Research Article

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Infinitely Many Solutions for a Non-homogeneous Differential Inclusion with Lack of Compactness

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Abstract: In this paper, we consider the following class of differential inclusion problems in $\mathbb{R}^N$ involving the $p(x)$-Laplacian:

$$-\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u \in a(x)\partial F(x,u) \quad \text{in } \mathbb{R}^N.$$

We are concerned with a multiplicity property, and our arguments combine the variational principle for locally Lipschitz functions with the properties of the generalized Lebesgue–Sobolev space. Applying the nonsmooth symmetric mountain pass lemma and the fountain theorem, we establish conditions such that the associated energy functional possesses infinitely many critical points, and then we obtain infinitely many solutions.

Keywords: $p(x)$-Laplacian, Differential Inclusion Problem, Locally Lipschitz Function, Infinitely many Solutions, Variational Method

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1 Introduction

Since many free boundary problems and obstacle problems may be reduced to partial differential equations with discontinuous nonlinearities, the existence of multiple solutions of the problems with discontinuous nonlinearities has been widely investigated in recent years. In 1981, Chang [1] extended the variational methods to a class of nondifferentiable functionals, and directly applied the variational methods for nondifferentiable functionals to prove some existence theorems for a PDE with discontinuous nonlinearities (see also [23, 24]). In 2000, Kourogenis and Papageorgiou [17] obtained some nonsmooth critical point theorems and applied these results to nonlinear elliptic equations at resonance, involving the p-Laplacian with discontinuous nonlinearities. Later on, we refer to the nonsmooth version of the three critical points theorem and the nonsmooth Ricceri-type variational principle established by Marano and Motreanu [20], who gave an application to elliptic problems involving the p-Laplacian with discontinuous nonlinearities. By using the Ekeland variational principle and a deformation theorem, Kandilakis, Kourogenis and Papageorgiou [16] obtained the local linking theorem for locally Lipschitz functions. Dai [4] elaborated a nonsmooth version of the fountain theorem and gave an application to a Dirichlet-type differential inclusion involving the $p(x)$-Laplacian.

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Recently, Ge and Liu [12] obtained a nonsmooth version of the principle of symmetric criticality by using the genus properties.

In the latest years, the study of nonlinear partial differential equations with non-standard growth conditions has been the object of an increasing amount of attention. A comprehensive treatment of the variational analysis of nonlinear partial differential equations with variable exponent can be found in the recent monographs [22, 27].

Motivated by the above references, we study in this paper the differential inclusion problem

$$-\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u \in a(x)\partial F(x,u) \quad \text{in } \mathbb{R}^N, \tag{P}$$

where $p\colon \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and $1 < p^- := \inf_{\mathbb{R}^N} p(x) \leq \sup_{\mathbb{R}^N} p(x) =: p^+ < N$, V is the new potential function, $a \in L^1_+(\mathbb{R}^N) \cap L^{\frac{N}{p(x)-1}}(\mathbb{R}^N)$, $F(x,t)$ is locally Lipschitz in the t-variable integrand (in general, it can be nonsmooth) and $\partial F(x,t)$ is the subdifferential with respect to the t-variable in the sense of Clarke [2].

In the last few decades, the existence and multiplicity of solutions to problem (P) have been investigated in many papers via variational methods, and several results have been obtained based on various hypotheses on the potential function F; we refer the reader to [3, 12–14] and the references therein for details. For example, when $a(x) \equiv 1$, $V(x) \equiv 1$ and p is radially symmetric on $\mathbb{R}^N$ with $2 \leq N < p^- \leq p^+ < +\infty$, the authors in [3] proved the existence of infinitely many radially symmetric solutions.

The case of $a(x) \equiv 1$, p, V are radially symmetric on $\mathbb{R}^N$ with $1 < p^- \leq p^+ < N$ and $V^- > 1$ was discussed in [13], where the existence of at least two nontrivial solutions is established.

For the case when $a(x) \equiv 1$, $V \in C(\mathbb{R}^N)$ with $0 < V^-$, and

$$\lim_{|y|\to\infty} \mu(\{x \in \mathbb{R}^N : V(x) \leq b\} \cap B_r(y)) = 0 \quad \text{for all } b > 0,$$

for some $r > 0$, Ge, Zhou and Xue [14] proved the existence of infinitely many distinct positive solutions whose $W^{1,p(x)}(\mathbb{R}^N)$-norms tend to zero (respectively, to infinity) whenever the nonlinearity oscillates at zero (respectively, at infinity). The same for asymptotically linear and coercive problems is studied by Ge, Zhou and Xue in [15]. The authors established the existence of at least two nontrivial solutions by using the Weierstrass theorem and the mountain pass theorem. Recently, for the case that $F(x,u)$ satisfies sublinear growth condition in u at infinity, Ge and Liu [12] used the genus properties in nonsmooth critical point theory in order to prove the existence of infinitely many solutions.

Inspired by the above-mentioned papers, we study problem (P) in two distinct situations. In this new framework, we will show the existence of infinitely many nontrivial solutions for problem (P). These results are beneficial supplement and development of the above-mentioned achievements. Besides, our results are also new, even in the smooth case. Before stating our main results, we first make some assumptions on the functions V and F. For the potential V, we make the following assumptions:

(V_1) $V \in C(\mathbb{R}^N)$, $0 < V^-$.

(V_2) There exists $r > 0$ such that, for any $b > 0$,

$$\lim_{|y|\to\infty} \mu(\{x \in \mathbb{R}^N : V(x) \leq b\} \cap B_r(y)) = 0,$$

where μ is the Lebesgue measure on $\mathbb{R}^N$.

Note that if $V \in C(\mathbb{R}^N)$ is coercive, namely,

(V_3) $\lim_{|x|\to\infty} V(x) = +\infty$,

then (V_2) is satisfied. For the nonlinearity F, we suppose the following hypotheses:

The function $F\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is such that $F(x,0) = 0$ a.e. on $\mathbb{R}^N$, and

(f_0) F is a Carathéodory function, that is, for all $t \in \mathbb{R}$, the mapping $x \mapsto F(x,t)$ is measurable and, for almost all $x \in \mathbb{R}^N$, the function $t \mapsto F(x,t)$ is locally Lipschitz;

(f_1) for almost all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}$ and all $w \in \partial F(x,t)$, we have

$$|w| \leq c(1 + |t|^{p(x)-1});$$

(f$_2$) there exist $\delta > 0$ and $\vartheta \in L^\infty_-(\mathbb{R}^N)$ such that, for almost all $x \in \mathbb{R}^N$, we have

$$\sup_{0<|t|<\delta} F(x,t) \leq \vartheta(x) < 0,$$

where $L^\infty_-(\mathbb{R}^N) = \{\eta \in L^\infty(\mathbb{R}^N) : \eta(x) < 0 \text{ for all } x \in \mathbb{R}^N\}$;

(f$_3$) there exists $q \in C_+(\mathbb{R}^N)$ with

$$p^+ < q^- \leq q(x) < \frac{Np(x) - p(x)(p(x)-1)}{N - p(x)} < p^*(x) \quad \text{for all } x \in \mathbb{R}^N$$

such that, for almost all $x \in \mathbb{R}^N$, we have

$$\liminf_{|t|\to\infty} \frac{F(x,t)}{|t|^{q(x)}} > 0;$$

(f$_4$) $F(x,-t) = F(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Now we are ready to state our first result.

Theorem 1.1. *Suppose that* (f$_0$) *and* (f$_1$)–(f$_4$) *hold. Then problem* (P) *has infinitely many nontrivial solutions.*

Moreover, if condition (f$_2$) is completely removed, then we have another group of solutions on (P), which reads as follows:

Theorem 1.2. *Suppose that* (f$_0$), (f$_1$), (f$_3$) *and* (f$_4$) *hold. Then problem* (P) *has infinitely many nontrivial solutions.*

The remainder of the paper is organized as follows. In Section 2, we will recall the definitions and some properties of variable exponent Sobolev spaces. In Section 3, the proof of the main results is given.

2 Preliminary Results

2.1 Variable Exponent Sobolev Spaces

We start with some preliminary basic results on the theory of Lebesgue–Sobolev spaces with variable exponent. For more details, we refer the reader to the books [5, 21] and the papers [6, 8, 9, 18, 25, 26].

Throughout this paper, we always assume $p(x) > 1$ and $p \in C(\mathbb{R}^N)$. Set

$$C_+(\mathbb{R}^N) = \{h \in C(\mathbb{R}^N) : h(x) > 1 \text{ for all } x \in \mathbb{R}^N\}.$$

For any $h \in C_+(\mathbb{R}^N)$, we will denote

$$h^- = \min_{x\in\mathbb{R}^N} h(x), \quad h^+ = \max_{x\in\mathbb{R}^N} h(x)$$

and denote by $h_1 \ll h_2$ the fact that $\inf_{x\in\mathbb{R}^N}(h_2(x) - h_1(x)) > 0$.

For $p(x) \in C_+(\mathbb{R}^N)$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\mathbb{R}^N) = \Big\{u : u \text{ is measurable and } \int_{\mathbb{R}^N} |u(x)|^{p(x)}\,dx < +\infty\Big\}$$

with the norm

$$|u|_{L^{p(x)}(\mathbb{R}^N)} = |u|_{p(x)} = \inf\Big\{\Lambda > 0 : \int_{\mathbb{R}^N} \Big|\frac{u(x)}{\Lambda}\Big|^{p(x)}\,dx \leq 1\Big\},$$

and we define the variable exponent Sobolev space

$$W^{1,p(x)}(\mathbb{R}^N) = \{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\},$$

with the norm $\|u\| = \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}$. We recall that spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.

Define $J\colon W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ by

$$J(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx \quad \text{for all } u \in W^{1,p(x)}(\mathbb{R}^N).$$

Then $J \in C^1(W^{1,p(x)}(\mathbb{R}^N, \mathbb{R})$. If we denote

$$A = J'\colon W^{1,p(x)}(\mathbb{R}^N) \to (W^{1,p(x)}(\mathbb{R}^N))^*,$$

then

$$\langle A(u), v\rangle = \int_{\mathbb{R}^N} [|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v + V(x)|u|^{p(x)-2}uv]\,dx \quad \text{for all } u, v \in W^{1,p(x)}(\mathbb{R}^N),$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $(W^{1,p(x)}(\mathbb{R}^N))^*$ and $W^{1,p(x)}(\mathbb{R}^N)$.

Proposition 2.1 ([1]). *Set $G = W^{1,p(x)}(\mathbb{R}^N)$, and A is as above. Then $A\colon G \to G^*$ is*
(1) *a convex, bounded and strictly monotone operator,*
(2) *a mapping of type $(S)_+$, i.e., $u_n \rightharpoonup u$ in G and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$ implies $u_n \to u$ in G, where $\rightharpoonup$ and $\to$ denote the weak and the strong convergence in G, respectively,*
(3) *a homeomorphism.*

Denote by $L^{q(x)}(\mathbb{R}^N)$ the conjugate space of $L^{p(x)}(\mathbb{R}^N)$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Then the Hölder-type inequality

$$\int_{\mathbb{R}^N} |uv|\,dx \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{L^{p(x)}(\mathbb{R}^N)}|v|_{L^{q(x)}(\mathbb{R}^N)}, \quad u \in L^{p(x)}(\mathbb{R}^N),\ v \in L^{q(x)}(\mathbb{R}^N),$$

holds. Furthermore, if we define the mapping $\rho\colon L^{p(x)}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\rho(u) = \int_{\mathbb{R}^N} |u|^{p(x)}\,dx,$$

then the following relations hold:

$$\text{for } u \neq 0, \quad |u|_{p(x)} = \mu \iff \rho\Big(\frac{u}{\mu}\Big) = 1, \tag{2.1}$$
$$|u|_{p(x)} < 1\ (= 1,\ > 1) \iff \rho(u) < 1\ (= 1,\ > 1), \tag{2.2}$$
$$|u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}, \tag{2.3}$$
$$|u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}. \tag{2.4}$$

Proposition 2.2 ([7]). *Let $p(x)$, $q(x)$ be measurable functions such that $p(x) \in L^\infty(\mathbb{R}^N)$ and $1 \le p(x)q(x) \le \infty$ almost everywhere in $\mathbb{R}^N$. Let $u \in L^{q(x)}(\mathbb{R}^N)$, $u \neq 0$. Then*

$$|u|_{p(x)q(x)} \ge 1 \implies |u|_{p(x)q(x)}^{p^-} \le \big||u|^{p(x)}\big|_{q(x)} \le |u|_{p(x)q(x)}^{p^+},$$
$$|u|_{p(x)q(x)} \le 1 \implies |u|_{p(x)q(x)}^{p^+} \le \big||u|^{p(x)}\big|_{q(x)} \le |u|_{p(x)q(x)}^{p^-}.$$

In particular, if $p(x) = p$ is a constant, then $\big||u|^p\big|_{q(x)} = |u|_{pq(x)}^p$.

Next we consider the case that V satisfies (V_1) and (V_2). We equip the linear subspace

$$E = \Big\{u \in W^{1,p(x)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx < +\infty\Big\}$$

with the norm

$$\|u\|_E = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^N} \Big(\Big|\frac{\nabla u}{\lambda}\Big|^{p(x)} + V(x)\Big|\frac{u}{\lambda}\Big|^{p(x)}\Big)\,dx \le 1\Big\}.$$

Then $(E, \|\cdot\|_E)$ is continuously embedded into $W^{1,p(x)}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(E, \|\cdot\|_E)$ is also a separable reflexive Banach space. It is easy to see that with the norm $\|\cdot\|_E$, Proposition 2.1 remains valid, that is, the following properties hold true.

Proposition 2.3. *Set* $I(u) = \int_{\mathbb{R}^N}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx$. *If* $u \in W^{1,p(x)}(\mathbb{R}^N)$, *then*
(i) *for* $u \neq 0$, $\|u\|_E = \lambda$ *if and only if* $I(\frac{u}{\lambda}) = 1$,
(ii) $\|u\|_E < 1$ $(= 1, > 1)$ *if and only if* $I(u) < 1$ $(= 1, > 1)$,
(iii) $\|u\|_E > 1$ *implies* $\|u\|_E^{p^-} \leq I(u) \leq \|u\|_E^{p^+}$,
(iv) $\|u\|_E < 1$ *implies* $\|u\|_E^{p^+} \leq I(u) \leq \|u\|_E^{p^-}$.

Proposition 2.4 ([8]). *If V satisfies* (V_1) *and* (V_2), *then*
(i) *we have a compact embedding* $E \hookrightarrow L^{p(x)}(\mathbb{R}^N)$,
(ii) *for any measurable function* $q\colon \mathbb{R}^N \to \mathbb{R}$ *with* $p < q \ll p^*$, *we have a compact embedding* $E \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.

2.2 Generalized Gradient

Let $(Y, \|\cdot\|)$ be a real Banach space and Y^* its topological dual. A function $\varphi\colon Y \to \mathbb{R}$ is called locally Lipschitz if each point $u \in Y$ possesses a neighborhood N_u such that $|f(u_1) - f(u_2)| \leq \|u_1 - u_2\|$ for all $u_1, u_2 \in N_u$, for a constant $L > 0$ depending on N_u. The generalized directional derivative of φ at the point $u \in Y$ in the direction $h \in Y$ is

$$\varphi^0(u; h) = \liminf_{\substack{w \to u \\ t \to 0^+}} \frac{\varphi(w + th) - \varphi(w)}{t}.$$

The generalized gradient of φ at $u \in Y$ is defined by

$$\partial\varphi(u) = \{u^* \in Y^* : \langle u^*, h\rangle \leq \varphi^0(u; h) \text{ for all } h \in Y\},$$

which is a nonempty, convex and w^*-compact subset of Y, where $\langle\cdot,\cdot\rangle$ is the duality pairing between Y^* and Y. We say that $u \in Y$ is a critical point of φ if $0 \in \partial\varphi(u)$. For further details, we refer the reader to [1].

Definition 2.5. Let Y be a real Banach space, and $\varphi\colon Y \to \mathbb{R}$ is a locally Lipschitz function. We say that φ satisfies the nonsmooth $(\mathrm{PS_c})$ condition if any sequence $\{u_n\} \subset Y$ such that $\varphi(u_n) \to c$ and $m(u_n) \to 0$ as $n \to +\infty$ has a strongly convergent subsequence, where $m(u_n) = \inf\{\|u^*\|_{Y^*} : u^* \in \partial\varphi(u_n)\}$. If this property holds at every level $c \in \mathbb{R}$, then we simply say that φ satisfies the nonsmooth (PS) condition.

In order to prove Theorem 1.1, we need the following lemma.

Lemma 2.6 ([11, Theorem 2.1.7]). *Assume that X is an infinite-dimensional Banach space, and let* $\varphi\colon X \to \mathbb{R}$ *be a locally Lipschitz function that satisfies the nonsmooth* $(\mathrm{PS_c})$ *condition for every* $c > 0$. *Assume* $\varphi(u) = \varphi(-u)$ *for all* $u \in X$ *and* $\varphi(0) = 0$. *Suppose* $X = X_1 \oplus X_2$, *where* X_1 *is finite-dimensional, and assume the following conditions:*
(a) $\alpha > 0$, $\gamma > 0$ *such that* $\|u\| = \gamma$ *with* $u \in X_2$ *implies* $\varphi(u) \geq \alpha$.
(b) *For any finite-dimensional subspace* $W \subset X_1$, *there is* $R = R(W)$ *such that* $\varphi(u) \leq 0$ *for* $u \in W$ *with* $\|u\| \geq R$.
Then φ *possesses an unbounded sequence of critical values.*

Finally, in order to prove Theorem 1.2 in the next section, we introduce the following notions and the fountain theorem.

Let X be a reflexive and separable Banach space. Then there exist (see [28, Section 17]) $e_j \subset X$ and $e_j^* \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_j : j = 1, 2, \ldots\}}, \quad X^* = \overline{\operatorname{span}\{e_j^* : j = 1, 2, \ldots\}} \quad \text{and} \quad \langle e_i^*, e_j\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we write

$$X_j = \operatorname{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j \quad \text{and} \quad Z_k = \bigoplus_{j=k}^{\infty} X_j. \tag{2.5}$$

Lemma 2.7 ([4, Theorem 3.1]). *Assume that*

(A_1) *X is a Banach space, $\varphi\colon X \to \mathbb{R}$ is a locally Lipschitz even function, the subspaces X_k, Y_k and Z_k are defined by* (2.5).

We also assume that, for any $k = 1, 2, \dots$, there exists $\rho_k > r_k > 0$ such that

(A_2) $\inf_{u \in Z_k, \|u\| = r_k} \varphi(u) \to +\infty$ *as* $k \to +\infty$,

(A_3) $\max_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \le 0$,

(A_4) *φ satisfies the nonsmooth* (PS_c) *condition for every $c > 0$.*

Then φ has an unbounded sequence of critical values.

3 Proof of the Main Results

Now we introduce the energy functional $\varphi\colon E \to \mathbb{R}$ associated with problem (P), which is defined by

$$\varphi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx.$$

From the hypotheses on F, it is standard to check that φ is locally Lipschitz on E, and (see [1])

$$\partial\varphi(u) \subseteq A(u) - \partial F(x,u)$$

for all $u \in E$.

By a solution of (P), we mean a function $u \in E$ to which there corresponds a mapping $\mathbb{R}^N \ni x \to w(x)$ with $w(x) \in \partial F(x,u)$ for almost every $x \in \mathbb{R}^N$ having the property $x \to w(x)h(x) \in L^1(\mathbb{R}^N)$ for every $h \in E$, and

$$\int_{\mathbb{R}^N} (|\nabla u|^{p(x)-2}\nabla u \cdot \nabla h + V(x)|u|^{p(x)-2}uh)\,dx = \int_{\mathbb{R}^N} w(x)h(x)\,dx.$$

Thus weak solutions of problem (P) are exactly the critical points of the functional φ.

Lemma 3.1. *Assume that all conditions of Theorem 1.1 are satisfied. Then the energy functional φ satisfies the nonsmooth* (PS) *condition in E.*

Proof. Suppose that $\{u_n\} \subset E$ is a (PS_c) sequence for φ, that is, $\varphi(u_n) \to c$ and $m(u_n) \to 0$ as $n \to +\infty$, which shows that

$$c = \varphi(u_n) + o(1), \quad m(u_n) = o(1), \tag{3.1}$$

where $o(1) \to 0$ as $n \to +\infty$.

We claim that the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded. Suppose that this is not the case. By passing to a subsequence if necessary, we may assume that $\|u_n\|_E \to +\infty$ as $n \to +\infty$. Without loss of generality, we assume $\|u_n\|_E \ge 1$. Let $u_n^* \in \partial\varphi(u_n)$ be such that $m(u_n) = \|u_n^*\|_{E^*}$, $n \in N$. We have $u_n^* = A(u_n) - w_n$, $w_n(x) \in \partial F(x, u_n(x))$, $w_n \in L^{p'(x)}(\mathbb{R}^N)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. By (3.1), there is a constant $M_1 > 0$ such that

$$|\varphi(u_n)| \le M_1 \quad \text{for all } n \ge 1, \tag{3.2}$$

and there is a constant $C > 0$ such that

$$\begin{aligned} C\|u_n\|_E \ge \langle u_n^*, u_n\rangle &= \langle A(u_n), u_n\rangle - \int_{\mathbb{R}^N} a(x) w_n u_n\,dx \\ &= \int_{\mathbb{R}^N} (|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)})\,dx - \int_{\mathbb{R}^N} a(x) w_n u_n\,dx. \end{aligned} \tag{3.3}$$

Then, for (3.2) and (3.3), we have

$$M_1 p^- + C\|u_n\|_E \ge \int_{\mathbb{R}^N} a(x)(p^- F(x,u_n) - w_n u_n)\,dx. \tag{3.4}$$

Next we estimate (3.4). By virtue of (f_3), there exists $c_1 > 0$ and $M_2 > 0$ such that, for almost all $x \in \mathbb{R}^N$ and all $|t| \geq M_2$, we have $F(x,t) \geq c_1|t|^{q(x)}$. On the other hand, from (f_1), for almost all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$ such that $|t| < M_2$, we have $|F(x,t)| \leq C$, where $C = C(M_2) > 0$. Therefore, for almost all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$, the above two inequalities imply

$$F(x,t) \geq c_1|t|^{q(x)} - c_2 \quad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{3.5}$$

where $c_2 = C + \max\{M_2^{q^-}, M_2^{q^+}\}c_1$. Using ($f_1$) again, we deduce another estimate:

$$|wt| \leq c(|t| + |t|^{p(x)}) \leq 2c(1 + |t|^{p(x)}). \tag{3.6}$$

From (3.5) and (3.6), we have

$$p^- F(x,t) - wt \geq p^- c_1|t|^{q(x)} - p^- c_2 - 2c(1 + |t|^{p(x)}). \tag{3.7}$$

Due to (3.4) and (3.7),

$$p^- c_1 \int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}\,dx - 2c \int_{\mathbb{R}^N} a(x)|u_n|^{p(x)}\,dx \leq M_1 p^- + C\|u_n\|_E + (p^- c_2 + 2c)|a|_1. \tag{3.8}$$

Note that $q^- > p^+$. Then, applying Young's inequality with ε, we get

$$\begin{aligned}|u|^{p(x)} = 1 \times |u|^{p(x)} = \varepsilon^{-\frac{p(x)}{q(x)}} \times \varepsilon^{\frac{p(x)}{q(x)}}|u|^{p(x)} &\leq (\varepsilon^{-\frac{p(x)}{q(x)}})^{\frac{q(x)}{q(x)-p(x)}} + \varepsilon\big||u|^{p(x)}\big|^{\frac{q(x)}{p(x)}} = \varepsilon^{-\frac{p(x)}{q(x)-p(x)}} + \varepsilon|u|^{q(x)}\\ &\leq \varepsilon^{-\frac{p^+}{q^- - p^+}} + \varepsilon|u|^{q(x)}.\end{aligned} \tag{3.9}$$

Hence, by (3.8) and (3.9), we obtain

$$(p^- c_1 - 2c\varepsilon) \int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}\,dx \leq M_1 p^- + C\|u_n\|_E + (p^- c_2 + 2c)|a|_1 + 2c\varepsilon^{-\frac{p^+}{q^- - p^+}}|a|_1.$$

Then, choosing ε_0 small enough such that $0 < \varepsilon_0 < \frac{p^- c_1}{2c}$, we obtain

$$\int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}\,dx \leq \frac{M_1 p^- + (p^- c_2 + 2c)|a|_1 + 2c\varepsilon^{-\frac{p^+}{q^- - p^+}}|a|_1}{p^- c_1 - 2c\varepsilon_0} + \frac{C}{p^- c_1 - 2c\varepsilon_0}\|u_n\|_E \quad \text{for all } n \geq 1, \tag{3.10}$$

On the other hand, using (f_1) again, we deduce another estimate:

$$|F(x,u_n)| \leq 2c(1 + |u_n|^{p(x)}). \tag{3.11}$$

Hence we obtain from (3.2), (3.11) and $p^+ < q(x)$ that

$$\begin{aligned}\frac{1}{p^+}\|u_n\|_E^{p^-} &\leq \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)})\,dx = \varphi(u_n) + \int_{\mathbb{R}^N} a(x)F(x,u)\,dx\\ &\leq M_1 + 2c|a|_1 + 2c\int_{\mathbb{R}^N} a(x)|u_n|^{p(x)}\,dx\\ &\leq M_1 + 2c|a|_1 + 2c\int_{\mathbb{R}^N} a(x)(1 + |u_n|^{q(x)})\,dx = M_1 + 4c|a|_1 + 2c\int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}\,dx.\end{aligned} \tag{3.12}$$

Therefore, combining (3.10) and (3.12), the boundedness of $\{u_n\}_{n=1}^{\infty}$ immediately follows, that is, there is constant $C > 0$ such that $\|u_n\|_E \leq C$. Thus, passing to a subsequence if necessary, we assume that $u_n \rightharpoonup u_0$ in E, so it follows from (3.1) that

$$\langle A(u_n), u_n - u_0\rangle - \int_{\mathbb{R}^N} a(x)w_n(u_n - u_0)\,dx \leq \varepsilon_n, \tag{3.13}$$

with $\varepsilon_n \downarrow 0$, $w_n(x) \in \partial F(x, u_n(x))$.

Next we prove that $\int_{\mathbb{R}^N} a(x)w_n(u_n - u_0)\,dx$ as $n \to +\infty$. Clearly, by hypothesis (f_1), we have

$$\int_{\mathbb{R}^N} a(x)|w_n||u_n - u_0|\,dx \le c \int_{\mathbb{R}^N} a(x)|u_n - u_0|\,dx + c \int_{\mathbb{R}^N} a(x)|u_n|^{p(x)-1}|u_n - u_0|\,dx. \tag{3.14}$$

On the one hand, using Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} a(x)|u_n|^{p(x)-1}|u_n - u_0|\,dx &\le 3|a|_{L^{\frac{N}{p(x)-1}}(\mathbb{R}^N)}\big||u_n|^{p(x)-1}\big|_{L^{\frac{p^*(x)}{p(x)-1}}(\mathbb{R}^N)}|u_n - u_0|_{L^{p(x)}(\mathbb{R}^N)}\\ &\le 3|a|_{L^{\frac{p(x)}{p(x)-r(x)}}(\mathbb{R}^N)}|u_n - u_0|_{L^{p(x)}(\mathbb{R}^N)}\big||u_n|^{p(x)-1}\big|_{L^{\frac{p^*(x)}{p(x)-1}}(\mathbb{R}^N)}.\end{aligned} \tag{3.15}$$

We claim that

$$\big||u_n|^{p(x)-1}\big|_{L^{\frac{p^*(x)}{p(x)-1}}(\mathbb{R}^N)} \le |u_n|_{p^*(x)}^{p^+-1} + 2. \tag{3.16}$$

Indeed, we have that

$$\text{if}\quad |u_n|_{p^*(x)} \ge 1, \quad \text{then}\quad \big||u_n|^{p(x)-1}\big|_{L^{\frac{p^*(x)}{p(x)-1}}(\mathbb{R}^N)} \le |u_n|_{p^*(x)}^{p^+-1}. \tag{I}$$

This is seen as follows: According to (2.1), to prove (I), this is equivalent to proving that $|u_n|_{p^*(x)} \ge 1$ implies

$$\int_{\mathbb{R}^N} \frac{|u_n(x)|^{(p(x)-1)\frac{p^*(x)}{p(x)-1}}}{|u_n|_{p^*(x)}^{(p^+-1)\frac{p^*(x)}{p(x)-1}}}\,dx = \int_{\mathbb{R}^N} \frac{|u_n(x)|^{p^*(x)}}{|u_n|_{p^*(x)}^{(p^+-1)\frac{p^*(x)}{p(x)-1}}}\,dx \le 1.$$

This inequality is justified as follows: since $|u_n|_{p^*(x)} \ge 1$ and

$$\begin{aligned}(p^+ - 1)\frac{p^*(x)}{p(x)-1} - p^*(x) &= p^+\frac{p^*(x)}{p(x)-1} - \Big(p^*(x) + \frac{p^*(x)}{p(x)-1}\Big)\\ &= p^+\frac{p^*(x)}{p(x)-1} - p(x)\frac{p^*(x)}{p(x)-1}\\ &= \frac{p^*(x)}{p(x)-1}(p^+ - p(x))\\ &\ge 0,\end{aligned}$$

we infer that

$$\frac{|u_n(x)|^{p^*(x)}}{|u_n|_{p^*(x)}^{(p^+-1)\frac{p^*(x)}{p(x)-1}}} = \frac{|u_n(x)|^{p^*(x)}}{|u_n|_{p^*(x)}^{p^*(x)}}\frac{1}{|u_n|_{p^*(x)}^{(p^+-1)\frac{p^*(x)}{p(x)-1}-p^*(x)}} \le \frac{|u_n(x)|^{p^*(x)}}{|u_n|_{p^*(x)}^{p^*(x)}},$$

which implies

$$\int_{\mathbb{R}^N} \frac{|u_n(x)|^{(p(x)-1)p'(x)}}{|u_n|_{p(x)}^{(p^+-1)p'(x)}}\,dx \le \int_{\mathbb{R}^N} \frac{|u_n(x)|^{p^*(x)}}{|u_n|_{p^*(x)}^{p^*(x)}}\,dx = 1,$$

and the prove of (I) is complete.

$$\text{If}\quad |u_n|_{p^*(x)} < 1, \quad \text{then}\quad \big||u_n|^{p(x)-1}\big|_{\frac{p^*(x)}{p(x)-1}} < 2. \tag{II}$$

Indeed, by $|u_n|_{p^*(x)} < \int_{\mathbb{R}^N}|u_n(x)|^{p^*(x)}\,dx + 1$ and (2.4), we obtain

$$\big||u_n|^{p(x)-1}\big|_{\frac{p^*(x)}{p(x)-1}} < \int_{\mathbb{R}^N} |u_n(x)|^{(p(x)-1)\frac{p^*(x)}{p(x)-1}}\,dx + 1 = \int_{\mathbb{R}^N} |u_n(x)|^{p^*(x)}\,dx + 1 < 1 + 1 = 2.$$

Clearly, (3.16) is a consequence of (I) and (II).

By the Sobolev embedding theorem, the inclusion $E \hookrightarrow L^{p^*(x)}(\mathbb{R}^N)$ is continuous, and hence there exists $C_1 > 0$ such that

$$|u_n|_{p^*(x)} \le C_1\|u_n\|_E \le CC_1. \tag{3.17}$$

From Proposition 2.4, the embedding $E \hookrightarrow L^{p(x)}(\mathbb{R}^N)$ is compact, and $u_n \rightharpoonup u$ in E implies $u_n \to u$ in $L^{p(x)}(\mathbb{R}^N)$. Hence, using (3.15), (3.16) and (3.17), we have

$$\int_{\mathbb{R}^N} a(x)|u_n|^{p(x)-1}|u_n - u_0|\,dx \to 0 \quad \text{as } n \to +\infty. \tag{3.18}$$

Choose $\theta(x) = \frac{p(x)}{p(x)-1}$. Then $\theta \in C_+(\mathbb{R}^N)$, $1 < \theta(x) < \frac{N}{p(x)-1}$ for all $x \in \mathbb{R}^N$, and there exists $\lambda\colon \mathbb{R}^N \to (0,1)$ such that

$$\frac{1}{\theta(x)} = \frac{\lambda(x)}{1} + \frac{1-\lambda(x)}{\frac{N}{p(x)-1}} \quad \text{a.e. } x \in \mathbb{R}^N.$$

Then, for $x \in \mathbb{R}^N$, we have

$$s(x) = \frac{1}{\theta(x)\lambda(x)} > 1, \quad t(x) = \frac{N}{\theta(x)(p(x)-1)(1-\lambda(x))} > 1.$$

Using $a \in L^1_+(\mathbb{R}^N) \cap L^{\frac{N}{p(x)-1}}(\mathbb{R}^N)$, we deduce

$$\begin{aligned}\int_{\mathbb{R}^N} |a|^{\theta(x)}\,dx &= \int_{\mathbb{R}^N} |a|^{\frac{1}{s(x)}} |a|^{\frac{\frac{N}{p(x)-1}}{t(x)}}\,dx \\ &\leq 2\bigg[\bigg(\int_{\mathbb{R}^N} |a|\,dx\bigg)^{\frac{1}{s^+}} + \bigg(\int_{\mathbb{R}^N} |a|\,dx\bigg)^{\frac{1}{s^-}}\bigg]\bigg[\bigg(\int_{\mathbb{R}^N} |a|^{\frac{N}{p(x)-1}}\,dx\bigg)^{\frac{1}{t^+}} + \bigg(\int_{\mathbb{R}^N} |a|^{\frac{N}{p(x)-1}}\,dx\bigg)^{\frac{1}{t^-}}\bigg] < +\infty. \end{aligned} \tag{3.19}$$

This implies $a \in L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N)$. Hence

$$\int_{\mathbb{R}^N} a(x)|u_n - u_0|\,dx \leq 2|a|_{\frac{p(x)}{p(x)-1}} |u_n - u_0|_{p(x)} \to 0 \quad \text{as } n \to +\infty.$$

Combining (3.13), (3.14), (3.18) and (3.19), we get $\lim_{n\to\infty}\langle A(u_n), u_n - u\rangle = 0$. By Proposition 2.1 (2), we get $u_n \to u_0$ in E. This proves that $\varphi(u)$ satisfies the nonsmooth (PS) condition on E. □

Lemma 3.2. *Assume that all conditions of Theorem 1.1 are satisfied. Then there exist $\alpha > 0$ and $\nu > 0$ such that, for any $u \in E$ with $\|u\|_E = \nu$, we have $\varphi(u) \geq \alpha$.*

Proof. Firstly, choose $q \in C_+(\mathbb{R}^N)$ (q is mentioned in (f_3)). Then

$$1 < \frac{p^*(x)}{p^*(x) - q(x)} = \frac{Np(x)}{Np(x) - q(x)(N-p(x))} < \frac{N}{p(x)-1}, \quad x \in \mathbb{R}^N. \tag{3.20}$$

By Proposition 2.4, the embedding $E \hookrightarrow L^{p^*(x)}(\mathbb{R}^N)$ is continuous, and there is constant $c_5 > 0$ such that

$$|u|_{p^*(x)} \leq c_5\|u\|_E \quad \text{for all } u \in E. \tag{3.21}$$

Now choose $\gamma > 0$ such that $\gamma < \min\{1, \frac{1}{c_5}\}$. Then, for such a fixed γ, we have

$$|u|_{p^*(x)} < 1 \quad \text{for all } u \in E \text{ with } \|u\|_E = \gamma. \tag{3.22}$$

Moreover, by virtue of hypothesis (f_2), we obtain

$$F(x,t) \leq \vartheta(x) \tag{3.23}$$

for any $x \in \mathbb{R}^N$ and $0 < |t| < \delta$.

On the other hand, for all $x \in \mathbb{R}^N$ and all $|t| \geq \delta$, (f_1) implies

$$|F(x,t)| \leq c_6|t|^{p(x)}, \tag{3.24}$$

where $c_6 = (1 + \frac{1}{\tau})c$ and $\tau = \min\{|\delta|^{p^+}, |\delta|^{p^-}\}$.

From (3.23) and (3.24), for all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$, we have $F(x,t) \leq \vartheta(x) + c_7|t|^{p(x)}$, where $c_7 = c_6 + \frac{|\vartheta|_\infty}{\tau}$. Thus, for all $u \in E$ with $\|u\|_E = \gamma$, we have

$$\begin{aligned}\varphi(u) &= \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx - \int_{\mathbb{R}^N} a(x)F(x,u)\,dx \\ &\geq \frac{1}{p^+}\|u\|_E^{p^+} - c_7\int_{\mathbb{R}^N} a(x)|u|^{p(x)}\,dx - \int_{\mathbb{R}^N} a(x)\vartheta(x)\,dx \\ &\geq \frac{1}{p^+}\|u\|_E^{p^+} - c_7\int_{\mathbb{R}^N} a(x)|u|^{p(x)}\,dx + \int_{\mathbb{R}^N} a(x)\vartheta(x)\,dx. \end{aligned} \tag{3.25}$$

Applying Young's inequality with ε, we get

$$|u|^{p(x)} = 1 \times |u|^{p(x)} \le \varepsilon \times 1^{\frac{q(x)}{q(x)-p(x)}} + \varepsilon^{-\frac{q(x)-p(x)}{p(x)}} \big||u|^{p(x)}\big|^{\frac{q(x)}{p(x)}} = \varepsilon + \varepsilon^{-\frac{q(x)-p(x)}{p(x)}} |u|^{q(x)}$$
$$\le \varepsilon + \varepsilon^{-\frac{q^+-p^-}{p^-}} |u|^{q(x)}. \tag{3.26}$$

So, returning to (3.25) and using (3.26), for all $u \in E$ with $\|u\|_E = \gamma$, we obtain

$$\varphi(u) \ge \frac{1}{p^+}\|u\|^{p^+} - \varepsilon^{-\frac{q^+-p^-}{p^-}} c_7 \int_{\mathbb{R}^N} a(x)|u|^{q(x)}\,dx - c_7\varepsilon \int_{\mathbb{R}^N} a(x)\,dx - \int_{\mathbb{R}^N} a(x\vartheta(x)\,dx. \tag{3.27}$$

Since $\vartheta \in L^\infty_-(\mathbb{R}^N)$, there exists some $c_8 > 0$ such that $-\vartheta(x) > c_8$. We can choose an ε_0 small enough such that $c_8|a|_1 - \varepsilon_0 c_7 |a|_1 > 0$, and then (3.27) immediately implies

$$\varphi(u) \ge \frac{1}{p^+}\|u\|_E^{p^+} - c_7\varepsilon_0^{-\frac{q^+-p^-}{p^-}} \int_{\mathbb{R}^N} a(x)|u|^{q(x)}\,dx. \tag{3.28}$$

Similarly to the proof of (3.19), and combining inequality (3.20), we have $a \in L^{\frac{p*(x)}{p^*(x)-q(x)}}(\mathbb{R}^N)$. Using Proposition 2.2, (3.21) and (3.22), for all $u \in E$ with $\|u\|_E = \gamma$, we obtain

$$\int_{\mathbb{R}^N} a(x)|u|^{q(x)}\,dx \le |a|_{L^{\frac{p^*(x)}{p^*(x)-q(x)}}(\mathbb{R}^N)} \big||u|^{q(x)}\big|_{L^{\frac{p^*(x)}{q(x)}}(\mathbb{R}^N)}$$
$$\le |a|_{L^{\frac{p^*(x)}{p^*(x)-q(x)}}(\mathbb{R}^N)} |u|_{p^*(x)}^{q^-}$$
$$\le |a|_{L^{\frac{p^*(x)}{p^*(x)-q(x)}}(\mathbb{R}^N)} c_5^{q^-}\|u\|_E^{q^-}. \tag{3.29}$$

Using (3.29) in (3.28), we see that, for any $u \in E$ with $\|u\|_E = \gamma$, we have

$$\varphi(u) \ge \frac{1}{p^+}\|u\|_E^{p^+} - \varepsilon_0^{-\frac{q^+-p^-}{p^-}} c_5^{q^-} |a|_{L^{\frac{p^*(x)}{p^*(x)-q(x)}}(\mathbb{R}^N)} \|u\|_E^{q^-},$$

which implies that there exist $\alpha > 0$ and $\nu > 0$ such that $\varphi(u) \ge \alpha$ for any $u \in E$ with $\|u\|_E = \nu$. □

Lemma 3.3. *Assume that all conditions of Theorem 1.1 are satisfied. Then $\varphi(u) \to -\infty$ as $\|u\|_E \to +\infty$ for all $u \in \mathcal{F}$, where $\mathcal{F}$ is an arbitrary finite-dimensional subspace of E.*

Proof. By virtue of hypothesis (f_3), we can find $M_4 > 0$ such that

$$F(x,t) \ge c_9|t|^{q(x)} \quad \text{for all } x \in \mathbb{R}^N,\ |t| \ge M_4. \tag{3.30}$$

In addition, from hypothesis (f_1), for almost all $x \in \mathbb{R}^N$ and $|t| < M_4$, we have

$$|F(x,t)| \le c_3, \tag{3.31}$$

where $c_3 = (1 + M_4^{p^+} + M_4^{p^-})c$. Thus, using (3.30) and (3.31), we obtain

$$F(x,t) \ge c_9|t|^{q(x)} - c_4 \quad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{3.32}$$

where $c_4 = c_3 + c_9 \max\{M_4^{q^+}, M_4^{q^-}\}$.

Since $a \in L^1_+(\mathbb{R}^N)$, then $a(x)$ is a measurable, nonnegative real-valued function for $x \in \mathbb{R}^N$. Define

$$L^{q(x)}_{a(x)}(\mathbb{R}^N) = \Big\{u \text{ is measurable and } \int_{\mathbb{R}^N} a(x)|u|^{q(x)}\,dx < +\infty\Big\}$$

with the norm

$$|u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} = \inf\Big\{\sigma > 0 : \int_{\mathbb{R}^N} a(x)\Big|\frac{u}{\sigma}\Big|^{q(x)}\,dx \le 1\Big\}.$$

Then $L^{q(x)}_{a(x)}(\mathbb{R}^N)$ is a Banach space. The space $L^{q(x)}_{a(x)}(\mathbb{R}^N)$, which is called weighted variable exponent Lebesgue space, is introduced in [19].

Claim. *The embedding* $E \hookrightarrow W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}_{a(x)}(\mathbb{R}^N)$ *is continuous.*

Set $h(x) = q(x)\frac{N}{N-p(x)+1}$, where $q(x)$ is mentioned in (f_3). Then $p^+ < h^-$ and $p(x) < h(x) < p^*(x)$ for all $x \in \mathbb{R}^N$. Hence, by Proposition 2.4, there is a continuous embedding $E \hookrightarrow L^{h(x)}(\mathbb{R}^N)$. Thus, for $u \in E$, we have $|u(x)|^{q(x)} \in L^{\frac{N}{N-p(x)+1}}(\mathbb{R}^N)$. By the Hölder inequality,

$$\int_{\mathbb{R}^N} a(x)|u|^{q(x)}\,dx \le 2|a|_{L^{\frac{N}{p(x)-1}}(\mathbb{R}^N)} \big||u|^{q(x)}\big|_{L^{\frac{N}{N-p(x)+1}}(\mathbb{R}^N)} < +\infty. \tag{3.33}$$

It follows that $u \in L^{q(x)}_{a(x)}(\mathbb{R}^N)$, and hence the embedding $E \hookrightarrow W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}_{a(x)}(\mathbb{R}^N)$ is continuous. The proof of this claim is complete.

Moreover, similarly to (2.3) and (2.4), we get

$$\begin{aligned} |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} > 1 &\implies |u|^{q^-}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} a(x)|u|^{q(x)}\,dx \le |u|^{q^+}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)}, \\ |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} < 1 &\implies |u|^{q^+}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} a(x)|u|^{q(x)}\,dx \le |u|^{q^-}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)}. \end{aligned} \tag{3.34}$$

Because W is a finite-dimensional subspace of E, all norms are equivalent, so we can find $0 < C = C(\mathcal{F}) < 1$ such that

$$C\|u\|_E \le |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \le \frac{1}{C}\|u\|_E \quad \text{for all } u \in \mathcal{F}. \tag{3.35}$$

Taking into account (3.32), (3.34) and (3.35), for every $u \in \mathcal{F}$ with $\|u\|_E > 1$ and $|u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} > 1$, we have

$$\begin{aligned} \varphi(u) &= \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx - \int_{\mathbb{R}^N} a(x)F(x,u)\,dx \\ &\le \frac{1}{p^-}\|u\|^{p^+} - c_9 \int_{\mathbb{R}^N} a(x)|u|^{q(x)}\,dx + c_4 \int_{\mathbb{R}^N} a(x)\,dx \\ &\le \begin{cases} \frac{1}{p^-}\|u\|^{p^+} + c_4|a|_1 - c_9 C^{q^-}\|u\|_E^{q^-} & \text{if } |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} > 1, \\ \frac{1}{p^-}\|u\|^{p^+} + c_4|a|_1 - c_9 C^{q^+}\|u\|_E^{q^+} & \text{if } |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} < 1, \end{cases} \end{aligned} \tag{3.36}$$

Because of $q^+ \ge q^- > p^+$, we see that $\varphi(u) \to -\infty$ as $\|u\|_E \to +\infty$. □

Proof of Theorem 1.1. It is obvious that φ is even and $\varphi(0) = 0$. Besides, Lemmas 3.1, 3.2 and 3.3 permit the application of Lemma 2.6 with $X = E$, $X_1 = \mathcal{F}$ (see Lemma 3.3) and $X_2 = E \ominus \mathcal{F}$ (see Lemma 3.2). Therefore, we obtain that the functional φ has an unbounded sequence of critical values, so problem (P) possesses infinitely many nontrivial solutions. □

At the end of this section, we prove Theorem 1.2.

Proof of Theorem 1.2. Since E is also a reflexive and separable Banach space, we can give the decomposition to E as (2.5). In what follows, we will prove that the functional φ satisfies all the conditions of Lemma 2.7 in E. By virtue of hypothesis (f_4) and Lemma 3.1, we deduce that φ is an even functional and satisfies the (PS) condition in E. Next we will prove that if k is large enough, then there exists $\rho_k > r_k > 0$ such that (A_2) and (A_3) hold in E. Let $u \in Z_k \subset E$ with $\|u\|_E > 1$. Then, using (3.11), we obtain

$$\begin{aligned} \varphi(u) &= \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx - \int_{\mathbb{R}^N} a(x)F(x,u)\,dx \\ &\ge \frac{1}{p^+}\|u\|_E^{p^-} - 2c\int_{\mathbb{R}^N} a(x)|u|^{p(x)}\,dx - 2c|a|_1. \end{aligned}$$

Choose $q \in C_+(\overline{\Omega})$, where $q(x)$ is mentioned in (f_3). Then, according to [10, Lemma 4.9], we have

$$\alpha_k = \sup\{|u|_{h(x)} : \|u\|_E = 1,\ u \in Z_k\} \to 0 \quad \text{as } k \to +\infty. \tag{3.37}$$

Thus, for $u \in Z_k \subset E$ with $\|u\|_E > 1$, by (3.33), we have

$$\begin{aligned}
\varphi(u) &\geq \frac{1}{p^+}\|u\|_E^{p^-} - 2c\int_{\mathbb{R}^N} a(x)(1+|u|^{q(x)})\,dx - 2c|a|_1 \\
&\geq \frac{1}{p^+}\|u\|_E^{p^-} - 2c\int_{\mathbb{R}^N} a(x)|u|^{q(x)}\,dx - 4c|a|_1 \\
&\geq \frac{1}{p^+}\|u\|_E^{p^-} - 4c|a|_{L^{\frac{N}{p(x)-1}}(\mathbb{R}^N)}\big||u|^{q(x)}\big|_{L^{\frac{N}{N-p(x)+1}}(\mathbb{R}^N)} - 4c|a|_1 \\
&\geq \frac{1}{p^+}\|u\|_E^{p^-} - 4c|a|_{L^{\frac{N}{p(x)-1}}(\mathbb{R}^N)}\max\{|u|_{h(x)}^{q^+}, |u|_{h(x)}^{q^-}\} - 4c|a|_1 \\
&\geq \frac{1}{p^+}\|u\|_E^{p^-} - 4c|a|_{L^{\frac{N}{p(x)-1}}(\mathbb{R}^N)}\max\{\alpha_k^{q^+}\|u\|_E^{q^+}, \alpha_k^{q^-}\|u\|_E^{q^-}\} - 4c|a|_1 \\
&\geq \frac{1}{p^+}\|u\|_E^{p^-} - 4c|a|_{L^{\frac{N}{p(x)-1}}(\mathbb{R}^N)}\alpha_k^{q^-}\|u\|_E^{q^+} - 4c|a|_1.
\end{aligned}$$

Take $r_k = (4q^+c|a|_{L^{\frac{N}{p(x)-1}}(\mathbb{R}^N)}\alpha_k^{q^-})^{\frac{1}{p^- - q^+}}$. Then, for any $u \in Z_k$ with $\|u\|_E = r_k$, we have

$$\varphi(u) \geq \left[\frac{1}{p^+} - \frac{1}{q^+}\right] r_k^{p^-} - 4c|a|_1.$$

Recall that $1 < p^- \leq p^+ < q^- \leq q^+$. Using this fact and (3.37), we obtain

$$\frac{1}{p^+} - \frac{1}{q^+} > 0 \quad \text{and} \quad \lim_{k\to+\infty} r_k = +\infty.$$

Hence $\varphi(u) \to +\infty$ with $u \in Z_k$ and $\|u\|_E = r_k$ as $k \to +\infty$. So (A_2) of Lemma 2.7 is satisfied. Furthermore, by (3.36), it is easy to see that

$$\varphi(u) \to -\infty \quad \text{as } \|u\|_E \to +\infty \quad \text{for all } u \in Y_k.$$

That is, (A_3) is also satisfied. From the proof of (A_2) and (A_3), we can choose $\rho_k > r_k > 0$. At this point, all assumptions of Lemma 2.7 are satisfied. Therefore, φ has an unbounded sequence of critical values, which implies that φ has infinitely many nontrivial critical points in E. □

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