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Nikolaos S. Papageorgiou & Vicențiu D. Rădulescu

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Noncoercive Resonant $(p, 2)$-Equations

Nikolaos S. Papageorgiou[1] · Vicenţiu D. Rădulescu[2,3]

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Abstract We consider a nonlinear Dirichlet problem driven by the sum of p-Laplacian and a Laplacian (a $(p, 2)$-equation) which is resonant at $\pm\infty$ with respect to the principal eigenvalue $\hat{\lambda}_1(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ and resonant at zero with respect to any nonprincipal eigenvalue of $(-\Delta, H_0^1(\Omega))$. At $\pm\infty$ the resonance occurs from the right of $\hat{\lambda}_1(p)$ and so the energy functional of the problem is indefinite. Using critical groups, we show that the problem has at least one nontrivial smooth solution. The result complements the recent work of Papageorgiou and Rădulescu (Appl Math Optim 69:393–430, 2014), where resonant $(p, 2)$-equations were examined with the resonance occurring from the left of $\hat{\lambda}_1(p)$ (coercive problem).

Keywords Resonance at zero and at infinity · C-condition · Critical groups · Regularity theory

Mathematics Subject Classification 35J20 · 35J60 · 58E05

✉ Vicenţiu D. Rădulescu
vicentiu.radulescu@imar.ro

1 Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece

2 Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

3 Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear elliptic equation

$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,\ 2 < p. \tag{1}$$

In this problem Δ_p denotes the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

The reaction term $f(z, x)$ is a measurable function which is C^1 in the x-variable. We assume that $f(z, \cdot)$ is asymptotically as $x \to \pm\infty$ and $x \to 0$, resonant (double resonance situation). For this doubly resonant problem, we prove an existence theorem producing a nontrivial smooth solution.

Elliptic equations driven by the sum of a p-Laplacian and a Laplacian, are known as $(p, 2)$-equations and arise in problems of mathematical physics. We mention the papers of Benci et al. [2] (quantum physics) and Cherfils and Ilyasov [4] (plasma physics). Recently there have been some existence and multiplicity results for such equations. We mention the works of Aizicovici et al. [1], Cingolani and Degiovanni [5], Mugnai and Papageorgiou [13], Papageorgiou and Rădulescu [15,16], Sun [18] and Sun et al. [19].

Our work here is closely related to the paper of Papageorgiou and Rădulescu [15] and in fact it complements it. In that paper, the authors examined $(p, 2)$-equations which at $\pm\infty$ are resonant with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$ of the Dirichlet p-Laplacian $(-\Delta_p, W_0^{1,p}(\Omega))$. The resonance occurs from the left of $\hat{\lambda}_1(p)$ in the sense that $\hat{\lambda}_1(p)|x|^p - pF(z, x) \to +\infty$ as $x \to \pm\infty$ (here $F(z, x) = \int_0^x f(z, s)ds$). Then the corresponding energy functional of the problem is coercive and this permits the use of the direct method of the calculus of variations. So, using this method together with suitable truncation techniques and Morse theory (critical groups), Papageorgiou and Rădulescu [15] proved multiplicity theorems producing three or four nontrivial solutions, all with sign information.

It is natural to ask what happens when the resonance at $\pm\infty$ with respect to $\hat{\lambda}_1(p) > 0$ occurs from the right, in the sense that $\hat{\lambda}_1|x|^p - pF(z, x) \to -\infty$ as $x \to \pm\infty$. In this case, the energy functional is no longer coercive and so the direct method fails and it is not clear if we can have a nontrivial solution.

In this paper we study this case and using Morse theory (critical groups), we show the existence of at least one nontrivial solution.

2 Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). We say that $\varphi \in C^1(X, \mathbb{R})$ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $(1+||u_n||)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
admits a strongly convergent subsequence.″

Let us briefly recall some basic definitions and facts concerning critical groups (Morse theory) which we will need in the sequel. So, let $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets

- $\varphi^c = \{u \in X : \varphi(u) \leqslant c\}$ (the sublevel set of φ at c),
- $K_\varphi = \{u \in X : \varphi'(u) = 0\}$ (the critical set of φ),
- $K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}$ (the critical set of φ at the level c).

Suppose (Y_1, Y_2) is a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the kth singular homology group for the pair (Y_1, Y_2) with integer coefficients. Suppose that $u \in K_\varphi^c$ is isolated. The critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U\backslash\{u\}) \quad \text{for all } k \in \mathbb{N}_0.$$

Here U is an open neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the neighborhood U of u.

Suppose that φ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see, for example, [9, p. 628]), implies that the above definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. We know that if for some $k \in \mathbb{N}_0$, $C_k(\varphi, \infty) \neq 0$, then we can find $u \in K_\varphi$ such that $C_k(\varphi, u) \neq 0$. Also, if $X = H =$ Hilbert space, $\varphi \in C^2(H, \mathbb{R})$ and $u \in K_\varphi$, then the "Morse index" of u, denoted by $\mu(u)$, is defined to be the supremum of the dimensions of the vector subspaces of H on which $\varphi''(u)$ is negative definite. The "nullity" of u, denoted by $\nu(u)$, is the dimension of $\ker \varphi''(u)$. We say that u is "nondegenerate", if $\varphi''(u)$ is invertible (that is, $\nu(u) = 0$). If $\varphi \in C^2(H, \mathbb{R})$ and $u \in K_\varphi$ is nondegenerate (hence by the inverse function theorem automatically isolated) with Morse index $\mu(u) = \mu$, then

$$C_k(\varphi, u) = \delta_{k,\mu}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Hereafter, by $\delta_{k,\mu}$ we denote the Kronecker symbol defined by

$$\delta_{k,\mu} = \begin{cases} 1 \text{ if } k = \mu \\ 0 \text{ if } k \neq \mu. \end{cases}$$

In our analysis of problem (1) we will use some basic facts about the spectrum of the p-Laplacian and the Laplacian.

So, consider the following nonlinear eigenvalue problem

$$-\Delta_q u(z) = \hat{\lambda}|u(z))|^{q-2}u(z) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0 \ (1 < q < \infty). \tag{2}$$

We say that $\hat{\lambda}$ is an eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$, if problem (2) admits a nontrivial solution $\hat{u} \in W_0^{1,q}(\Omega)$, which is an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. There exists a smallest eigenvalue $\hat{\lambda}_1(q) > 0$ with the following properties:

- $\hat{\lambda}_1(q)$ is isolated (that is, there exists $\epsilon > 0$ such that $(\hat{\lambda}_1(q), \hat{\lambda}_1(q) + \epsilon)$ contains no eigenvalues of $(-\Delta_q, W_0^{1,q}(\Omega))$);
- $\hat{\lambda}_1(q)$ is simple (that is, if $\hat{u}, \hat{v} \in W_0^{1,q}(\Omega)$ are eigenfunction corresponding to $\hat{\lambda}_1$, then $\hat{u} = \xi\hat{v}$ with $\xi \in \mathbb{R}\backslash\{0\}$);
- $$\hat{\lambda}_1(q) = \inf\left[\frac{||Du||_q^q}{||u||_q^q} : u \in W_0^{1,q}(\Omega), u \neq 0\right] > 0. \tag{3}$$

The infimum in (3) is realized on the corresponding one dimensional eigenspace. It is clear from (3) that the elements of this eigenspace do not change sign. By $\hat{u}_1(q)$ we denote the L^q-normalized positive eigenfunction corresponding to $\hat{\lambda}_1(q) > 0$. From the nonlinear regularity theory and the nonlinear maximum principle (see, for example, [9, pp. 737–738]), we have

$$\hat{u}_1(q) \in \operatorname{int} C_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega, \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0\right\}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$ (C_+ denotes the positive cone of the ordered Banach space $C_0^1(\overline{\Omega})$, defined by $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}$ and int C_+ is its interior). Using the Ljusternik–Schnirelmann minimax scheme, we can produce a whole strictly increasing sequence $\{\hat{\lambda}_k(q)\}_{k\geqslant 1}$ of eigenvalues (known as LS-eigenvalues) such that $\hat{\lambda}_k(q) \to +\infty$. We can have at least three such sequences of LS-eigenvalues depending on the index used in the minimax scheme. All three coincide in the first two eigenvalues, but we do not know if this is also true for the higher eigenvalues. We only know that their elements are ordered. In general, the spectrum of $(-\Delta_q, W_0^{1,q}(\Omega))$ is far from fully described. Note that the isolation of $\hat{\lambda}_1(q) > 0$ implies he second eigenvalue $\hat{\lambda}_2^*(q)$ is well-defined by

$$\hat{\lambda}_2^*(q) = \inf[\hat{\lambda} : \hat{\lambda} \text{ is an eigenvalue of (2)}, \ \hat{\lambda} > \hat{\lambda}_1(q)].$$

We know that $\hat{\lambda}_2^*(q) = \hat{\lambda}_2(q)$ (that is, the second eigenvalue coincides with the second LS-eigenvalue). So, the Ljusternik–Schnirelmann minimax scheme provides a variational characterization of $\hat{\lambda}_2(q)$. However, for our purposes more convenient is an alternative minimax characterization due to Cuesta et al. [7]. Let

$$\partial B_1^{L^q} = \{u \in L^q(\Omega) : ||u||_q = 1\}, \ M = W_0^{1,q}(\Omega) \cap \partial B_1^{L^q},$$
$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1, 1], M) : \hat{\gamma}(-1) = -\hat{u}_1(q), \hat{\gamma}(1) = \hat{u}_1(q)\}.$$

Proposition 1 *We have* $\hat{\lambda}_2(q) = \inf_{\hat{\gamma}\in\hat{\Gamma}} \max_{-1\leqslant t\leqslant 1} ||D\hat{\gamma}(t)||_q^q$.

When $q = 2$ (linear eigenvalue problem), we have a complete description of the spectrum of $(-\Delta, H_0^1(\Omega))$. This consists of a sequence of distinct eigenvalues $\{\hat{\lambda}_k(2)\}_{k\geqslant 1}$ such that $\hat{\lambda}_k(2) \to +\infty$. By $E(\hat{\lambda}_k(2))$ we denote the eigenspace for the eigenvalue $\hat{\lambda}_k(2)$. In this case we have the orthogonal direct sum decomposition

$$H_0^1(\Omega) = \overline{\underset{k\geqslant 1}{\oplus} E(\hat{\lambda}_k(2))}.$$

These eigenspaces exhibit the unique continuation property, which says that if $u \in E(\hat{\lambda}_k(2))$ vanishes on a set of positive measure, then $u = 0$. Also, each $E(\hat{\lambda}_k(2))$ is finite dimensional and $E(\hat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$ (regularity theory).

Now we have variational characterizations for all the eigenvalues, namely

$$\hat{\lambda}_1(2) = \inf\left[\frac{||Du||_2^2}{||u||_2^2} : u \in H_0^1(\Omega), u \neq 0\right] \text{ (see (3) with } q = 2) \tag{4}$$

$$\begin{aligned}\hat{\lambda}_k(2) &= \inf\left[\frac{||Du||_2^2}{||u||_2^2} : u \in \overline{\underset{i\geqslant k}{\oplus} E(\hat{\lambda}_i(2))},\ u \neq 0\right] \\ &= \sup\left[\frac{||Du||_2^2}{||u||_2^2} : u \in \overset{k}{\underset{i=1}{\oplus}} E(\hat{\lambda}_i(2)),\ u \neq 0\right] \text{ for all } k \geqslant 2.\end{aligned} \tag{5}$$

In (4) and (5) the infima and suprema are realized on the corresponding eigenspaces.

We mention that for all $1 < q < \infty$ and all $k \geqslant 2$, the eigenfunctions corresponding to $\hat{\lambda}_k(q)$ are nodal (that is, sign changing).

We introduce the following linear subspace of $W_0^{1,q}(\Omega)$ $(1 < q < \infty)$:

$$V_q = \{u \in W_0^{1,q}(\Omega) : \int_\Omega \hat{u}_1(q)^{q-1} u dz = 0\}.$$

We have the following direct sum decomposition

$$W_0^{1,q}(\Omega) = \mathbb{R}\hat{u}_1(q) \oplus V_q.$$

We define

$$\tilde{\lambda}(q) = \inf\left[\frac{||Du||_q^q}{||u||_q^q} : u \in V_q,\ u \neq 0\right]. \tag{6}$$

Proposition 2 *We have* $\hat{\lambda}_1(q) < \tilde{\lambda}(q) \leqslant \hat{\lambda}_2(q)$.

Proof From (3) and (6), we have

$$\hat{\lambda}_1(q) \leqslant \tilde{\lambda}(q).$$

Suppose that equality holds. So, we can find $\{u_n\}_{n\geqslant 1} \subseteq V_q$ such that

$$||u_n||_q = 1 \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad ||Du||_q^q \to \hat{\lambda}_1(q) = \tilde{\lambda}(q) \text{ as } n \to \infty. \tag{7}$$

From (3) it is clear that $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,q}(\Omega)$ is bounded and so we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,q}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^q(\Omega). \tag{8}$$

Then $u \in V_q$ and $||u||_q = 1$. Also, from (6) and (8), we have

$$\begin{aligned} &\hat{\lambda}_1(q) \leqslant ||Du||_q^q \leqslant \liminf_{n\to\infty} ||Du_n||_q^q = \tilde{\lambda}(q) = \hat{\lambda}_1(q), \\ \Rightarrow\ &\hat{\lambda}_1(q) = ||Du||_q^q \text{ with } ||u||_q = 1, \\ \Rightarrow\ &u = \pm\hat{u}_1(q), \end{aligned}$$

a contradiction to the fact that $u \in V_q$. Therefore, we have

$$\hat{\lambda}_1(q) < \tilde{\lambda}(q).$$

Next we show that

$$\tilde{\lambda}(q) \leqslant \hat{\lambda}_2(q).$$

Again we argue by contradiction. So, suppose that $\hat{\lambda}_2(q) < \tilde{\lambda}(q)$. From Proposition 1 we know that we can find $\hat{\gamma} \in \hat{\Gamma}$ such that

$$||D\hat{\gamma}(t)||_q^q < \tilde{\lambda}(q) \quad \text{for all } t \in [-1, 1]. \tag{9}$$

From the definition of $\hat{\Gamma}$, we have

$$\hat{\gamma}(-1) = -\hat{u}_1(q) \in -\text{int}\, C_+ \quad \text{and} \quad \hat{\gamma}(1) = \hat{u}_1(q) \in \text{int}\, C_+.$$

Consider the function $\xi : [-1, 1] \to \mathbb{R}$ defined by

$$\xi(t) = \int_\Omega \hat{\gamma}(t)(z)\hat{u}_1(q)(z)^{q-1} dz.$$

Then $\xi(-1) = -||\hat{u}_1(q)||_q^q = -1 < 0 < 1 = ||\hat{u}_1(q)||_q^q = \xi(1)$. Since $\xi(\cdot)$ is continuous, it follows from Bolzano's theorem that we can find $t_0 \in (-1, 1)$ such that

$$\begin{aligned} &\xi(t_0) = 0, \\ \Rightarrow &\int_\Omega \hat{\gamma}(t_0)\hat{u}_1(q)^{q-1} dz = 0, \end{aligned}$$

$$\Rightarrow \hat{\gamma}(t_0) \in V_q,$$
$$\Rightarrow \tilde{\lambda}(q) \leqslant ||D\hat{\gamma}(t_0)||_q^q \text{ (see (6))}$$

and this contradicts (9). Therefore we conclude that $\tilde{\lambda}(q) \leqslant \hat{\lambda}_2(q)$. □

Remark 1 In general we do not know if the inequality $\tilde{\lambda}(q) \leqslant \hat{\lambda}_2(q)$ can be strict. Note that if $q = 2$, then $\tilde{\lambda}(2) = \hat{\lambda}_2(2)$.

In the sequel for $1 < q < \infty$, by $A_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)^*$ $\left(\frac{1}{q} + \frac{1}{q'} = 1\right)$, we denote the map defined by

$$\langle A_q(u), v\rangle = \int_\Omega |Du|^{q-2}(Du, Dv)_{\mathbb{R}^N} dz \quad \text{for all } u, v \in W_0^{1,q}(\Omega). \tag{10}$$

From Gasinski and Papageorgiou [9, p. 746], we have:

Proposition 3 *If $1 < q < \infty$ and $A_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ is defined by (10), then A_q is monotone, continuous (hence maximal monotone), bounded (that is, maps bounded sets to bounded sets) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,q}(\Omega)$ and*

$$\limsup_{n\to\infty} \langle A_q(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \to u$ in $W_0^{1,q}(\Omega)$.

If $q = 2$, we write $A = A_2 \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$.

Finally let us fix our notation. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Also, if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function, then we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } W_0^{1,q}(\Omega),$$

the Nemytskii operator corresponding to h. By $||\cdot||$ we denote the norm for the Sobolev space $W_0^{1,p}(\Omega)$. The Poincaré inequality implies that

$$||u|| = ||Du||_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Let p^* denote the critical Sobolev exponent, that is,

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leqslant p. \end{cases}$$

3 Existence of Nontrivial Solutions

The hypotheses on the reaction term $f(z,x)$ are the following:

$H: f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, $f(z,0) = 0,\ f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \leqslant a(z)(1+|x|^{r-1})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+,\ p \leqslant r < p^*$;

(ii) $\lim\limits_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = \hat{\lambda}_1(p)$ uniformly for almost all $z \in \Omega$;

(iii) if $F(z,x) = \int_0^x f(z,s)ds$, then there exist $\tau > 2$ and $\beta_0 > 0$ such that

$$\limsup_{x\to\pm\infty} \frac{f(z,x)x - pF(z,x)}{|x|^\tau} \leqslant -\beta_0 \text{ uniformly for almost all } z \in \Omega;$$

(iv) there exist $l \in \mathbb{N},\ l \geqslant 2,\ \delta > 0$ and $\eta \in L^\infty(\Omega)$ such that $\hat{\lambda}_l(2) \leqslant \eta(z)$ for almost all $z \in \Omega$, $\eta \not\equiv \hat{\lambda}_l(2)$,

$$\eta(z)x^2 \leqslant f(z,x)x \leqslant \hat{\lambda}_{l+1}(2)x^2 \text{ for almost all } z \in \Omega,\ \text{all } 0 < |x| \leqslant \delta,$$

and the second inequality is strict on a set of positive measure.

Remark 2 Hypothesis $H(ii)$ implies that asymptotically as $x \to \pm\infty$, we have resonance with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$. Hypothesis $H(iii)$ implies that the resonance occurs from the right of $\hat{\lambda}_1(p)$. Indeed, as will see in the proof of Proposition 5, this hypothesis implies that

$$\hat{\lambda}_1(p)|x|^p - F(z,x) \to -\infty \text{ as } x \to \pm\infty.$$

Hypothesis $H(iv)$ implies that we can have resonance also at zero (double resonance). Resonance is possible with respect to $\hat{\lambda}_{l+1}(2)$ as $x \to 0$, since we can have

$$\lim_{x\to 0} \frac{f(z,x)}{x} = \hat{\lambda}_{l+1}(2) \quad \text{uniformly for almost all } z \in \Omega.$$

Symmetrically, we may assume that

$$\hat{\lambda}_l(2)x^2 \leqslant f(z,x)x \leqslant \hat{\eta}(z)x^2 \quad \text{for almost all } z \in \Omega,\ \text{all } 0 < |x| \leqslant \delta$$

with $\hat{\eta} \in L^\infty(\Omega)$, $\hat{\eta}(z) \leqslant \hat{\lambda}_{l+1}(2)$ for almost all $z \in \Omega$, $\hat{\eta} \not\equiv \hat{\lambda}_{l+1}(2)$ and the first inequality is strict on a set of positive measure. Note that hypothesis $H(iv)$ is in contrast to the situation in Papageorgiou and Rădulescu [15], where the hypotheses on the reaction term do not permit resonance at zero.

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \int_\Omega F(z, u(z))dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\varphi \in C^2(W_0^{1,p}(\Omega))$.

Proposition 4 *If hypotheses $H(i), (ii), (iii)$ hold, then the functional φ satisfies the C-condition.*

Proof Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leqslant M_1 \text{ for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{11}$$

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{12}$$

From (11) we have

$$\left|\left\langle A_p(u_n), h\right\rangle + \langle A(u_n), h\rangle - \int_\Omega f(z, u_n)hdz\right| \leqslant \frac{\epsilon_n||h||}{1 + ||u_n||}$$
$$\text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \epsilon_n \to 0^+. \tag{13}$$

We claim that $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. We argue indirectly. So, suppose that the sequence is unbounded. By passing to a subsequence if necessary, we may assume that

$$||u_n|| \to \infty \text{ as } n \to \infty. \tag{14}$$

Let $y_n = \frac{u_n}{||u_n||}$, $n \in \mathbb{N}$. Then $||y_n|| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{15}$$

From (13) we have

$$\left|\left\langle A_p(y_n), h\right\rangle + \frac{1}{||u_n||^{p-2}}\langle A(y_n), h\rangle - \int_\Omega \frac{N_f(u_n)}{||u_n||^{p-1}}hdz\right|$$
$$\leqslant \frac{\epsilon_n||h||}{(1 + ||u_n||)||u_n||^{p-1}} \quad \text{for all } n \in \mathbb{N}. \tag{16}$$

Hypotheses $H(i), (ii)$ imply that

$$|f(z, x)| \leqslant c_1(1 + |x|^{p-1}) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_1 > 0. \tag{17}$$

From (17) it follows that

$$\left\{\frac{N_f(u_n)}{||u_n||^{p-1}}\right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{18}$$

So, by passing to a subsequence if necessary and using hypothesis $H(ii)$, we have

$$\frac{N_f(u_n)}{||u_n||^{p-1}} \xrightarrow{w} \hat{\lambda}_1(p)|y|^{p-2}y \text{ in } L^{p'}(\Omega) \tag{19}$$

(see Filippakis and Papageorgiou [8], proof of Proposition 4.4). In (16) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (14), (15), (18) and the fact that $2 < p$. Then

$$\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0,$$
$$\Rightarrow y_n \to y \text{ in } W_0^{1,p}(\Omega), \text{ hence } ||y|| = 1. \tag{20}$$

So, if in (16) we pass to the limit as $n \to \infty$ and use (14), (19), (20) and the fact that $2 < p$, then

$$\langle A_p(y), h\rangle = \hat{\lambda}_1(p) \int_\Omega |y|^{p-2} yhdz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$
$$\Rightarrow -\Delta_p y(z) = \hat{\lambda}_1(p)|y(z)|^{p-2}y(z) \quad \text{for almost all } z \in \Omega, \; y|_{\partial\Omega} = 0,$$
$$\Rightarrow y = \xi \hat{u}_1(p) \text{ with } \xi \neq 0 \text{ (see (20))}.$$

Recall that $\hat{u}_1(p) \in \operatorname{int} C_+$, hence $|y(z)| > 0$ for all $z \in \Omega$ and so

$$|u_n(z)| \to +\infty \quad \text{for all } z \in \Omega, \text{ as } n \to \infty.$$

Hypothesis $H(iii)$ implies that

$$\limsup_{n\to\infty} \frac{f(z, u_n(z))u_n(z) - pF(z, u_n(z))}{|u_n(z)|^\tau} \leqslant -\beta_0 < 0 \quad \text{for almost all } z \in \Omega. \tag{21}$$

By hypothesis $H(iii)$ we see that we can find $\beta_1 \in (0, \beta_0)$ and $M > 0$ such that

$$\frac{f(z, x)x - pF(z, x)}{|x|^\tau} \leqslant -\beta_1 \quad \text{for almost all } z \in \Omega, \text{ all } |x| \geqslant M.$$

Then we have

$$\begin{aligned}\frac{1}{||u_n||^\tau} \int_\Omega [f(z, u_n)u_n - pF(z, u_n)]dz = &\int_{\{|u_n| \geqslant M\}} \frac{f(z, u_n)u_n - pF(z, u_n)}{|u_n|^\tau} |y_n|^\tau dz \\ &+ \frac{1}{||u_n||^\tau} \int_{\{|u_n|<M\}} [f(z, u_n)u_n \\ &- pF(z, u_n)]dz.\end{aligned}$$

Recalling that $||u_n|| \to \infty$ (see (14)), we see that

$$\frac{1}{||u_n||^\tau} \int_{\{|u_n|<M\}} [f(z, u_n)u_n - pF(z, u_n)]dz \to 0.$$

On the other hand from (21), Fatou's lemma and because $y \neq 0$ (see (20)), we infer that

$$\limsup_{n\to\infty} \int_{\{|u_n|\geqslant M\}} \frac{f(z,u_n)u_n - pF(z,u_n)}{|u_n|^\tau} |y_n|^\tau dz < 0,$$
$$\Rightarrow \limsup_{n\to\infty} \frac{1}{||u_n||^\tau} \int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz < 0. \tag{22}$$

On the other hand from (11) we have

$$-pM_1 \leqslant ||Du_n||_p^p + \frac{p}{2}||Du_n||_2^2 - \int_\Omega pF(z,u_n)dz \quad \text{for all } n \in \mathbb{N}. \tag{23}$$

Also in (13) we choose $h = u_n \in W_0^{1,p}(\Omega)$. We obtain

$$-\epsilon_n \leqslant -||Du_n||_p^p - ||Du_n||_2^2 + \int_\Omega f(z,u_n)u_n dz \quad \text{for all } n \in \mathbb{N}. \tag{24}$$

Adding (23) and (24) we have for all $n \in \mathbb{N}$

$$-M_2 \leqslant \left(\frac{p}{2} - 1\right) ||Du_n||_2^2 + \int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz,$$
$$\Rightarrow -\frac{M_2}{||u_n||^\tau} \leqslant \left(\frac{p}{2} - 1\right) \frac{||Dy_n||_2^2}{||u_n||^{\tau-2}} + \frac{1}{||u_n||^\tau} \int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz,$$
$$\Rightarrow -\frac{M_2}{||u_n||^\tau} \leqslant \frac{c_2}{||u_n||^{\tau-2}} + \frac{1}{||u_n||^\tau} \int_\Omega [f(z,u_n)u_n - pf(z,u_n)]dz \tag{25}$$

for some $c_2 > 0$, all $n \in \mathbb{N}$ (since $\{Dy_n\}_{n\geqslant 1} \subseteq L^2(\Omega, \mathbb{R}^N)$ is bounded and $2 < p$).

Passing to the limit as $n \to \infty$ in (25), using (14) and recalling that $\tau > 2$ (see hypothesis $H(iii)$), we have

$$0 \leqslant \liminf_{n\to\infty} \frac{1}{||u_n||^\tau} \int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz. \tag{26}$$

Comparing (22) and (26), we reach a contradiction. This proves that

$$\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{27}$$

From (17) and (27) it is clear that

$$\{N_f(u_n)\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{28}$$

If in (13) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (27), (28), then

$$\begin{aligned}
&\lim_{n\to\infty} \left[\left\langle A_p(u_n), u_n - u\right\rangle + \langle A(u_n), u_n - u\rangle\right] = 0,\\
&\quad \Rightarrow \limsup_{n\to\infty} \left[\left\langle A_p(u_n), u_n - u\right\rangle + \langle A(u), u_n - u\rangle\right] \leqslant 0\\
&(\text{recall } A \text{ is monotone, see Proposition3}),\\
&\quad \Rightarrow \limsup_{n\to\infty} \left\langle A_p(u_n), u_n - u\right\rangle \leqslant 0,\\
&\quad \Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition3)},\\
&\quad \Rightarrow \varphi \text{ satisfies the } C\text{-condition.}
\end{aligned}$$

The proof is now complete. □

In the sequel we assume that K_φ is finite or otherwise we already have an infinity of solutions for problem (1), which belong to $C_0^1(\overline{\Omega})$ (see the proof of Theorem 7). So, we are done.

Proposition 5 *If hypotheses $H(i)$, (ii), (iii) hold, then $C_1(\varphi, \infty) \neq 0$.*

Proof Recall that

$$W_0^{1,p}(\Omega) = \mathbb{R}\hat{u}_1(p) \oplus V_p.$$

□

Claim 1 *$\varphi|_{\mathbb{R}\hat{u}_1(p)}$ is anticoercive (that is, $\varphi(t\hat{u}_1(p)) \to -\infty$ as $t \to \pm\infty$).*

It is clear from hypothesis $H(iii)$ that we can always assume that $\tau < p$.
Hypothesis $H(iii)$ implies that we can find $\beta_1 \in (0, \beta_0)$ and $M_3 > 0$ such that

$$f(z,x)x - pF(z,x) \leqslant -\beta_1|x|^\tau \quad \text{for almost all } z \in \Omega, \text{ all } |x| \geqslant M_3. \tag{29}$$

We have

$$\begin{aligned}
\frac{d}{dx}\,\frac{F(z,x)}{|x|^p} &= \frac{f(z,x)|x|^p - p|x|^{p-2}xF(z,x)}{|x|^{2p}}\\
&= \frac{f(z,x)x - pF(z,x)}{|x|^p x}\\
&\leqslant -\beta_1\frac{1}{|x|^{p-\tau+1}} \text{ if } x \geqslant M_3 \text{ and } \geqslant -\beta_1\frac{1}{|x|^{p-\tau}x} \text{ if } x \leqslant -M_3,\\
&\Rightarrow \frac{F(z,x)}{|x|^p} - \frac{F(z,y)}{|y|^p} \leqslant \frac{\beta_1}{p-\tau}\left[\frac{1}{|x|^{p-\tau}} - \frac{1}{|y|^{p-\tau}}\right]\\
&\quad \text{for all } z \in \Omega, \text{ all } |x| \geqslant |y| \geqslant M_3.
\end{aligned}\tag{30}$$

Note that hypothesis $H(ii)$ implies that

$$\lim_{x\to\pm\infty} \frac{pF(z,x)}{|x|^p} = \hat{\lambda}_1(p) \text{ uniformly for almost all } z \in \Omega. \tag{31}$$

So, if in (30) we let $|x| \to +\infty$ and use (31), then

$$\begin{aligned}
&\frac{1}{p}\hat{\lambda}_1(p) - \frac{F(z,y)}{|y|^p} \leqslant -\frac{\beta_1}{p-\tau}\frac{1}{|y|^{p-\tau}} \quad \text{for almost all } z \in \Omega,\ \text{all } |y| \geqslant M_3 \\
&\quad (\text{recall } \tau < p), \\
&\Rightarrow \frac{\hat{\lambda}_1(p)}{p}|y|^p - F(z,y) \leqslant -\frac{\beta_1}{p-\tau}|y|^\tau \quad \text{for almost all } z \in \Omega,\ \text{all } |y| \geqslant M_3.
\end{aligned} \tag{32}$$

Hypothesis $H(i)$ and (32) imply that we can find $c_3 > 0$ such that

$$\frac{\hat{\lambda}_1(p)}{p}|y|^p - F(z,y) \leqslant -\frac{\beta_1}{p-\tau}|y|^\tau + c_3 \quad \text{for almost all } z \in \Omega,\ \text{all } y \in \mathbb{R}. \tag{33}$$

Then we have

$$\begin{aligned}
\varphi(t\hat{u}_1(p)) &= \int_\Omega \left[\frac{\hat{\lambda}_1(p)}{p}|t\hat{u}_1(p)|^p - F(z, t\hat{u}_1(p))\right] dz + \frac{t^2}{2}||D\hat{u}_1(p)||_2^2 \\
&\leqslant -\frac{\beta_1|t|^\tau}{p-\tau}||\hat{u}_1(p)||_\tau^\tau + \frac{t^2}{2}||D\hat{u}_1||_2^2 + c_3|\Omega|_N \text{ (see(33))}.
\end{aligned} \tag{34}$$

Since $\tau > 2$ (see hypothesis $H(iii)$), from (34) we infer that

$$\begin{aligned}
&\varphi(t\hat{u}_1(p)) \to -\infty \text{ as } t \to \pm\infty, \\
&\Rightarrow \varphi|_{\mathbb{R}\hat{u}_1(p)} \text{ is anticoercive.}
\end{aligned}$$

This proves Claim 1.

Claim 2 *$\varphi|_{V_p}$ is bounded below.*

From (31) and hypothesis $H(i)$, we see that given $\epsilon > 0$, we can find $c_4 = c_4(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\hat{\lambda}_1(p)+\epsilon}{p}|x|^p + c_4 \quad \text{for almost all } z \in \Omega,\ \text{all } x \in \mathbb{R}. \tag{35}$$

For $u \in V_p$, we have

$$\varphi(u) \geqslant \frac{1}{p}\left[\hat{\lambda}(p) - (\hat{\lambda}_1(p)+\epsilon)\right]||u||_p^p - c_4|\Omega|_N \text{ (see(6) and (35))}. \tag{36}$$

Choosing $\epsilon \in (0, \hat{\lambda}(p) - \hat{\lambda}_1(p))$ (see Proposition 1), from (36) we see that

$$\varphi|_{V_p} \text{ is bounded below.}$$

This proves Claim 2.

Recall that K_φ is finite or otherwise we already have infinitely many nontrivial solutions for problem (1) and so we are done. Then Claims 1, 2 and Proposition 4, permit the use of Proposition 6.63, p. 160, of Motreanu et al. [12] and have $C_1(\varphi, \infty) \neq 0$.

Next we compute the critical group of φ at the origin.

Proposition 6 *If hypotheses $H(i), (iv)$ hold, then $C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}_0$ with $d_l = dim \overset{l}{\underset{\text{i}=1}{\oplus}} E(\hat{\lambda}_i(2))$.*

Proof Consider the C^2-functional $\hat{\psi} : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}(u) = \frac{1}{2}||Du||_2^2 - \int_\Omega F(z, u)dz \quad \text{for all } u \in H_0^1(\Omega).$$

$\square$

Let $\psi = \hat{\psi}|_{W_0^{1,p}(\Omega)}$ (recall $2 < p$).

Claim 3 *We have $C_k(\psi, 0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.*

Let $\eta_0 \in (\hat{\lambda}_l(2), \hat{\lambda}_{l+1}(2))$ and consider the C^2-functional $\sigma : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\sigma(u) = \frac{1}{2}||Du||_2^2 - \frac{\eta_0}{2}||u||_2^2 \quad \text{for all } u \in H_0^1(\Omega).$$

Evidently $u = 0$ is a nondegenerate critical point of σ with Morse index d_l. Therefore

$$C_k(\sigma, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{37}$$

We consider the homotopy $h(t, u)$ defined by

$$h(t, u) = (1 - t)\hat{\psi}(u) + t\sigma(u) \quad \text{for all } (t, u) \in [0, 1] \times H_0^1(\Omega).$$

Let $t \in (0, 1]$ and $u \in C_0^1(\overline{\Omega})$ with $||u||_{C_0^1(\overline{\Omega})} \leqslant \delta$ (here $\delta > 0$ is as postulated in hypothesis $H(iv)$). Let $\langle \cdot, \cdot \rangle_0$ denote the duality brackets for the pair $(H^{-1}(\Omega), H_0^1(\Omega))$. We have

$$\langle h'_u(t, u), y \rangle_0 = (1 - t)\left\langle \hat{\psi}'(u), y \right\rangle_0 + t\langle \sigma'(u), y \rangle_0 \quad \text{for all } y \in H_0^1(\Omega). \tag{38}$$

We consider the following orthogonal direct sum decomposition of the Hilbert space $H_0^1(\Omega)$:

$$H_0^1(\Omega) = \bar{H}_l \oplus \hat{H}_l \text{ with } \bar{H}_l = \mathop{\oplus}_{i=1}^{l} E(\hat{\lambda}_i(2)), \ \hat{H}_l = \overline{\mathop{\oplus}_{i \geqslant l+1} E(\hat{\lambda}_i(2))}\,.$$

Then every $v \in H_0^1(\Omega)$ admits a unique sum decomposition as

$$v = \bar{v} + \hat{v} \text{ with } \bar{v} \in \bar{H}_l \text{ and } \hat{v} \in \hat{H}_l.$$

In (38) we choose $y = \hat{u} - \bar{u} \in H_0^1(\Omega)$. Exploiting the orthogonality of the component spaces in the above decomposition, we have

$$\left\langle \hat{\psi}'(u), \hat{u} - \bar{u} \right\rangle_0 = ||D\hat{u}||_2^2 - ||D\bar{u}||_2^2 - \int_\Omega f(z, u)(\hat{u} - \bar{u})dz. \tag{39}$$

Note that hypothesis $H(iv)$ implies that

$$\eta(z) \leqslant \frac{f(z, x)}{x} \leqslant \hat{\lambda}_{l+1}(2) \quad \text{for almost all } z \in \Omega, \ \text{all } 0 < |x| \leqslant \delta.$$

Let $y = \hat{u} - \bar{u} \in H_0^1(\Omega)$. Then

$$\begin{aligned} f(z, u)(\hat{u} - \bar{u}) = f(z, u)y &= \frac{f(z, u)}{u} uy \\ &\leqslant \begin{cases} \hat{\lambda}_{l+1}(2)(\hat{u}^2 - \bar{u}^2) & \text{if } uy > 0 \\ \eta(z)(\hat{u}^2 - \bar{u}^2) & \text{if } uy < 0 \end{cases} \\ &\leqslant \hat{\lambda}_{l+1}(2)\hat{u}^2 - \eta(z)\bar{u}^2. \end{aligned}$$

So, we have proved that

$$f(z, u(z))(\hat{u} - \bar{u})(z) \leqslant \hat{\lambda}_{l+1}(2)\hat{u}(z)^2 - \eta(z)\bar{u}(z)^2 \quad \text{for almost all } z \in \Omega\,. \tag{40}$$

Using (40) in (39), we obtain

$$\begin{aligned} \left\langle \hat{\psi}'(u), \hat{u} - \bar{u} \right\rangle_0 &= ||D\hat{u}||_2^2 - \hat{\lambda}_{l+1}(2)||\hat{u}||_2^2 - \left[||D\bar{u}||_2^2 - \hat{\lambda}_l(2)||\bar{u}||_2^2 \right] \\ &\geqslant 0 \text{ (see (5))}. \end{aligned} \tag{41}$$

Also we have

$$\begin{aligned} \langle \sigma'(u), \hat{u} - \bar{u} \rangle_0 &= ||D\hat{u}||_2^2 - \eta_0||\hat{u}||_2^2 - \left[||D\bar{u}||_2^2 - \eta_0||\bar{u}||_2^2 \right] \\ &\geqslant c_5||Du||^2 \text{ for some } c_5 > 0 \text{ (recall that } \eta_0 \in (\hat{\lambda}_l(2), \hat{\lambda}_{l+1}(2))). \end{aligned} \tag{42}$$

Returning to (39) and using (41) and (42), we have

$$\left\langle h'_u(t,u), \hat{u}-\bar{u}\right\rangle_0 \geqslant tc_5||Du||^2 > 0 \text{ (recall that } t \in (0,1]). \tag{43}$$

We claim that $0 \in K_{\hat{\psi}}$ is isolated. Arguing by contradiction, suppose that the claim is false. Then we can find $\{u_n\}_{n\geqslant 1} \subseteq H_0^1(\Omega)$ such that

$$u_n \to 0 \text{ in } H_0^1(\Omega) \text{ and } \hat{\psi}'(u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{44}$$

From the equality in (44), we have

$$-\Delta u_n(z) = f(z, u_n(z)) \text{ for almost all } z \in \Omega,\ u_n|_{\partial\Omega} = 0 \quad \text{for all } n \in \mathbb{N} \tag{45}$$

Standard regularity theory (the Calderon–Zygmund estimates), implies that we can find $\alpha \in (0,1)$ and $M_4 > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad ||u_n||_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant M_4 \quad \text{for all } n \in \mathbb{N}. \tag{46}$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$, from (44) and (46) we have

$$\begin{aligned} & u_n \to 0 \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty, \\ \Rightarrow\ & \eta(z)u_n(z)^2 \leqslant f(z,u_n(z))u_n(z) \leqslant \hat{\lambda}_{l+1}(2)u_n(z)^2 \\ & \text{for almost all } z \in \Omega,\ \text{all } n \geqslant n_0 \text{ (see hypothesis } H(iv)). \end{aligned} \tag{47}$$

Relation (47) implies that

$$\begin{aligned} & f(z,u_n(z))(\hat{u}_n(z)-\bar{u}_n(z)) \leqslant \hat{\lambda}_{l+1}(2)\hat{u}_n(z)^2 - \eta(z)\bar{u}_n(z)^2 \\ & \quad \text{for almost all } z \in \Omega,\ \text{all } n \geqslant n_0. \end{aligned}$$

Multiplying (45) with $(\hat{u}_n - \bar{u}_n)(z)$, integrating over Ω and using Green's identity, the orthogonality of the component spaces and (47), we have

$$\begin{aligned} & ||D\hat{u}_n||_2^2 - ||D\bar{u}_n||_2^2 \\ & = \int_\Omega (Du_n, D\hat{u}_n - D\bar{u}_n)_{\mathbb{R}^N} dz \text{ (since } u_n = \hat{u}_n + \bar{u}_n) \\ & = \int_\Omega f(z,u_n)(\hat{u}_n - \bar{u}_n)dz \text{ (see(45))} \\ & \leqslant \int_\Omega [\hat{\lambda}_{l+1}(2)\hat{u}_n^2 - \eta(z)\bar{u}_n^2]dz \\ & \Rightarrow 0 \leqslant ||D\hat{u}_n||_2^2 - \hat{\lambda}_{l+1}(2)||\hat{u}_n||_2^2 \leqslant ||D\bar{u}_n||_2^2 - \int_\Omega \eta(z)\bar{u}_n^2 dz \leqslant -\hat{\xi}_0||\bar{u}_n||^2 \end{aligned}$$

$$\text{for some } \hat{\xi}_0 > 0 \text{ (see [17])},$$
$$\Rightarrow \bar{u}_n = 0 \text{ and } u_n = \hat{u}_n \in E(\hat{\lambda}_{l+1}(2)) \text{ for all } n \geqslant n_0.$$

The unique continuation property of the eigenspaces implies that

$$\hat{u}_n(z) \neq 0 \quad \text{for almost all } z \in \Omega.$$

Then

$$\hat{\lambda}_{l+1}(2)||u_n||_2^2 = ||Du_n||_2^2 = \int_\Omega f(z, u_n)u_n dz < \hat{\lambda}_{l+1}(2)||u_n||_2^2 \quad \text{for all } \geqslant n_0,$$

since $u_n(z) \neq 0$ for almost all $z \in \Omega$, see hypothesis $H(iv)$. This contradiction proves that $0 \in K_{\hat{\psi}}$ is isolated.

Since $h(0, \cdot) = \hat{\psi}(\cdot)$, using (43) we see that $u = 0$ is an isolated critical point of $h(t, \cdot)$ for all $t \in [0, 1]$. Invoking Theorem 5.2 of Corvellec and Hantoute [6], we have

$$C_k(\hat{\psi}, 0) = C_k(\sigma, 0) \quad \text{for all } k \in \mathbb{N}_0,$$
$$\Rightarrow C_k(\hat{\psi}, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (37))}. \tag{48}$$

Because $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$, it follows that

$$C_k(\hat{\psi}, 0) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0 \text{ (see [3, p.14] and [14])},$$
$$\Rightarrow C_k(\psi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (48))}. \tag{49}$$

This proves the Claim.

We have

$$|\varphi(u) - \psi(u)| \leqslant \frac{1}{p}||u||^p \tag{50}$$
$$|\langle \varphi'(u) - \psi'(u), h\rangle| \leqslant c_6||u||^{p-1}||h|| \text{ for some } c_6 > 0,$$
$$\Rightarrow ||\varphi'(u) - \psi'(u)||_* \leqslant c_6||u||^{p-1}. \tag{51}$$

From (50), (51) and the continuity of critical groups in the C^1-topology (see Corvellec and Hantoute [6, Theorem 5.1]), we have

$$C_k(\varphi, 0) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0,$$
$$\Rightarrow C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see(49)}s).$$

Now we are ready for the existence theorem.

Theorem 7 *Assume that hypotheses H hold. Then problem* (1) *admits a nontrivial solution* $u_0 \in C_0^1(\overline{\Omega})$.

Proof From Proposition 5 we know that

$$C_1(\varphi, \infty) \neq 0.$$

So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$u_0 \in K_\varphi \quad \text{and} \quad C_1(\varphi, u_0) \neq 0. \tag{52}$$

Since $\varphi \in C^2(W_0^{1,p}(\Omega))$ from (52) and Papageorgiou and Rădulescu [15], it follows that

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{53}$$

From Proposition 6, we have

$$C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{54}$$

Comparing (53) and (54) we see that $u_0 \neq 0$ (recall $l \geqslant 2$). Since $u_0 \in K_\varphi$ (see (52)) it follows that u_0 is a nontrivial solution of (1). Moreover, from Ladyzhenskaya and Uraltseva [10, Theorem 7.1, p. 286], we have $u_0 \in L^\infty(\Omega)$ and so we can apply Theorem 1 of Lieberman [11] and conclude that $u_0 \in C_0^1(\overline{\Omega})$. $\square$

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