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Finitely many solutions for a class of boundary value problems with superlinear convex nonlinearity

By

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Abstract. We consider the nonlinear Sturm-Liouville problem $-u'' = f(u) + h$ in $(0, 1)$, $u(0) = u(1) = 0$, where $h \in L^2(0, 1)$ and f is a positive convex nonlinearity with superlinear growth at infinity. Our main result establishes that the above boundary value problem admits a finite number of solutions but it cannot have infinitely many solutions.

1. Introduction and the main result. Consider the linear Sturm-Liouville problem

$$\text{(S-L)} \qquad \begin{cases} -u'' = u + h, & \text{in } (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

where $h \in L^2(0, 1)$. The Fredholm alternative establishes that only one of the following situations can occur: (i) if 1 is not an eigenvalue of the operator $Lu = -u''$ in $H_0^1(0, 1)$ then problem (S-L) has a unique solution in $H^2(0, 1) \cap H_0^1(0, 1)$, for any $h \in L^2(0, 1)$; or (ii) if 1 is an eigenvalue of L in $H_0^1(0, 1)$ (say, $\lambda_n = 1$) then problem (S-L) has a weak solution if and only if $\int_0^1 he_n dx = 0$, where e_n denotes an eigenvalue corresponding to λ_n. Moreover, in this case, there exists a one-parameter family of solutions, given by $u(x) = Ce_n(x) + \sum_{k \neq n} \frac{(h,e_k)_{L^2}}{\lambda_k - 1} e_k(x)$.

In this paper we establish the key role of the *linear* term u in the right-hand side of (S-L). In particular, we deduce that if $p \geqq 2$ is an integer then the nonlinear Sturm-Liouville problem

$$\begin{cases} -u'' = u^p + h, & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

has a *finite* number of solutions, for *any* $h \in L^2(0, 1)$.

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There are many results in the literature related to the multiplicity of solutions to Sturm-Liouville problems. We refer to [6], [8], [9], [10], [11], [13], [15] for the existence of a finite number of solutions, to [12] for the existence of arbitrarily many solutions, subject to a small perturbation, and to [1], [4], [5] for the exact number of solutions to several classes of Sturm-Liouville problems. The methods employed in these papers are based on Leray-Schauder topological degree, Morse theory, variational tools, Sturm comparison-type theorems, bifurcation techniques or critical point theory. The above mentioned papers do not say anything about the possibility that the considered Sturm-Liouville problems have a finite number of solutions, but they cannot admit infinitely many solutions. In this spirit, our approach is different.

We are concerned in this paper with the autonomous superlinear problem

$$\begin{cases} -u'' = f(u) + h, & \text{in } (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

Developing some ideas of McKean and Scovel [7], we obtain that problem (1) has a finite number of solutions, provided that f is analytic, convex, nonnegative and satisfies superlinear growth assumptions at $+\infty$. Our arguments are characteristic for one-dimensional differential problems and the proof of the main result is carried out by using analytical and monotonicity methods.

Throughout the paper we assume that $f : \mathbb{R} \to [0, +\infty)$ is an analytic function which fulfills the following hypotheses:

(f_1) f is increasing in $[0, +\infty)$;
(f_2) $\lim\limits_{t\to\infty} \frac{f(t)}{t} = +\infty$;
(f_3) $f'' > 0$;
(f_4) there exists positive numbers α, β and t_0 such that $\alpha + \beta < \frac{1}{2}$ and, for any $t \geqq t_0$,

$$\alpha F^{-1}(\beta t^2) \geqq f^{-1}(t),$$

where $F(t) = \int\limits_0^t f(s)ds$.

We establish in this paper the following result.

Theorem 1. *Assume that f satisfies conditions (f_1)–(f_4). Then problem* (1) *has a finite number of solutions, for any $h \in L^2(0, 1)$.*

The main idea of the proof is that if problem (1) has many solutions, then it has too many solutions. More precisely, if problem (1) has infinitely many solutions, then it admits an unbounded curve of solutions. This will be seen to contradict our technical hypothesis (f_4).

In the sequel, $\lambda_j(g)$ and $\varphi_j(g)$, where $g \in L^\infty(0, 1)$ and $j = 1, 2, \ldots$, denote the jth eigenvalue (eigenfunction, resp.) of $-u'' - gu$ in $H_0^1(0, 1)$. When $g = 0$ we simply write λ_j or φ_j. Following the idea in [7], $\varphi_j(g)$ will be normalized by $\varphi_j'(g)(0) = 1$. Throughout this paper we assume, without loss of generality, that condition (f_4) in Theorem 1 is fulfilled for $t_0 = 0$.

2. Auxiliary results. Let us first define (in the weak sense) the operator $A : H_0^1 \to H^{-1}$, $Au = -u'' - f(u)$.

Proposition 1. *Let* $f : \mathbb{R} \to \mathbb{R}_+$ *satisfy* (f_1), (f_4) *and such that*

i) $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$;

ii) $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \infty$.

Then A is a proper mapping.

Proof. Let $T : H_0^1(0,1) \to H^{-1}(0,1)$ be the operator defined by $Tu = -u''$ and set $E = T^{-1}$, $B = EA$. It is enough to prove that B is proper. Note that for any $u \in L^1(0,1)$ we can write

$$Eu(x) = (1-x)\int_0^x yu(y)dy + x\int_x^1 (1-y)u(y)dy. \tag{2}$$

We have $Bu = u - Ef(u)$, $B : H_0^1(0,1) \to H_0^1(0,1)$. Let K be a compact subset of $H_0^1(0,1)$. We must prove that if $Bu \in K$ then u moves into a compact set in $H_0^1(0,1)$.

It is enough to show that $f(u)$ is bounded in $L^1(0,1)$. Indeed, it is easy to see that

$$(Eu)'(x) = -\int_0^x yu(y)dy + \int_x^1 (1-y)u(y)dy,$$

so that E maps $L^1(0,1)$ continuously to $W_0^{2,1}(0,1) \cap H_0^1(0,1)$, which is compactly embedded in $H_0^1(0,1)$, because $W^{1,1}(0,1)$ is compactly embedded in $L^2(0,1)$.

If we prove that $f(u)$ is bounded in L^1, then the desired result follows via a reasoning with convergent subsequences.

Since $H_0^1(0,1) \hookrightarrow L^\infty(0,1)$, there exists $a > 0$ such that

$$Bu \in K \Longrightarrow -a \leqq u - Ef(u) \leqq a. \tag{3}$$

Note that this implies $u \geqq -a$ and $\int_0^1 Ef(u) \leqq \int_0^1 u + a$. Fix arbitrarily $b \in (0, 1/2)$. We have

$$\begin{aligned}\int_0^1 u + a &\geqq \int_0^1 Ef(u) = \int_0^1 1 \cdot Ef(u) = \int_0^1 E1 \cdot f(u)\\ &= \int_0^1 \frac{x(1-x)}{2} f(u) \geqq \frac{b(1-b)}{2} \int_b^{1-b} f(u).\end{aligned}$$

Hence

$$\int_b^{1-b} f(u) \leqq \frac{2}{b(1-b)}\left(\int_0^1 u + a\right), \qquad \forall b \in \left(0, \frac{1}{2}\right). \tag{4}$$

We also have, by (3),

$$\begin{aligned}\int_0^b f(u) &= \int_{[0,b]\cap\{u\leqq 0\}} f(u) + \int_{[0,b]\cap\{u>0\}} f(u) \leqq M + \int_{[0,b]\cap\{u>0\}} f(u)\\ &\leqq M + \int_{[0,b]\cap\{u>0\}} f(Ef(u)+a),\end{aligned}$$

where $M = \sup\limits_{[-a,0]} f$. Relation (2) yields

$$\begin{aligned}\int_0^b f(u) &\leqq M + \int_0^b f\left(x\int_0^1 f(u) + a\right)\\ &\leqq M + \frac{1}{\int_0^1 f(u)} F\left(b\int_0^1 f(u) + a\right),\end{aligned} \tag{5}$$

provided that $\int_0^1 f(u) > 0$. Obviously, a similar estimation holds for $\int_{1-b}^1 f(u)$, so relations (4) and (5) yield, for any $b \in (0, 1/2)$,

$$\begin{aligned}\left(\int_0^1 f(u)\right)^2 \leqq 2\Bigg[& F\left(b\int_0^1 f(u) + a\right) + M\int_0^1 f(u)\\ &+ \frac{\int_0^1 f(u)}{b(1-b)}\int_0^1 u + \frac{a}{b(1-b)}\int_0^1 f(u)\Bigg]\end{aligned} \tag{6}$$

Notice that $\int_0^1 u \leqq \int_0^1 \max(u,0)$ and, by Jensen's inequality,

$$f\left(\int_0^1 \max(u,0)\right) \leqq \int_0^1 f(\max(u,0)).$$

Thus we obtain

$$\int_0^1 u \leqq f^{-1}\left(\int_0^1 f(u) + f(0)\right).$$

Set $x = \int_0^1 f(u)$ and $c = f(0)$. Hence relation (6) becomes, for any $b \in (0, 1/2)$,

$$x^2 \leqq 2\left[F(bx+a) + Mx + \frac{1}{b(1-b)}xf^{-1}(x+c) + \frac{ax}{b(1-b)}\right]. \tag{7}$$

We will finish the proof by showing that (7) is violated for large x. For this purpose we choose $\alpha_0, \beta_0, \epsilon, x_1 > 0$ such that

$$\alpha_0 + \beta_0 + \epsilon(\beta_0 + 1) < \frac{1}{2} \quad \text{and} \quad \frac{\alpha_0 F^{-1}(\beta_0 x^2)}{f^{-1}(x)} \geqq 1 \quad \text{for} \quad x \geqq x_1.$$

We have, by (f_2), $\lim\limits_{x\to\infty} \frac{f^{-1}(x)}{x} = 0$, that is, $b = \frac{f^{-1}(x+c)}{\alpha_0 x}$ is as small as we want for large x. Then for large x and b as above we have

$$\frac{2}{b(1-b)}xf^{-1}(x+c) + \frac{2ax}{b(1-b)} + 2Mx < 2(\alpha_0 + \epsilon)x^2.$$

Now for large x and the above b we have

$$F(bx+a) = F\left(\frac{f^{-1}(x)}{\alpha_0} + a\right) \leqq \beta_0(1+\epsilon)x^2.$$

The last inequality can be written, if $\frac{f^{-1}(x+c)}{\alpha_0} \geqq 0$ (this begins to happen, because $\lim\limits_{x\to\infty} f^{-1}(x) = \infty$), as

$$1 \leqq \frac{\alpha_0 F^{-1}(\beta_0(1+\epsilon)x^2)}{f^{-1}(x+c) + \alpha_0 a} = \frac{\alpha_0 F^{-1}(\beta_0(1+\epsilon)x^2)}{f^{-1}(\sqrt{1+\epsilon}x)} \cdot \frac{f^{-1}(\sqrt{1+\epsilon}x)}{f^{-1}(x+c) + \alpha_0 a}$$

which is obviously true because of the monotonicity of f^{-1}. $\square$

All the other results in this section are concerned with analytic regularity. We shall use the implicit function theorem for analytic functions (see Theorem 4.B and Corollary 4.23 in [14]) and the following useful observation: if X, Y are Banach spaces, then $H : X \to Y$ is analytic if and only if for each $x_0 \in X$ there exists a ball B centered in 0 and continuous mappings $T_n : B + \{x_0\} \to \sum_n (X, Y)$, for $n = 1, 2, \ldots$, such that

$$x \in B + \{x_0\},\ h \in B \Longrightarrow H(x+h) - H(x) = \sum_{n \geqq 1} T_n(x)(h, \ldots, h).$$

Here $\sum_n (X, Y) = \{T : X^n \to Y : \ T \text{ is symmetric, continuous and } n - \text{linear}\}$.

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function and K a compact subset of $\mathbb{R}$. Then the formal series $\sum \frac{c_n}{n!} x^n$, with $c_n = \max\limits_K |f^{(n)}|$, has a positive convergence radius.*

Proof. Let Ω be a complex neighborhood of $\mathbb{R}$ in which f can be extended to a holomorphic function g. Choose $R > 0$ such that $d(K, C\Omega) > R$. Define $L = \{z \in \Omega : d(z, K) \leqq R\}$ and $M = \max\limits_L |g|$. Then $c_n \leqq \frac{n!M}{R^n}$ by the Cauchy inequalities. □

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Then the mapping*

$$H_0^1(0, 1) \ni u \longmapsto f(u) \in C([0, 1])$$

is analytic.

Proof. Take $K = \overline{u(\mathbb{R})}$ and, for all $n \geqq 1$, $T_n(u)(h_1, \ldots, h_n) = f^{(n)}(u) h_1 ... h_n$. It follows that $\|T_n(u)\| \leqq c_n$, with c_n as in the above lemma, and obviously, if we set $T_0(u) = f(u)$, then

$$\sum_{n \geqq 0} \frac{T_n(u)(h, \ldots, h)}{n!} = f(u + h),$$

provided that the series is convergent. □

The following comparison principle is well-known in the literature (see, e. g., Theorem 1.2, p. 210 in [14]).

Lemma 3. *Let u be the solution of the problem*

$$\begin{cases} -u'' = gu + h, \text{ in } (0, 1) \\ u(0) = a \\ u'(0) = b \end{cases}$$

and v be the solution of

$$\begin{cases} v'' = Mv + N, \quad \text{in } (0, 1) \\ v(0) = |a| \\ v'(0) = |b|. \end{cases} \tag{8}$$

If $|g| \leqq M$ and $|h| \leqq N$, then $|u| \leqq v$.

In the sequel, $C^2([0, 1])$ will be renormed with the equivalent norm

$$\|u\|_{C^2([0,1])} = \max_{x \in [0,1]} |u''(x)| + |u(0)| + |u'(0)|.$$

For $g \in C([0, 1])$ and $\lambda \in \mathbb{R}$, let $w(g, \lambda)$ be the unique solution $w \in C^2([0, 1])$ of

$$\begin{cases} -w'' = gw + \lambda w, \quad \text{in } (0, 1) \\ w(0) = 0 \\ w'(0) = 1. \end{cases} \tag{9}$$

Proposition 2. *The mapping*

$$X \times \mathbb{R} = C([0,1]) \times \mathbb{R} \ni (g, \lambda) \longmapsto w(g, \lambda) \in C^2([0,1]) = Y$$

is analytic.

Proof. Define for $g \in X$ the sequence $(w_n = w_n(g)) \subset Y$ by

$$\begin{cases} -w_0'' = g w_0, & \text{in } (0,1) \\ w_0(0) = 0 \\ w_0'(0) = 1 \end{cases}$$

and, for any $n \geqq 1$,

$$\begin{cases} -w_n'' = g w_n + w_{n-1}, & \text{in } (0,1) \\ w_n(0) = 0 \\ w_n'(0) = 0. \end{cases} \tag{10}$$

We will prove below that the series $\sum\limits_{n \geqq 0} \lambda^n w_n$ is entire. Taking this for granted, note that $w = \sum\limits_{n \geqq 0} \lambda^n w_n$ is the solution of (9).

Now, for fixed $n_0 \in \mathbb{N}$, $\lambda_0 \in \mathbb{R}$, $g_0 \in X$, take $g = g_0 + \lambda_0$ and define the sequence $(T_{n_0,n})_{n \geqq 0}$ as follows: $T_{n_0,0} = w_{n_0}(g)$ and, if $n \geqq 1$, then for each $h_1, \dots, h_n \in X$, $T_{n_0,n}(h_1, \dots, h_n)$ is the unique solution $w \in Y$ of the problem

$$\begin{cases} -w'' = g w + \frac{1}{n} \sum\limits_c T_{n_0,n-1}(h_1, \dots, h_{n-1}) h_n + T_{n_0-1,n}(h_1, \dots, h_n) \\ w(0) = 0 \\ w'(0) = 0. \end{cases}$$

Here $\sum\limits_c$ denotes the cyclic sum. It will be showed below that the series $\sum\limits_{n_0,n} \|T_{n_0,n}\| \lambda^{n_0} \|h\|^n$ has positive convergence radius. If we note that $\sum\limits_n T_{n_0,n}(h, \dots, h) = w_{n_0}(g+h)$ when the series is convergent, we obtain the desired result. Hence we only need to obtain the desired estimations. This will be proved in the following three lemmas. $\square$

Lemma 4. *There exists $M_n \geqq 0$, $n = 1, 2, \dots$ such that for any positive integer n and all $t \in [0,1]$,*

$$|w_n(t)| \leqq M_n \frac{t^{2n+1}}{(2n+1)!}. \tag{11}$$

Proof. Relation (11) is obvious for $n = 0$, because $w_0(0) = 0$. Let $n \geqq 1$ and suppose that (11) has been already proved for $n-1$ instead of n. By (10) we get $|w_n(t)| \leqq Ct^2$, so that (10) gives by integration $|w_n(t)| \leqq Ct^3$ if $n = 1$ and $|w_n(t)| \leqq Ct^4$ if $n \geqq 2$. For $n \geqq 2$ we continue until we obtain the existence of the desired constant. $\square$

Lemma 5. *Let $M = \max|g|$ and M_n be the best constants in* (11). *Then there exists n_0 such that, for any $n \geqq n_0$,*

$$M_n \leqq e^{\frac{M\pi^2}{6}} M_{n_0}.$$

In particular, M_n are bounded, so that $\sum \lambda^n w_n$ is entire.

Proof. It will be enough to find n_0 such that, for any $n \geqq n_0$,

$$M_{n+1} \leqq \left(1 + \frac{M}{n^2}\right) M_n. \tag{12}$$

If M_n is the best constant, then for given $\epsilon > 0$ there is $t \in (0, 1]$ such that

$$(M_{n+1} - \epsilon)\frac{t^{2n+3}}{(2n+3)!} \leqq M M_{n+1}\frac{t^{2n+5}}{(2n+5)!} + M_n\frac{t^{2n+3}}{(2n+3)!}.$$

Relation (12) follows now easily for large n. □

Lemma 6. *The following inequality is satisfied:*

$$\|T_{n_0,n}\| \leqq a^{n+n_0} b, \quad n, n_0 \in \mathbb{N},$$

where $a = e^{\sqrt{M}}$ and $b = \sum\limits_{n_0} \|T_{n_0,0}\|$.

Proof. Applying Lemma 3 we obtain $\|T_{n_0,n}(h_1, \ldots, h_n)\|_Y \leqq \|u\|_Y$, where u is the solution of

$$\begin{cases} u'' = Mu + N, & \text{in } (0,1) \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

and $N = (\|T_{n_0,n-1}\| + \|T_{n_0-1,n}\|)\|h_1\| \ldots \|h_n\|$ (if we have negative indices, the norms are considered to be 0). Since $u(t) = \frac{N}{2M}(e^{\sqrt{M}t} + e^{-\sqrt{M}t} - 2)$, we obtain that $u(t) \leqq \frac{Na}{2}$, so that $\|T_{n_0,n}\| \leqq \frac{a}{2}(\|T_{n_0,n-1}\| + \|T_{n_0-1,n}\|)$ and the desired inequalities follow easily by induction on $n_0 + n$. □

Corollary 1. *The mapping $X \times \mathbb{R} \ni (g, \lambda) \overset{T}{\longmapsto} w(g, \lambda, 1) \in \mathbb{R}$ is analytic.*

Proposition 3. *For $n \geqq 1$, the mapping $X \ni g \longmapsto \lambda_n(g)$ is analytic.*

Proof. Let T be as in Corollary 1. Note that $\frac{\partial T}{\partial \lambda}(g, \lambda) = \frac{\partial w}{\partial \lambda}(g, \lambda, 1)$. Hence $T(g, \lambda) = 0$ implies $\frac{\partial T}{\partial \lambda}(g, \lambda) \neq 0$. Indeed, if we examine the power series in the Proof of the Proposition 1, we get that $\frac{\partial w}{\partial \lambda}$ is the solution w_1 of

$$\begin{cases} -w_1'' = (g + \lambda)w_1 + w_0, & \text{in } (0,1) \\ w_1(0) = 0 \\ w_1'(0) = 0, \end{cases} \tag{13}$$

where w_0 is the solution of

$$\begin{cases} -w_0'' = (g+\lambda)w_0, & \text{in } (0,1) \\ w_0(0) = 0 \\ w_0'(0) = 1. \end{cases} \tag{14}$$

Now $T(g,\lambda) = 0$ and $\frac{\partial T}{\partial \lambda}(g,\lambda) = 0$ would imply that (13) and (14) have solutions in $H_0^1(0,1)$. Eq. (13) multiplied by w_0 and Eq. (14) multiplied by w_1 give by substraction the contradiction $\int_0^1 w_0^2 = 0$.

Note that $\lambda = \lambda_n(g)$ for some n if and only if $T(g,\lambda) = 0$. The above proof shows that the implicit function theorem can be applied near each zero of T. Fix n and g_0 and let $\lambda(g)$ be the analytic function that satisfies

$$\begin{cases} T(g,\lambda(g)) = 0 \ \text{ near } g_0 \\ \lambda(g_0) = \lambda_n(g_0). \end{cases}$$

All it remains to prove is that $\lambda = \lambda_n$ near g_0. It is obvious enough to show that λ_n depends continuously on g. This will be done in the next result. $\square$

Lemma 7. *The mapping $X \ni g \longmapsto \lambda_n(g) \in \mathbb{R}$ is continuous.*

Proof. Take $g_k \to g$. Let $a \leqq \liminf_k \lambda_n(g_k)$. We may suppose $\lambda_n(g_k) \geqq a$ for each k. Let V be a fixed n-dimensional subspace of $H_0^1(0,1)$. Then there exists $w_k \in V$ such that

$$1 = \int_0^1 w_k'^2 \geqq \int_0^1 (g_k + a) w_k^2. \tag{15}$$

It is obvious that (15) gives the existence of some $w \in H_0^1(0,1)$ such that $w \in V$ (because V is convex and closed) and $1 = \int_0^1 w'^2 \geqq \int_0^1 (g+a)w^2$.

Choose $a \geqq \limsup_k \lambda_n(g_k)$. Suppose that $\lambda_n(g_k) \leqq a$ for all k. There exists $e_{k,1}, \ldots, e_{k,n} \in H_0^1(0,1)$, of L^2-norm 1 and mutually orthogonal in $L^2(0,1)$ such that, for any $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$,

$$\int_0^1 (\alpha_1 e_{k,1}' + \cdots + \alpha_n e_{k,n}')^2 \leqq \int_0^1 (g_k + a)(\alpha_1 e_{k,1} + \cdots + \alpha_n e_{k,n})^2. \tag{16}$$

If we take $\alpha = (\delta_{jl})$ we get that $(e_{k,j})$ is bounded in $H_0^1(0,1)$. We may suppose that $e_{k,j} \stackrel{k}{\to} e_j$ in $* - H_0^1$, L^2 and $a.e.$ Then e_j are mutually orthogonal and (16) gives yields $\int_0^1 u'^2 \leqq \int_0^1 (g+a)u^2$, for any $u \in Sp\{e_1, \ldots, e_n\}$. This means that λ_n is both upper and lower semicontinuous. $\square$

Proposition 4. *Let $g : \mathbb{R} \to \mathbb{R}$ be an analytic function with $g' > 0$. Let g^{-1} be the inverse of g on the image. Then the mapping*

$$H : H_0^1(0,1) \times H_0^1(0,1) \to H_0^1(0,1), \quad H(u,v) = g^{-1}\left(\int_0^1 g(u+tv)dt\right)$$

is well-defined and analytic.

Proof. By a translation argument, we may suppose that $g(0) = 0$. Then $U = \{g \circ u;\ u \in H_0^1(0,1)\}$ is an open subset of $H_0^1(0,1)$. Indeed, let $J = g(\mathbb{R})$ and $u_0 \in H_0^1(0,1)$. Take $R > 0$ such that

$$g(u_0(I)) + [-R, R] \subset J. \tag{17}$$

We claim that

$$h \in H_0^1(0,1) \text{ and } \|g(u) - h\|_{H_0^1(0,1)} \leqq R \text{ imply } h \in U. \tag{18}$$

Indeed, h is bounded, $g^{-1}(0) = 0$, g^{-1} is a Lipschitz map on compact sets, so that (18) will be proved using Proposition VIII.3 in [2] if we show that $h(\mathbb{R}) \subset J$. But $\|g(u) - h\|_{L^\infty} \leqq \|g(u) - h\|_{H_0^1}$ and we use (17).

Moreover, $H_0^1(0,1) \ni u \overset{H_1}{\longmapsto} g(u) \in U$ defines an analytic diffeomorphism. The analyticity follows from Lemma 2 and it is obvious that it is one-to-one and onto and that the inverse is of the same form.

Since $H = H_1^{-1} \circ H_2$, with

$$H_2(u,v) = \int_0^1 g(u+tv)dt, \quad H_2 : H_0^1(0,1) \times H_0^1(0,1) \to U \tag{19}$$

all it remains to prove is that H_2 is well defined and analytic. The fact that H_2 is well defined follows from the following characterization of U : $u \in U$ if and only if $u \in H_0^1(0,1)$ and $u(\mathbb{R}) \subset J$. We also have to use the fact that $g(u + tv) \in H_0^1(0,1)$ for each t, so the "Riemann sums" associated to the integrals in (19) are also there. A limit argument concludes the proof.

The analyticity can be read off from the following diagrams:

$$H_0^1(0,1) \times H_0^1(0,1) \times [0,1] \ni (u,v,t) \longmapsto u+tv \longmapsto g(u+tv) \in$$

$$H_0^1(0,1) C(K, H_0^1(\Omega)) \ni h_t(\cdot) \longmapsto \int_K h_t(\cdot)dt \in H_0^1(\Omega).$$

□

Proposition 5. *For each $R > 0$, there exists $M > 0$ such that for any $a \in X$ with $|a| \leqq R$, it follows that $\int_0^1 w^2(a,0) \geqq M$.*

Proof. Lemma 3 shows that $|w(a,0)| \leqq M_1 = \frac{2}{\sqrt{R}} e^{\sqrt{R}}$. Then $|w''(a,0)| \leqq M_1$, so that $w'(a,0,t) \geqq 1 - M_1 t$ and $w(a,0,t) \geqq 1 - \frac{M_1}{2} t^2$. It follows that $|w(a,0,t)| \geqq (1 - \frac{M_1}{2} t^2)^+$. So, we can take $M = \frac{2}{3}\sqrt{\frac{2}{M_1}}$. □

3. Proof of Theorem 1. Suppose that problem (1) has the distinct solutions $u_1, u_2, \ldots$. Using Proposition 1, we may suppose that (u_n) is convergent to a solution u_0 of problem (1), and $u_n \neq u_0$, for any $n \geqq 1$.

Let $v_n,\ w_n \in H_0^1(0,1)$ be uniquely determined by

$$w_n = u_n - u_0$$

and

$$f'(v_n) = \begin{cases} \frac{f(u_n)-f(u_0)}{u_n-u_0}, & \text{in } \{u_n \neq u_0\} \\ f'(u_0), & \text{in } \{u_n = u_0\}. \end{cases}$$

An equivalent definition for v_n is

$$v_n = H(u_0, u_n - u_0), \qquad \text{where } g = f'. \tag{20}$$

Then problem (1) can be rewritten as $-w_n'' = f'(v_n) w_n$. This shows that if u_n is a solution different from u_0, then the operator $-u'' - f'(v_n)I$ is not invertible, where v_n is associated to u_n as in (20). Note that v_n is between u_n and u_0, so that $f'(v_n)$ is bounded by some $M > 0$ so that $\lambda_j(f'(v_n)) \geqq \pi^2 j^2 - M > 0$ for large j. We may thus suppose that there exists k such that, for any n,

$$\lambda_k(f'(v_n)) = 0. \tag{21}$$

This implies $\lambda_k(f'(u_0)) = 0$. As the spectrum is simple in dimension 1, it also follows from (21) that, for some a_n,

$$w_n = a_n \varphi_k(f'(v_n)). \tag{22}$$

In fact, u is a solution of (1) different from u_0 if and only if there exists $a,\ j$ and v such that

$$H(u_0, a\varphi_j(f'(v))) = v \tag{23}$$

and

$$\lambda_j(f'(v)) = 0. \tag{24}$$

We now consider (23), with $j = k$, as an equation of v with parameter a. Relation (23) is equivalent with

$$\mathcal{K}(a, v) = 0, \tag{25}$$

where $\mathcal{K}(a,v) = H_2(u_0, a\varphi_k(g(v))) - g(v)$ and $g = f'$. Notice that $\mathcal{K}(0,v) = g(u_0) - g(v)$, so that $\frac{\partial\mathcal{K}}{\partial v}(0,v)w = -g'(v)w$ is an invertible operator in $\mathcal{L}(H_0^1(0,1))$ because v is bounded, so that $g'(v)$ is bounded from above and below. This shows that for a near 0, there is a unique $v(a)$ near u_0 such that (25) is satisfied by $v = v(a)$. Moreover, the dependence on a is analytical.

Now (22) yields $\int_0^1 a_n^2\varphi_k^2(f'(v_n)) \to 0$. Proposition 5 shows that $\int_0^1 \varphi_k^2(f'(v_n))$ is bounded from below, so that $a_n \to 0$. This shows that for large n we have $v_n = v(a_n)$. Since $\lambda_k(f'(v_n)) = 0$, it follows that $\lambda_k(f'(v(a))) = 0$ for an infinity of a near 0, that is $\lambda_k(f'(v(a))) = 0$ for all a for which $v(a)$ is defined. Now (23) and (24) are satisfied with $j = k$, so that $u(a) = u_0 + a\varphi_k(f'(v(a)))$ is a solution of problem (1) for all such a.

If we examine the proof, we have shown that if u_0 is a cluster point of the set of all solutions of (1) then there exists an analytic arc $a \longmapsto v(a)$ through u_0 such that

i) $\lambda_k(f'(v(a))) = 0$, k being the same for all a.
ii) $u(a) = u_0 + a\varphi_k(f'(v(a)))$ is a solution of (1).

An equivalent statement is the existence of an analytic arc $a \longmapsto u(a)$ through u_0 such that

i)′ $u(a)$ is a solution of (1).
ii)′ $\lambda_k(f'(v(a))) = 0$, where $v(a) = H(u_0, u(a) - u_0)$.
iii)′ $u(a)$ is proportional to $\varphi_k(f'(v(a)))$.

We claim that this arc can be defined for all a. Indeed, let J be the maximal interval in which the implicit function theorem can be applied to $\mathcal{K}$. Suppose $J \neq \mathbb{R}$. We may suppose, for example, $\sup J = b_0 < \infty$. Take $b_n \nearrow b_0$, $b_n \in J$. Then $u(b_n)$ may be supposed to be convergent to and different from some "$u(b_0)$" (indeed, by ii) $u'(a,0)$ are different for different a). We may apply the reasoning made for u_0 to $u(b_0)$. The second arc obtained overlaps with the previous one in $u(b_n)$. This means that they overlap on a whole arc, contradicting by this the fact that b_0 is finite.

In view of the analyticity, ii) is true for all a. Using Proposition 5 and 1, we obtain that $\int_0^1 \varphi_k^2(f'(v(a)))$ is bounded from below. Then ii) implies $\|u(a)\|_{L^2} \to \infty$. This contradiction concludes the proof.

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