Research Article

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A Critical Point Theorem for Perturbed Functionals and Low Perturbations of Differential and Nonlocal Systems

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Abstract: In this paper we establish a new critical point theorem for a class of perturbed differentiable functionals without satisfying the Palais–Smale condition. We prove the existence of at least one critical point to such functionals, provided that the perturbation is sufficiently small. The main abstract result of this paper is applied both to perturbed nonhomogeneous equations in Orlicz–Sobolev spaces and to nonlocal problems in fractional Sobolev spaces.

Keywords: Critical Point Theorem, Perturbed Functional, Orlicz–Sobolev Space, Fractional Sobolev Space, Existence of Solutions

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1 Introduction and Main Results

The present paper is motivated by a seminal work by Ahmad, Lazer and Paul [3], where it is studied the solvability of a nonlinear elliptic equation with Dirichlet boundary condition under the assumption that the associated homogeneous problem has nontrivial solutions. More precisely, Ahmad, Lazer and Paul [3] were concerned with the existence of weak solutions of the problem

$$\begin{cases} (Lu)(x) = f(x, u(x)) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and L is a self-adjoint second order operator defined on Ω with real symmetric coefficients. If the linear homogeneous problem

$$\begin{cases} (Lu)(x) = 0 & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{1.2}$$

has only the trivial weak solution, then the solvability of problem (1.1) follows from a straightforward application of the Leray–Schauder theory. The interesting case, therefore, is when problem (1.1) has nontrivial

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solutions. This abstract setting corresponds to the resonance case. Ahmad, Lazer and Paul [3] established a sufficient condition (which is sharp for a certain class of nonlinearities f) for the existence of solutions of problem (1.1), provided that problem (1.2) admits at least one nontrivial solution. Our purpose is to complement this pioneering result in the framework of perturbed energy functionals associated to nonlinear differential systems or nonlocal problems with a variational structure.

In the first part of the present paper we develop a new variational method to prove the existence of critical points of perturbed differentiable functionals defined on a real Banach space. In contrast to other works, we do not suppose that the functional satisfies the well-known Palais–Smale condition which is a crucial assumption in applying critical point theory. Our study is motivated by several works showing the existence of solutions for a class of perturbed nonlinear partial differential equations, see, for example, Bahri [5], Bahri and Berestycki [6], Bahrouni, Ounaies and Rădulescu [8–10], Bartsch and Willem [13], Gonçalves and Miyagaki [18], Kajikiya [22], Mihăilescu and Rădulescu [23], Struwe [34] and the references therein. Since the pioneer work of Ambrosetti and Rabinowitz [4] (see also [28, 29, 35, 36]), different variants of critical point theorems have been developed. In [4], in order to avoid the boundedness condition of a Palais–Smale sequence, Ambrosetti and Rabinowitz introduced a new type of assumption called a superquadratic growth condition, that is,

$$B'(u)u \geq \alpha B(u)$$

for some functional B on a Banach space and for a constant $\alpha > 2$. This type of condition remained for a long time a crucial assumption to obtain bounded Palais–Smale sequences for functionals $A(u) - B(u)$ with A quadratic and exhibiting a mountain pass geometry. We refer to Jeanjean [20] for a thorough analysis of bounded Palais–Smale sequences and applications to nonlinear problems.

In the symmetric case, which corresponds to even energy functionals, there is a large literature on the existence of multiple and infinitely many solution (critical points), see, for example, Ambrosetti and Rabinowitz [4], Bartsch [12], Bartsch and Willem [13], Kajikiya [21], Willem [36], Zou [37] and the references therein. In the celebrated paper of Ambrosetti and Rabinowitz [4] the existence of infinitely many critical points for a class of symmetric functionals I is proved by taking the max-min and the min-max of I over certain dual families of subsets of a real Banach space. In [12] and [13], new critical point results called fountain theorems are established. These are effective tools for studying the existence of infinitely many large or small energy solutions. It should be noted that the Palais–Smale condition on the functional plays an important role for these theorems and their applications.

Motivated by the results above, our aim in this paper is to prove a general critical point theorem for a class of perturbed functionals without satisfying the Palais–Smale condition. More precisely, we want to give an answer to the important question about the existence of critical points of functionals of the type $I = I_1 + I_2$, provided that I_1 has at least one critical point. Our abstract results are motivated by the existence of solutions of the following class of nonlinear equation in Orlicz–Sobolev spaces:

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x,u) + \lambda g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is supposed to be a bounded domain, $\lambda \in \mathbb{R}$ and f, g are two continuous functions. Another motivation comes from nonlocal problems for the fractional Laplacian given in the form

$$\begin{cases} (-\Delta)^s u = \lambda g(x,u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.4}$$

where $(-\Delta)^s$ stands for the fractional Laplacian, Ω is an open bounded smooth domain, $N \geq 3$, $s \in (0, 1)$ and f, g are continuous functions.

Our paper is organized as follows. In Section 2 we give some definitions and fundamental properties of the Orlicz–Sobolev spaces. In Section 3 we establish two critical point theorems which asserts the existence of at least one critical point of certain functionals. In the last sections, namely Sections 4 and 5, we are going to apply our abstract results to certain classes of nonlinear equations in Orlicz–Sobolev spaces like (1.3) and to nonlocal equations driven by the fractional Laplacian as written in (1.4), respectively.

Some of the methods used in this paper have been widely described in the recent monographs [26, 27].

2 Preliminary Results

We denote by $L^p(\mathbb{R}^N)$ the usual Lebesgue space equipped with the norm

$$\|u\|_m = \Big(\int_{\mathbb{R}^N} |u|^m \, dx \Big)^{\frac{1}{m}} \quad \text{for all } 1 \le m < \infty.$$

Furthermore, $L^\infty(\mathbb{R}^N, \mathbb{R})$ stands for the space of all essentially bounded (measurable) functions from $\mathbb{R}^N$ into $\mathbb{R}$ endowed with the norm

$$\|u\|_\infty = \operatorname{ess\,sup}\{|u(x)| : x \in \mathbb{R}^N\}.$$

By $B(0, R)$ we denote the open ball of radius R centered at zero and $B^c(0, R) = \mathbb{R}^N \setminus B(0, R)$ is the complement of $B(0, R)$ in $\mathbb{R}^N$.

The following definition is important in our treatment.

Definition 2.1. Let X be a real Banach space, let $c \in \mathbb{R}$ and let $F \subset X$ be a closed subset. We say that $I \in C^1(X, \mathbb{R})$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ on F ($(\mathrm{PS})_{F,c}$ for short), if any subsequence $(u_n)_{n\in\mathbb{N}} \subseteq F$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in X^*, has a convergent subsequence to some $u \in F$. If $F = X$, we write $(\mathrm{PS})_c$.

This compactness-type condition on I is crucial in deriving the minimax theory of the critical values.

Let us recall the following version of Ekeland's variational principle established by Ekeland [16] or Gonçalves and Miyagaki [18].

Theorem 2.2. *Let X be a real Banach space. If $I \in C^1(X, \mathbb{R})$ is bounded from below on a closed subset $F \subset X$ with a nonempty interior and if*

$$I(v) < 0 < \inf_{u\in\partial F} I(u) \quad \text{for some } v \in F^\circ, \tag{2.1}$$

then

$$c := \inf_{u\in F} I(u) \tag{2.2}$$

is a critical value provided that $(\mathrm{PS})_{F,c}$ holds.

Definition 2.3. Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$.

(1) We say that u is a c-Ekeland solution of I if $I(u) = 0$ and $I'(u) = c$, where c is given in (2.2).

(2) We say that I has the Ekeland geometry if I satisfies property (2.1).

(3) We say that I has a mountain pass geometry if there exist $u_1 \in X$ and constants $r, \rho > 0$ such that

$$I(u_1) < 0, \quad \|u_1\| > r \quad \text{and} \quad I(u) \ge \rho \quad \text{when} \quad \|u\| = r.$$

(4) We say that u is a c-mountain pass solution of I if it has a mountain pass geometry, $I(u) = c$ and $I'(u) = c$, where

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0,\ \gamma(1) = u\}.$$

Let X be a real Banach space and let $I \in C^1(X, \mathbb{R})$. We denote by $\tilde{I}$ the functional of class $C^1(X, \mathbb{R})$ defined by

$$\tilde{I}(u) = I(u) \quad \text{for } \|u\| \le 2M \qquad \text{and} \qquad \tilde{I}(u) = \alpha \quad \text{for } \|u\| \ge 4M, \tag{2.3}$$

where M is a positive constant and $\alpha \in \mathbb{R}$.

Now, we recall some basic facts about Orlicz and Orlicz–Sobolev spaces. For more details we refer to Adams and Hedberg [1], Adams [2], Gossez [19], Mihăilescu and Rădulescu [23], Rao and Ren [30] and the references therein.

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \ge 3$, with smooth boundary $\partial\Omega$.

Assume that $a\colon (0,\infty) \to \mathbb{R}$ is a function such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(t) = a(|t|)t \quad \text{for } t \neq 0 \qquad \text{and} \qquad \varphi(t) = 0 \quad \text{for } t = 0,$$

is an odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. We define

$$\phi(t) = \int_0^t \varphi(s)\,ds \quad \text{and} \quad \phi^*(t) = \int_0^t \varphi^{-1}(s)\,ds \quad \text{for all } t \in \mathbb{R}.$$

Observe that ϕ is a Young function, that is,

$$\phi(0) = 0, \quad \phi \text{ is convex} \quad \text{and} \quad \lim_{x\to\infty} \phi(x) = +\infty.$$

Moreover, since $\phi(x) = 0$ if and only if $x = 0$, we obtain

$$\lim_{x\to 0} \frac{\phi(x)}{x} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\phi(x)}{x} = +\infty.$$

Then ϕ is called a N-function. The function ϕ^* is called the complementary function of ϕ. We observe that ϕ^* is also an N-function and Young's inequality holds, that is,

$$t \le \phi(s) + \phi^*(t) \quad \text{for all } s, t \ge 0.$$

The Orlicz space $L_\phi(\Omega)$ defined by the N-function is the space of all measurable functions $u\colon \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\phi} = \sup\left\{\int_\Omega uv\,dx : \int_\Omega \phi^*(|v|)\,dx \le 1\right\} < \infty.$$

Then $(L_\phi(\Omega), \|\cdot\|_{L_\phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\phi := \inf\left\{k > 0 : \int_\Omega \phi\left(\frac{u}{k}\right) dx \le 1\right\}.$$

In Orlicz spaces we also have Hölder's inequality in the form

$$\int_\Omega uv\,dx \le 2\|u\|_{L_\phi}\|u\|_{L_{\phi^*}} \quad \text{for all } u \in L_\phi(\Omega) \text{ and for all } v \in L_{\phi^*}(\Omega),$$

see, for example, Rao and Ren [30].

We denote by $W^1L_\phi(\Omega)$ the Orlicz–Sobolev space defined by

$$W^1L_\phi(\Omega) = \left\{u \in L_\phi : \frac{\partial u}{\partial x_i} \in L_\phi(\Omega) \text{ for } i = 1,\dots,N\right\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\phi} := \|u\|_\phi + \||\nabla u|\|_\phi.$$

Furthermore, we define the Orlicz–Sobolev space $W_0^1L_\phi$ as the closure of $C_0^\infty(\Omega)$ in $W^1L_\phi(\Omega)$. By Lemma 5.7 in Gossez [19] we may consider on $W_0^1L_\phi$ the equivalent norm

$$\|u\| := \||\nabla u|\|_\phi.$$

We define

$$\varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\phi(t)} \quad \text{and} \quad \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\phi(t)}.$$

In this paper we assume that

$$1 \le \varphi_0 < \frac{t\varphi(t)}{\phi(t)} \le \varphi^0 < \infty \quad \text{for all } t \ge 0. \tag{2.4}$$

Condition (2.4) implies that ϕ satisfies the Δ_2-condition, that is,

$$\phi(2t) \le K\phi(t) \quad \text{for all } t \ge 0, \tag{2.5}$$

where K is a positive constant. In addition, we have the following statements:

$$\|u\|^{\varphi^0} \le \int_\Omega \phi(|\nabla u|)\,dx \le \|u\|^{\varphi_0} \quad \text{for all } u \in W_0^1L_\phi(\Omega) \text{ with } \|u\| < 1, \tag{2.6}$$

$$\|u\|^{\varphi_0} \le \int_\Omega \phi(|\nabla u|)\,dx \le \|u\|^{\varphi^0} \quad \text{for all } u \in W_0^1L_\phi(\Omega) \text{ with } \|u\| > 1, \tag{2.7}$$

see Mihăilescu and Rădulescu [23].

Furthermore, in this paper we assume that the function ϕ satisfies the following condition:

$$[0, +\infty) \ni t \mapsto \phi(\sqrt{t}) \text{ is convex.} \tag{2.8}$$

Conditions (2.5) and (2.8) guarantee that the Orlicz–Sobolev space $W_0^1L_\phi(\Omega)$ is a reflexive Banach space, see Mihăilescu and Rădulescu [24].

3 Critical Point Theory for Perturbed Functionals

We are now ready to state our main abstract result.

Theorem 3.1. *Let X be a real Banach space and let I_λ be a real-valued functional on X such that*

$$I_\lambda = I_1 + \lambda I_2,$$

with $\lambda \in \mathbb{R}$ and $I_1, I_2 \in C^1(X, \mathbb{R})$. We suppose that:

(I) *I_1 has an Ekeland geometry, I_1 satisfies the $(\mathrm{PS})_{F,c}$-condition and $\widetilde{I}_2$ as well as $\widetilde{I}'_2$ are bounded, where $\widetilde{I}_2$ and $\widetilde{I}'_2$ are defined in* (2.3).

(II) *For a c-Ekeland solution u of I_1, there exists a positive constant M such that*

$$\|u\| \le M,$$

where M is given in (2.3).

(III) *For every $\lambda > 0$, $\widetilde{I}_\lambda = I_1 + \lambda\widetilde{I}_2 \in C^1(X, \mathbb{R})$ and it satisfies the* $(\mathrm{PS})_{F,c}$*-condition.*

Then there exists $\lambda_0 > 0$ such that, for each $|\lambda| < \lambda_0$, I_λ has a critical point.

Proof. By assumption (I) along with Theorem 2.2, we can conclude that there exist $c \in \mathbb{R}$, a closed subset $F \subset X$ and $u_1 \in X$ such that

$$c = \inf_{u\in F} I_1(u) = I_1(u_1) < 0.$$

Step 1: $\widetilde{I}_\lambda$ admits a critical point $u_2 \in X$. Because of the boundedness of $\widetilde{I}_2$, we get

$$I_1(u) - C\lambda \le \widetilde{I}_\lambda(u) \le I_1(u) + C\lambda \quad \text{for all } u \in X, \tag{3.1}$$

where $C > 0$ is independent of λ and u. From (3.1) and condition (I) it follows that, for $|\lambda|$ small enough,

$$-\infty < \inf_{u\in F} \widetilde{I}_\lambda(u) < 0$$

and

$$0 < \inf_{u\in\partial F} I_1(u) - C\lambda < \inf_{u\in\partial F} \widetilde{I}_\lambda(u).$$

This implies that, for $|\lambda|$ small enough, $\widetilde{I}_\lambda$ has the Ekeland geometry and satisfies the $(\mathrm{PS})_{F,c}$-condition. These facts in combination with Theorem 2.2 show that, for $|\lambda|$ small enough, $\widetilde{I}_\lambda$ admits a critical point $u_2 \in X$ such that

$$c_\lambda = \inf_{u\in F} \widetilde{I}_\lambda(u) = \widetilde{I}_\lambda(u_2).$$

Step 2: $c_\lambda \to c$ as $\lambda \to 0$. In view of (3.1), we infer that

$$c - |\lambda| C \le c_\lambda \le c + |\lambda| C \quad \text{for all } \lambda \in \mathbb{R},$$

which proves the claim.

Step 3. There exists $\lambda_0 > 0$ such that, for $|\lambda| < \lambda_0$, any c_λ-Ekeland solution u of $\widetilde{I}_\lambda$ satisfies

$$\|u\| \le 2M.$$

Suppose that the claim is not satisfied. Then there exist sequences $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $(u_n)_{n\in\mathbb{N}} \subseteq X$ such that $(\lambda_n)_{n\in\mathbb{N}}$ converges to zero, u_n is a c_{λ_n}-Ekeland solution of $\widetilde{I}_{\lambda_n}$ and

$$\|u_n\| > 2M. \tag{3.2}$$

We are going to shows that $u_n \to u_0$ in X (for a subsequence if necessary, still denoted by $(u_n)_{n\in\mathbb{N}}$), where u_0 is a c-Ekeland solution of I_1. By definition, $\widetilde{I}_{\lambda_n}(u_n) = c_{\lambda_n}$ and $\widetilde{I}'_{\lambda_n}(u_n) = 0$. Hence, using Step 2, the boundedness of $\widetilde{I}_2$ and the fact that $\lambda_n \to 0$ leads to

$$I_1(u_n) \to c \quad \text{and} \quad I_1'(u_n) \to 0 \quad \text{as } n \to +\infty.$$

By assumption (I) it follows that there exists a subsequence of (u_n) (still denoted by u_n) and $u_0 \in X$ such that $u_n \to u_0$ in X. Hence, we obtain

$$I_1(u_0) = c \quad \text{and} \quad I_1'(u_0) = 0.$$

This shows that u_0 is a c-Ekeland solution of I_1. Therefore, with a view of (II), we finally get

$$\|u_n\| = \|u_n - u_0 + u_0\| \le \|u_n - u_0\| + \|u_0\| < 2M$$

for all n large enough. This is a contradiction to (3.2) and so, Step 3 has been proved.

Now, combining the steps above, we conclude, for $|\lambda|$ small enough, that

$$\widetilde{I}_\lambda(u_2) = I_\lambda(u_2) = c_\lambda \quad \text{and} \quad \widetilde{I}'_\lambda(u_2) = I'_\lambda(u_2) = 0.$$

This finishes the proof. □

Remark 3.2. (1) Under the conditions of Theorem 3.1, we point out that the functional I_λ does not have to satisfy the (PS)-condition.

(2) Let X be a Banach space of real-valued functions which has the following property: If $u_n \rightharpoonup u$ in X, then there exists a subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that $u_{\varphi(n)} \to u$ in $L^\infty(X)$. In this case we can replace $\|u\| \le M$ in (2.3) by $\|u\|_\infty \le M$. Moreover, we can replace assumption (II) in Theorem 3.1 by

$$\|u\|_\infty \le M,$$

where M is given in (2.3) and u being a c-Ekeland solution of I_1.

As a direct consequence of Theorem (3.1) we can state the following results.

Theorem 3.3. *Let X be a real Banach space and let I_λ be a real-valued functional on X such that*

$$I_\lambda = I_1 + \lambda I_2$$

with $\lambda \in \mathbb{R}$ and $I_1, I_2 \in C^1(X, \mathbb{R})$. We suppose the following assumptions:
(1) *I_1 has a mountain pass geometry, I_1 satisfies the (PS)-condition, $\widetilde{I}_2$ and $\widetilde{I}'_2$ are bounded.*
(2) *The exists a positive constant M such that*

$$\|u\| \le M,$$

where M is given in (2.3) for every c-mountain pass solution u of I_1.
(3) *For every $\lambda > 0$, $\widetilde{I}_\lambda = I_1 + \lambda \widetilde{I}_2$ satisfies the (PS)-condition.*
Then there exists $\lambda_0 > 0$ such that, for all $|\lambda| < \lambda_0$, I_λ has a critical point.

Proof. Applying the mountain pass theorem, see, for example, Rabinowitz [28], the proof can be done in a similar way to that of Theorem 3.1. □

4 Nonhomogeneous Nonlinear Equations in Orlicz–Sobolev Spaces

In this section we are going to apply the abstract critical point results from Section 3 to nonhomogeneous nonlinear equations defined in Orlicz–Sobolev spaces. We denote by E the generalized Orlicz–Sobolev space $W_0^1L_\phi(\Omega)$ where we assume (2.5) and (2.8) introduced in Section 2.

We are interested in weak solutions to the nonhomogeneous equation

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = |u|^{p-2}u + \lambda g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$. We suppose the following hypotheses:

(A) $1 < p < \varphi_0$ and $N < \varphi_0$.

(B) $g(\cdot,\cdot)$ is continuous on $\overline{\Omega} \times [0,\infty)$.

The main result of this section is given in the next theorem.

Theorem 4.1. *Assume that conditions* (A) *and* (B) *are fulfilled. Then there exists* $\lambda_0 > 0$ *such that for all* $|\lambda| < \lambda_0$ *problem* (4.1) *has at least one weak solution.*

Remark 4.2. Problem (4.1) has been widely studied in the literature, see, for example, Mihăilescu and Rădulescu [23] and Rădulescu and Repovš [32] just to mention the main ones. The main novelty in treating problem (4.1) is the fact that we assume the main term to be in the equation in order to have an Ekeland structure. We do not assume any other conditions for the perturbed term.

We first introduce the variational setting for problem (4.1). We denote by $I_\lambda\colon E \to \mathbb{R}$ the energy function of problem defined by

$$I_\lambda(u) = \int_\Omega \phi(|\nabla u|)\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx - \lambda\int_\Omega G(x,u)\,dx,$$

where $G(x,s) = \int_0^s g(x,t)\,dt$. Note that under assumptions (A) and (B), the functional $I_\lambda\colon E \to \mathbb{R}$ is well-defined, of class C^1 on E and any critical point of I_λ is a weak solution of problem (4.1).

We introduce the functionals $I_1, I_2, I_3\colon E \to \mathbb{R}$ defined by

$$I_1(u) = \int_\Omega \phi(|\nabla u|)\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx,$$
$$I_2(u) = \int_\Omega G(x,u)\,dx,$$
$$I_3(u) = \int_\Omega \phi(|\nabla u|)\,dx.$$

In order to prove Theorem 4.1 we will use Theorem 3.1 in combination with Ekeland's variational principle which is the nonlinear version of the Bishop–Phelps theorem.

Lemma 4.3. *Suppose that assumption* (A) *is satisfied. Then* I_1 *has an Ekeland geometry property.*

Proof. First, we note that by condition (2.4) we know that E is continuously embedded in the classical Sobolev space $W_0^{1,\varphi_0}(\Omega)$ and consequently, E is continuously embedded in $L^\infty(\Omega)$. Hence, there exists $\alpha > 0$ such that

$$\|u\|_\infty \leq \alpha\|u\| \quad \text{for all } u \in E.$$

The inequality above along with (2.7) show that for $\|u\| > 1$ we have

$$I_1(u) = \int_\Omega \phi(|\nabla u|)\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx \geq \|u\|^{\varphi_0} - \alpha^p|\Omega|\|u\|^p \geq \|u\|^p(\|u\|^{\varphi_0-p} - \alpha^p|\Omega|).$$

We set $\rho > \max(1, (\alpha^p|\Omega|)^{\frac{1}{\varphi_0-p}})$. Therefore, since $p < \varphi_0$, we obtain

$$I_1(u) \geq \rho^{\varphi_0} - \alpha^p|\Omega|\rho^p = \gamma > 0 \quad \text{for all } \|u\| = \rho. \tag{4.2}$$

Let $v \in E \setminus \{0\}$. Then, from (2.6) and (A), we get

$$I_1(tv) = \int_\Omega \phi(t|\nabla v|)\,dx - \frac{t^p}{p}\int_\Omega |v|^p\,dx \le t^{\varphi_0}\|v\|^{\varphi_0} - \frac{t^p}{p}\int_\Omega |v|^p\,dx < 0 \tag{4.3}$$

for all $t > 0$ small enough. Taking $F = B(0,\rho)$ and using (4.2) as well as (4.3), the desired result is shown. □

Now we will show that I_1 satisfies the (PS)-condition.

Lemma 4.4. *Assume that assumption* (A) *is fulfilled. Then I_1 satisfied the* (PS)*-condition.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subseteq E$ be a (PS)-sequence of I_1, that is,

$$|I_1(u_n)| \le C \quad \text{for all } n \in \mathbb{N} \qquad \text{and} \qquad I_1'(u_n) \to 0 \quad \text{as } n \to \infty$$

for some $C > 0$.

We claim that $(u_n)_{n\in\mathbb{N}}$ is bounded in E. Arguing by contradiction, suppose that the sequence $(u_n)_{n\in\mathbb{N}}$ is unbounded in E. Without loss of any generality, we can assume that $\|u_n\| > 1$ for all $n \ge 1$. By relation (2.7) we conclude that

$$C \ge I_1(u_n) \ge \|u_n\|^{\varphi_0} - C\|u_n\|^p. \tag{4.4}$$

From (4.4) and assumption (A) we see that $(u_n)_{n\in\mathbb{N}}$ must be bounded. This proves the claim.

As before, by condition (2.4), we know that E is compactly embedded in $L^\infty(\Omega)$. Using this fact and since $I_3' : E \to E^*$ is of type (S_+), we conclude that $u_n \to u_0$ in E, which shows that the (PS)-condition is satisfied. □

Lemma 4.5. *Assume that* (A) *holds. Then there exists $M > 0$ such that*

$$\|u\|_\infty \le M$$

for every c-Ekeland solution $u \in E$ of I_1.

Proof. Let $u \in E$ be a c-Ekeland solution of I_1. Then

$$I_1(u) = c \quad \text{and} \quad I_1'(u) = 0.$$

Applying the same argument as in the proof of Lemma 4.4, we obtain

$$c \ge \min(\|u\|^{\varphi^0}, \|u\|^{\varphi_0}) - C\|u\|^p,$$

where C is a positive constant. It follows that there exists a positive constant β independent of u such that $\|u\| \le \beta$, by condition (A). Consequently, since E is continuously embedded in $L^\infty(\Omega)$, there exists $M > 0$ independent of u such that $\|u\|_\infty \le M$. □

Now, we choose a function $h \in D(\mathbb{R}^N, \mathbb{R})$ with $0 \le h \le 1$ in $\mathbb{R}^N$ such that

$$h(x) = 1 \quad \text{for } |x| \le 2M \qquad \text{and} \qquad h(x) = 0 \quad \text{for } |x| \ge 4M,$$

where M is given in Lemma 4.5. Then the function

$$\overline{G}(t,u) := h(u(t))G(t,u(t))$$

is of class C^1 in $\Omega \times \mathbb{R}$ and by (B) we know that $\overline{G}(t,u)$ and $\overline{G}_u(t,u)$ are bounded on $\Omega \times \mathbb{R}$.

Next, we define $\widetilde{I}_\lambda, \widetilde{I}_2 : E \to \mathbb{R}$ by

$$\widetilde{I}_\lambda(u) = \int_\Omega \phi(|\nabla u|)\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx - \lambda\int_\Omega h(u(x))G(x,u(x))\,dx,$$

$$\widetilde{I}_2(u) = \int_\Omega h(u(x))G(x,u(x))\,dx.$$

Lemma 4.6. *Assume that* (A) *and* (B) *are fulfilled. Then, for every* $\lambda \in \mathbb{R}$, $\tilde{I}_\lambda$ *satisfies the* (PS)*-condition.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subseteq E$ be a (PS)-sequence of $\tilde{I}_\lambda$, that is,

$$|\tilde{I}_\lambda(u_n)| \leq C \quad \text{for all } n \in \mathbb{N} \qquad \text{and} \qquad \tilde{I}'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty$$

for some $C > 0$.

We claim that $(u_n)_{n\in\mathbb{N}}$ is bounded in E. Arguing by contradiction, we suppose that the sequence $(u_n)_{n\in\mathbb{N}}$ is unbounded in E. Without loss of any generality, we can assume that $\|u_n\| > \max(4M, 1)$ for all n ≥ 1. Due to the boundedness of $\overline{G}$, we obtain

$$C \geq \tilde{I}_\lambda(u_n) \geq \|u_n\|^{\varphi_0} - C\|u_n\|^p.$$

This shows the claim.

As before, by using the boundedness of $\overline{G}$ and $\overline{G}_u$, the rest of the proof is similar to that in Lemma 4.4. □

Proof of Theorem 4.1. By the boundedness of $\overline{G}$ and $\overline{G}'$, we know that $\tilde{I}_2$ and $\tilde{I}'_2$ are bounded. Then, from Lemmas 4.3, 4.4, 4.5 and 4.6, we see that assumptions (I)–(III) of Theorem 3.1 are satisfied. Therefore, the proof is finished. □

5 Fractional Nonlinear Equations

In this section we are interested in weak solutions to nonlinear fractional problems. Precisely, we study the fractional equation:

$$\begin{cases} (-\Delta)^s u = \lambda g(x,u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.1}$$

where $(-\Delta)^s$ stands for the fractional Laplacian, Ω is a bounded domain with smooth boundary, $s \in (0,1)$, $N < 2s$ and λ is a parameter to be specified.

In the following we suppose the subsequent hypotheses:

(F) $f \in L^\infty(\Omega)$ and $f > 0$.

Our main result in this section reads as follows.

Theorem 5.1. *Assume that conditions* (B) *and* (F) *are satisfied. Then there exists* $\lambda_0 > 0$ *such that for all* $|\lambda| < \lambda_0$, *problem* (5.1) *has at least one weak solution.*

Let

$$E = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$$

be endowed with the norm

$$\|u\|_E = \Bigg(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Bigg)^{\frac{1}{2}},$$

which is a uniformly convex Banach space. The embedding $X \to L^r(\Omega)$ is continuous for $r \in [1, 2_s^*]$ and compact for $r \in [1, 2_s^*[$, where $2_s^* = \frac{2N}{N-2s}$. Further details about the space E can be found in the monograph of Molica Bisci, Radulescu and Servadei [26]

The energy function $I_\lambda : E \to \mathbb{R}$ concerning problem (5.1) is defined by

$$I_\lambda(u) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \lambda \int_\Omega G(x,u)\,dx - \int_\Omega f(x)u\,dx,$$

where

$$G(x,s) = \int_0^s g(x,t)\,dt.$$

The functional I_λ is well-defined, of class C^1 on E and any critical point of I_λ is a weak solution of problem (5.1).

We define the functionals $I_1, I_2 \colon E \to \mathbb{R}$ by

$$I_1(u) = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\,dx\,dy - \int_{\Omega} f(x)u\,dx,$$

$$I_2(u) = \int_{\Omega} G(x, u)\,dx.$$

Remark 5.2. Nonlocal problems like (5.1) has been studied by several authors in the past years and it became a large interest due to its applications. Knowing that we cannot refer to all papers dealing with fractional problems, we mention the works of Bahrouni [7], Bahrouni and Rădulescu [11], Caffarelli, Salsa and Silvestre [15], Caffarelli and Silvestre [14], El-Manouni, Hajaiej and Winkert [17], Molica Bisci and Rădulescu [25], Molica Bisci, Rădulescu and Servadei [26], Ros, Oton and Serra [31], Sire and Valdinoci [33] and the references therein. The main novelty in our case is the fact that we do not need to assume any further condition on the nonlinear g.

The key role in the proof of Theorem 5.1 is the application of the abstract critical point theorem developed in Section 3. We start with a simple observation.

Lemma 5.3. *Assume that the conditions of Theorem 5.1 are fulfilled. Then I_1 satisfied the* (PS)*-condition.*

Proof. The proof follows by standard arguments and is omitted. □

Lemma 5.4. *Assume that condition* (F) *is satisfied. Then it holds:*
(i) *I_1 has an Ekeland geometry property.*
(ii) *There exists a positive constant β such that*

$$\|u\|_\infty \leq M = \beta\|f\|_\infty,$$

with u being a c-Ekeland solution of I_1.

Proof. (i) By Hölder's inequality and condition (F) we derive

$$I_1(u) = \|u\|^2 - \int_{\Omega} f(x)u\,dx \geq \|u\|^2 - \|f\|_2\|u\|_2 \geq \|u\|^2 - \beta\|f\|_2\|u\| \geq \|u\|(\|u\| - \beta\|f\|_2)$$

for some $\beta > 0$. Setting $\rho > \beta\|f\|_2$ gives

$$I_1(u) \geq \rho^2 - \beta\|f\|_2\rho = \gamma > 0 \quad \text{for all } \|u\| = \rho. \tag{5.2}$$

Let $v \in E \setminus \{0\}$ be such that $v > 0$. Then, by condition (F), we get

$$I_1(tv) \leq t^2\|v\|^2 - t\int_{\Omega} f(x)v\,dx < 0 \quad \text{for } t \text{ small enough.} \tag{5.3}$$

Taking $F = B(0, \rho)$ and using (5.2) as well as (5.3), the assertion follows.

(ii) Let u be any c-Ekeland solution of I_1. We construct the barrier $w \in E$ such that

$$\begin{aligned} (-\Delta)^s w &\geq 1 \quad \text{in } \Omega,\\ w &\geq 0 \quad \text{in } \Omega^c,\\ w &\leq C \quad \text{in } \Omega, \end{aligned}$$

where C is a positive constant depending only on $\operatorname{diam}(\Omega)$.

Now, let $v(x) = \|f\|_\infty w(x)$. Then we clearly have

$$(-\Delta)^s u \leq (-\Delta)^s v \quad \text{and} \quad u = 0 \leq v \quad \text{in } \Omega^c.$$

Thus, by the comparison principle, we have $u \leq v$ in Ω. In particular,

$$u \leq C\|f\|_\infty \quad \text{in } \Omega. \tag{5.4}$$

Applying the same argument to $(-u)$, we infer that

$$-u \le C\|f\|_\infty \quad \text{in } \Omega. \tag{5.5}$$

Combining (5.4) and (5.5), we get our desired result. □

As we did in Section 4 we now choose a function $h \in D(\mathbb{R}^N, \mathbb{R})$ with $0 \le h \le 1$ in $\mathbb{R}^N$ such that

$$h(x) = 1 \quad \text{for } |x| \le 2M \qquad \text{and} \qquad h(x) = 0 \quad \text{for } |x| \ge 4M,$$

Then the function

$$\overline{G}(t, u) := h(u(t))G(t, u(t))$$

is of class C^1 in $\Omega \times \mathbb{R}$. Moreover, by assumption (B), we see that $\overline{G}(t, u)$ and $\overline{G}_u(t, u)$ are bounded on $\Omega \times \mathbb{R}$.

Now we introduce the functionals $\widetilde{I}_\lambda, \widetilde{I}_2 : E \to \mathbb{R}$ defined by

$$\widetilde{I}_\lambda(u) = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy - \lambda \int_\Omega h(u(x))G(x, u)\, dx - \int_\Omega f(x)u\, dx,$$

$$\widetilde{I}_2(u) = \int_\Omega h(u(x))G(x, u)\, dx.$$

Lemma 5.5. *Suppose that* (B) *and* (F) *are satisfied. Then, for each* $\lambda \in \mathbb{R}$, $\widetilde{I}_\lambda$ *satisfies the* (PS)*-condition.*

Proof. In view of the boundedness of $\overline{G}(x, u)$ and $\overline{G}'(x, u)$, the proof is similar to that in Lemma 4.6. □

Proof of Theorem 5.1. By the boundedness of $\overline{G}$ and $\overline{G}'$, we conclude that $\widetilde{I}_2$ and $\widetilde{I}_2'$ are bounded. Then, from Lemmas 5.3, 5.4 and 5.5, we verify that conditions (I)–(III) of Theorem 3.1 are satisfied. Therefore, we have shown the existence of a weak solution of problem (5.1). □

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