Advanced Nonlinear Studies **14** (2014), 719–736

Qualitative Properties of Anisotropic Elliptic Schrödinger Equations

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Received 31 August 2013
Communicated by Ireneo Peral

Abstract

We consider a class of nonlinear stationary Schrödinger-type equations and we are concerned with sufficient properties that guarantee the existence of multiple solutions in a suitable Sobolev space with variable exponents. We first establish that in the case of small perturbations, the problem admits at least two weak solutions. Next, in the case of convex-concave nonlinearities, we obtain conditions for the existence of infinitely many solutions with high (resp., small) energies. The arguments combine variational techniques with a careful analysis of the energy levels.

2010 Mathematics Subject Classification. 35J62, 35J70, 46E35, 58E05.
Key words. Degenerate anisotropic Sobolev spaces, Variable exponent, Mountain pass theorem, Ekeland variational principle, Fountain theorem, Dual fountain theorem.

1 Introduction

The Schrödinger equation has a central place in quantum mechanics and it plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future behaviour of a dynamic system. The linear Schrödinger equation provides a thorough description of a particle in a non-relativistic setting. The structure of the nonlinear form of this equation is much more complicated. The most common applications of the nonlinear Schrödinger equation vary from Bose-Einstein condensates and nonlinear optics, stability of Stokes waves in water, propagation of the electric field in optical fibers to the self-focusing and collapse of Langmuir waves in plasma physics and the behaviour of deep water waves and freak waves (or rogue waves) in the ocean. The nonlinear Schrödinger equation also describes various phenomena arising in the theory of Heisenberg ferromagnets and magnons, self-channelling of a high-power ultra-short laser in matter, condensed matter theory, dissipative quantum mechanics, electromagnetic fields, plasma physics (e.g., the Kurihara superfluid film equation). We refer to Ablowitz, Prinari and Trubatch [1], Cazenave [7], Sulem [24] for a modern overview and relevant applications.

The German physicist Werner Heisenberg, 1932 Nobel Prize in Physics, said:
" ... the progress of physics will to a large extent depend on the progress of nonlinear mathematics, of methods to solver nonlinear equations ... and therefore we can learn by comparing different nonlinear problems."

Our main purpose is to consider the nonlinear Schrödinger equation in a new setting corresponding to anisotropic spaces of Sobolev-type, in which different space directions have different roles. We first establish that in the case of small perturbations, the associated Dirichlet problem admits at least two solutions. Next, in the case of convex-concave type nonlinearities, we establish necessary conditions to guarantee the existence of infinitely many solutions with high (resp., small) energies.

2 Abstract framework

In the present paper we are interested in the study of the anisotropic nonlinear problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+ -2} u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $b \in L^\infty(\Omega)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions fulfilling some natural hypotheses. The differential operator $\sum_{i=1}^N \partial_{x_i} a_i(x, \partial_{x_i} u)$ is a $\overrightarrow{p}(\cdot)$–Laplace type operator, $\overrightarrow{p}(x) = (p_1(x), p_2(x), \ldots, p_N(x))$, and $P_+^+ = \max_{i\in\{1,\ldots,N\}} \sup_{x\in\Omega} p_i(x)$. We also assume that $p_i(x)$ is a continuous function on $\overline{\Omega}$ for all $i = 1, \ldots, N$. We denote by $a_i(x, \eta)$ the continuous derivative with respect to η of the mapping $A_i : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $A_i = A_i(x, \eta)$, that is, $a_i(x, \eta) = \frac{\partial}{\partial\eta} A_i(x, \eta)$. Throughout this paper we assume that the following hypotheses are fulfilled:

($\mathbf{A_0}$) $A_i(x, 0) = 0$ for a.e. $x \in \Omega$.

($\mathbf{A_1}$) There exists a positive constant $\overline{c}_i$ such that a_i satisfies the growth condition

$$|a_i(x, \eta)| \leq \overline{c}_i(1 + |\eta|^{p_i(x)-1}),$$

for all $x \in \Omega$ and $\eta \in \mathbb{R}^N$.

($\mathbf{A_2}$) The inequalities

$$|\eta|^{p_i(x)} \leq a_i(x, \eta)\eta \leq p_i(x) A_i(x, \eta)$$

hold for all $x \in \Omega$ and $\eta \in \mathbb{R}^N$.

($\mathbf{A_3}$) There exists $k_i > 0$ such that

$$A_i\left(x, \frac{\eta + \xi}{2}\right) \leq \frac{1}{2} A_i(x, \eta) + \frac{1}{2} A_i(x, \xi) - k_i |\eta - \xi|^{p_i(x)},$$

for all $x \in \Omega$ and $\eta, \xi \in \mathbb{R}^N$, with equality if and only if $\eta = \xi$.

($\mathbf{A_4}$) The mapping A_i is even with respect to its second variable, that is,

$$A_i(x, -\eta) = A_i(x, \eta),$$

for all $x \in \Omega$ and $\eta \in \mathbb{R}^N$.

(**B**) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.

The differential operator in problem (2.1) is the anisotropic $\overrightarrow{p}(x)$–Laplace type operator (where $\overrightarrow{p}(x) = (p_1(x), \ldots, p_N(x))$ because if we take

$$a_i(x, \eta) = |\eta|^{p_i(x)-2}\eta,$$

for all $i \in \{1, \ldots, N\}$, then $A_i(x, \eta) = \frac{1}{p_i(x)}|\eta|^{p_i(x)}$ for all $i \in \{1, \ldots, N\}$, that is,

$$\Delta_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^N \partial_{x_i}\Big(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\Big).$$

Obviously, there are many other operators deriving from $\sum_{i=1}^N \partial_{x_i} a_i(x, \partial_{x_i}u)$. Indeed, if we take

$$a_i(x, \eta) = (1 + |\eta|^2)^{\frac{p_i(x)-2}{2}}\eta,$$

for all $i \in \{1, \ldots, N\}$, we have $A_i(x, \eta) = \frac{1}{p_i(x)}[(1 + |\eta|^2)^{\frac{p_i(x)}{2}} - 1]$ for all $i \in \{1, \ldots, N\}$ and we obtain the anisotropic variable mean curvature operator

$$\sum_{i=1}^N \partial_{x_i}\Big[(1 + |\partial_{x_i}u|^2)^{\frac{p_i(x)-2}{2}}\partial_{x_i}u\Big].$$

Kone, Ouaro, and Traore [19] established the existence and uniqueness of a weak energy solution to the following boundary value problem

$$\begin{cases} -\displaystyle\sum_{i=1}^N \partial_{x_i} a_i(x, \partial_{x_i}u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

In [20], the author considered (2.2) where $f = \lambda(x)|u|^{q(x)-2}u$. Combining the mountain-pass theorem of Ambrosetti and Rabinowitz [3] with the Ekeland's variational principle, they proved that under suitable conditions, problem (2.2) has two nontrivial weak solutions. Boureanu [5] proved that problem (2.2) has a sequence of weak solutions by means of the symmetric mountain-pass theorem. In this paper, we consider the perturbed problem (2.1) in two cases, corresponding to the growth rate of f.

First, we recall some definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. We will also introduce an adequate functional space where problems of type (2.1) can be studied. Such a space will be called an anisotropic variable exponent Sobolev space and it can be characterized as a functional space of Sobolev's type in which different space directions have different roles.

For any $\Omega \subset \mathbb{R}^N$, we set

$$C_+(\overline{\Omega}) = \{h(x) \in C(\overline{\Omega}), 1 < \min_{x\in\overline{\Omega}} h(x) < \max_{x\in\overline{\Omega}} h(x) < \infty\},$$

and we define

$$h^+ = \sup\{h(x);\ x \in \overline{\Omega}\}, \qquad h^- = \inf\{h(x);\ x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \Big\{u;\ u \text{ is measurable real-valued and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\},$$

endowed with the *Luxemburg norm*

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\Big\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\Big\}.$$

As established by Kováčik and Rákosník [18], $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a Banach space.

Proposition 2.1 (see [10, 14]) (**i**) *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, uniformly convex Banach space and its dual space is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)}+\frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\Big|\int_\Omega uvdx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

(**ii**) *If $p_1(x),\ p_2(x) \in C_+(\overline{\Omega}),\ p_1(x) \le p_2(x),\ \forall x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 2.2 (see [13]) *Denote $\rho(u) = \int_\Omega |u|^{p(x)}dx$. Then for $u \in L^{p(x)}(\Omega)$, $(u_n) \subset L^{p(x)}(\Omega)$ we have*
(1) $|u|_{p(x)} < 1$ (respectively $= 1;\ > 1) \Longleftrightarrow \rho(u) < 1$ (respectively $= 1;\ > 1$);
(2) for $u \ne 0,\ |u|_{p(x)} = \lambda \Longleftrightarrow \rho(\frac{u}{\lambda}) = 1$;
(3) if $|u|_{p(x)} > 1$, then $|u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$;
(4) if $|u|_{p(x)} < 1$, then $|u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}$;
(5) $|u_n|_{p(x)} \to 0$ (respectively $\to \infty) \Longleftrightarrow \rho(u_n) \to 0$ (respectively $\to \infty$).

The Sobolev space with variable exponent $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ \partial_{x_i} u \in L^{p(x)}(\Omega),\ i \in \{1, \dots, N\}\}.$$

Then $W^{1,p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{p(x)} = |u(x)|_{p(x)} + |\nabla u(x)|_{p(x)}.$$

As shown by Zhikov [27, 28] the smooth functions are in general not dense in $W^{1,p(x)}(\Omega)$, but if the exponent variable p in $C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is,

$$|p(x) - p(y)| \le \frac{-M}{\log(|x-y|)} \quad \text{for all } \ x, y \in \Omega \ \text{ such that } \ |x-y| \le \frac{1}{2},$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$. The Sobolev space with zero boundary values $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p(x)}$. Of course, also the norms $\|u\|_{p(x)} = |\nabla u|_{p(x)}$ and $\|u\|_{p(x)} = \sum_{i=1}^N |\partial_{x_i} u|_{p(x)}$ are equivalent norms in $W_0^{1,p(x)}(\Omega)$. Note that when $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \ge N$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact. Details, extensions and further references can be found in [15]-[18].

Finally, we introduce a natural generalization of the function space $W_0^{1,p(x)}(\Omega)$, which will enable us to study with sufficient accuracy problem (2.1). For this purpose, let $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$ be the vectorial function $\overrightarrow{p}(x) = (p_1(x), p_2(x), ..., p_N(x))$ with $p_i \in C_+(\overline{\Omega}),\ i \in \{1, \dots, N\}$. We define $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, the *anisotropic variable exponent Sobolev space*, as the closure of $C_0^\infty(\Omega)$, with respect to the norm

$$\|u\| = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}.$$

As it was pointed out in [21], $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is a reflexive Banach space.

The above definition shows that the anisotropic variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$ is a function space of Sobolev's type in which different space directions have different roles.

We refer to the books [2, 6, 8, 9, 23] for related results and complements.

3 Main results

A central role in our analysis will be played by the vectors $\overrightarrow{P}_+, \overrightarrow{P}_- \in \mathbb{R}^N$ and by the positive numbers $P^+_+,\ P^+_-,\ P^-_+,\ P^-_-$ defined as follows:

$$\overrightarrow{P}_+ = (p_1^+, p_2^+, \ldots, p_N^+), \qquad \overrightarrow{P}_- = (p_1^-, p_2^-, \ldots, p_N^-),$$

$$P^+_+ = \max\{p_1^+, p_2^+, \ldots, p_N^+\}, \qquad P^+_- = \max\{p_1^-, p_2^-, \ldots, p_N^-\},$$

$$P^-_+ = \min\{p_1^+, p_2^+, \ldots, p_N^+\}, \qquad P^-_- = \min\{p_1^-, p_2^-, \ldots, p_N^-\}.$$

Throughout this paper, we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1. \tag{3.3}$$

This condition ensures that the anisotropic space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is embedded into some Lebesgue space $L^r(\Omega)$. If hypothesis (3.3) is no longer fulfilled, then one has embeddings into Orlicz or Hölder spaces.

Define $P^*_- \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P^*_- = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \qquad P_{-,\infty} = \max\{P^+_-, P^*_-\}.$$

First, we consider the case when $f(x,u) = \lambda(|u|^{q(x)-2}u + |u|^{\gamma(x)-2}u)$ in which the parameter λ is positive and $q(x), \gamma(x)$ are continuous functions on $\overline{\Omega}$. Problem (2.1) then becomes

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P^+_+-2}u = \lambda(|u|^{q(x)-2}u + |u|^{\gamma(x)-2}u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.4}$$

Definition 3.1 *A function $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is said to be a weak solution of problem (3.4) if and only if*

$$\int_\Omega \Big\{ \sum_{i=1}^{N} a_i(x, \partial_{x_i}) \partial_{x_i}\varphi + b(x)|u|^{P^+_+-2}u\varphi - \lambda|u|^{q(x)-2}u\varphi - \lambda|u|^{\gamma(x)-2}u\varphi \Big\}\, dx = 0,$$

for all $\varphi \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

Our first result establishes that problem (3.4) has at least two distinct solutions in the case of small perturbations. The exact statement of this result is given in what follows.

Theorem 3.1 *Let $q(x), \gamma(x) \in C_+(\overline{\Omega})$ with $P^+_+ < q^- \le q(x) \le q^+ < P^*_-$, $\gamma^+ < P^-_-$. Then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, problem (3.4) has at least two distinct, nontrivial weak solutions.*

Next, we consider problem (2.1) in the case when $f(x,u) = \lambda|u|^{q(x)-2}u + \mu|u|^{\gamma(x)-2}u$. We assume that λ, μ are parameters such that $\lambda^2 + \mu^2 \neq 0$. Thus, we study the nonlinear problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P^+_+-2}u = \lambda|u|^{q(x)-2}u + \mu|u|^{\gamma(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.5}$$

Definition 3.2 *A function $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is said to be a weak solution of problem (3.5) if and only if*

$$\int_\Omega \Big\{ \sum_{i=1}^N a_i(x, \partial_{x_i})\partial_{x_i}\varphi + b(x)|u|^{P_+^+-2}u\varphi - \lambda|u|^{q(x)-2}u\varphi - \mu|u|^{\gamma(x)-2}u\varphi \Big\}\, dx = 0,$$

for all $\varphi \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

In what follows, similarly to the result named "concave and convex nonlinearities" for the Laplace operator in [25], we establish the following multiplicity property.

Theorem 3.2 *Let $q, \gamma \in C_+(\overline{\Omega})$ with $P_+^+ < q^- \le q(x) \le q^+ < P_-^*$, $\gamma^+ < P_-^-$. Then the following properties hold:*

(i) *for every $\lambda > 0$, $\mu \in \mathbb{R}$, problem (3.5) has a sequence of weak solutions $(\pm u_k)$ with high energy solutions;*

(ii) *for every $\mu > 0$, $\lambda \in \mathbb{R}$, problem (3.5) has a sequence of weak solutions $(\pm v_k)$ with small energy solutions.*

It should be noticed that by conditions in Theorems 3.1 and 3.2, we have

$$P_{-,\infty} = \max\{P_-^+, P_-^*\} = P_-^* \,.$$

4 Notations and auxiliary results

We start by recalling a theorem concerning the embedding of the anisotropic Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ into the Lebesgue space with variable exponent $L^{q(x)}(\Omega)$.

Proposition 4.1 (see [21]) *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 3)$ is a bounded domain with smooth boundary. Assume relation (3.3) is satisfied and that $q \in C(\overline{\Omega})$ verifies*

$$1 < q(x) < P_{-,\infty}, \qquad \text{for all} \quad x \in \overline{\Omega}. \tag{4.6}$$

Then the embedding

$$W_0^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact.

Since $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is a reflexive and separable Banach space, then $(W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$ is too. There exist (see [26]) $\{e_j\} \subset W_0^{1,\overrightarrow{p}(x)}(\Omega)$ and $\{e_j^*\} \subset (W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$ such that

$$W_0^{1,\overrightarrow{p}(x)}(\Omega) = \overline{\text{span}\,\{e_j \,:\, j = 1, 2, ...\}},$$

$$(W_0^{1,\overrightarrow{p}(x)}(\Omega))^* = \overline{\text{span}\,\{e_j^* \,:\, j = 1, 2, ...\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

where $\langle \cdot \rangle$ denote the duality product between $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ and $(W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$. We define

$$X_j = \text{span}\,\{e_j\}, \qquad Y_k = \bigoplus_{j=1}^k X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^\infty X_j}.$$

Lemma 4.1 (see [12]) *Assume that $q(x), \gamma(x) \in C_+(\overline{\Omega})$ and $q(x), \gamma(x) < P_{-,\infty}$, for all $x \in \overline{\Omega}$. Denote*

$$\beta_k = \sup\{|u|_{L^{q(x)}(\Omega)};\ \|u\| = 1,\ u \in Z_k\}, \quad \theta_k = \sup\{|u|_{L^{\gamma(x)}(\Omega)};\ \|u\| = 1,\ u \in Z_k\}.$$

Then $\lim_{k\to\infty} \beta_k = 0,\ \lim_{k\to\infty} \theta_k = 0.$

A central role in our arguments will be played by the fountain theorem, which is due to Bartsch [4]. This result is nicely presented in Willem [25] by using the quantitative deformation lemma. We also point out that the dual version of the fountain theorem is due to Bartsch and Willem, see [25]. Both the fountain theorem and its dual form are effective tools for studying the existence of infinitely many large or small energy solutions. It should be noted that the Palais-Smale condition plays an important role for these theorems and their applications.

Lemma 4.2 *(Fountain theorem, see [25]). Let $I \in C^1(X, \mathbb{R})$ be an even functional, where $(X, \|\,.\,\|)$ is a separable and reflexive Banach space. Suppose that for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(**H1**) $\inf\{I(u)\ :\ u \in Z_k,\ \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$.

(**H2**) $\max\{I(u)\ :\ u \in Y_k,\ \|u\| = \rho_k\} \le 0$.

(**H3**) *I satisfies the Palais-Smale condition for every $c > 0$.*

Then I has an unbounded sequence of critical values.

Lemma 4.3 *(Dual fountain theorem, see [25]). Assume* (**H1**) *is satisfied and there is $k_0 > 0$ so that, for each $k \ge k_0$, there exist $\rho_k > r_k > 0$ such that*

(**J1**) $a_k = \inf\{I(u)\ :\ u \in Z_k,\ \|u\| = \rho_k\} \ge 0$.

(**J2**) $b_k = \max\{I(u)\ :\ u \in Y_k,\ \|u\| = r_k\} < 0$.

(**J3**) $d_k = \inf\{I(u)\ :\ u \in Z_k,\ \|u\| \le \rho_k\}\ \to 0$ *as* $k \to +\infty$.

(**J4**) *I satisfies the $(PS)^*_c$ condition for every $c \in [d_{k_0}, 0)$.*

Then I has a sequence of negative critical values converging to 0.

Definition 4.1 *We say that the functional I satisfies the $(PS)^*_c$ condition (with respect to (Y_n)), if any sequence $(u_{n_j}) \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $I(u_{n_j}) \to c$ and $(I|_{Y_{n_j}})'(u_{n_j}) \to 0$, contain a subsequence converging to a critical point of I.*

In the present paper we choose $X = W_0^{1,\vec{p}(x)}(\Omega)$. To apply the fountain theorem and the dual fountain theorem, we will operate with energy functionals and rely on the critical point theory. This is why we provide properties of some of the functionals that will be involved in our future calculus. We start with $\Lambda_i : X \to \mathbb{R}$, $i \in \{1, \ldots, N\}$, defined by

$$\Lambda_i(u) = \int_\Omega A_i(x, \partial_{x_i} u)\, dx.$$

for all $u \in X$.

Proposition 4.2 (see [19]) *For $i \in \{1, \ldots, N\}$,*

(i) *the functional Λ_i is well-defined on X,*

(ii) *the functional Λ_i is of class $C^1(X, \mathbb{R})$ and*

$$\langle \Lambda_i'(u), \varphi \rangle = \int_\Omega a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi dx,$$

for all $u, \varphi \in X$. In addition Λ_i' is continuous, bounded and strictly monotone.

(iii) Λ_i *is weakly lower semi-continuous.*

Denote by $\Lambda : X \to \mathbb{R}$ the functional

$$\Lambda(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u) dx.$$

We recall the following result concerning the functional Λ.

Lemma 4.4 (see [20]) *Assume that the sequence (u_n) converges weakly to u in X and*

$$\limsup_{n\to\infty} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u)\, dx \le 0.$$

Then (u_n) converges strongly to u in X.

In the sequel, we use c_i, to denote the general nonnegative or positive constant (the exact value may change from line to line).

5 Existence of two solutions

This section is devoted to the proof of Theorem 3.1, which is essentially based on the mountain pass theorem [3] combined with the Ekeland variational principle [11]. We refer to the recent paper [22] for several relevant applications of the mountain pass theorem.

Let us define the functional $I_\lambda : X \to \mathbb{R}$ by

$$I_\lambda(u) = \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+} |u|^{P_+^+} - \frac{\lambda}{q(x)} |u|^{q(x)} - \frac{\lambda}{\gamma(x)} |u|^{\gamma(x)} \right\} dx.$$

Then, the functional I_λ associated with problem (3.4) is well defined and of C^1 class on X. Moreover, we have

$$\begin{aligned}\langle I'_\lambda(u), \varphi\rangle = \int_\Omega \Big\{ & \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i}\varphi + b(x)|u|^{P_+^+-2} u\varphi - \\ & \lambda |u|^{q(x)-2} u\varphi - \lambda |u|^{\gamma(x)-2} u\varphi \Big\}\, dx,\end{aligned}$$

for all $u, \varphi \in X$. Thus, weak solutions of problem (3.4) are exactly the critical points of the functional I_λ. Due to the Proposition 4.2, we can show that I_λ is weakly lower semi-continuous in X. The following lemma plays an important role in our arguments.

Lemma 5.1 *The following assertions hold:*

- **(i)** *there exist $\lambda^* > 0$ and $\delta, r > 0$ such that for any $\lambda \in (0, \lambda^*)$, we have $I_\lambda(u) \ge \delta$ for all $u \in X$ with $\|u\| = r$;*
- **(ii)** *there exists $\varphi \in X$, $\varphi \ne 0$, such that $\lim_{t\to\infty} I_\lambda(t\varphi) = -\infty$;*
- **(iii)** *there exists $\psi \in X$, $\psi \ge 0$, $\psi \ne 0$ such that $I_\lambda(t\psi) < 0$ for all $t > 0$ small enough.*

Proof. (**i**) Since $q(x)$ and $\gamma(x)$ fulfil (4.6), by Proposition 4.1, we deduce that X is continuously embedded in $L^{q(x)}(\Omega)$ and $L^{\gamma(x)}(\Omega)$. It follows that there exist two positive constants c_1 and c_2 such that

$$\int_\Omega |u|^{q(x)} dx \le c_1\Big(\|u\|^{q^-} + \|u\|^{q^+}\Big), \tag{5.7}$$

and

$$\int_\Omega |u|^{\gamma(x)} dx \le c_2\Big(\|u\|^{\gamma^-} + \|u\|^{\gamma^+}\Big), \tag{5.8}$$

for all $u \in X$. Using the hypothesis (**A2**), (**B**), relations (5.7) and (5.8) yield

$$\begin{aligned}
I_\lambda(u) &= \int_\Omega \Big\{ \sum_{i=1}^{N} A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - \frac{\lambda}{q(x)}|u|^{q(x)} - \frac{\lambda}{\gamma(x)}|u|^{\gamma(x)} \Big\} dx \\
&\geq \frac{1}{P_+^+} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)} dx + \frac{b_0}{P_+^+}|u|^{P_+^+}_{L^{P_+^+}(\Omega)} - \frac{\lambda c_1}{q^-}\Big(\|u\|^{q^-} + \|u\|^{q^+}\Big) \\
&\quad - \frac{\lambda c_2}{\gamma^-}\Big(\|u\|^{\gamma^-} + \|u\|^{\gamma^+}\Big).
\end{aligned} \tag{5.9}$$

Next, we focus our attention on the case when $u \in X$ and $\|u\| < 1$. For such an element u, we have $|\partial_{x_i} u|_{p_i(x)} < 1,\ i \in \{1, \ldots, N\}$ and by Proposition 2.2, we obtain

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)} dx &\geq \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i^+}_{p_i(x)} \geq \sum_{i=1}^{N} |\partial_{x_i} u|^{P_+^+}_{p_i(x)} \\
&\geq N\Big(\frac{\sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(x)}}{N}\Big)^{P_+^+} = \frac{\|u\|^{P_+^+}}{N^{P_+^+ - 1}}.
\end{aligned} \tag{5.10}$$

Using (**B**), we can write

$$\frac{1}{P_+^+} \int_\Omega b(x)|u|^{P_+^+} dx \geq \frac{b_0}{P_+^+}|u|^{P_+^+}_{L^{P_+^+}(\Omega)} \geq 0. \tag{5.11}$$

Taking into consideration (5.10) and (5.11), the inequality (5.9) reduces to

$$\begin{aligned}
I_\lambda(u) &\geq \frac{\|u\|^{P_+^+}}{P_+^+ N^{P_+^+ - 1}} - \frac{\lambda c_1}{q^-}\Big(\|u\|^{q^-} + \|u\|^{q^+}\Big) - \frac{\lambda c_2}{\gamma^-}\Big(\|u\|^{\gamma^-} + \|u\|^{\gamma^+}\Big) \\
&\geq \Big[c_3\|u\|^{P_+^+} - c_4\lambda(\|u\|^{q^-} + \|u\|^{q^+})\Big] \\
&\quad + \Big[c_3\|u\|^{P_+^+} - c_5\lambda(\|u\|^{\gamma^-} + \|u\|^{\gamma^+})\Big].
\end{aligned}$$

Since the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(t) = c_3 - c_4 t^{q^+ - P_+^+} - c_4 t^{q^- - P_+^+}$$

is positive in a neighborhood of the origin, it follows that there exists $r \in (0, 1)$ such that $g(r) > 0$. On the other hand, defining

$$\lambda^* = \min\Big\{1, \frac{c_3}{2c_5} \min\{r^{P_+^+ - \gamma^-}, r^{P_+^+ - \gamma^+}\}\Big\},$$

we deduce that, provided $\lambda < \lambda^*$, there exists $\delta > 0$ such that for any $u \in X$ with $\|u\| = r$ we have $I_\lambda(u) \geq \delta$.

(**ii**) From ($\mathbf{A_0}$) and ($\mathbf{A_1}$), we have

$$A_i(x, \eta) = \int_0^1 a_i(x, t\eta)\eta dt \leq c_6(|\eta| + \frac{1}{p_i(x)}|\eta|^{p_i(x)}),$$

for all $x \in \overline{\Omega}$ and $\eta \in \mathbb{R}^N$, where $c_6 = \max_{i \in \{1, \ldots, N\}} \overline{c}_i$. Therefore

$$\int_\Omega \sum_{i=1}^{N} A_i(x, \partial_{x_i} u) dx \leq c_6 \sum_{i=1}^{N} \int_\Omega \Big(|\partial_{x_i} u| + \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\Big) dx.$$

Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \neq 0$. For any $t > 1$, we find

$$\begin{aligned} I_\lambda(t\varphi) &= \int_\Omega \Big\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(t\varphi)) + \frac{b(x)}{P_+^+}|t\varphi|^{P_+^+} - \frac{\lambda}{q(x)}|t\varphi|^{q(x)} - \frac{\lambda}{\gamma(x)}|t\varphi|^{\gamma(x)} \Big\}\, dx \\ &\leq c_6 \sum_{i=1}^N \int_\Omega \Big(|\partial_{x_i}(t\varphi)| + \frac{|\partial_{x_i}(t\varphi)|^{p_i(x)}}{p_i(x)} \Big) dx + \frac{1}{P_+^+}\int_\Omega b(x)|t\varphi|^{P_+^+}\, dx \\ &-\lambda \int_\Omega \frac{1}{q(x)}|t\varphi|^{q(x)}\, dx - \lambda \int_\Omega \frac{1}{\gamma(x)}|t\varphi|^{\gamma(x)}\, dx \\ &\leq c_6 t^{P_+^+} \sum_{i=1}^N \int_\Omega \Big(|\partial_{x_i}\varphi| + \frac{1}{P_-^-}|\partial_{x_i}\varphi|^{p_i(x)} \Big)\, dx + \frac{t^{P_+^+}}{P_+^+}\int_\Omega b(x)|\varphi|^{P_+^+} dx \\ &-\frac{\lambda t^{q^-}}{q^+}\int_\Omega |\varphi|^{q(x)}\, dx. \end{aligned}$$

Since $P_+^+ < q^-$, we infer that $\lim_{t\to\infty} I_\lambda(t\varphi) = -\infty$.

(**iii**) Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$, $\psi \neq 0$, $t \in (0,1)$. By ($\mathbf{A_0}$) and ($\mathbf{A_1}$), we find

$$\begin{aligned} I_\lambda(t\psi) &= \int_\Omega \Big\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(t\psi)) + \frac{b(x)}{P_+^+}|t\psi|^{P_+^+} - \frac{\lambda}{q(x)}|t\psi|^{q(x)} - \frac{\lambda}{\gamma(x)}|t\psi|^{\gamma(x)} \Big\}\, dx \\ &\leq c_6 \sum_{i=1}^N \int_\Omega \Big(|\partial_{x_i}(t\psi)| + \frac{|\partial_{x_i}(t\psi)|^{p_i(x)}}{p_i(x)} \Big)\, dx + \frac{1}{P_+^+}\int_\Omega b(x)|t\psi|^{P_+^+}\, dx \\ &-\lambda \int_\Omega \frac{1}{q(x)}|t\psi|^{q(x)}\, dx - \lambda \int_\Omega \frac{1}{\gamma(x)}|t\psi|^{\gamma(x)}\, dx \\ &\leq c_6 t^{P_-^-} \sum_{i=1}^N \int_\Omega \Big(|\partial_{x_i}\psi| + \frac{1}{P_-^-}|\partial_{x_i}\psi|^{p_i(x)} \Big) dx + \frac{t^{P_+^+}}{P_+^+}\int_\Omega b(x)|\psi|^{P_+^+}\, dx \\ &-\frac{\lambda t^{\gamma^+}}{\gamma^+}\int_\Omega |\psi|^{\gamma(x)} dx < 0, \end{aligned}$$

for all $t < \rho^{\frac{1}{P_+^+ - \gamma^+}}$, with

$$0 < \rho < \min\left\{ 1, \frac{\lambda \int_\Omega |\psi|^{\gamma(x)} dx}{\gamma^+ \Big[c_6 \sum_{i=1}^N \int_\Omega \Big(|\partial_{x_i}\psi| + \frac{1}{P_-^-}|\partial_{x_i}\psi|^{p_i(x)} \Big)\, dx + \frac{1}{P_+^+}\int_\Omega b(x)|\psi|^{P_+^+}\, dx \Big]} \right\}.$$

The proof of Lemma 5.1 is complete.

Lemma 5.2 *The functional I_λ satisfies the Palais-Smale condition in X.*

Proof. Let $(u_n) \subset X$ be a sequence such that

$$I_\lambda(u_n) \to c_7 \quad \text{and} \quad I_\lambda'(u_n) \to 0 \text{ as } \quad n \to \infty. \tag{5.12}$$

We claim that (u_n) is bounded. Arguing by contradiction. We assume that, passing eventually to a subsequence still denote by (u_n), $\|u_n\| \to \infty$ as $n \to \infty$. Using relation (5.12), for n large enough, we have

$$
\begin{aligned}
1 + c_7 + \|u_n\| \quad &\geq I_\lambda(u_n) - \frac{1}{q^-}\langle I'_\lambda(u_n), u_n\rangle \\
&\geq \sum_{i=1}^{N}\int_\Omega \Big[A_i(x, \partial_{x_i}u_n) + \frac{1}{P_+^+}b(x)|u_n|^{P_+^+} - \frac{\lambda}{q(x)}|u_n|^{q(x)} - \frac{\lambda}{\gamma(x)}|u_n|^{\gamma(x)}\Big]\,dx \\
&-\frac{1}{q^-}\sum_{i=1}^{N}\int_\Omega \Big[a_i(x, \partial_{x_i}u_n)\partial_{x_i}u_n + b(x)|u_n|^{P_+^+} - \lambda|u_n|^{q(x)} - \lambda|u_n|^{\gamma(x)}\Big]\,dx \\
&\geq \sum_{i=1}^{N}\int_\Omega \Big[A_i(x, \partial_{x_i}u_n) - \frac{1}{q^-}a_i(x, \partial_{x_i}u_n)\partial_{x_i}u_n\Big]\,dx \\
&+\Big(\frac{1}{P_+^+} - \frac{1}{q^-}\Big)\int_\Omega b(x)|u|^{P_+^+}\,dx + \lambda\int_\Omega\Big(\frac{1}{q^-} - \frac{1}{q(x)}\Big)|u_n|^{q(x)}\,dx \\
&-\lambda\int_\Omega\Big(\frac{1}{\gamma(x)} - \frac{1}{q^-}\Big)|u_n|^{\gamma(x)}\,dx.
\end{aligned}
\tag{5.13}
$$

From $(\mathbf{A_2})$, for all $x \in \Omega$ and $i \in \{1, \ldots, N\}$ we have

$$a_i(x, \partial_{x_i}u_n)\partial_{x_i}u_n \leq p_i(x)A_i(x, \partial_{x_i}u_n) \leq P_+^+A_i(x, \partial_{x_i}u_n),$$

which implies

$$-\frac{1}{q^-}a_i(x, \partial_{x_i}u_n)\partial_{x_i}u_n \geq -\frac{P_+^+}{q^-}A_i(x, \partial_{x_i}u_n).$$

Joining together the previous inequality into relation (5.13) we obtain

$$1 + c_7 + \|u_n\| \geq \Big(1 - \frac{P_+^+}{q^-}\Big)\sum_{i=1}^{N}\int_\Omega A_i(x, \partial_{x_i}u_n)dx - c_8\|u_n\|^{\gamma^+}.$$

Again from $(\mathbf{A_2})$ we have

$$A_i(x, \partial_{x_i}u_n) \geq \frac{1}{p_i(x)}|\partial_{x_i}u_n|^{p_i(x)} \geq \frac{1}{P_+^+}|\partial_{x_i}u_n|^{p_i(x)},$$

for all $x \in \Omega$ and $i \in \{1, \ldots, N\}$, thus

$$1 + c_7 + \|u_n\| \geq \Big(\frac{1}{P_+^+} - \frac{1}{q^-}\Big)\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u_n|^{p_i(x)}\,dx - c_8\|u_n\|^{\gamma^+}. \tag{5.14}$$

For each $i \in \{1, 2, \ldots, N\}$ and n we define

$$\alpha_{i,n} = \begin{cases} P_+^+ & \text{if } |\partial_{x_i}u_n|_{p_i(x)} < 1, \\ P_-^- & \text{if } |\partial_{x_i}u_n|_{p_i(x)} > 1. \end{cases}$$

Using Proposition 2.2 and Jensen's inequality (applied to the convex function $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(t) = t^{P_-^-}$, $P_-^- > 1$) or the generalized mean inequality, for n large enough we have

$$
\begin{aligned}
\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u_n|^{p_i(x)}dx \quad &\geq \sum_{i=1}^{N}|\partial_{x_i}u_n|_{p_i(x)}^{\alpha_{i,n}} \\
&\geq \sum_{i=1}^{N}|\partial_{x_i}u_n|_{p_i(x)}^{P_-^-} - \sum_{\{i;\alpha_{i,n}=P_+^+\}}\Big(|\partial_{x_i}u_n|_{p_i(x)}^{P_-^-} - |\partial_{x_i}u_n|_{p_i(x)}^{P_+^+}\Big) \\
&\geq N\Big(\frac{\sum_{i=1}^{N}|\partial_{x_i}u_n|^{p_i(x)}}{N}\Big)^{P_-^-} - N \\
&= \frac{\|u_n\|^{P_-^-}}{N^{P_-^- -1}} - N.
\end{aligned}
\tag{5.15}
$$

Taking into consideration (5.15), we get

$$1 + c_7 + \|u_n\| \geq \Big(\frac{1}{P_+^+} - \frac{1}{q^-}\Big)\Big(\frac{\|u_n\|^{P_-^-}}{N^{P_-^- - 1}} - N\Big) - c_8\|u_n\|^{\gamma^+}.$$

Dividing the above inequality by $\|u_n\|^{P_-^-}$ and passing to the limit as $n \to \infty$ we obtain a contradiction, since $q^- > P_+^+$ and $P_-^- > \gamma^+$. It follows that (u_n) is bounded in X. This information combined with the fact that X is reflexive implies that there exists a subsequence, still denote by (u_n), and $u_0 \in X$ such that (u_n) converges weakly to u_0 in X.

Using (5.12) we infer that

$$\lim_{n\to\infty} \langle I'_\lambda(u_n), u_n - u_0\rangle = 0,$$

more precisely,

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \Big[\quad & \sum_{i=1}^N a_i(x, \partial_{x_i}u_n)(\partial_{x_i}u_n - \partial_{x_i}u_0) + \\ & b(x)|u_n|^{P_+^+-2}u_n(u_n - u_0) - \\ & \lambda|u_n - u_0|^{q(x)} - \lambda|u_n - u_0|^{\gamma(x)}\Big]dx = 0.\end{aligned} \tag{5.16}$$

Note that $q(x), \gamma(x)$ and P_+^+ fulfill (4.6), hence Proposition 4.1 yields that the embeddings $X \hookrightarrow L^{q(x)}(\Omega)$, $X \hookrightarrow L^{\gamma(x)}(\Omega)$ and $X \hookrightarrow L^{P_+^+}(\Omega)$ are compact. Thus (u_n) converges strongly to u_0 in $L^{q(x)}(\Omega), L^{\gamma(x)}(\Omega)$ and also in $L^{P_+^+}(\Omega)$. By these facts and using Propositions 2.1-2.2, relation (5.16) reduces to

$$\lim_{n\to\infty}\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i}u_n)(\partial_{x_i}u_n - \partial_{x_i}u_0)\, dx = 0.$$

Using Lemma 4.4, we deduce that (u_n) converges strongly to u_0 in X, in other words I_λ satisfies the Palais-Smale condition.

5.1 Proof of Theorem 3.1 concluded

By Lemmas 5.1 and 5.2, all assumptions of the mountain pass theorem in [3] are satisfied. Then we deduce u_0 as a nontrivial critical point of the functional I_λ with $I_\lambda(u_0) = c_7$ and thus a nontrivial weak solution of problem (3.4).

We now prove that there exists a second weak solution $u_1 \in X$ such that $u_1 \neq u_0$. Indeed, let λ^* as in the proof of Lemma 5.1(**i**) and assume that $\lambda < \lambda^*$. By Lemma 5.1(**i**), it follows that on the boundary of the ball centered at the origin and of radius r in X, denoted by $B_r(0) = \{\omega \in X;\ \|\omega\| < r\}$, we have

$$\inf_{\partial B_r(0)} I_\lambda(u) > 0.$$

On the other hand, by Lemma 5.1(**iii**), there exists $\varphi \in X$ such that

$$I_\lambda(t\varphi) < 0 \quad \text{for} \quad t > 0 \quad \text{small enough}.$$

Moreover, for $u \in B_r(0)$,

$$I_\lambda(u) \geq \Big[c_3\|u\|^{P_+^+} - c_4\lambda(\|u\|^{q^-} + \|u\|^{q^+})\Big] + \Big[c_3\|u\|^{P_+^+} - c_5\lambda(\|u\|^{\gamma^-} + \|u\|^{\gamma^+})\Big].$$

It follows that

$$-\infty < c_9 = \inf_{\overline{B_r(0)}} I_\lambda(u) < 0.$$

We let now $0 < \varepsilon < \inf_{\partial B_r(0)} I_\lambda - \inf_{B_r(0)} I_\lambda$. Applying Ekeland variational principle [11] to the functional $I_\lambda : \overline{B_r(0)} \to \mathbb{R}$, we find $u_\varepsilon \in \overline{B_r(0)}$ such that

$$\begin{aligned} I_\lambda(u_\varepsilon) &< \inf_{\overline{B_r(0)}} I_\lambda + \varepsilon \\ I_\lambda(u_\varepsilon) &< I_\lambda(u) + \varepsilon\|u - u_\varepsilon\|, \quad u \neq u_\varepsilon. \end{aligned}$$

Since

$$I_\lambda(u_\varepsilon) \le \inf_{\overline{B_r(0)}} I_\lambda + \varepsilon \le \inf_{B_r(0)} I_\lambda + \varepsilon < \inf_{\partial B_r(0)} I_\lambda,$$

we deduce that $u_\varepsilon \in B_r(0)$. Now, we define $K : \overline{B_r(0)} \to \mathbb{R}$ by $K(u) = I_\lambda(u) + \varepsilon\|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of K and thus

$$\frac{K(u_\varepsilon + tv) - K(u_\varepsilon)}{t} \ge 0,$$

for small $t > 0$ and $v \in B_r(0)$. The above relation yields

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \ge 0.$$

Letting $t \to 0$ it follows that $\langle I'_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\| > 0$ and we infer that $\|I'_\lambda(u_\varepsilon)\| \le \varepsilon$. We deduce that there exists a sequence $(v_n) \subset B_r(0)$ such that

$$I_\lambda(v_n) \to c_9 \quad \text{and} \quad I'_\lambda(v_n) \to 0. \tag{5.17}$$

It is clear that (v_n) is bounded in X. Thus, there exists $u_1 \in X$ such that, up to a subsequence, (v_n) converges weakly to u_1 in X. Actually, with similar arguments as those used in the proof that the sequence $u_n \to u_0$ in X we can show that $v_n \to u_1$ in X. Thus, by relation (5.17),

$$I_\lambda(u_1) = c_9 < 0 \quad \text{and} \quad I'_\lambda(u_1) = 0,$$

hence u_1 is a nontrivial weak solution for problem (3.4). □

Finally, since

$$I_\lambda(u_0) = c_7 > 0 > c_9 = I_\lambda(u_1),$$

we see that $u_0 \neq u_1$. Thus, problem (3.4) has two nontrivial weak solutions.

6 Infinitely many high or small energy solutions

Since in this section, we are concerned with the existence of multiple weak solutions of problem (3.5), we begin by giving the definition of such solutions.

Definition 6.1 *A function $u \in X$ which verifies*

$$\int_\Omega \Big\{ \sum_{i=1}^N a_i(x, \partial_{x_i})\partial_{x_i}\varphi + b(x)|u|^{P_+^+-2}u\varphi - \lambda|u|^{q(x)-2}u\varphi - \mu|u|^{\gamma(x)-2}u\varphi \Big\}\, dx = 0,$$

for all $\varphi \in X$ is called a weak solution of problem (3.5).

We associate to problem (3.5) the energy functional $I_{\lambda,\mu} : X \to \mathbb{R}$ defined by

$$I_{\lambda,\mu}(u) = \int_\Omega \Big\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - \frac{\lambda}{q(x)}|u|^{q(x)} - \frac{\mu}{\gamma(x)}|u|^{\gamma(x)} \Big\}\, dx.$$

Due to Proposition 4.2, by a standard calculus it can be shown that $I_{\lambda,\mu}$ is well defined and $I_{\lambda,\mu} \in C^1(X, \mathbb{R})$ with

$$\langle I'_{\lambda,\mu}(u), \varphi\rangle = \int_\Omega \Big\{ \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i}\varphi + b(x)|u|^{P_+^+-2}u\varphi - \lambda|u|^{q(x)-2}u\varphi - \mu|u|^{\gamma(x)-2}u\varphi \Big\}\, dx,$$

for all $u, \varphi \in X$. Hence any critical point $u \in X$ of $I_{\lambda,\mu}$ is a weak solution of problem (3.5).

We will use Lemma 4.2 to prove Theorem 3.2 (**i**) and Lemma 4.3 to prove Theorem 3.2 (**ii**), respectively. We will show that hypotheses (**H1**)-(**H3**) and (**J1**)-(**J4**) are fulfilled. To this end we will prove the following auxiliary results.

Lemma 6.1 *For every $k \in \mathbb{N}$, there exists $r_k > 0$ such that $\inf_{u\in Z_k,\ \|u\|=r_k} I_{\lambda,\mu}(u) \to +\infty$, when $k \to +\infty$.*

Proof. By (**A2**) and (**B**) for any $u \in Z_k,\ \|u\| = r_k > 1$ (r_k will be specified below), when $\lambda > 0, \mu \in \mathbb{R}$ we have

$$\begin{aligned} I_{\lambda,\mu}(u) &= \int_\Omega \Big\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - \frac{\lambda}{q(x)}|u|^{q(x)} - \frac{\mu}{\gamma(x)}|u|^{\gamma(x)}\Big\}\, dx \\ &\geq \frac{1}{P_+^+}\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\, dx + \frac{b_0}{P_+^+}\int_\Omega |u|^{P_+^+}\, dx - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}\, dx - |\mu| \int_\Omega \frac{1}{\gamma(x)}|u|^{\gamma(x)}\, dx \\ &\geq \frac{1}{P_+^+}\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\, dx + \frac{b_0}{P_+^+}|u|^{P_+^+}_{L^{P_+^+}(\Omega)} - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}\, dx - \frac{c_{10}|\mu|}{\gamma^-}\|u\|^{\gamma^+}. \end{aligned} \tag{6.18}$$

Taking into consideration relations (5.11) and (5.15), the inequality (6.18) reduces to

$$I_{\lambda,\mu}(u) \geq \frac{\|u\|^{P_-^-}}{P_+^+ N^{P_-^- - 1}} - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)} dx - \frac{c_{10}|\mu|}{\gamma^-}\|u\|^{\gamma^+}.$$

Since $P_-^- > \gamma^+$, there exists $r_0 > 0$ large enough such that $\frac{c_{10}|\mu|}{\gamma^-}\|u\|^{\gamma^+} \leq \frac{1}{2P_+^+ N^{P_-^- -1}}\|u\|^{P_-^-}$ as $r = \|u\| \geq r_0$. If $|u|_{q(x)} \leq 1$ then $\int_\Omega |u|^{q(x)}dx \leq |u|^{q^-}_{q(x)} \leq 1$. However, if $|u|_{q(x)} > 1$ then $\int_\Omega |u|^{q(x)}dx \leq |u|^{q^+}_{q(x)} \leq (\beta_k\|u\|)^{q^+}$. So, we conclude that

$$I_{\lambda,\mu}(u) \geq \begin{cases} \frac{1}{2P_+^+ N^{P_-^- -1}}\|u\|^{P_-^-} - \frac{\lambda c_{11}}{q^-} & \text{if } |u|_{q(x)} \leq 1, \\ \frac{1}{2P_+^+ N^{P_-^- -1}}\|u\|^{P_-^-} - \frac{\lambda}{q^-}(\beta_k\|u\|)^{q^+} & \text{if } |u|_{q(x)} > 1, \end{cases}$$

$$\geq \frac{1}{2P_+^+ N^{P_-^- -1}}\|u\|^{P_-^-} - \frac{\lambda}{q^-}(\beta_k\|u\|)^{q^+} - c_{12}.$$

Choose $r_k = \left(\frac{\lambda}{q^-}N^{P_-^- -1}q^+\beta_k^{q^+}\right)^{\frac{1}{P_-^- - q^+}}$. Then

$$I_{\lambda,\mu}(u) \geq \frac{1}{N^{P_-^- -1}}\Big(\frac{1}{P_+^+} - \frac{1}{q^+}\Big)r_k^{P_-^-} - c_{12} \to +\infty \quad \text{as} \quad k \to +\infty,$$

because $P_+^+ < q^- \leq q^+$ and $\beta_k \to 0$ as $k \to +\infty$. So, we can deduce for $u \in Z_k$ with $\|u\| = r_k > 1$, $I_{\lambda,\mu}(u) \to +\infty$ as $k \to +\infty$ and the proof is complete.

Lemma 6.2 *For every $k \in \mathbb{N}$ there exists $\rho_k > r_k$ (r_k given by Lemma 6.1) such that*

$$\max_{u\in Y_k,\ \|u\|=\rho_k} I_{\lambda,\mu}(u) \leq 0.$$

Proof. From ($\mathbf{A_0}$) and ($\mathbf{A_1}$), for any $u \in Y_k\backslash\{0\}$ with $\|u\| = 1$ and $1 < \rho_k = t_k$ with $t_k \to \infty$, $\lambda > 0,\ \mu \in \mathbb{R}$, we have

$$\begin{aligned} I_{\lambda,\mu}(t_k u) &= \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i}(t_k u))dx + \frac{1}{P_+^+}\int_\Omega b(x)|t_k u|^{P_+^+}dx - \lambda\int_\Omega \frac{1}{q(x)}|t_k u|^{q(x)}\,dx \\ &-\mu\int_\Omega \frac{1}{\gamma(x)}|t_k u|^{\gamma(x)}\,dx \\ &\le c_6\sum_{i=1}^N\int_\Omega\Big(|\partial_{x_i}(t_k u)| + \frac{|\partial_{x_i}(t_k u)|^{p_i(x)}}{p_i(x)}\Big)dx + \frac{1}{P_+^+}\int_\Omega b(x)|t_k u|^{P_+^+}\,dx \\ &-\frac{\lambda}{q^+}\int_\Omega |t_k u|^{q(x)}dx + \frac{|\mu|}{\gamma^-}\int_\Omega |t_k u|^{\gamma(x)}\,dx \\ &\le c_6 t_k^{P_+^+}\sum_{i=1}^N\int_\Omega\Big(|\partial_{x_i}u| + \frac{|\partial_{x_i}u|^{p_i(x)}}{P_-^-}\Big)dx + \frac{t_k^{P_+^+}}{P_+^+}\int_\Omega b(x)|u|^{P_+^+}\,dx \\ &-\frac{\lambda t_k^{q^-}}{q^+}\int_\Omega |u|^{q(x)}\,dx + \frac{|\mu| t_k^{\gamma^+}}{\gamma^-}\int_\Omega |u|^{\gamma(x)}\,dx. \end{aligned}$$

Since $\dim Y_k < \infty$ and all norms are equivalent in the finite dimensional space, it is easy to see that $I(t_k u) \to -\infty$ as $k \to +\infty$ for $u \in Y_k$, due to $P_+^+ < q^-$ and $\gamma^+ < P_-^-$. Therefore, we deduce that for ρ_k large enough ($\rho_k > r_k$), $\max_{u\in Y_k,\ \|u\|=\rho_k} I_{\lambda,\mu}(u) \le 0$.

Lemma 6.3 *The energy functional $I_{\lambda,\mu}$ satisfies the Palais-Smale condition.*

Proof. The proof is similar to that of Lemma 5.2 if we use the following inequality

$$1 + c_{13} + \|u_n\| \ge I_{\lambda,\mu}(u_n) - \frac{1}{q^-}\langle I'_{\lambda,\mu}(u_n), u_n\rangle.$$

Now, we want to construct the geometry of the dual fountain theorem. We will show that when $\mu > 0,\ \lambda \in \mathbb{R}$, hypotheses (**J1**)-(**J4**) are fulfilled.

Lemma 6.4 *There is $k_0 > 0$ so that, for each $k \ge k_0$, there exists $\rho_k > 0$ such that* $\inf_{u\in Z_k,\ \|u\|=\rho_k} I_{\lambda,\mu}(u) \ge 0$.

Proof. By (**A2**), relations (5.10) and (5.11), for any $u \in Z_k$ we have

$$\begin{aligned} I_{\lambda,\mu}(u) &= \int_\Omega\Big\{\sum_{i=1}^N A_i(x, \partial_{x_i}u) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - \frac{\lambda}{q(x)}|u|^{q(x)} - \frac{\mu}{\gamma(x)}|u|^{\gamma(x)}\Big\}dx \\ &\ge \frac{1}{P_+^+}\sum_{i=1}^N\int_\Omega |\partial_{x_i}u|^{p_i(x)}dx + \frac{b_0}{P_+^+}|u|^{P_+^+}_{L^{P_+^+}(\Omega)} - \frac{c_{14}|\lambda|}{q^-}\|u\|^{q^-}\,dx - \frac{\mu}{\gamma^-}\int_\Omega |u|^{\gamma(x)}\,dx \\ &\ge \frac{1}{P_+^+N^{P_+^+-1}}\|u\|^{P_+^+} - \frac{c_{14}|\lambda|}{q^-}\|u\|^{q^-} - \frac{\mu}{\gamma^-}\int_\Omega |u|^{\gamma(x)}dx. \end{aligned}$$

Since $P_+^+ < q^-$, there exists $\rho_0 > 0$ small enough such that $\frac{c_{14}|\lambda|}{q^-}\|u\|^{q^-} \le \frac{1}{2P_+^+N^{P_+^+-1}}\|u\|^{P_+^+}$ as $0 < \rho = \|u\| \le \rho_0$. Then from the proof above, we have

$$I_{\lambda,\mu}(u) \ge \begin{cases} \frac{1}{2P_+^+N^{P_+^+-1}}\|u\|^{P_+^+} - \frac{c_{15}\mu}{\gamma^-} & \text{if } |u|_{\gamma(x)} \le 1, \\ \frac{1}{2P_+^+N^{P_+^+-1}}\|u\|^{P_+^+} - \frac{\mu}{\gamma^-}(\theta_k\|u\|)^{\gamma^+} & \text{if } |u|_{\gamma(x)} > 1. \end{cases} \tag{6.19}$$

Choose $\rho_k = \Big(\frac{2P_+^+N^{P_+^+-1}\mu\theta_k^{\gamma^+}}{\gamma^-}\Big)^{\frac{1}{P_+^+-\gamma^+}}$. Then

$$I_{\lambda,\mu}(u) \ge \frac{1}{2P_+^+N^{P_+^+-1}}(\rho_k)^{P_+^+} - \frac{1}{2P_+^+N^{P_+^+-1}}(\rho_k)^{P_+^+} = 0.$$

Since $P_+^+ \geq P_-^- > \gamma^+$, $\theta_k \to 0$, we know $\rho_k \to 0$ as $k \to \infty$. So we can deduce for $u \in Z_k$ with $\|u\| = \rho_k$, $I_{\lambda,\mu}(u) \geq 0$.

Lemma 6.5 *For every $k \geq k_0$ there exists $r_k < \rho_k$ (ρ_k given by Lemma 6.4) such that*

$$\max_{u \in Y_k,\ \|u\|=r_k} I_{\lambda,\mu}(u) < 0 \quad as \quad k \to +\infty.$$

Proof. From ($\mathbf{A_0}$) and ($\mathbf{A_1}$), for $v \in Y_k$ with $\|v\| = 1$ and $0 < t < \rho_k < 1$, we have

$$\begin{aligned}
I_{\lambda,\mu}(tv) &\leq c_6 \sum_{i=1}^{N} \int_\Omega \Big(|\partial_{x_i}(tv)| + \frac{|\partial_{x_i}(tv)|^{p_i(x)}}{p_i(x)} \Big) dx + \frac{1}{P_+^+} \int_\Omega b(x)|tv|^{P_+^+}\, dx \\
&+ \frac{|\lambda|}{q^-} \int_\Omega t^{q(x)}|v|^{q(x)}\, dx - \frac{\mu}{\gamma^+} \int_\Omega t^{\gamma(x)}|v|^{\gamma(x)}\, dx \\
&\leq c_6 t^{P_-^-} \sum_{i=1}^{N} \int_\Omega \Big(|\partial_{x_i} v| + \frac{|\partial_{x_i} v|^{p_i(x)}}{P_-^-} \Big) dx + \frac{t^{P_+^+}}{P_+^+} \int_\Omega b(x)|v|^{P_+^+}\, dx \\
&+ \frac{|\lambda| t^{q^-}}{q^-} \int_\Omega |v|^{q(x)}\, dx - \frac{\mu t^{\gamma^+}}{\gamma^+} \int_\Omega |v|^{\gamma(x)}\, dx.
\end{aligned}$$

Since $\dim Y_k = k$, conditions $\gamma^+ < P_-^-$ and $P_+^+ < q^-$ imply that there exists a $r_k \in (0, \rho_k)$ such that $I_{\lambda,\mu}(u) < 0$ when $\|u\| = r_k$. Hence $b_k = \max\{I_{\lambda,\mu}(u) \ :\ u \in Y_k,\ \|u\| = r_k\} < 0$ and the proof is complete.

Lemma 6.6 *For every $k \geq k_0$ and ρ_k given by Lemma 6.4, we have*

$$\inf_{u \in Z_k,\ \|u\| \leq \rho_k} I_{\lambda,\mu}(u) \to 0.$$

Proof. Because $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$d_k = \inf\{I_{\lambda,\mu}(u) \ :\ u \in Z_k,\ \|u\| \leq \rho_k\} \ \leq b_k = \max\{I_{\lambda,\mu}(u) \ :\ u \in Y_k,\ \|u\| = r_k\} < 0.$$

From (6.19), for $u \in Z_k$, $\|u\| \leq \rho_k$ small enough we can write

$$\begin{aligned}
I_{\lambda,\mu}(u) &\geq \tfrac{1}{2P_+^+ N^{P_+^+ - 1}} \|u\|^{P_+^+} - \tfrac{\mu}{\gamma^-} \theta_k^{\gamma^+} \|u\|^{\gamma^+} \\
&\geq -\tfrac{\mu}{\gamma^-} \theta_k^{\gamma^+} \|u\|^{\gamma^+},
\end{aligned}$$

hence $d_k \to 0$, due to $\theta_k \to 0$ and $\rho_k \to 0$ as $k \to +\infty$,

Lemma 6.7 *The functional $I_{\lambda,\mu}$ satisfies the $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0)$.*

Proof. Suppose $(u_{n_j}) \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$ and $(I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}) \to 0$. Assume $\|u_{n_j}\| > 1$ for convenience. If $\lambda \geq 0$, for n large enough, we have

$$\begin{aligned}
1 + c_{16} + \|u_{n_j}\| &\geq I_{\lambda,\mu}(u_{n_j}) - \frac{1}{q^-} \langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j} \rangle \\
&\geq \Big(\frac{1}{P_+^+} - \frac{1}{q^-} \Big) \Big(\frac{\|u_{n_j}\|^{P_-^-}}{N^{P_-^- - 1}} - N \Big) - c_{17} \|u_{n_j}\|^{\gamma^+}.
\end{aligned}$$

Since $P_-^- > \gamma^+$ and $q^- > P_+^+$, we deduce that (u_{n_j}) is bounded in X.
If $\lambda < 0$, for n large enough, we can consider the inequality below to get the boundedness of (u_{n_j}).

$$1 + c_{18} + \|u_{n_j}\| \geq I_{\lambda,\mu}(u_{n_j}) - \frac{1}{q^+} \langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j} \rangle.$$

Passing if necessary to a subsequence, we can assume $u_{n_j} \rightharpoonup u$ in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$\begin{aligned} \lim_{n_j\to+\infty}\langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j} - u\rangle &= \lim_{n_j\to+\infty}\langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j} - v_{n_j}\rangle \\ &+ \lim_{n_j\to+\infty}\langle I'_{\lambda,\mu}(u_{n_j}), v_{n_j} - u\rangle \\ &= \lim_{n_j\to+\infty}\left\langle (I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j}\right\rangle \\ &= 0. \end{aligned}$$

Similar to the process of verifying the Palais-Smale condition in the proof of Lemma 5.2, we conclude $u_{n_j} \to u$, furthermore we have $I'_{\lambda,\mu}(u_{n_j}) \to I'_{\lambda,\mu}(u)$.
Let us prove $I'_{\lambda,\mu}(u) = 0$ below. Taking $\omega_k \in Y_k$, notice that when $n_j \geq k$ we have

$$\begin{aligned} \langle I'_{\lambda,\mu}(u), \omega_k\rangle &= \langle I'_{\lambda,\mu}(u) - I'_{\lambda,\mu}(u_{n_j}), \omega_k\rangle + \langle I'_{\lambda,\mu}(u_{n_j}), \omega_k\rangle \\ &= \langle I'_{\lambda,\mu}(u) - I'_{\lambda,\mu}(u_{n_j}), \omega_k\rangle + \left\langle (I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}), \omega_k\right\rangle. \end{aligned}$$

Going to the limit on the right side of the above equation reaches

$$\langle I'_{\lambda,\mu}(u), \omega_k\rangle = 0, \qquad \text{for all} \quad \omega_k \in Y_k,$$

so $I'_{\lambda,\mu}(u) = 0$, this show that $I_{\lambda,\mu}$ satisfies the $(PS)^*_c$ condition for every $c \in \mathbb{R}$.

6.1 Proof of Theorem 3.2 concluded

(**i**) The fact that the mapping A_i is even in η implies that $I_{\lambda,\mu}$ is even. The proof follows immediately from Lemmas 6.1-6.3 and Lemma 4.2.

(**ii**) This follows by combining Lemmas 6.4–6.7 and Lemma 4.3. □

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