0'(\xi_2) = h_2^* \quad \text{with } h_k^* \in a(x_0(\xi_k)) \text{ for } k = 1, 2.$$

So, using Proposition 2.2, we have

$$\frac{1}{2}|x_0(\xi_1)(t) - x_0(\xi_2)(t)|^2 + \int_0^t \langle h_1^*(s) - h_2^*(s), x_0(\xi_1)(s) - x_0(\xi_2)(s)\rangle\, ds = \frac{1}{2}|\xi_1 - \xi_2|^2 \quad \text{for all } t \in T,$$

and hence (recall that $A(t,\cdot)$ is monotone)

$$\|x_0(\xi_1) - x_0(\xi_2)\|_{C(T,H)} \le |\xi_1 - \xi_2|. \tag{3.45}$$

We consider the multifunction $\Gamma_0 : H \to P_{wkc}(L^1(T,H))$ defined by

$$\Gamma_0(\xi) = S^1_{F(\cdot, x_0(\xi)(\cdot))} \quad \text{for all } \xi \in H.$$

We have

$$\begin{aligned}
h(\Gamma_0(\xi), \Gamma_0(\xi_2)) &= \sup[|\sigma(g, \Gamma_0(\xi_1)) - \sigma(g, \Gamma_0(\xi_2))| : g \in L^\infty(T,H) = L^1(T,H)^*,\ \|g\|_{L^\infty(T,H)} \le 1] \\
&\le \int_0^b \sup_{|v|\le 1}\big[\big|\sigma(v, F(t, x_0(\xi_1)(t))) - \sigma(v, F(t, x_0(\xi_2)(z)))\big|\big]\, dt \quad \text{(see [34, Theorem 6.4.16])} \\
&= \int_0^b h\big(F(t, x_0(\xi_1)(t)), F(t, x_0(\xi_2)(t))\big)\, dt \\
&\le \int_0^b k(t)|x_0(\xi_1)(t) - x_0(\xi_2)(t)|\, dt \\
&\le \int_0^b k(t)|\xi_1 - \xi_2|\, dt \quad \text{(see (3.45))} \\
&= \|k\|_1 |\xi_1 - \xi_2|.
\end{aligned}$$

Therefore, $\xi \mapsto \Gamma_0(\xi)$ is h-Lipschitz.

Also, $\Gamma_0(\cdot)$ has decomposable values. So, we can apply the selection theorem of Bressan and Colombo [5] (see also [22, Theorem 8.7, p. 245]) and find a continuous map $\gamma_0 : H \to L^1(T,H)$ such that $\gamma_0(\xi) \in \Gamma_0(\xi)$ for all $\xi \in H$. Evidently, $\gamma_0(\xi) \in L^2(T,H)$ for all $\xi \in H$.

We consider the following auxiliary Cauchy problem:

$$-x'(t) \in A(t, x(t)) + \gamma_0(\xi)(t) \quad \text{for almost all } t \in T, \qquad x(0) = \xi.$$

This problem has a unique solution $x_1(\xi) \in W_p(0,b)$. By induction we will produce two sequences

$$\{x_n(\xi)\}_{n\ge 1} \subseteq W_p(0,b) \quad \text{and} \quad \{\gamma_n(\xi)\}_{n\ge 1} \subseteq L^2(T,H),$$

which satisfy the following:

(a) $x_n(\xi) \in W_p(0,b)$ is the unique solution of the Cauchy problem

$$-x'(t) \in A(t, x(t)) + \gamma_{n-1}(\xi)(t) \quad \text{for almost all } t \in T, \qquad x(0) = \xi. \tag{3.46}$$

(b) $\xi \mapsto \gamma_n(\xi)$ is continuous from H into $C(T,H)$.
(c) $\gamma_n(\xi)(t) \in F(t, x_n(\xi)(t))$ for almost all $t \in T$ and all $\xi \in H$.
(d) $|\gamma_n(\xi)(t) - \gamma_{n-1}(\xi)(t)| \le k(t)\beta_n(\xi)(t)$ for almost all $t \in T$ and all $\xi \in H$, where

$$\beta_n(\xi)(t) = 2\int_0^t \eta(\xi)(s)\frac{(\tau(t) - \tau(s))^{n-1}}{(n-1)!}\, ds + 2b\bigg(\sum_{k=1}^n \frac{\epsilon}{2^{k+1}}\bigg)\frac{\tau(t)^{n-1}}{(n-1)!}$$

with $\epsilon > 0$, $\eta(\xi)(t) = a_2(t) + c_2|x_0(\xi)(t)|$ and $\tau(t) = \int_0^t k(s)\, ds$.

Note that the maps $\xi \mapsto \eta(\xi)$ and $\xi \mapsto \beta_n(\xi)$ are continuous from H into $L^1(T)$. So, suppose we have produced $\{x_k(\xi)\}_{k=1}^n$ and $\{\gamma_k(\xi)\}_{k=1}^n$ (induction hypothesis). Let $x_{n+1}(\xi) \in W_p(0,b)$ be the unique solution of the Cauchy problem

$$-x'(t) \in A(t, x(t)) + \gamma_n(\xi)(t) \quad \text{for almost all } t \in T, \qquad x(0) = \xi. \tag{3.47}$$

By (3.46) and (3.47) we have

$$-x_n'(\xi) = h_n^* + y_{n-1}(\xi) \quad \text{and} \quad -x_{n+1}'(\xi) = h_{n+1}^* + y_n(\xi) \quad \text{in } L^{p'}(T, X^*) \tag{3.48}$$

with

$$h_n^* \in a(x_n(\xi)), \quad h_{n+1}^* \in a(x_{n+1}(\xi)). \tag{3.49}$$

Using (3.48) we can write

$$\begin{aligned}
&\langle x_{n+1}'(\xi)(t) - x_n'(\xi)(t), x_{n+1}(\xi)(t) - x_n(\xi)(t)\rangle \\
&\quad = \langle h_n^*(\xi)(t) - h_{n+1}^*(\xi)(t), x_{n+1}(\xi)(t) - x_n(\xi)(t)\rangle + (y_{n-1}(\xi)(t) - y_n(\xi)(t), x_{n+1}(\xi)(t) - x_n(\xi)(t)) \\
&\quad \le (y_{n-1}(\xi)(t) - y_n(\xi)(t), x_{n+1}(\xi)(t) - x_n(\xi)(t)) \quad \text{for almost all } t \in T
\end{aligned}$$

(see hypothesis (HA2) (ii) and (3.49)). Therefore, from Proposition 2.2,

$$\frac{1}{2}\frac{d}{dt}|x_{n+1}(\xi)(t) - x_n(\xi)(t)|^2 \le |y_{n-1}(\xi)(t) - y_n(\xi)(t)||x_{n+1}(\xi)(t) - x_n(\xi)(t)| \quad \text{for almost all } t \in T,$$

and thus

$$\frac{1}{2}|x_{n+1}(\xi)(t) - x_n(\xi)(t)|^2 \le \int_0^t |y_{n-1}(\xi)(s) - y_n(\xi)(s)|\,|x_{n+1}(\xi)(s) - x_n(\xi)(s)|\,ds \quad \text{for all } t \in T. \tag{3.50}$$

By (3.50) and [6, Lemma A.5], we infer that

$$\begin{aligned}
|x_{n+1}(\xi)(t) - x_n(\xi)(t)| &\le \int_0^t |y_{n-1}(\xi)(s) - y_n(\xi)(s)|\,ds \\
&\le \int_0^t k(s)\beta_n(\xi)(s)\,ds \quad \text{(by the induction hypothesis, see (d))} \\
&= 2\int_0^t k(s)\int_0^s \eta(\xi)(r)\frac{(\tau(s)-\tau(r))^{n-1}}{(n-1)!}\,dr\,ds + 2b\Big(\sum_{k=0}^n \frac{\epsilon}{2^{k+1}}\Big)\int_0^t k(s)\frac{\tau(s)^{n-1}}{(n-1)!}\,ds \\
&= 2\int_0^t \eta(\xi)(s)\int_s^t k(r)\frac{(\tau(r)-\tau(s))^{n-1}}{(n-1)!}\,dr\,ds + 2b\Big(\sum_{k=0}^n \frac{\epsilon}{2^{k+1}}\Big)\int_0^t \frac{d}{ds}\frac{\tau(s)^n}{n!}\,ds \\
&= 2\int_0^t \eta(\xi)(s)\int_s^t \frac{d}{dr}\frac{(\tau(r)-\tau(s))^n}{n!}\,dr\,ds + 2b\Big(\sum_{k=0}^n \frac{\epsilon}{2^{k+1}}\Big)\frac{\tau(t)^n}{n!} \\
&= 2\int_0^t \eta(\xi)(s)\frac{(\tau(t)-\tau(s))^n}{n!}\,ds + 2b\Big(\sum_{k=0}^n \frac{\epsilon}{2^{k+1}}\Big)\frac{\tau(t)^n}{n!} \\
&< \beta_{n+1}(\xi)(t) \quad \text{for almost all } t \in T
\end{aligned} \tag{3.51}$$

(see (d)). Using the induction hypothesis (see (c)) and hypothesis (HF2) (ii), we have

$$\begin{aligned}
d(y_n(\xi)(t), F(t, x_{n+1}(\xi)(t))) &\le h(F(t, x_n(\xi)(t)), F(t, x_{n+1}(\xi)(t))) \\
&\le k(t)|x_n(\xi)(t) - x_{n+1}(\xi)(t)| \\
&< k(t)\beta_{n+1}(\xi)(t) \quad \text{for almost all } t \in T
\end{aligned} \tag{3.52}$$

(see (3.51)).

Consider the multifunction $\Gamma_{n+1}\colon H \to 2^{L^1(T,H)}$ defined by

$$\Gamma_{n+1}(\xi) = \{f \in S^1_{F(\cdot, x_{n+1}(\xi)(\cdot))} : |y_n(\xi)(t) - f(t)| < k(t)\beta_{n+1}(\xi)(t) \text{ for almost all } t \in T\}.$$

By (3.52) and [22, Lemma 8.3, p 239], we have that $\xi \mapsto \Gamma_{n+1}(\xi)$ has nonempty decomposable values and it is LSC. Thus, $\xi \mapsto \overline{\Gamma_{n+1}(\xi)}$ is LSC with decomposable values.

We can apply the selection theorem of Bressan and Colombo [5, Theorem 3] to find a continuous map $y_{n+1} : H \to L^1(T, H)$ such that $y_{n+1}(\xi) \in \overline{\Gamma_{n+1}(\xi)}$ for all $\xi \in H$. This completes the induction process and we have produced two sequences $\{x_n(\xi)\}_{n\geq 1}$, $\{y_n(\xi)\}_{n\geq 1}$ which satisfy properties (a)–(d) stated earlier.

From (3.51) we have

$$\begin{aligned}\int_0^b |y_n(\xi)(t) - y_{n-1}(\xi)(t)|\,dt &< \int_0^b \beta_{n+1}(\xi)(t)\,dt\\ &< \int_0^b \eta(\xi)(t)\frac{(\tau(b)-\tau(t))^n}{n!}\,dt + 2b\epsilon\frac{\tau(b)^n}{n!}\\ &\leq \frac{\tau(b)^n}{n!}[\|\eta(\xi)\|_1 + 2b\epsilon].\end{aligned} \tag{3.53}$$

Recall that $\xi \mapsto \eta(\xi)$ is continuous from H into $L^1(H)$ and maps bounded sets to bounded sets. So, from (3.53) it follows that $\{y_n(\xi)\}_{n\geq 1} \subseteq L^1(T)$ is Cauchy, uniformly on bounded sets of H. Moreover, from (3.51) and (3.53), we have

$$\|x_{n+1}(\xi) - x_n(\xi)\|_{C(T,H)} \leq \|y_n(\xi) - y_{n-1}(\xi)\|_{L^1(T,H)} \leq \frac{\tau(b)^n}{n!}[\|\eta(\xi)\|_1 + 2b\epsilon].$$

Thus, $\{x_n(\xi)\}_{n\geq 1} \subseteq C(T, H)$ is Cauchy, uniformly on bounded sets. Therefore, we have

$$x_n(\xi) \to x(\xi) \quad \text{in } C(T,H) \qquad \text{and} \qquad y_n(\xi) \to y(\xi) \quad \text{in } L^1(T,H). \tag{3.54}$$

Evidently, $\xi \mapsto x(\xi)$ is continuous from H into $C(T, H)$, while because of hypothesis (HF2) (iii), we have $y_n(\xi) \to y(\xi)$ in $L^2(T, H)$. Let $\hat{x}(\xi) \in W_p(0, b)$ be the unique solution of

$$-y'(t) \in A(t, y(t)) + y(\xi)(t) \quad \text{for almost all } t \in T, \qquad y(0) = 0.$$

As before, exploiting the monotonicity of $A(t, \cdot)$ (see hypothesis (HA2) (ii)), we have

$$\frac{1}{2}|x_{n+1}(\xi)(t) - \hat{x}(\xi)(t)|^2 \leq \int_0^t |y_n(\xi)(s) - y(\xi)(s)|\,|x_{n+1}(\xi)(s) - \hat{x}(\xi)(s)|\,ds \quad \text{for all } t \in T,$$

which implies (see [6, Lemma A.5])

$$\|x_{n+1}(\xi) - \hat{x}(\xi)\|_{C(T,H)} \leq \|y_n(\xi) - y(\xi)\|_{L^1(T,H)}.$$

Therefore, from (3.54), $x(\xi) = \hat{x}(\xi)$.

So, $x(\xi) \in S(\xi)$ and the map $\vartheta : H \to C(T, H)$ defined by $\vartheta(\xi) = x(\xi)$ is a continuous selection of the solution multifunction $\xi \mapsto S(\xi)$. □

An easy but useful consequence of Proposition 3.8 and its proof is a parametric version of the Filippov–Gronwall inequality (see [1, Theorem 1, pp. 120–121] and [19]) for differential inclusions. So, we consider the following parametric version of problem (3.1):

$$-x'(t) \in A(t, x(t)) + F(t, x(t), \lambda) \quad \text{for almost all } t \in T, \qquad x(0) = \xi(\lambda).$$

The parameter space D is a complete metric space. The hypotheses on the parametric vector field $F(t, x, \lambda)$ and the initial condition $\xi(\lambda)$ are as follows.

(HF2)′ $F\colon T \times H \times D \to P_{f_c}(H)$ is a multifunction such that the following hold:

(i) $t \mapsto F(t, x, \lambda)$ is graph measurable for all $(x, \lambda) \in H \times D$.

(ii) For almost all $t \in T$, all $x, y \in H$ and all $\lambda \in D$, we have

$$h(F(t, x, \lambda), F(t, y, \lambda)) \leq k(t)|x - y|$$

with $k \in L^1(T)_+$.

(iii) For almost all $t \in T$, all $x \in H$, all $\lambda \in D$ and all $h \in F(t, x)$, we have

$$|h| \leq a_3(t) + c_3|x|$$

with $a_3 \in L^2(T)_+$, $c_3 > 0$, and if $p = 2$, then $\beta^2 c_3 < c_2$ (see (2.1)).

(iv) For almost all $t \in T$ and all $x \in H$, the multifunction $\lambda \mapsto F(t, x, \lambda)$ is LSC.

(H0) The mapping $\lambda \mapsto \xi(\lambda)$ is continuous from D into H.

Assume that $\lambda \mapsto (u(\lambda), h(\lambda))$ is a continuous map from D into $C(T, H) \times L^2(T, H)$. We can find a continuous map $p\colon D \to L^2(T)$ such that

$$d\big(h(\lambda)(t), F(t, u(\lambda)(t), \lambda)\big) \leq p(\lambda)(t) \quad \text{for almost all } t \in T,$$

see hypothesis (HF2)′ (iii).

In what follows, by $e(h, \lambda) \in W_p(0, b)$ we denote the unique solution of the Cauchy problem

$$-u'(t) \in A(t, u(t)) + h(t) \quad \text{for almost all } t \in T, \qquad u(0) = \xi(\lambda)$$

with $h \in L^2(T, H)$.

We have the following approximation result.

Proposition 3.9. *Assume that hypotheses* (HA2), (HF2)′, (H0) *hold,* $\lambda \mapsto (u(\lambda), h(\lambda))$ *is a continuous map from* D *into* $C(T, H) \times L^2(T, H)$ *with* $u(\lambda) = e(h(\lambda), \lambda)$, $\epsilon > 0$ *and* $p\colon D \to L^2(T)_+$ *is a continuous map such that*

$$d\big(h(\lambda)(t), F(t, u(\lambda)(t), \lambda)\big) \leq p(\lambda)(t) \quad \textit{for almost all } t \in T.$$

Then there exists a continuous map $\lambda \mapsto (x(\lambda), f(\lambda))$ *from* D *into* $C(T, H) \times L^2(T, H)$ *such that*

$$x(\lambda) = e(f(\lambda), \lambda) \quad \textit{with } f(\lambda) \in S^2_{F(\cdot, x(\lambda)(\cdot), \lambda)}$$

and

$$|x(\lambda)(t) - u(\lambda)(t)| \leq b\epsilon e^{\tau(t)} + \int_0^t p(\lambda)(s) e^{\tau(t) - \tau(s)}\, ds \quad \textit{for all } t \in T$$

with $\tau(t) = \int_0^t k(s)\, ds$.

Proof. Consider the multifunction $R_\epsilon\colon D \to 2^{L^1(T,H)}$ defined by

$$R_\epsilon(\lambda) = \{v \in S^1_{F(\cdot, u(\lambda)(\cdot), \lambda)} : |v(t) - h(\lambda)(t)| < p(\lambda)(t) + \epsilon \text{ for almost all } t \in T\}.$$

This multifunction has nonempty, decomposable values and it is LSC (see [22, Lemma 8.3, p 239]). Hence, $\lambda \mapsto \overline{R_\epsilon(\lambda)}$ has the same properties. So, we can find a continuous map $y_0\colon D \to L^1(T, H)$ such that

$$y_0(\lambda) \in \overline{R_\epsilon(\lambda)} \quad \text{for all } \lambda \in D.$$

Let $x_1(\lambda) \in W_p(0, b)$ be the unique solution of the following Cauchy problem:

$$-x'(t) \in A(t, x(t)) + y_0(\lambda)(t) \quad \text{for almost all } t \in T, \qquad x(0) = \xi(\lambda).$$

Then as in the proof of Proposition 3.8, we can generate by induction two sequences

$$\{x_n(\lambda)\}_{n\geq 1} \subseteq W_p(0, b) \quad \text{and} \quad \{y_n(\lambda)\}_{n\geq 1} \subseteq L^2(T, H),$$

which satisfy properties (a)–(d) listed in the proof of Proposition 3.8. As before (see the proof of Proposition 3.8), we have

$$|x_{n+1}(\lambda)(t) - x_n(\lambda)(t)| \leq \|y_n(\lambda) - y_{n-1}(\lambda)\|_{L^1(T,H)} \quad \text{for all } (\lambda, t) \in D \times T.$$

From this inequality and property (d) of the sequences (see the proof of Proposition 3.8), we infer that

$$\{x_n(\lambda)\}_{n\geq 1} \subseteq C(T, H) \quad \text{and} \quad \{y_n(\lambda)\}_{n\geq 1} \subseteq L^1(T, H)$$

are both Cauchy uniformly in $\lambda \in K \subseteq D$ compact (recall that $\lambda \mapsto p(\lambda)$ is continuous, hence locally bounded). So, we have

$$x_n(\lambda) \to \hat{x}(\lambda) \quad \text{in } C(T,H) \qquad \text{and} \qquad y_n(\lambda) \to \hat{y}(\lambda) \quad \text{in } L^1(T,H),$$

and both maps $D \ni \lambda \mapsto \hat{x}(\lambda) \in C(T,H)$ and $D \ni \lambda \mapsto \hat{y}(\lambda) \in L^1(T,H)$ are continuous. Moreover, we have $\hat{y}(\lambda) \in S^2_{F(\cdot,\hat{x}(\lambda)(\cdot),\lambda)}$ (see the proof of Theorem 3.5) and that $\lambda \mapsto \hat{y}(\lambda)$ is continuous from D into $L^2(T,H)$. If $x(\lambda) = e(\hat{y}(\lambda),\lambda)$, then

$$|x_n(\lambda)(t) - x(\lambda)(t)| \le \int_0^b |y_{n-1}(\lambda)(s) - \hat{y}(\lambda)(s)|\,ds \to 0 \quad \text{for all } t \in T,$$

which implies

$$\hat{x}(\lambda) = x(\lambda) = e(\hat{y}(\lambda),\lambda) \quad \text{for all } \lambda \in D.$$

From the triangle inequality, we have

$$|u(\lambda)(t) - x_n(\lambda)(t)| \le |u(\lambda)(t) - x_1(\lambda)(t)| + \sum_{k=1}^{n-1} |x_k(\lambda)(t) - x_{k+1}(\lambda)(t)| \quad \text{for all } t \in T.$$

Using property (d) (see the proof of Proposition 3.8), we have

$$|x_k(\lambda)(t) - x_{k+1}(\lambda)(t)| \le \frac{1}{k!}\int_0^t p(\lambda)(s)(\tau(t)-\tau(s))^k\,ds + \frac{b\epsilon}{k!}\tau(t)^k \quad \text{for all } t \in T.$$

So, finally we can write that

$$|u(\lambda)(t) - x(\lambda)(t)| \le b\epsilon e^{\tau(b)} + \int_0^t p(\lambda)(s)e^{\tau(t)-\tau(s)}\,ds \quad \text{for all } t \in T \text{ and all } \lambda \in D.$$ □

We want to strengthen Proposition 3.8, and require that the selection $\vartheta(\cdot)$ passes through a preassigned solution. We mention that an analogous result for differential inclusions in $\mathbb{R}^N$ with $A \equiv 0$, was proved by Cellina and Staicu [10].

We start with a simple technical lemma.

Lemma 3.10. *If $\{u_k\}_{k=0}^N \subseteq L^1(T,H)$ and $\{T_k(\xi)\}_{k=0}^N$ is a partition of $T = [0,b]$ with endpoints which depend continuously on $\xi \in H$, then there exists $\hat{d} \in L^1(T)_+$ for which the following holds: "Given $\epsilon > 0$, we can find $\delta > 0$ such that for $|\xi - \xi'| \le \delta$,*

$$\left|\sum_{k=0}^N \chi_{T_k(\xi)}(t)u_k(t) - \sum_{k=0}^N \chi_{T_k(\xi')}(t)u_k(t)\right| \le \hat{d}(t)\chi_C(t)$$

with $C \subseteq T$ measurable and $|C|_1 \le \epsilon$".

Proof. We have

$$\begin{aligned}\left|\sum_{k=0}^N \chi_{T_k(\xi)}(t)u_k(t) - \sum_{k=0}^N \chi_{T_k(\xi')(t)}(t)u_k(t)\right| &\le \sum_{k=0}^N |\chi_{T_k(\xi)}(t) - \chi_{T_k(\xi')}||u_k(t)| \\ &= \sum_k \chi_{T_k(\xi)\Delta T_k(\xi')}(t)|u_k(t)|. \end{aligned} \tag{3.55}$$

We set $\hat{d}(t) = \sum_{k=0}^N |u_k(t)| \in L^1(T)_+$. From the hypothesis concerning the partition $\{T_k(\xi)\}_{k=0}^N$ of T, we see that given $\epsilon > 0$, we can find $\delta > 0$ such that for $|\xi - \xi'| \le \delta$,

$$\chi_{T_k(\xi)\Delta T_k(\xi')}(t) \le \chi_C(t) \quad \text{for almost all } t \in T \text{ and all } k \in \{0,\dots,N\} \tag{3.56}$$

with $C \subseteq T$ measurable, $|C|_1 \le \epsilon$. Then, from (3.55) and (3.56),

$$\left|\sum_{k=0}^N \chi_{T_k(\xi)}(t)u_k(t) - \sum_{k=0}^N \chi_{T_k(\xi')}(t)u_k(t)\right| \le \chi_C(t)\sum_{k=0}^N |u_k(t)| = \hat{d}(t)\chi_C(t) \quad \text{for almost all } t \in T.$$

The proof is now complete. □

With this lemma, we can produce a continuous selection of the solution multifunction $\xi \mapsto S(\xi)$, which passes through a preassigned point.

Proposition 3.11. *If hypotheses* (HA2), (HF2) *hold, $K \subseteq H$ is compact, $\xi_0 \in K$ and $v \in S(\xi_0)$, then there exists a continuous map $\psi\colon K \to C(T,H)$ such that*

$$\psi(\xi) \in S(\xi) \quad \textit{for all } \xi \in K \qquad \textit{and} \qquad \psi(\xi_0) = v.$$

Proof. Since $v \in S(\xi_0)$, we have

$$-v'(t) \in A(t, v(t)) + f(t) \quad \text{for almost all } t \in T, \qquad v(0) = \xi_0 \tag{3.57}$$

with $f \in S^2_{F(\cdot, v(\cdot))}$. Given $g \in L^2(T,H)$, we consider the unique solution of the Cauchy problem

$$-y'(t) \in A(t, y(t)) + g(t) \quad \text{for almost all } t \in T, \qquad y(0) = \xi \in H. \tag{3.58}$$

In what follows, by $e(g,\xi) \in W_p(0,b)$ we denote the unique solution of problem (3.58) and we set $\mu_0(\xi) = e(f,\xi)$. An easy application of the Yankov–von Neumann–Aumann selection theorem (see [22, Theorem 2.14, p. 158]) gives $\gamma_0(\xi) \in L^2(T,H)$ such that

$$\gamma_0(\xi)(t) \in F(t, \mu_0(\xi)(t)) \quad \text{for almost all } t \in T$$

and

$$\begin{aligned} |f(t) - \gamma_0(\xi)(t)| &= d(f(t), F(t, \mu_0(\xi)(t))) \\ &\le k(t)|v(t) - \mu_0(\xi)(t)| \quad \text{(see hypothesis (H}F\text{2) (ii))} \\ &= k(t)|e(f,\xi_0)(t) - e(f,\xi)(t)| \\ &\le k(t)|\xi_0 - \xi| \quad \text{for almost all } t \in T, \end{aligned}$$

see (3.45).

Let $\vartheta > 0$. We define

$$\delta(\xi) = \begin{cases} \min\left\{2^{-3}\vartheta, \dfrac{|\xi - \xi_0|}{2}\right\} & \text{if } \xi \neq \xi_0, \\ 2^{-3}\vartheta & \text{if } \xi = \xi_0. \end{cases}$$

The family $\{B_{\delta(\xi)}(\xi)\}_{\xi \in K}$ is an open cover of the compact set K. So, we can find $\{\xi_k\}_{k=0}^N \subseteq K$ such that $\{B_{\delta(\xi_k)}(\xi_k)\}_{k=0}^N$ is a finite subcover of K. Let $\{\eta_k\}_{k=0}^N$ be a locally Lipschitz partition of unity subordinated to the finite subcover. We define

$$T_0(\xi) = [0, \eta_0(\xi)b] \quad \text{and} \quad T_k(\xi) = \left[\left(\sum_{i=0}^{k-1} \eta_i(\xi)\right)b, \left(\sum_{i=0}^{k} \eta_i(\xi)\right)(b)\right] \quad \text{for all } k \in \{1, \dots, N\}.$$

The endpoints in these intervals are continuous functions of ξ. We consider the following Cauchy problem:

$$-y'(t) \in A(t, y(t)) + \sum_{k=0}^{N} \chi_{T_k(\xi)}(t)\gamma_0(\xi_k)(t) \quad \text{for almost all } t \in T, \qquad y(0) = \xi \in K. \tag{3.59}$$

Problem (3.59) has a unique solution $\mu_1(\xi) \in W_p(0,b)$. Let

$$\lambda_0(\xi)(\cdot) = \sum_{k=0}^{N} \chi_{T_k(\xi)}(\cdot)\gamma_0(\xi_k)(\cdot) \in L^2(T,H).$$

Using Lemma 3.10, we can find $\hat{d} \in L^1(T)_+$ such that, for any given $\epsilon > 0$, we can find $\delta > 0$ for which we have

$$\xi, \xi' \in K, \quad |\xi - \xi'|_1 \le \delta \Rightarrow |\lambda_0(\xi)(t) - \lambda_0(\xi')(t)| \le \hat{d}(t)\chi_C(t) \quad \text{for almost all } t \in T \tag{3.60}$$

with $C \subseteq T$ measurable, $|C|_1 \le \epsilon$. We have $\mu_1(\xi') = e(\lambda_0(\xi'), \xi')$. As before, exploiting the monotonicity of $A(t, \cdot)$ (see hypothesis (HA2) (ii)) and using [6, Lemma A.5], we have

$$|\mu_1(\xi)(t) - \mu_1(\xi')(t)| \le |\xi - \xi'| + \int_0^t |\lambda_0(\xi)(s) - \lambda_0(\xi')(s)|\, ds \quad \text{for all } t \in T. \tag{3.61}$$

Let $\epsilon > 0$ be given. By the absolute continuity of the Lebesgue integral, we can find $\delta_1 > 0$ such that

$$\int_C \hat{d}(s)\, ds \le \frac{\epsilon}{2} \quad \text{for all } C \subseteq T \text{ measurable with } |C|_1 \le \delta_1. \tag{3.62}$$

Also, using (3.60), we can find $\delta \in (0, \epsilon/2)$ such that

$$\xi, \xi' \in K, \quad |\xi - \xi'|_1 \le \delta \ \Rightarrow\ |\lambda_0(\xi)(t) - \lambda_0(\xi')(t)| \le \hat{d}(t)\chi_{C_1}(t) \quad \text{for almost all } t \in T \tag{3.63}$$

with $C_1 \subseteq T$ measurable, $|C_1|_1 \le \delta_1$. So, returning to (3.61) and using (3.62) and (3.63), we see that

$$\xi, \xi' \in K, \quad |\xi - \xi'| \le \delta \ \Rightarrow\ |\mu_1(\xi)(t) - \mu_1(\xi')(t)| \le \frac{\epsilon}{2} + \int_0^t \hat{d}(s)\chi_{C_1}(s)\, ds \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } t \in T.$$

Therefore, $\xi \mapsto \mu_1(\xi)$ is continuous from H into $C(T, H)$. Again, with an application of the Yankov–von Neumann–Aumann selection theorem, we obtain $\gamma_1(\xi) \in L^2(T, H)$ such that

$$\gamma_1(\xi)(t) \in F(t, \mu_1(\xi)(t)) \quad \text{for almost all } t \in T$$

and

$$|\gamma_0(\xi)(t) - \gamma_1(\xi)(t)| = d\big(\gamma_0(\xi)(t), F(t, \mu_1(\xi)(t))\big) \quad \text{for almost all } t \in T \text{ and all } \xi \in K.$$

As in the proof of Proposition 3.8, we produce inductively two sequences

$$\{\mu_n(\xi)\}_{n\ge0} \subseteq W_p(0, b) \quad \text{and} \quad \{\gamma_n(\xi)\}_{n\ge0} \subseteq L^2(T, H), \quad \xi \in K,$$

which satisfy the following properties:
(a) $\mu_n(\xi) = e(\lambda_{n-1}(\xi), \xi)$ with $\lambda_{n-1}(\xi) = \sum_{k=0}^N \chi_{T_k(\xi)}\gamma_{n-1}(\xi_k)(t)$, $\gamma_{-1}(\xi) = f$,
(b) $\xi \mapsto \mu_n(\xi)$ is continuous from K into $C(T, H)$,
(c) $|\mu_n(\xi)(t) - \mu_{n-1}(\xi)(t)| \le \frac{\vartheta}{2^{n+2} n!}(\int_0^t k(s)\, ds)^n$ for all $\xi \in K$,
(d) $\gamma_n(\xi)(t) \in F(t, \mu_n(\xi)(t))$ for almost all $t \in T$ and

$$|\gamma_{n-1}(\xi)(t) - \gamma_n(\xi)(t)| = d(\gamma_{n-1}(\xi)(t), F(t, \mu_n(\xi)(t))) \quad \text{for almost all } t \in T.$$

So, by the induction hypothesis, suppose that we have produced

$$\{\mu_k(\xi)\}_{k=0}^n \subseteq W_p(0, b) \quad \text{and} \quad \{\gamma_k(\xi)\}_{k=0}^n \subseteq L^2(T, H),$$

which satisfy properties (a)–(d) stated above. We set

$$\mu_{n+1}(\xi) = e(\lambda_n(\xi), \xi) \quad \text{with} \quad \lambda_n(\xi)(t) = \sum_{k=0}^n \chi_{T_k(\xi)}(t)\gamma_n(\xi_k)(t).$$

As above (see in the first part of the proof the argument concerning the map $\xi \mapsto \mu_1(\xi)$), we can show that $\xi \mapsto \mu_{n+1}(\xi)$ is continuous from K into $C(T, H)$. Also, by the monotonicity of $A(t, \cdot)$ (see hypothesis (HA2) (ii) and [6, Lemma A.5]), we have (see hypothesis (HF2) (ii) and property (d) of the induction hypothesis)

$$\begin{aligned} |\mu_{n+1}(\xi)(t) - \mu_n(\xi)(t)| &\le \sum_{k=0}^n \int_0^t \chi_{T_k(\xi)}(s)k(s)|\mu_n(\xi)(s) - \mu_{n-1}(\xi)(s)|\, ds \\ &\le \sum_{k=0}^n \int_0^t \chi_{T_k(\xi)}(s)k(s)\frac{\vartheta}{2^{n+2} n!}\bigg(\int_0^s k(s)\, dr\bigg)^n ds \end{aligned}$$

$$= \int_0^t \frac{\vartheta}{2^{n+2}(n+2)!} \frac{d}{ds} \left(\int_0^s k(r)\,dr \right)^{n+1} ds$$
$$= \frac{\vartheta}{2^{n+2}(n+1)!} \left(\int_0^t k(s)\,ds \right)^{n+1}.$$

Note that for the second inequality we used property (c) of the induction hypothesis. Moreover, a standard measurable selection argument produces a measurable map $\gamma_{n+1}(\xi)\colon T \to H$, $\xi \in K$, such that

$$\gamma_{n+1}(\xi)(t) \in F(t, \mu_{n+1}(\xi)(t)) \quad \text{and} \quad |\gamma_n(\xi)(t) - \gamma_{n+1}(\xi)(t)| = d(\gamma_n(\xi)(t), F(t, \mu_{n+1}(\xi)(t)))$$

for almost all $t \in T$. This completes the induction process.

Note that from property (c),

$$\|\gamma_{n+1}(\xi) - \mu_n(\xi)\|_{C(T,H)} \le \frac{\vartheta}{2^{n+3}} e^{\|k\|_1}.$$

Therefore, we can say that

$$\mu_n(\xi) \to \psi(\xi) \quad \text{in } C(T, H) \text{ as } n \to \infty, \text{ uniformly in } \xi \in K. \tag{3.64}$$

It follows that $\xi \mapsto \psi(\xi)$ is continuous from K into $C(T, H)$.

Note that $T_0(\xi_0) = T = [0, b]$ and so $\mu_0(\xi_0) = e(f, \xi_0) = v$ (see (3.57)). Hence, $\psi(\xi_0) = v$. It remains to show that ψ is a selection of the solution multifunction $\xi \mapsto S(\xi)$. By property (d) and hypothesis (HF2) (ii), we have

$$|\gamma_n(\xi)(t) - \gamma_{n+1}(\xi)(t)| \le k(t)|\mu_n(\xi)(t) - \mu_{n+1}(\xi)(t)| \quad \text{for almost all } t \in T,$$

and thus

$$\gamma_{n+1}(\xi) \to \hat{\gamma}(\xi) \quad \text{in } L^2(T, H). \tag{3.65}$$

Let

$$\hat{\mu}(\xi) = e\left(\sum_{k=0}^{N} \int_0^t \chi_{T_k(\xi)}(s)\hat{\gamma}(\xi_k)(s)\,ds, \xi \right).$$

Since, from (3.65),

$$\sum_{k=0}^{N} \chi_{T_k(\xi)}\gamma_n(\xi_k) \to \sum_{k=0}^{N} \chi_{T_k(\xi)}\hat{\gamma}(\xi_k) \quad \text{in } L^2(T, H),$$

we have $\mu_n(\xi) \to \hat{\mu}(\xi)$ in $C(T, H)$. Therefore, from (3.64),

$$\psi(\xi) = \hat{\mu}(\xi) \in S(\xi) \quad \text{for all } \xi \in K.$$ □

4 Optimal control problems

In this section we deal with the sensitivity analysis of the optimal control problem (1.1).

Let $Q(\xi, \lambda) \subseteq W_p(0, b) \times L^2(T, Y)$ be the admissible "state-control" pairs. First we investigate the dependence of this set on the initial condition $\xi \in H$ and the parameter $\lambda \in E$. Recall that the control space Y is a separable reflexive Banach space and the parameter space E is a compact metric space. To have a useful result on the dependence of $Q(\xi, \lambda)$ on $(\xi, \lambda) \in H \times E$, we introduce the following conditions on the data of the evolution inclusion in problem (1.1) (the dynamical constraint of the problem).

(HA3) $A\colon T \times X \times E \to 2^{X^*} \setminus \{\emptyset\}$ is a multifunction such that
- (i) $t \mapsto A_\lambda(t, x)$ is graph measurable for every $(x, \lambda) \in X \times E$.
- (ii) $x \mapsto A_\lambda(t, x)$ is maximal monotone for almost all $t \in T$, all $\lambda \in E$.

(iii) For almost all $t \in T$, all $x \in X$, all $\lambda \in E$ and all $h^* \in A_\lambda(t, x)$, we have

$$\|h^*\|_* \leq a_\lambda(t) + c_\lambda \|x\|^{p-1}$$

with $\{a_\lambda\}_{\lambda \in E} \subseteq L^{p'}(T)$ bounded, $\{c_\lambda\}_{\lambda \in E} \subseteq (0, +\infty)$ bounded and $2 \leq p < \infty$.

(iv) For almost all $t \in T$, all $x \in X$, all $\lambda \in E$ and all $h^* \in A_\lambda(t, x)$, we have

$$\langle h^*, x \rangle \geq \hat{c}\|x\|^p - \hat{a}(t)$$

with $\hat{c} > 0$, $\hat{a} \in L^1(T)_+$.

(v) If $\lambda_n \to \lambda$ in E, then $\frac{d}{dt} + a_{\lambda_n} \xrightarrow{\text{PG}} \frac{d}{dt} + a_\lambda$ as $n \to \infty$.

Hypotheses (HA3) (i)–(iv) are the same as hypotheses (HA2) (i)–(iv) for every map A_λ, $\lambda \in E$. The new condition is hypothesis (HA3) (v), which requires elaboration. In the examples that follow, we present characteristic situations where this hypothesis is satisfied.

Example 4.1. (a) First, we present a situation which will be used in Section 5. So, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $X = W_0^{1,p}(\Omega)$ $(2 \leq p < \infty)$, $H = L^2(\Omega)$ and $X^* = W^{-1,p'}(\Omega)$. Evidently, (X, H, X^*) is an evolution triple (see Definition 2.1) with compact embeddings. We consider a map $a(t, z, \xi)$ satisfying the following conditions:

(Ha) $a\colon T \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a map such that the following hold:

(i) $|a(t, z, 0)| \leq c_0$ for almost all $(t, z) \in T \times \Omega$.

(ii) $(t, z) \mapsto a(t, z, \xi)$ is measurable for every $\xi \in \mathbb{R}^N$.

(iii) For almost all $(t, z) \in T \times \Omega$ and all $\xi_1, \xi_2 \in \mathbb{R}^N$, we have

$$|a(t, z, \xi_1) - a(t, z, \xi_2)| \leq \hat{c}(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha}|\xi_1 - \xi_2|^\alpha$$

with $\hat{c}_1 > 0$, $\alpha \in (0, 1]$.

(iv) For almost all $(t, z) \in T \times \Omega$ and all $\xi_1, \xi_2 \in \mathbb{R}^N$, $\xi_1 \neq \xi_2$, we have

$$(a(t, z, \xi_1) - a(t, z, \xi_2), \xi_1 - \xi_2)_{\mathbb{R}^N} \geq \hat{c}_2|\xi_1 - \xi_2|^p$$

with $\hat{c}_2 > 0$.

We consider the operator $A\colon T \times X \to X^*$ defined by

$$\langle A(t, x), h \rangle = \int_\Omega (a(t, z, Dx), Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } (t, x, h) \in T \times X \times X.$$

Using the nonlinear Green's identity (see [20, p. 210]), we have

$$A(t, x) = -\operatorname{div}(B(t, x)),$$

with $B(t, x)(\cdot) = a(t, \cdot, Dx(\cdot)) \in L^{p'}(\Omega, \mathbb{R}^N)$ for all $(t, x) \in T \times X$.

Now consider a sequence $\{a_n(t, z, \xi)\}_{n \geq 1}$ of such maps satisfying

$$|a_n(t, z, \xi) - a_n(s, z, \xi)| \leq \vartheta(t - s)(1 + |\xi|^{p-1})$$

for almost all $z \in \Omega$, all $t, s \in T$, all $\xi \in \mathbb{R}^N$ and all $n \in \mathbb{N}$, with $\vartheta\colon \mathbb{R}_+ \to \mathbb{R}_+$ being an increasing function which is continuous at $r = 0$ and $\vartheta(0) = 0$. We assume that for almost all $t \in T$, $a_n(t, \cdot, \cdot) \xrightarrow{\text{G}} a(t, \cdot, \cdot)$ in the sense of Defranceschi [12]. By [40] we have

$$\frac{d}{dt} + a_n \xrightarrow{\text{PG}} \frac{d}{dt} + a.$$

(b) We can allow multivalued maps, provided that we drop the t-dependence. So, we consider multivalued maps $a(z, \xi)$ which satisfy the following conditions:

(Ha)′ $a\colon \Omega \times \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ is a measurable map such that the following hold:

(i) $a(\cdot, \cdot)$ is measurable.

(ii) $\xi \mapsto a(z,\xi)$ is maximal monotone for almost all $z \in \Omega$.
(iii) For almost all $z \in \Omega$, all $\xi \in \mathbb{R}^N$ and all $y \in a(z,\xi)$, we have

$$|y|^{p'} \leq m_1(z) + \tilde{c}_1(y,\xi) \quad \text{with } m_1 \in L^1(\Omega),\ \tilde{c}_1(y,\xi) > 0,$$
$$|\xi|^p \leq m_2(z) + \tilde{c}_2(y,\xi) \quad \text{with } m_2 \in L^1(\Omega),\ \tilde{c}_2(y,\xi) > 0\,(2 \leq p < \infty).$$

We again consider the evolution triple

$$X = W_0^{1,p}(\Omega), \quad H = L^2(\Omega), \quad X^* = W^{-1,p'}(\Omega)\,(2 \leq p < \infty),$$

and consider the multivalued map $A\colon X \to 2^{X^*} \setminus \{\emptyset\}$ defined by

$$A(x) = \{-\operatorname{div} g : g \in S^{p'}_{a(\cdot,Dx(\cdot))}\}.$$

We consider a sequence $\{a_n(z,\xi)\}_{n\geq 1}$ of such maps and assume that $a_n \xrightarrow{\text{G}} a$ in the sense of Defranceschi [12]. Then by [16] we have

$$\frac{d}{dt} + a_n \xrightarrow{\text{PG}} \frac{d}{dt} + a.$$

(c) A third situation leading to hypothesis (HA3) (v) is the following one. We consider maps $A_\lambda(t,x)$ satisfying the following conditions:

(HA3)′ $A\colon T \times X \times E \to X^*$ is a map such that the following hold:
(i) For all $t, t+\tau \in T$, all $x \in X$ and all $\lambda \in E$, we have

$$\|A_\lambda(t+\tau,x) - A_\lambda(t,x)\| \leq O(\tau)(1 + \|x\|^{p-1}).$$

(ii) $x \mapsto A_\lambda(t,x)$ is semicontinous for all $(t,\lambda) \in T \times E$.
(iii) For all $t \in T$, all $x, u \in X$ and all $\lambda \in E$, we have

$$\langle A_\lambda(t,x) - A_\lambda(t,u), x-u\rangle \geq \tilde{c}\|x-u\|^p$$

with $\tilde{c} > 0$.
(iv) If $\lambda_n \to \lambda$ in E, then $A_{\lambda_n}(t,\cdot) \xrightarrow{\text{G}} A_\lambda(t,\cdot)$ for all $t \in T$. (This means that $A^{-1}_{\lambda_n}(t,x^*) \xrightarrow{w} A^{-1}_\lambda(t,x)$ for all $x^* \in X^*$, see [14, Definition 3.8.20].)

Under these conditions, by [26], we have

$$\frac{d}{dt} + a_{\lambda_n} \xrightarrow{\text{PG}} \frac{d}{dt} + a_\lambda.$$

Next, we introduce the conditions on the multifunctions F and G involved in the dynamics of (1.1).

(HF3) $F\colon T \times H \times E \to P_{f_c}(H)$ is a multifunction such that the following hold:
(i) $t \mapsto F(t,x,\lambda)$ is graph measurable for all $(x,\lambda) \in H \times E$.
(ii) for almost all $t \in T$, all $x, y \in H$ and all $\lambda \in E$, we have

$$h\big(F(t,x,\lambda), F(t,y,\lambda)\big) \leq k(t)|x-y|$$

with $k \in L^1(T)_+$.
(iii) For almost all $t \in T$, all $x \in H$ and all $\lambda \in E$, we have

$$|F(t,x,\lambda)| \leq a_\lambda(t) + c_\lambda|x|$$

with $\{a_\lambda\}_{\lambda\in E} \subseteq L^2(T)$ and $\{c_\lambda\}_{\lambda\in E} \subseteq (0,+\infty)$ bounded.
(iv) For almost all $t \in T$, all $x \in H$ and all $\lambda, \lambda' \in E$, we have

$$h\big(F(t,x,\lambda), F(t,x,\lambda')\big) \leq \beta(d(\lambda,\lambda'))w(t,|x|)$$

with $\beta(r) \to 0^+$ as $r \to 0^+$ and $w(t,\cdot)$ bounded on bounded sets.

(HG) $G: T \times Y \times E \to P_{f_c}(H)$ is a multifunction such that the following hold:
(i) $t \mapsto G(t, u, \lambda)$ is graph measurable for all $(u, \lambda) \in Y \times E$.
(ii) For almost all $t \in T$, all $\lambda \in E$, $u \mapsto G(t, u, \lambda)$ is concave (that is, $\mathrm{Gr}\, G(t, \cdot, \lambda) \subseteq Y \times H$ is concave, see [22, Definition 1.1 and Remark 1.2, p. 585]) and $(u, \lambda) \mapsto G(t, u, \lambda)$ is h-continuous.
(iii) For almost all $t \in T$, all $u \in U(t, \lambda)$ and all $\lambda \in E$, we have

$$|G(t, u, \lambda)| \le \hat{a}_\lambda(t)$$

with $\{\hat{a}_\lambda\}_{\lambda \in E} \subseteq L^2(T)$ bounded.

Remark 4.2. A typical situation resulting to a concave multifunction $u \mapsto G(t, u, \lambda)$ is when

$$G(t, u, \lambda) = B_\lambda(t)u + C(t, \lambda) \quad \text{for all } (t, u, \lambda) \in T \times Y \times E$$

with $B_\lambda(t) \in \mathcal{L}(Y, H)$ and $C(t, \lambda) \in P_{f_c}(H)$ for all $(t, \lambda) \in T \times E$.

Another situation, leading to the concavity of $G(t, \cdot, \lambda)$, is when H is an ordered Hilbert space and $g_\lambda, \tilde{g}_\lambda : T \times Y \to H$ are two Carathéodory maps such that for almost all $t \in T$

$$g_\lambda(t, \cdot) \text{ is order convex and } \tilde{g}_\lambda(t, \cdot) \text{ is order concave.}$$

We set $G(t, u, \lambda) = \{h \in H : g_\lambda(t, u) \le h \le \tilde{g}_\lambda(t, u)\}$. Then $G(t, \cdot, \lambda)$ is concave.

Finally, we impose conditions on the control constraint $U(t, \lambda)$.
(HU) $U: T \times E \to P_{f_c}(Y)$ is a multifunction such that the following hold:
(i) $t \mapsto U(t, \lambda)$ is graph measurable for all $\lambda \in E$.
(ii) $\lambda \mapsto U(t, \lambda)$ is h-continuous for almost all $t \in T$.
(iii) $|U(t, \lambda)| \le \tilde{a}_\lambda(t)$ for almost all $t \in T$ and all $\lambda \in E$ with $\{\tilde{a}_\lambda\}_{\lambda \in E} \subseteq L^2(T)$ bounded.

Proposition 4.3. *If hypotheses* (HA3), (HF3), (HG), (HU) *hold and* $(\xi_n, \lambda_n) \to (\xi, \lambda)$ *in* $H \times E$, *then*

$$K_{\mathrm{seq}}(s \times w) - \limsup_{n\to\infty} Q(\xi_n, \lambda_n) \subseteq Q(\xi, \lambda) \quad \text{in } L^p(T, H) \times L^2(T, Y),$$
$$K(s \times s) - \liminf_{n\to\infty} Q(\xi_n, \lambda_n) \supseteq Q(\xi, \lambda) \quad \text{in } C(T, H) \times L^2(T, Y).$$

Proof. Let $(x, u) \in K_{\mathrm{seq}}(s \times w) - \limsup_{n\to\infty} Q(\xi_n, \lambda_n)$. By definition (see Section 2), we can find a subsequence $\{m\}$ of $\{n\}$ and $(x_m, u_m) \in Q(\xi_m, \lambda_m)$, $m \in \mathbb{N}$ such that

$$x_m \to x \quad \text{in } L^p(T, H) \qquad \text{and} \qquad u_m \xrightarrow{w} u \quad \text{in } L^2(T, Y) \qquad \text{as } m \to \infty. \tag{4.1}$$

For every $m \in \mathbb{N}$, we have

$$-x_m'(t) \in A_{\lambda_m}(t, x_m(t)) + f_m(t) + g_m(t) \quad \text{for almost all } t \in T, \qquad x_m(0) = \xi_m, \tag{4.2}$$

with $f_m, g_m \in L^2(T, H)$ such that

$$f_m(t) \in F(t, x_m(t), \lambda_m) \quad \text{and} \quad g_m(t) \in G(t, u_m(t), \lambda_m) \quad \text{for almost all } t \in T. \tag{4.3}$$

We deduce by hypotheses (HF3) (iii), (HG) (iii) and Theorem 3.5 and its proof that $\{x_m\}_{m\in\mathbb{N}} \subseteq W_p(0, b)$ is bounded and $\{x_m\}_{m\in\mathbb{N}} \subseteq C(T, H)$ is relatively compact. So, from (4.1) we obtain

$$x_m \xrightarrow{w} x \quad \text{in } W_p(0, b) \qquad \text{and} \qquad x_m \to x \quad \text{in } C(T, H) \qquad \text{as } m \to \infty. \tag{4.4}$$

By (4.3) and hypotheses (HF3) (iii), (HG) (iii), it is clear that $\{f_m\}_{m\in\mathbb{N}}, \{g_m\}_{m\in\mathbb{N}} \subseteq L^2(T, H)$ are bounded. Hence, we may assume (at least for a subsequence), that

$$f_m \xrightarrow{w} f \qquad \text{and} \qquad g_m \xrightarrow{w} g \quad \text{in } L^2(T, H) \qquad \text{as } m \to \infty. \tag{4.5}$$

Proposition 6.6.33 of [34], implies that

$$f(t) \in \overline{\mathrm{conv}}\, w - \limsup_{m\to\infty} F(t, x_m(t), \lambda_m) \quad \text{for all } t \in T \setminus N,\ |N|_1 = 0. \tag{4.6}$$

Fix $t \in T \setminus N$ and let $y \in w - \limsup_{m\to\infty} F(t, x_m(t), \lambda_m)$. By definition, we know that there exists a subsequence $\{k\}$ of $\{m\}$, and $y_k \in F(t, x_k(t), \lambda_k)$ for all $k \in \mathbb{N}$ such that $y_k \xrightarrow{w} y$ in H as $k \to \infty$. The function $v \mapsto d(v, F(t, x(t), \lambda))$ is continuous and convex, hence weakly lower semicontinuous. Therefore,

$$d(y, F(t, x(t), \lambda)) \le \liminf_{k\to\infty} d(y_k, F(t, x(t), \lambda)). \tag{4.7}$$

On the other hand, we have

$$d(y_k, F(t, x(t), \lambda)) \le h\big(F(t, x_k(t), \lambda_k), F(t, x(t), \lambda)\big). \tag{4.8}$$

Using hypotheses (HF3) (ii) and (iv), we have

$$\begin{aligned} h\big(F(t, x_k(t), \lambda_k), F(t, x(t), \lambda)\big) &\le h\big(F(t, x_k(t), \lambda_k), F(t, x(t), \lambda_k)\big) + h\big(F(t, x(t), \lambda_k), F(t, x(t), \lambda)\big) \\ &\le k(t)|x_k(t) - x(t)| + \beta(d(\lambda_k, \lambda))w(t, |x(t)|), \end{aligned}$$

and so, from (4.4),

$$h\big(F(t, x_k(t), \lambda_k), F(t, x(t), \lambda)\big) \to 0 \quad \text{as } k \to \infty.$$

Then, from (4.7) and (4.8), we obtain $d(y, F(t, x(t), \lambda)) = 0$, hence $y \in F(t, x(t), \lambda)$. Therefore,

$$w - \limsup_{m\to\infty} F(t, x_m(t), \lambda_m) \subseteq F(t, x(t), \lambda) \quad \text{for all } t \in T \setminus N,\ |N|_1 = 0,$$

which implies (see (4.6) and recall that F is convex-valued)

$$f(t) \in F(t, x(t), \lambda) \quad \text{for all } t \in T \setminus N,\ |N|_1 = 0.$$

Next, for each $m \in \mathbb{N}$, we have

$$g_m \in S^2_{G(\cdot, u_m(\cdot), \lambda_m)}.$$

Let $h \in L^2(T, H)$ and let $(\cdot, \cdot)_{L^2(T,H)}$ denote the inner product of $L^2(T, H)$ (recall that $L^2(T, H)^* = L^2(T, H)$). Then (see [34, Theorem 6.4.16])

$$(h, g_m)_{L^2(T,H)} \le \sigma\big(h, S^2_{G(\cdot, u_m(\cdot), \lambda_m)}\big) = \int_0^b \sigma(h(t), G(t, u_m(t), \lambda_m))\, dt. \tag{4.9}$$

The concavity of $G(t, \cdot, \lambda)$ (see hypothesis (HG) (ii)), implies that the function $u \mapsto \sigma(h(t), G(t, u, \lambda))$ is concave. Since E is a complete metric space, it can be isometrically embedded, by the Arens–Eells theorem (see [21, Theorem 4.143]), as a closed subset of a separable Banach space (recall that E is compact). So, by [3], we have (see hypothesis (HG) (ii))

$$\limsup_{m\to\infty} \int_0^b \sigma\big(h(t), F(t, u_m(t), \lambda_m)\big)\, dt \le \int_0^b \sigma(h(t), F(t, u(t), \lambda))\, dt,$$

and thus

$$\limsup_{m\to\infty} \sigma\big(h, S^2_{G(\cdot, u_m(\cdot), \lambda_m)}\big) \le \sigma\big(h, S^2_{G(\cdot, u(\cdot), \lambda)}\big).$$

Therefore, from (4.5) and (4.9),

$$(h, g)_{L^2(T,H)} \le \sigma\big(h, S^2_{G(\cdot, u(\cdot), \lambda)}\big).$$

Since $h \in L^2(T, H)$ is arbitrary, it follows that $g \in S^2_{G(\cdot, u(\cdot), \lambda)}$, hence

$$g(t) \in G(t, u(t), \lambda) \quad \text{for almost all } t \in T.$$

Let $y_m \in W_p(0, b)$ be the unique solution of the Cauchy problem

$$-y_m'(t) \in A_{\lambda_m}(t, y_m(t)) + f(t) + g(t) \quad \text{for almost all } t \in T, \qquad y_m(0) = \xi. \tag{4.10}$$

Hypothesis (HA3) (v) implies that

$$y_m \xrightarrow{w} y \quad \text{in } W_p(0,b) \tag{4.11}$$

with $y \in W_p(0,b)$ being the unique solution of the Cauchy problem

$$-y'(t) \in A_\lambda(t, y(t)) + f(t) + g(t) \quad \text{for almost all } t \in T, \qquad y(0) = \xi, \tag{4.12}$$

see Section 2. From (4.2), (4.10) and the monotonicity of $A_{\lambda_m}(t,\cdot)$ (see hypothesis (HA3) (ii)), we have

$$\langle x_m'(t) - y_m'(t), x_m(t) - y_m(t)\rangle \le (f(t) + g(t) - f_m(t) - g_m(t), x_m(t) - y_m(t)) \quad \text{for almost all } t \in T.$$

Therefore, by Proposition 2.2,

$$\frac{1}{2}|x_m(t) - y_m(t)|^2 \le \frac{1}{2}|\xi_m - \xi|^2 + \int_0^t (f(s) + g(s) - f_m(s) - g_m(s), x_m(s) - y_m(s))\,ds \quad \text{for all } t \in T,$$

which yields $\|x_m - y_m\|_{C(T,H)} \to 0$ as $m \to \infty$, and hence, by (4.4) and (4.11), $x = y$.

Recalling that

$$f(t) \in F(t, x(t), \lambda) \quad \text{and} \quad g(t) \in G(t, u(t), \lambda) \quad \text{for almost all } t \in T,$$

it follows from (4.12) that $(x,u) \in Q(\xi,\lambda)$, which implies

$$K_{\mathrm{seq}}(s \times w) - \limsup_{n\to\infty} Q(\xi_n, \lambda_n) \subseteq Q(\xi,\lambda) \quad \text{in } L^p(T,H) \times L^2(T,Y).$$

Next, we will prove the second convergence of the proposition. So, let $(x,u) \in Q(\xi,\lambda)$. By definition we have

$$-x'(t) \in A_\lambda(t, x(t)) + F(t, x(t), \lambda) + g(t) \quad \text{for almost all } t \in T, \qquad x(0) = \xi$$

with $g \in L^2(T,H)$ satisfying

$$g(t) \in G(t, u(t), \lambda) \quad \text{for almost all } t \in T.$$

For every $v \in L^2(T,Y)$, we have (see [34, Theorem 6.4.16])

$$d(v, S^2_{U(\cdot,\lambda_n)}) = \int_0^b d(v(t), U(t,\lambda_n))\,dt$$

Hypothesis (HU) (ii) and the dominated convergence theorem imply that

$$\int_0^b d(v(t), U(t,\lambda_n))\,dt \to \int_0^b d(v(t), U(t,\lambda))\,dt,$$

and so

$$d(v, S^2_{U(\cdot,\lambda_n)}) \to d(v, S^2_{U(\cdot,\lambda)}).$$

Hence, [34, Proposition 6.6.22] implies that we can find $u_n \in S^2_{U(\cdot,\lambda_n)}$ $(n \in \mathbb{N})$ such that

$$u_n \to u \quad \text{in } L^2(T,Y) \text{ as } n \to \infty.$$

Then hypothesis (HG) (ii) guarantees that we can find

$$g_n \in L^2(T,H),\ g_n(t) \in G(t, u_n(t), \lambda_n) \quad \text{for almost all } t \in T \text{ and all } n \in \mathbb{N}$$

such that

$$g_n \to g \quad \text{in } L^2(T,H) \text{ as } n \to \infty.$$

Given $\xi' \in H$, let $S(\xi') \subseteq W_p(0,b)$ be the set of solutions of the Cauchy problem

$$-y'(t) \in A_{\lambda_n}(t, y(t)) + F(t, y(t), \lambda) + g(t) \quad \text{for almost all } t \in T, \qquad y(0) = \xi'.$$

Let $K = \{\xi_n, \xi\}_{n\geq 1} \subseteq H$. This is a compact set in H. Invoking Proposition 3.11 (with $\xi_0 = \xi$), we produce a continuous map $\psi\colon K \to C(T, H)$ such that

$$\hat{y} = \psi(\hat{\xi}) \in S(\hat{\xi}) \quad \text{for all } \hat{\xi} \in H, \qquad \psi(\xi) = x. \tag{4.13}$$

Let $y_n = \psi(\xi_n)$ ($n \in \mathbb{N}$) and use Proposition 3.9 to find $x_n \in W_p(0, b)$ solution of the Cauchy problem

$$-x_n'(t) \in A_{\lambda_n}(t, x_n(t)) + F(t, x_n(t), \lambda_n) + g_n(t) \quad \text{for almost all } t \in T, \qquad x_n(0) = \epsilon_n,$$

for which, we have

$$|x_n(t) - y_n(t)| \leq b\epsilon e^{\tau(t)} + \int_0^t \eta_n(s) e^{\tau(t)-\tau(s)}\, ds \quad \text{for all } t \in T \tag{4.14}$$

with $\epsilon > 0$, $\tau(t) = \int_0^t k(s)\, ds$, $\eta_n \in L^1(T)$, $\eta_n \to 0$ in $L^1(T)$. So, we obtain (see (4.14))

$$\limsup_{n\to\infty} \|x_n - y_n\|_{C(T,H)} \leq b\epsilon e^{\tau(b)}.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\|x_n - y_n\|_{C(T,H)} \to 0 \quad \text{as } n \to \infty. \tag{4.15}$$

Finally, from (4.13) we have

$$\|x_n - x\|_{C(T,H)} \leq \|x_n - y_n\|_{C(T,H)} + \|y_n - x\|_{C(T,H)} = \|x_n - y_n\|_{C(T,H)} + \|\psi(\xi_n) - \psi(\xi)\|_{C(T,H)}.$$

Therefore, from (4.15) and the fact that $\psi(\cdot)$ is continuous, $\|x_n - x\|_{C(T,H)} \to 0$. Since $(x_n, u_n) \in Q(\xi_n, \lambda_n)$ ($n \in \mathbb{N}$) and $u_n \to u$ in $L^2(T, Y)$, we conclude that

$$Q(\xi, \lambda) \subseteq K(s \times s) - \liminf_{n\to\infty} Q(\xi_n, \lambda_n) \text{ in } C(T, H) \times L^2(T, Y). \qquad \square$$

An immediate consequence of the above proposition is the following corollary concerning the multifunction $(\xi, \lambda) \mapsto Q(\xi, \lambda)$ of admissible state-control pairs.

Corollary 4.4. *If hypotheses* (HA3), (HF3), (HG), (HU) *hold, then the multifunction*

$$Q\colon H \times E \to 2^{C(T,H)\times L^2(T,Y)} \setminus \{\emptyset\}$$

is LSC and sequentially closed in $C(T, H) \times L^2(T, Y)_w$ *(that is,* $\operatorname{Gr} Q \subseteq H \times E \times C(T, H) \times L^2(T, Y)_w$ *is sequentially closed).*

Now we bring the cost functional into the picture. The hypotheses on the integrands $L(t, x, \lambda)$ and $H(t, u, \lambda)$ are as follows.

(HL) $L\colon T \times H \times E \to \mathbb{R}$ is an integrand such that the following hold:
- (i) $t \mapsto L(t, x, \lambda)$ is measurable for every $(x, \lambda) \in H \times E$.
- (ii) If $\lambda_n \to \lambda$ in E, then for all $x \in H$, we have $L(\cdot, x, \lambda_n) \xrightarrow{w} L(\cdot, x, \lambda)$ in $L^1(T)$.
- (iii) For almost all $t \in T$, all $x, y \in H$ and all $\lambda \in E$, we have

$$|L(t, x, \lambda) - L(t, y, \lambda)| = (1 + |x| \vee |y|)\rho(t, |x - y|),$$

where $|x| \vee |y| = \max\{|x|, |y|\}$ and $\rho(t, r)$ is a Carathéodory function on $T \times \mathbb{R}_+$ with values in $(0, +\infty)$ such that

$$\rho(t, 0) = 0 \qquad \text{for almost all } t \in T$$

and

$$\sup_{0\leq r\leq\vartheta} [\rho(t, r)] \leq \beta_\vartheta(t) \quad \text{for almost all } t \in T$$

with $\beta_\vartheta \in L^1(T)_+$, $\vartheta > 0$.

(HH) $H: T \times Y \times E \to \mathbb{R}$ is an integrand such that the following hold:
(i) $t \mapsto H(t, u, \lambda)$ is measurable for all $(u, \lambda) \in Y \times E$.
(ii) $u \mapsto H(t, u, \lambda)$ is convex for almost all $t \in T$ and all $v \in E$, and $\lambda \mapsto H(t, u, \lambda)$ is continuous for almost all $t \in T$ and all $u \in Y$.
(iii) For almost all $t \in T$ and all $(u, \lambda) \in Y \times E$, we have

$$H(t, u, \lambda) \leq a(t)(1 + \|u\|_Y^2) \quad \text{with } a \in L^\infty(T).$$

(H$\hat{\psi}$) $\hat{\psi}: H \times E \to \mathbb{R}$ is a continuous function.

Using the direct method of the calculus of variations, we can produce optimal admissible state-control pairs for problem (1.1).

Proposition 4.5. *If hypotheses* (HA3), (HF3), (HG), (HU), (HL), (HH) *and* (H$\hat{\psi}$) *hold, then for every* $(\xi, \lambda) \in H \times E$ *we can find* $(x^*, u^*) \in Q(\xi, \lambda)$ *such that* $J(x^*, u^*, \xi, \lambda) = m(\xi, \lambda)$.

Proof. Let $\{(x_n, u_n)\}_{n\geq 1} \subseteq Q(\xi, \lambda)$ be a minimizing sequence for problem (1.1). So, we have

$$J(x_n, u_n, \xi, \lambda) \downarrow m(\xi, \lambda) \quad \text{as } n \to \infty.$$

Theorem 3.5 and hypothesis (HU) imply that

$$\{(x_n, u_n)\}_{n\geq 1} \subseteq W_p(0, b) \times L^2(T, Y) \quad (\text{respectively, } \subseteq C(T, H) \times L^2(T, Y))$$

is relatively $w \times w$-compact (respectively, $s \times w$-compact). So, by the Eberlein–Smulian theorem and by passing to a suitable subsequence if necessary, we can say that

$$x_n \xrightarrow{w} x^* \quad \text{in } W_p(0, b), \qquad x_n \to x^* \quad \text{in } C(T, H), \qquad u_n \xrightarrow{w} u^* \quad \text{in } L^2(T, Y). \tag{4.16}$$

Then (4.16) and Proposition 4.3 imply that

$$(x^*, u^*) \in Q(\xi, \lambda). \tag{4.17}$$

Also, (4.16), hypothesis (HL) (iii) and the dominated convergence theorem, imply that

$$\int_0^b L(t, x_n(t), \lambda)\, dt \to \int_0^b L(t, x^*(t), \lambda)\, dt. \tag{4.18}$$

In addition, as before (see the proof of Proposition 4.3), using [3, Theorem 2.1], we obtain

$$\int_0^b H(t, u^*(t), \lambda)\, dt \leq \liminf_{n\to\infty} \int_0^b H(t, u_n(t), \lambda)\, dt. \tag{4.19}$$

Finally, (4.16) and hypothesis $H(\hat{\psi})$ imply that

$$\hat{\psi}(\xi, x_n(b), \lambda) \to \hat{\psi}(\xi, x^*(b), \lambda). \tag{4.20}$$

We deduce from (4.17), (4.18), (4.19) and (4.20) that

$$J(x^*, u^*, \xi, \lambda) = m(\xi, \lambda) \quad \text{with } (x^*, u^*) \in Q(\xi, \lambda).$$

This concludes the proof. □

We are now ready for the main sensitivity results concerning problem (1.1). The first one establishes the Hadamard well-posedness of the problem.

Theorem 4.6. *If hypotheses* (HA3), (HF3), (HG), (HL), (HH) *and* (H$\hat{\psi}$) *hold, then the value function* $m: H \times E \to \mathbb{R}$ *of problem* (1.1) *is continuous.*

Proof. Let $(\xi_n, \lambda_n) \to (\xi, \lambda)$ in $H \times E$. Let $(x, u) \in Q(\xi, \lambda)$ such that (see Proposition 4.5)

$$J(x, u, \xi, \lambda) = m(\xi, \lambda).$$

Invoking Proposition 4.3, we can find $(x_n, u_n) \in Q(\xi_n, \lambda_n)$ for all $n \in \mathbb{N}$ such that

$$x_n \to x \quad \text{in } C(T, H) \qquad \text{and} \qquad u_n \to u \quad \text{in } L^2(T, Y). \tag{4.21}$$

We claim that

$$\left| \int_0^b L(t, x_n(t), \lambda_n)\, dt - \int_0^b L(t, x(t), \lambda)\, dt \right| \to 0 \quad \text{as } n \to \infty. \tag{4.22}$$

To this end, note that

$$\begin{aligned} &\left| \int_0^b L(t, x_n(t), \lambda_n)\, dt - \int_0^b L(t, x(t), \lambda)\, dt \right| \\ &\quad \le \left| \int_0^b L(t, x_n(t), \lambda_n)\, dt - \int_0^b L(t, x(t), \lambda_n)\, dt \right| + \left| \int_0^b L(t, x(t), \lambda_n)\, dt - \int_0^b L(t, x(t), \lambda)\, dt \right| \end{aligned} \tag{4.23}$$

for all $n \in \mathbb{N}$.

First, we estimate the first summand in the right-hand side of (4.23). Using hypothesis (HL) (iii), we have

$$\left| \int_0^b L(t, x_n(t), \lambda_n) - L(t, x(t), \lambda_n)\, dt \right| \le \int_0^b (1 + |x_n(t)| \vee |x(t)|)\rho(t, |x_n(t) - x(t)|)\, dt.$$

Let $M = \sup_{n\ge1} \|x_n\|_{C(T,H)} < +\infty$ (see (4.21)). Then, from (4.21) and hypothesis (HL) (iii), we have

$$\left| \int_0^b L(t, x_n(t), \lambda_n)\, dt - \int_0^b L(t, x(t), \lambda_n)\, dt \right| \le (1 + M) \int_0^b \rho(t, |x_n(t) - x(t)|)\, dt \to 0 \quad \text{as } n \to \infty. \tag{4.24}$$

Next, we estimate the second term on the right-hand side of (4.23). Let $\vartheta > 2\|x\|_{C(T,H)}$ and let $\beta_\vartheta \in L^1(T)_+$ as postulated by hypothesis (HL) (iii). Given $\epsilon > 0$, we can find $\delta > 0$ such that

$$\text{“if } C \subseteq T \text{ is measurable with } |C|_1 \le \delta, \text{ then } \int_C \beta_\vartheta(t)\, dt \le \frac{\epsilon}{2(1 + \vartheta)}.\text{”} \tag{4.25}$$

Here, we use the absolute continuity of the Lebesgue integral. Invoking the Scorza–Dragoni theorem (see [34, Theorem 6.2.9]), we can find $T_1 \subseteq T$ closed with $|T \setminus T_1| \le \frac{\delta}{2}$ and such that $\rho|_{T_1\times\mathbb{R}_+}$ is continuous. Since $\rho(t, 0) = 0$, we can find $\delta_1 > 0$ such that

$$\text{“if } r \in [0, \delta_1], \text{ then } |\rho(t, r)| \le \frac{\epsilon}{2b(1 + \vartheta)} \text{ for all } t \in T_1.\text{”} \tag{4.26}$$

Recall that simple functions are dense in $L^p(T, H)$. Using this fact, the property that $L^p(T, H)$-convergence implies pointwise convergence for almost all $t \in T$ for at least a subsequence, and invoking Egorov's theorem, we can find $T_2 \subseteq T$ closed and a simple function $s\colon T \to H$ such that

$$\|s\|_\infty \le \|x\|_{C(T,H)}, \qquad |T \setminus T_2|_1 \le \frac{\delta}{2} \qquad \text{and} \qquad |x(t) - s(t)| \le \delta_1 \quad \text{for all } t \in T_2. \tag{4.27}$$

We set $T_3 = T_1 \cap T_2$. This is a closed subset of T with $|T \setminus T_3|_1 \le \delta$. We have (see hypothesis (HL) (iii) and (4.27))

$$\begin{aligned} \left| \int_0^b L(t, x(t), \lambda_n)\, dt - \int_0^b L(t, s(t), \lambda_n)\, dt \right| &\le (1 + \|x\|_{C(T,H)}) \int_0^b \rho(t, |x(t) - s(t)|)\, dt \\ &\le (1 + \vartheta)\left[\int_{T_3} \rho(t, |x(t) - s(t)|)\, dt + \int_{T\setminus T_3} \rho(t, |x(t) - s(t)|)\, dt \right] \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned} \tag{4.28}$$

see (4.25), (4.26) and (4.27).

Similarly, we show that

$$\left| \int_0^b L(t, s(t), \lambda)\, dt - \int_0^b L(t, x(t), \lambda)\, dt \right| \le \epsilon. \tag{4.29}$$

Let $s(t) = \sum_{k=1}^N v_k \chi_{C_k}(t)$ with $v_k \in H$, $C_k \subseteq T$ measurable. Using hypothesis (HL) (ii), we can find $n_0 \in \mathbb{N}$ such that

$$\left| \int_0^b (L(t, s(t), \lambda_n) - L(t, s(t), \lambda))\, dt \right| \le \sum_{k=1}^N \left| \int_{C_k} (L(t, v_k, \lambda_n) - L(t, v_k, \lambda))\, dt \right| \le \epsilon \quad \text{for all } n \ge n_0. \tag{4.30}$$

From (4.28), (4.29) and (4.30) it follows that

$$\int_0^b L(t, x(t), \lambda_n)\, dt \to \int_0^b L(t, x(t), \lambda)\, dt \quad \text{as } n \to \infty.$$

This convergence and (4.24) imply that (4.22) (our claim) is true.

Next, we consider the integral functional

$$\Phi(u, \lambda) = \int_0^b H(t, u(t), \lambda)\, dt \quad \text{for all } (u, \lambda) \in L^2(T, Y) \times E.$$

For every $\lambda \in E$, $u \mapsto \Phi(u, \lambda)$ is convex (see hypothesis (HH) (ii)). Also, hypothesis (HH) (iii) implies that in a neighborhood of every $u \in L^2(T, Y)$, $\{\Phi(\cdot, \lambda)\}_{\lambda \in E}$ is equibounded above, hence $\{\Phi(\cdot, \lambda)\}_{\lambda \in E}$ is equi-locally Lipschitz (see [34, Theorem 1.2.3]). Therefore, it follows that (see (4.21))

$$\Phi(u_n, \lambda_n) \to \Phi(u, \lambda) \quad \text{as } n \to \infty. \tag{4.31}$$

Finally, (4.21) and hypothesis (H$\hat{\psi}$) imply that

$$\hat{\psi}(\xi_n, x_n(b), \lambda_n) \to \hat{\psi}(\xi, x(b), \lambda). \tag{4.32}$$

By (4.22), (4.31), (4.32), we have

$$\int_0^b L(t, x_n(t), \lambda_n)\, dt + \int_0^b H(t, u_n(t), \lambda_n)\, dt + \hat{\psi}(\xi_n, x_n(b), \lambda_n) \to J(x, u, \xi, \lambda) = m(\xi, \lambda),$$

which implies

$$\limsup_{n \to \infty} m(\xi_n, \lambda_n) \le m(\xi, \lambda). \tag{4.33}$$

From Proposition 4.5 we know that for every $n \in \mathbb{N}$, we can find $(x_n, u_n) \in Q(\xi_n, \lambda_n)$ such that

$$J(x_n, u_n, \xi_n, \lambda_n) = m(\xi_n, \lambda_n). \tag{4.34}$$

As in the proof of Theorem 3.5, we can show that $\{x_n\}_{n \ge 1} \subseteq W_p(0, b)$ is bounded. In addition, hypothesis (HU) implies that $\{u_n\}_{n \ge 1} \subseteq L^2(T, Y)$ is bounded. So, by passing to a suitable subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_p(0, b) \qquad \text{and} \qquad u_n \xrightarrow{w} u \quad \text{in } L^2(T, Y) \qquad \text{as } n \to \infty. \tag{4.35}$$

By (4.35) and (2.3), we also have

$$x_n \to x \quad \text{in } L^p(T, H) \text{ as } n \to \infty. \tag{4.36}$$

Then (4.35), (4.36) and Proposition 4.3 imply that $(x, u) \in Q(\xi, \lambda)$. Moreover, reasoning as in the proof of Theorem 3.5, we can show that $\{x_n\}_{n \ge 1} \subseteq C(T, H)$ is relatively compact, hence (see (4.36))

$$x_n \to x \quad \text{in } C(T, H). \tag{4.37}$$

By (4.37) and the first part of the proof, we have

$$\int_0^b L(t, x_n(t), \lambda_n)\, dt \to \int_0^b L(t, x(t), \lambda)\, dt.$$

In addition, (4.35) and hypotheses (HH) (ii), (H$\hat{\psi}$) imply (see[3])

$$\int_0^b H(t, u(t), \lambda)\, dt \le \liminf_{n\to\infty} \int_0^b H(t, u_n(t), \lambda_n)\, dt,$$

and thus

$$\hat{\psi}(\xi_n, x_n(b), \lambda_n) \to \hat{\psi}(\xi, x(b), \lambda).$$

Therefore, from (4.34) we see that

$$\int_0^b L(t, x(t), \lambda)\, dt + \int_0^b H(t, u(t), \lambda)\, dt + \hat{\psi}(\xi, x(b), \lambda) \le \liminf_{n\to\infty} m(\xi_n, \lambda_n),$$

and thus

$$m(\xi, \lambda) \le \liminf_{n\to\infty} m(\xi_n, \lambda_n). \tag{4.38}$$

We infer from (4.33) and (4.38) that $m(\xi_n, \lambda_n) \to m(\xi, \lambda)$, hence $m\colon H \times E \to \mathbb{R}$ is continuous. □

For every $(\xi, \lambda) \in H \times E$, we introduce the set $\Sigma(\xi, \lambda)$ of optimal state-control pairs, that is,

$$\Sigma(\xi, \lambda) = \{(x, u) \in Q(\xi, \lambda) : J(x, u, \xi, \lambda) = m(\xi, \lambda)\}.$$

By Proposition 4.5, we know that $\Sigma(\xi, \lambda) \neq \emptyset$ for every $(\xi, \lambda) \in H \times E$. For this multifunction we can prove the following useful continuity property.

Theorem 4.7. *If hypotheses* (HA3), (HF3), (HU), (HL), (HH) *and* (H$\hat{\psi}$) *hold, then the multifunction*

$$\Sigma\colon H \times E \to 2^{C(T,H)\times L^2(T,Y)} \setminus \{\emptyset\}$$

is sequentially USC into $C(T, H) \times L^2(T, Y)_w$.

Proof. Let $C \subseteq C(T, H) \times L^2(T, Y)_w$ be sequentially closed. We need to show that

$$\Sigma^-(C) = \{(\xi, \lambda) \in H \times E : V(\xi, \lambda) \cap C \neq \emptyset\}$$

is closed in $H \times E$ (see Section 2). To this end, let $\{(\xi_n, \lambda_n)\}_{n\ge1} \subseteq \Sigma^-(C)$, and assume that

$$(\xi_n, \lambda_n) \to (\xi, \lambda) \quad \text{in } H \times E.$$

Let $(x_n, u_n) \in \Sigma(\xi_n, \lambda_n) \cap C$, $n \in \mathbb{N}$. We know from the proof of Theorem 4.6 that at least for a subsequence, we have

$$x_n \xrightarrow{w} x \quad \text{in } W_p(0, b), \qquad x_n \to x \quad \text{in } C(T, H), \qquad u_n \xrightarrow{w} u \quad \text{in } L^2(T, Y) \qquad \text{as } n \to \infty. \tag{4.39}$$

By (4.39) and Proposition 4.3, we have

$$(x, u) \in Q(\xi, \lambda). \tag{4.40}$$

Also, we know from the proof of Theorem 4.6 that

$$J(x, u, \xi, \lambda) \le \liminf_{n\to\infty} J(x_n, u_n, \xi_n, \lambda_n) = \liminf_{n\to\infty} m(\xi_n, \lambda) = m(\xi, \lambda).$$

Therefore, from (4.40), $J(x, u, \xi, \lambda) = m(\xi, \lambda)$, and thus $(x, u) \in \Sigma(\xi, \lambda)$. Moreover, from (4.39) and since $C \subseteq C(T, H) \times L^2(T, Y)_w$ is sequentially closed, we deduce that $(x, u) \in \Sigma(\xi, \lambda) \cap C$. Therefore, $\Sigma^-(C) \subseteq H \times E$ is closed and this proves the desired sequential upper semicontinuity of the multifunction $(\xi, \lambda) \mapsto \Sigma(\xi, \lambda)$. □

5 Application to distributed parameter systems

In this section we present an application to a class of multivalued parabolic optimal control problems.

So, let $T = [0, b]$ and let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We examine the following nonlinear, multivalued parabolic optimal control problem:

$$\begin{cases} J(x, u, \xi, \lambda) = \int_0^b \int_\Omega L_1(t, z, x(t, z))\, dz\, dt + \int_0^b \int_\Omega H_1(t, z, u(t, z))\, dz\, dt \to \inf = m(\xi, \lambda), \\ -\dfrac{\partial x}{\partial t} \in -\operatorname{div} a_\lambda(z, Du) + F_1(t, z, x(t, z), \lambda) + g(t, z, \lambda)u(t, z) \quad \text{on } T \times \Omega, \\ x|_{T\times\partial\Omega} = 0, \qquad x(0, z) = \xi(z) \quad \text{for almost all } z \in \Omega, \\ \|u(t, \cdot)\|_{L^2(\Omega)} \le r(t, \lambda) \quad \text{for almost all } t \in T. \end{cases} \tag{5.1}$$

Here, $a_\lambda \colon \Omega \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ $(\lambda \in E)$ is a family of multifunctions as in Example 4.1 (b). For the other data of problem (5.1), we introduce the following conditions:

(HF_1) $F_1 \colon T \times \Omega \times \mathbb{R} \times E \to P_{f_c}(\mathbb{R})$ is a multifunction such that the following hold:

(i) $(t, z) \mapsto F_1(t, z, x, \lambda)$ is measurable for all $(x, \lambda) \in \mathbb{R} \times E$.

(ii) For almost all $(t, z) \in T \times \Omega$, all $x, y \in \mathbb{R}$ and all $\lambda \in E$, we have

$$h\big(F_1(t, z, x, \lambda), F_1(t, z, y, \lambda)\big) \le k_1(t, z)|x - y|$$

with $k_1 \in L^1(T, L^\infty(\Omega))$.

(iii) For almost all $(t, z) \in T \times \Omega$, all $x \in \mathbb{R}$ and all $\lambda \in E$, we have

$$|F_1(t, z, x, \lambda)| \le \hat{a}_1(t, z) + \hat{c}|x|$$

with $\hat{a}_1 \in L^2(T \times \Omega)$, $\hat{c}_1 > 0$.

(iv) For almost all $(t, z) \in T \times \Omega$, all $x \in \mathbb{R}$ and all $\lambda, \lambda' \in E$, we have

$$h\big(F_1(t, z, x, \lambda), F_1(t, z, x, \lambda')\big) \le \beta(d(\lambda, \lambda'))w(z, |x|)$$

with $\beta(r) \to 0$ as $r \to 0^+$ and $w \in L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R}_+)$.

Remark 5.1. Consider the multifunction $F(t, z, x, \lambda)$ defined by

$$F(t, z, x, \lambda) = [f(t, z, x, \lambda), \hat{f}(t, z, x, \lambda)],$$

where $f, \hat{f} \colon T \times \Omega \times \mathbb{R} \times E \to \mathbb{R}$ are two functions with the following properties:

- $(t, z) \mapsto f(t, z, x, \lambda), \hat{f}(t, z, x, \lambda)$ are both measurable for all $(x, \lambda) \in \mathbb{R} \times E$.
- For almost all $(t, z) \in T \times \Omega$, all $x, x' \in \mathbb{R}$ and all $\lambda, \lambda' \in E$, we have

$$|f(t, z, x, \lambda) - f(t, z, x', \lambda')| \le k(t, z)[|x - x'| + d(\lambda, \lambda')]$$
$$|\hat{f}(t, z, x, \lambda) - \hat{f}(t, z, x', \lambda')| \le \hat{k}(t, z)[|x - x'| + d(\lambda, \lambda')],$$

with $k, \hat{k} \in L^1(T, L^\infty(\Omega))$.

Then this multifunction satisfies hypotheses (HF_1).

(Hg) $g \colon T \times \Omega \times E \to \mathbb{R}$ is a Carathéodory function (that is, $(t, z) \to g(t, z, \lambda)$ is measurable for all $\lambda \in E$ and $\lambda \to g(t, z, \lambda)$ is continuous for almost all $(t, z) \in T \times \Omega$) and for almost all $(t, z) \in T \times \Omega$ and all $\lambda \in E$, we have $|g(t, z, \lambda)| \le M$ with $M > 0$.

(Hr) $r \colon T \times E \to \mathbb{R}_+$ is a Carathéodory function (that is, $t \mapsto r(t, \lambda)$ is measurable for all $\lambda \in E$ and $\lambda \to r(t, \lambda)$ is continuous for almost all $t \in T$) and for almost all $t \in T$ and all $\lambda \in E$, we have

$$0 \le r(t, \lambda) \le a(t) \quad \text{with } a \in L^2(T).$$

Now, we introduce the conditions on the two integrands involved in the cost functional problem (5.1).

($\mathrm{H}L_1$) $L\colon T \times \Omega \times \mathbb{R} \times E \to \mathbb{R}$ is an integrand such that the following hold:

(i) $(t,z) \mapsto L_1(t,z,x,\lambda)$ is measurable for all $(x,\lambda) \in \mathbb{R} \times E$.

(ii) If $\lambda_n \to \lambda$ in E, then for all $x \in L^2(\Omega)$ we have $L_1(\cdot,\cdot,x(\cdot),\lambda_n) \xrightarrow{w} L_1(\cdot,\cdot,x(\cdot),\lambda)$ in $L^1(T \times \Omega)$.

(iii) For almost all $(t,z) \in T \times \Omega$, all $x, y \in \mathbb{R}$ and all $\lambda \in E$,

$$|L_1(t,z,x,\lambda) - L_1(t,z,y,\lambda)| \le c(1 + |x| \vee |y|)\rho(t,z,|x-y|),$$

with $\rho(t,z,r)$ Carathéodory, $\rho(t,z,0) = 0$ for almost all $(t,z) \in T \times \Omega$ and for almost all (t,z), all $r \in [0,\vartheta]$ we have

$$0 \le \rho(t,z,r) \le \beta_\vartheta(t,z)$$

with $\beta_\vartheta \in L^1(T \times \Omega)$.

($\mathrm{H}H_1$) $H_1\colon T \times \Omega \times \mathbb{R} \times E \to \mathbb{R}$ is an integrand such that the following hold:

(i) $(t,z) \mapsto H_1(t,z,x,\lambda)$ is measurable for all $(x,\lambda) \in \mathbb{R} \times E$.

(ii) For almost all $(t,z) \in T \times \Omega$, $u \mapsto H_1(t,z,u,\lambda)$ is convex for all $\lambda \in E$, while $\lambda \mapsto H_1(t,z,u,\lambda)$ is continuous for all $u \in \mathbb{R}$.

(iii) For almost all $(t,z) \in T \times \Omega$, all $|u| \le r_\lambda(t,z)$ and all $\lambda \in E$, we have

$$|H_1(t,z,u,\lambda)| \le \hat{a}_\lambda(t,z)$$

with $\{\hat{a}_\lambda\}_{\lambda \in E} \subseteq L^2(T \times \Omega)$ bounded.

We consider the following evolution triple:

$$X = W_0^{1,p}(\Omega), \quad H = L^2(\Omega), \quad X^* = W^{-1,p'}(\Omega).$$

Since $2 \le p < \infty$, the Sobolev embedding theorem implies that in this triple the embeddings are compact.

For every $\lambda \in E$, let $A_\lambda\colon X \to 2^{X^*} \setminus \{\emptyset\}$ be the multivalued map defined by

$$A_\lambda(x) = \{-\operatorname{div} g : g \in L^{p'}(\Omega,\mathbb{R}^N),\ g(z) \in a_\lambda(z, Dx(z)) \text{ for almost all } z \in \Omega\}.$$

This map is maximal monotone and if $\lambda_n \to \lambda$ in E, then (see Example 4.1 (b))

$$\frac{d}{dt} + a_{\lambda_n} \xrightarrow{\mathrm{PG}} \frac{d}{dt} + a_\lambda.$$

So, hypotheses ($\mathrm{H}A3$) hold. In fact, we can have t-dependence at the expense of assuming that a_λ is single-valued. So, we assume that $a_\lambda(t,z,\xi)$ satisfies the conditions of Example 4.1 (a). Then the map $A_\lambda\colon T \times X \to X^*$ is defined by

$$A_\lambda(t,x)(\cdot) = -\operatorname{div} a_\lambda(t,\cdot,Dx(\cdot)).$$

In fact, by the nonlinear Green's identity (see [20, p. 210]), we have

$$\langle A_\lambda(t,x), h\rangle = \int_\Omega (a_\lambda(t,z,Dx), Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } x, h \in W_0^{1,p}(\Omega).$$

As we have already mentioned in Example 4.1 (a), we know from [40] that if $\lambda_n \to \lambda$ in E, then

$$\frac{d}{dt} + a_{\lambda_n} \xrightarrow{\mathrm{PG}} \frac{d}{dt} + a_\lambda,$$

and so hypotheses ($\mathrm{H}A3$) hold.

As a special case of interest, we consider the situation where the elliptic differential operator is a weighted p-Laplacian, that is,

$$\operatorname{div}(a_\lambda(t,z)|Dx|^{p-2}Dx) \text{ for all } x \in W_0^{1,p}(\Omega).$$

Here, for every $\lambda \in E$, $a_\lambda\colon T \times \Omega \to \mathbb{R}$ is a measurable function with the following properties:

- $0 < \hat{c}_1 \le a_\lambda(t,z) \le \hat{c}_2$ for almost all $(t,z) \in T \times \Omega$ and all $\lambda \in E$.
- If $\lambda_n \to \lambda$ in E, then for almost all $t \in T$,

$$\frac{1}{a_{\lambda_n}(t,\cdot)^{p'-1}} \xrightarrow{w} \frac{1}{a_\lambda(t,\cdot)^{p'-1}} \quad \text{in } L^1(\Omega).$$

For this case we consider the following parametric (with parameter $\lambda \in E$) family of convex (in $\xi \in \mathbb{R}^N$) integrands:

$$\varphi_\lambda(t,z,\xi) = \frac{a_\lambda(t,z)}{p}|\xi|^p.$$

Then the convex conjugate of $\varphi_\lambda(t,z,\cdot)$ is given by

$$\varphi_\lambda^*(t,z,\xi^*) = \frac{1}{p' a_\lambda(t,z)^{p'-1}}|\xi^*|^{p'}.$$

By hypothesis we have that $\lambda_n \to \lambda$ in E, hence

$$\varphi_{\lambda_n}^*(t,\cdot,\xi^*) \to \varphi_\lambda^*(t,\cdot,\xi^*) \quad \text{in } L^1(\Omega) \text{ for almost all } t \in T \text{ and all } \xi^* \in \mathbb{R}^N. \tag{5.2}$$

We introduce the integral functional Φ_λ defined by

$$\Phi_\lambda(t,x) = \int_\Omega \varphi_\lambda(t,z,Dx)\,dz \quad \text{for all } (t,x) \in T \times W_0^{1,p}(\Omega).$$

By [29], we know that (5.2) implies

$$\Phi_\lambda(t,x) = \Gamma_{\mathrm{seq}}(w) - \Phi_{\lambda_n}(t,x)$$

with $\Gamma_{\mathrm{seq}}(w)$ denoting the sequential Γ-convergence of $\Phi_{\lambda_n}(t,\cdot)$ on $W_0^{1,p}(\Omega)_w$ (see [7]). Then, from [12, Theorem 3.3], it follows that

$$a_{\lambda_n}(t,\cdot,\cdot) \xrightarrow{\mathrm{G}} a_\lambda(t,\cdot,\cdot) \quad \text{for almost all } t \in T,$$

and so from [40], we conclude that

$$\frac{d}{dt} + a_{\lambda_n} \xrightarrow{\mathrm{PG}} \frac{d}{dt} + a_\lambda.$$

Also, let $Y = H = L^2(\Omega)$ and

$$F(t,x,\lambda) = S^2_{F_1(t,\cdot,x(\cdot),\lambda)}, \quad G(t,u,\lambda) = \{g(t,\cdot,\lambda)u(\cdot) : \|u\|_{L^2(\Omega)} \le r(t,\lambda)\}$$
$$U(t,\lambda) = \{u \in L^2(\Omega) : \|u\|_{L^2(\Omega)} \le r(t,\lambda)\}.$$

Then hypotheses (HF1), (Hg), (Hr) imply that conditions (HF3), (HG), (HU) hold. So, the dynamics of (5.1) are described by an evolution inclusion similar to the one in problem (1.1).

Finally, let

$$L(t,x,\lambda) = \int_\Omega L_1(t,z,x(z),\lambda)\,dz \quad \text{for all } x \in L^2(\Omega),$$
$$H(t,u,\lambda) = \int_\Omega H_1(t,z,u(z),\lambda)\,dz \quad \text{for all } u \in L^2(\Omega).$$

Hypotheses (HL_1), (HH_1) imply that conditions (HL),(HH), respectively, hold. So, we can apply Theorems 4.6 and 4.7 and obtain the following result concerning the variational stability of problem (5.1).

Proposition 5.2. *If the maps a_λ are as above and hypotheses* (HF_1), (Hg), (Hr), (HL_1), (HH_1) *hold, then for every $(\xi,\lambda) \in L^2(\Omega) \times E$, problem* (5.1) *admits optimal pairs (that is, $\Sigma(\xi,\lambda) \neq \emptyset$) and*

$$(\xi,\lambda) \mapsto m(\xi,\lambda) \text{ is continuous on } L^2(\Omega) \times E,$$
$$(\xi,\lambda) \mapsto \Sigma(\xi,\lambda) \text{ is sequentially USC from } L^2(\Omega) \times E \text{ into } C(T, L^2(\Omega)) \times L^2(T \times \Omega)_w.$$

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