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Existence and multiplicity of solutions for double-phase Robin problems

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Abstract

We consider a double phase Robin problem with a Carathéodory nonlinearity. When the reaction is superlinear but without satisfying the Ambrosetti–Rabinowitz condition, we prove an existence theorem. When the reaction is resonant, we prove a multiplicity theorem. Our approach is Morse theoretic, using the notion of homological local linking.

1. *Introduction*

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. In this paper, we study the following two phase Robin problem

$$\left\{\begin{array}{ll} -\operatorname{div}(a_0(z)|Du|^{p-2}Du) - \Delta_q u + \xi(z)|u|^{p-2}u = f(z,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_\theta} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{1}$$

where $1 < q < p \leqslant N$.

In this problem, the weight $a_0 : \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous and $a_0(z) > 0$ for all $z \in \Omega$. The potential function $\xi \in L^\infty(\Omega)$ satisfies $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$, while the reaction term $f(z,x)$ is Carathéodory (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$ the function $x \mapsto f(z,x)$ is continuous; the abbreviation 'a.a.' stands for 'almost all'). Let $F(z,\cdot)$ be the primitive of $f(z,\cdot)$, that is, $F(z,x) = \int_0^x f(z,s)ds$. We assume that for a.a. $z \in \Omega$, $F(z,\cdot)$ is q-linear near the origin. On the other hand, near $\pm\infty$, we consider two distinct cases for $f(z,\cdot)$.

(i) For a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-superlinear but without satisfying the Ambrosetti–Rabinowitz condition (the AR-condition for short), which is common in the literature when dealing with superlinear problems.

(ii) For a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-linear and possibly resonant with respect to the principal eigenvalue of the weighted p-Laplacian

$$u \mapsto -\operatorname{div}(a_0(z)|Du|^{p-2}Du)$$

with Robin boundary condition.

In the boundary condition, $\frac{\partial u}{\partial n_\theta}$ denotes the conormal derivative of u corresponding to the modular function $\theta(z,x) = a_0(z)x^p + x^q$ for all $z \in \Omega$, all $x \geqslant 0$. We interpret this derivative via the nonlinear Green identity (see [18, p. 34]) and

$$\frac{\partial u}{\partial n_\theta} = [a_0(z)|Du|^{p-2} + |Du|^{q-2}]\,\frac{\partial u}{\partial n} \text{ for all } u \in C^1(\overline{\Omega}),$$

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with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient β satisfies $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

The differential operator in problem (1) is a weighted (p,q)-Laplace operator and it corresponds to the energy functional

$$u \mapsto \int_\Omega [a_0(z)|Du|^p + |Du|^q]dz.$$

Since we do not assume that the weight function $a_0(z)$ is bounded away from zero, the continuous integrand $\theta_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ of this integral functional exhibits unbalanced growth, namely

$$|y|^q \leqslant \theta_0(z,y) \leqslant c_0(1+|y|^p) \text{ for a.a. } z \in \Omega, \text{ all } y \in \mathbb{R}^N \text{ and some } c_0 > 0.$$

Such integral functionals were first investigated by Marcellini [14] and Zhikov [22], in connection with problems in nonlinear elasticity theory. Recently, Baroni, Colombo and Mingione [3] and Colombo and Mingione [6, 7] revived the interest in them and produced important local regularity results for the minimizers of such functionals. A global regularity theory for such problems remains elusive.

In this paper, using tools from Morse theory (in particular, critical groups), we prove an existence theorem (for the superlinear case) and a multiplicity theorem (for the linear resonant case). Existence and multiplicity results for two phase problems were proved recently by Cencelj, Rădulescu and Repovš [4] (problems with variable growth), Colasuonno and Squassina [5] (eigenvalue problems), Liu and Dai [13] (existence of solutions for problems with superlinear reaction), Papageorgiou, Rădulescu and Repovš [19] (multiple solutions for superlinear problems), and Papageorgiou, Vetro and Vetro [20] (parametric Dirichlet problems). The approach in all the aforementioned works is different and the hypotheses on the reaction are more restrictive.

Finally, we mention that (p,q)-equations arise in many mathematical models of physical processes. We refer to the very recent works of Bahrouni, Rădulescu and Repovš [1, 2] and the references therein.

2. *Mathematical background*

The study of two-phase problems requires the use of Musielak–Orlicz spaces. So, let $\theta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be the modular function defined by

$$\theta(z,x) = a_0(z)x^p + x^q \text{ for all } z \in \Omega,\ x \geqslant 0.$$

This is a generalized N-function (see Musielak [16]) and it satisfies

$$\theta(z,2x) \leqslant 2^p\theta(z,x) \text{ for all } z \in \Omega,\ x \geqslant 0,$$

that is, $\theta(z,\cdot)$ satisfies the (Δ_2)-property (see [16, p. 52]). Using the modular function $\theta(z,x)$, we can define the Musielak–Orlicz space $L^\theta(\Omega)$ as follows:

$$L^\theta(\Omega) = \left\{ u : \Omega \to \mathbb{R};\ u \text{ is measurable and } \int_\Omega \theta(z,|u|)dz < \infty \right\}.$$

This space is equipped with the so-called 'Luxemburg norm' defined by

$$\|u\|_\theta = \inf\left\{ \lambda > 0 : \int_\Omega \theta(z, \frac{|u|}{\lambda})dz \leqslant 1 \right\}.$$

Using $L^\theta(\Omega)$, we can define the following Sobolev-type space $W^{1,\theta}(\Omega)$, by setting

$$W^{1,\theta}(\Omega) = \{u \in L^\theta(\Omega) :\ |Du| \in L^\theta(\Omega)\}.$$

We equip $W^{1,\theta}(\Omega)$ with the norm $\|\cdot\|$ defined by

$$\|u\| = \|u\|_\theta + \|Du\|_\theta,$$

where $\|Du\|_\theta = \|\,|Du|\,\|_\theta$. The spaces $L^\theta(\Omega)$ and $W^{1,\theta}(\Omega)$ are separable and uniformly convex (hence reflexive) Banach spaces.

Let $\hat{\theta}(z,x)$ be another modular function. We say that '$\hat{\theta}$ is weaker than θ' and write $\hat{\theta} \prec \theta$, if there exist c_1, $c_2 > 0$ and a function $\eta \in L^1(\Omega)$ such that

$$\hat{\theta}(z,x) \leqslant c_1\,\theta(z,c_2x) + \eta(z) \text{ for a.a. } z \in \Omega \text{ and all } x \geqslant 0.$$

Then we have

$$L^\theta(\Omega) \hookrightarrow L^{\hat{\theta}}(\Omega) \text{ and } W^{1,\theta}(\Omega) \hookrightarrow W^{1,\hat{\theta}}(\Omega) \text{ continuously.}$$

Combining this fact with the classical Sobolev embedding theorem, we obtain the following embeddings; see Propositions 2.15 and 2.18 of Colasuonno and Squassina [5].

PROPOSITION 1. *We assume that $1 < q < p < \infty$. Then the following properties hold.*

(a) *If $q \neq N$, then $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ continuously for all $1 \leqslant r \leqslant q^*$, where*

$$q^* = \begin{cases} \dfrac{Nq}{N-q} & \text{if } q < N \\ +\infty & \text{if } q \geqslant N. \end{cases}$$

(b) *If $q = N$, then $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ continuously for all $1 \leqslant r < \infty$.*
(c) *If $q \leqslant N$, then $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ compactly for all $1 \leqslant r < q^*$.*
(d) *If $q > N$, then $W^{1,\theta}(\Omega) \hookrightarrow L^\infty(\Omega)$ compactly.*
(e) *$W^{1,\theta}(\Omega) \hookrightarrow W^{1,q}(\Omega)$ continuously.*

We have

$$L^p(\Omega) \hookrightarrow L^\theta(\Omega) \hookrightarrow L^p_{a_0}(\Omega) \cap L^q(\Omega)$$

with both embeddings being continuous.

We consider the modular function

$$\rho_\theta(u) = \int_\Omega \theta(z,|Du|)dz = \int_\Omega [a_0(z)|Du|^p + |Du|^q]dz \text{ for all } u \in W^{1,\theta}(\Omega).$$

There is a close relationship between the norm $\|\cdot\|$ of $W^{1,\theta}(\Omega)$ and the modular functional $\rho_\theta(\cdot)$; see Proposition 2.1 of Liu and Dai [13].

PROPOSITION 2. (a) *If $u \neq 0$, then $\|Du\|_\theta = \lambda$ if and only if $\rho_\theta(\frac{u}{\lambda}) \leqslant 1$.*
(b) *$\|Du\|_\theta < 1$ (respectively, $= 1, > 1$) if and only if $\rho_\theta(u) < 1$ (respectively, $= 1, > 1$).*
(c) *If $\|Du\|_\theta < 1$, then $\|Du\|_\theta^p \leqslant \rho_\theta(u) \leqslant \|Du\|_\theta^q$.*
(d) *If $\|Du\|_\theta > 1$, then $\|Du\|_\theta^q \leqslant \rho_\theta(u) \leqslant \|Du\|_\theta^p$.*
(e) *$\|Du\|_\theta \to 0$ if and only if $\rho_\theta(u) \to 0$.*
(f) *$\|Du\|_\theta \to +\infty$ if and only if $\rho_\theta(u) \to +\infty$.*

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the 'boundary' Lebesgue spaces $L^s(\partial\Omega)$ for $1 \leqslant s \leqslant \infty$. It is well known that there exists a unique continuous linear map $\gamma_0 : W^{1,q}(\Omega) \to L^q(\partial\Omega)$, known as the 'trace map', such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in W^{1,q}(\Omega) \cap C(\overline{\Omega}).$$

We have

$$\operatorname{im}\gamma_0 = W^{\frac{1}{q'},q}(\Omega)\ \left(\frac{1}{q}+\frac{1}{q'}=1\right) \text{ and } \ker\gamma_0 = W_0^{1,q}(\Omega).$$

Moreover, the trace map $\gamma_0(\cdot)$ is compact into $L^s(\partial\Omega)$ for all $1 \leqslant s < (N-1)q/(N-q)$ if $q < N$, and for all $1 \leqslant s < \infty$ if $q \geqslant N$. In what follows, for the sake of notational simplicity, we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of the Sobolev functions on the boundary $\partial\Omega$ are understood in the sense of traces.

Let $\langle\cdot,\cdot\rangle$ denote the duality brackets for the pair $(W^{1,\theta}(\Omega), W^{1,\theta}(\Omega)^*)$ and $\langle\cdot,\cdot\rangle_{1,q}$ denote the duality brackets for the pair $(W^{1,q}(\Omega), W^{1,q}(\Omega)^*)$. We introduce the maps $A_p^{a_0} : W^{1,\theta}(\Omega) \to W^{1,\theta}(\Omega)^*$ and $A_q : W^{1,q}(\Omega) \to W^{1,q}(\Omega)^*$ defined by

$$\langle A_p^{a_0}(u), h\rangle = \int_\Omega a_0(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}dz \text{ for all } u,h \in W^{1,\theta}(\Omega),$$

$$\langle A_q(u), h\rangle_{1,q} = \int_\Omega |Du|^{q-2}(Du, Dh)_{\mathbb{R}^N}dz \text{ for all } u,h \in W^{1,q}(\Omega).$$

We have

$$\langle A_q(u), h\rangle_{1,q} = \langle A_q(u), h\rangle \text{ for all } u,h \in W^{1,\theta}(\Omega).$$

We introduce the following hypotheses on the weight $a_0(\cdot)$ and on the coefficients $\xi(\cdot)$ and $\beta(\cdot)$.

H_0: $a_0 : \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous, $a_0(z) > 0$ for all $z \in \Omega$, $\xi \in L^\infty(\Omega)$, $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$, $\xi \not\equiv 0$ or $\beta \not\equiv 0$ and $q > Np/(N+p-1)$.

REMARK 1. The latter condition on the exponent q implies that $W^{1,\theta}(\Omega) \hookrightarrow L^p(\partial\Omega)$ compactly and $q < p^*$.

We introduce the C^1-functional $\gamma_p : W^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_p(u) = \int_\Omega a_0(z)|Du|^p dz + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \text{ for all } u \in W^{1,\theta}(\Omega).$$

Then hypotheses H_0, Lemma 4.11 of Mugnai and Papageorgiou [**15**], and Proposition 2.4 of Gasinski and Papageorgiou [**10**], imply

$$c_1\,\|u\|^p \leqslant \gamma_p(u) \text{ for some } c_1 > 0, \text{ all } u \in W^{1,\theta}(\Omega). \tag{2}$$

We denote by $\hat{\lambda}_1(p)$ the first (principal) eigenvalue of the following nonlinear eigenvalue problem

$$\left\{\begin{array}{ll} -\operatorname{div}(a_0(z)|Du|^{p-2}Du) + \xi(z)|u|^{p-2}u = \hat{\lambda}|u|^{p-2}u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{3}$$

Here, $\frac{\partial u}{\partial n_p} = |Du|^{p-2}\frac{\partial u}{\partial n}$. The eigenvalue $\hat{\lambda}_1(p)$ has the following variational characterization

$$\hat{\lambda}_1(p) = \inf\left\{\frac{\gamma_p(u)}{\|u\|_p^p} :\ u \in W^{1,p}(\Omega)\setminus\{0\}\right\} \text{ (see [17]).} \tag{4}$$

Then by (2), we see that $\hat{\lambda}_1(p) > 0$. This eigenvalue is simple (that is, if $\hat{u}$, $\hat{v}$ are corresponding eigenfunctions, then $\hat{u} = \eta\hat{v}$ with $\eta \in \mathbb{R}\setminus\{0\}$) and isolated (that is, if $\hat{\sigma}(p)$ denotes the spectrum of (3), then we can find $\varepsilon > 0$ such that $(\hat{\lambda}_1(p), \hat{\lambda}_1(p)+\varepsilon) \cap \hat{\sigma}(p) = \emptyset$). The infimum in (4) is realized on the corresponding one-dimensional eigenspace, the elements of which have fixed sign. We denote by $\hat{u}_1(p)$ the corresponding positive, L^p-normalized (that is, $\|\hat{u}_1(p)\|_p = 1$) eigenfunction. We know that $\hat{u}_1(p) \in L^\infty(\Omega)$ (see [5, Section 3.2]) and $\hat{u}_1(p)(z) > 0$ for a.a. $z \in \Omega$ (see [19, Proposition 4]).

We will also use the spectrum of the following nonlinear eigenvalue problem

$$-\Delta_q u = \hat{\lambda}|u|^{q-2}u \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

It is well known that this problem has a sequence of variational eigenvalues $\{\hat{\lambda}_k(q)\}_{k\geqslant 1}$ such that $\hat{\lambda}_k(q) \to +\infty$ as $k \to \infty$. We have $\hat{\lambda}_1(q) = 0 < \hat{\lambda}_2(q)$ (see [9, Section 6.2]).

Let X be a Banach space and $\phi \in C^1(X, \mathbb{R})$. We denote by K_ϕ the critical set of ϕ, that is,

$$K_\phi = \{u \in X : \ \phi'(u) = 0\}.$$

Also, if $\eta \in \mathbb{R}$, then we set

$$\phi^\eta = \{u \in X : \ \phi(u) \leqslant \eta\}.$$

Consider a topological pair (A, B) such that $B \subseteq A \subseteq X$. Then for every $k \in \mathbb{N}_0$, we denote by $H_k(A, B)$ the kth-singular homology group for the pair (A, B) with coefficients in a field $\mathbb{F}$ of characteristic zero (for example, $\mathbb{F} = \mathbb{R}$). Then each $H_k(A, B)$ is an $\mathbb{F}$-vector space and we denote by $\dim H_k(A, B)$ its dimension. We also recall that the homeomorphisms induced by maps of pairs and the boundary homomorphism ∂, are all $\mathbb{F}$-linear.

Suppose that $u \in K_\phi$ is isolated. Then for every $k \in \mathbb{N}_0$, we define the 'k-critical group' of ϕ at u by

$$C_k(\phi, u) = H_k(\phi^c \cap U, \phi^c \cap U \setminus \{u\}),$$

where U is an isolating neighborhood of u, that is, $K_\phi \cap U \cap \phi^c = \{u\}$. The excision property of singular homology implies that this definition is independent of the choice of the isolating neighborhood U.

We say that ϕ satisfies the 'C-condition' if it has the following property:

'Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\phi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\phi'(u_n) \to 0$ in X^* as $n \to \infty$, has a strongly convergent subsequence'.

Suppose that $\phi \in C^1(X, \mathbb{R})$ satisfies the C-condition and that $\inf \phi(K_\phi) > -\infty$. Let $c < \inf \phi(K_\phi)$. Then the critical groups of ϕ at infinity are defined by

$$C_k(\phi, \infty) = H_k(X, \phi^c) \text{ for all } k \in \mathbb{N}_0.$$

On account of the second deformation theorem (see [18, Theorem 5.3.12, p. 386]) this definition is independent of the choice of the level $c < \inf \phi(K_\phi)$.

Our approach is based on the notion of local (m, n)-linking $(m, n \in \mathbb{N})$, sees [18, Definition 6.6.13, p. 534].

DEFINITION 3. Let X be a Banach space, $\phi \in C^1(X, \mathbb{R})$, and 0 an isolated critical point of ϕ with $\phi(0) = 0$. Let $m, n \in \mathbb{N}$. We say that ϕ has a 'local (m, n)-linking' near the origin if there exist a neighborhood U of 0 and nonempty sets E_0, $E \subseteq U$, and $D \subseteq X$ such that $0 \notin E_0 \subseteq E$, $E_0 \cap D = \emptyset$ and

(a) 0 is the only critical point of ϕ in $\phi^0 \cap U$;
(b) $\dim \operatorname{im} i_* - \dim \operatorname{im} j_* \geqslant n$, where

$$i_* : H_{m-1}(E_0) \to H_{m-1}(X \setminus D) \text{ and } j_* : H_{m-1}(E_0) \to H_{m-1}(E)$$

are the homomorphisms induced by the inclusion maps $i : E_0 \to X \setminus D$ and $j : E_0 \to E$;
(c) $\phi|_E \leqslant 0 < \phi|_{U\cap D\setminus\{0\}}$.

REMARK 2. The notion of 'local (m,n)-linking' was introduced by Perera [**21**] as a generalization of the concept of local linking due to Liu [**12**]. Here we introduce a slightly more general version of this notion.

3. *The superlinear case*

In this section, we treat the superlinear case, that is, we assume that the reaction $f(z,\cdot)$ exhibits $(p-1)$-superlinear growth near $\pm\infty$.

The hypotheses on $f(z,x)$ are the following.

H_1: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$ and

(i) $|f(z,x)|\leqslant \hat{a}(z)(1+|x|^{r-1})$ for a.a. $z\in\Omega$ and all $x\in\Omega$, with $\hat{a}\in L^\infty(\Omega)$, $p<r<q^*$;

(ii) if $F(z,x)=\int_0^x f(z,s)ds$, then $\lim_{x\to\pm\infty}\frac{F(z,x)}{|x|^p}=+\infty$ uniformly for a.a. $z\in\Omega$;

(iii) if $\eta(z,x)=f(z,x)x-pF(z,x)$, then there exists $e\in L^1(\Omega)$ such that

$$\eta(z,x)\leqslant\eta(z,y)+e(z) \text{ for a.a. } z\in\Omega \text{ and all } 0\leqslant x\leqslant y \text{ or } y\leqslant x\leqslant 0;$$

(iv) there exist $\delta>0$, $\theta\in L^\infty(\Omega)$ and $\hat{\lambda}>0$ such that

$$0\leqslant\theta(z) \text{ for a.a. } z\in\Omega,\ \theta\not\equiv 0,\ \hat{\lambda}\leqslant\hat{\lambda}_2(q),$$

$$\theta(z)|x|^q\leqslant qF(z,x)\leqslant\hat{\lambda}|x|^q \text{ for a.a. } z\in\Omega \text{ and all } |x|\leqslant\delta.$$

REMARK 3. Evidently, hypotheses H_1(ii) and (iii) imply that for a.a. $z\in\Omega$, the function $f(z,\cdot)$ is superlinear. However, to express this superlinearity, we do not invoke the usual AR-condition. We recall that the AR-condition says that there exist $\tau>p$ and $M>0$ such that

$$0<\tau F(z,x)\leqslant f(z,x)x \text{ for a.a. } z\in\Omega \text{ and all } |x|\geqslant M; \text{ and} \tag{5}$$

$$0<\operatorname{essinf}_\Omega F(\cdot,\pm M). \tag{6}$$

Integrating (5) and using (6), we obtain a weaker condition, namely

$$\begin{aligned} &c_2|x|^\tau\leqslant F(z,x) \quad \text{for a.a. } z\in\Omega,\text{ all } |x|\geqslant M \text{ and some } c_2>0,\\ \Rightarrow\ &c_3|x|^\tau\leqslant f(z,x)x \quad \text{for a.a. } z\in\Omega,\text{ all } |x|\geqslant M \text{ and with } c_3=\tau c_2>0.\end{aligned}$$

Therefore, the AR-condition implies that, eventually, $f(z,\cdot)$ has at least $(\tau-1)$-polynomial growth.

In the present work, instead of the AR-condition, we use the quasimonotonicity hypothesis H_1(iii), which is less restrictive and incorporates in our framework also $(p-1)$-superlinear nonlinearities with slower growth near $\pm\infty$ (see the examples below). Hypothesis H_1(iii) is a slight generalization of a condition which can be found in Li and Yang [**11**]. There are very natural ways to verify the quasimonotonicity condition. So, if there exists $M>0$ such that for a.a. $z\in\Omega$, either the function

$$x\mapsto\frac{f(z,x)}{|x|^{q-2}x} \text{ is increasing on } x\geqslant M \text{ and decreasing on } x\leqslant -M$$

or the mapping

$$x\mapsto\eta(z,x) \text{ is increasing on } x\geqslant M \text{ and decreasing on } x\leqslant -M,$$

then hypothesis H_1(iii) holds.

Hypothesis H_1(iv) implies that for a.a. $z\in\Omega$, the primitive $F(z,\cdot)$ is q-linear near 0.

Examples. The following functions satisfy hypotheses H_1. For the sake of simplicity, we drop the z-dependence:

$$f_1(x) = \begin{cases} \mu|x|^{q-2}x & \text{if } |x| \leqslant 1 \\ \mu|x|^{r-2}x & \text{if } |x| > 1 \ \ (\text{with } 0 < \mu \leqslant \hat{\lambda}_2(q) \text{ and } p < r < q^*) \end{cases}$$

$$f_2(x) = \begin{cases} \mu|x|^{q-2}x & \text{if } |x| \leqslant 1 \\ \mu|x|^{p-2}x \ln x + \mu|x|^{\tau-2}x & \text{if } |x| > 1 \ \ (\text{with } 0 < \mu \leqslant \hat{\lambda}_2(q) \text{ and } 1 < \tau < p). \end{cases}$$

Note that only f_1 satisfies the AR-condition, whereas the function f_2 does not satisfy this growth condition.

The energy functional for problem (1) is the C^1-functional $\varphi : W^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{p}\,\gamma_p(u) + \frac{1}{q}\,\|Du\|_q^q - \int_\Omega F(z,u)dz \text{ for all } u \in W^{1,\theta}(\Omega).$$

Next, we show that $\varphi(\cdot)$ satisfies the C-condition.

PROPOSITION 4. *If hypotheses H_0, H_1 hold, then the functional $\varphi(\cdot)$ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,\theta}(\Omega)$ such that

$$|\varphi(u_n)| \leqslant c_4 \text{ for some } c_4 > 0 \text{ and all } n \in \mathbb{N}, \tag{7}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W^{1,\theta}(\Omega)^* \text{ as } n \to \infty. \tag{8}$$

From (8), we have

$$\begin{aligned} \Big| \langle A_p^{a_0}(u_n), h\rangle &+ \langle A_q(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n h dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n h d\sigma \\ &- \int_\Omega f(z,u_n)hdz \Big| \leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|}, \end{aligned} \tag{9}$$

for all $h \in W^{1,\theta}(\Omega)$, with $\varepsilon_n \to 0$.

In (9), we choose $h = u_n \in W^{1,\theta}(\Omega)$ and obtain for all $n \in \mathbb{N}$

$$-\int_\Omega a_0(z)|Du_n|^p dz - \|Du_n\|_q^q - \int_\Omega \xi(z)|u_n|^p dz - \int_{\partial\Omega} \beta(z)|u_n|^p d\sigma + \int_\Omega f(z,u_n)u_n dz \leqslant \varepsilon_n. \tag{10}$$

Also, by (7) we have for all $n \in \mathbb{N}$,

$$\begin{aligned} &\int_\Omega a_0(z)|Du_n|^p dz + \frac{p}{q}\,\|Du_n\|_q^q + \frac{p}{q}\int_\Omega \xi(z)|u_n|^p dz \\ &\quad + \frac{p}{q}\int_{\partial\Omega} \beta(z)|u_n|^p d\sigma - \int_\Omega pF(z,u_n)dz \leqslant pc_4. \end{aligned} \tag{11}$$

We add relations (10) and (11). Since $q < p$, we obtain

$$\int_\Omega \eta(z,u_n)dz \leqslant c_5 \text{ for some } c_5 > 0 \text{ and all } n \in \mathbb{N}. \tag{12}$$

Claim. The sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,\theta}(\Omega)$ is bounded.

We argue by contradiction. Suppose that the claim is not true. We may assume that

$$\|u_n\| \to \infty \text{ as } n \to \infty. \tag{13}$$

We set $y_n = u_n/\|u_n\|$ for all $n \in \mathbb{N}$. Then $\|y_n\| = 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,\theta}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega), \tag{14}$$

see hypotheses H_0, Proposition 1, and Remark 1.

We first assume that $y \not\equiv 0$. Let

$$\Omega_+ = \{z \in \Omega : \ y(z) > 0\} \text{ and } \Omega_- = \{z \in \Omega : \ y(z) < 0\}.$$

Then at least one of these measurable sets has positive Lebesgue measure on $\mathbb{R}^N$. We have

$$u_n(z) \to +\infty \text{ for a.a. } z \in \Omega_+ \text{ and } u_n(z) \to -\infty \text{ for a.a. } z \in \Omega_-.$$

Let $\hat{\Omega} = \Omega_+ \cup \Omega_-$ and let $|\cdot|_N$ denote the Lebesgue measure on $\mathbb{R}^N$. We see that $|\hat{\Omega}|_N > 0$ and on account of hypothesis H_1(ii), we have

$$\begin{aligned} &\frac{F(z,u_n(z))}{\|u_n\|^p} = \frac{F(z,u_n(z))}{|u_n(z)|^p}\,|y_n(z)|^p \to +\infty \text{ for a.a. } z \in \hat{\Omega}, \\ \Rightarrow\ &\int_{\hat{\Omega}} \frac{F(z,u_n(z))}{\|u_n\|^p} dz \to +\infty \text{ by Fatou's lemma.} \end{aligned} \tag{15}$$

Hypotheses H_1(i) and (ii) imply

$$F(z,x) \geqslant -c_6 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_6 > 0. \tag{16}$$

Thus we obtain

$$\begin{aligned} \int_\Omega \frac{F(z,u_n)}{\|u_n\|^p} dz &= \int_{\hat{\Omega}} \frac{F(z,u_n)}{\|u_n\|^p} dz + \int_{\Omega\setminus\hat{\Omega}} \frac{F(z,u_n)}{\|u_n\|^p} dz \\ &\geqslant \int_{\hat{\Omega}} \frac{F(z,u_n)}{\|u_n\|^p} dz - \frac{c_6|\Omega|_N}{\|u_n\|^p} \text{ (see (16))}, \\ &\Rightarrow \lim_{n\to\infty} \int_\Omega \frac{F(z,u_n)}{\|u_n\|^p} dz = +\infty \text{ (see (15) and (13)).} \end{aligned} \tag{17}$$

By (7), we have

$$\int_\Omega \frac{pF(z,u_n)}{\|u_n\|^p} dz \leqslant \gamma_p(y_n) + \frac{p}{q}\,\frac{1}{\|u_n\|^{p-q}}\,\|Dy_n\|_q^q + \frac{c_4}{\|u_n\|^p} \leqslant c_7, \tag{18}$$

for some $c_7 > 0$ and all $n \in \mathbb{N}$ (see (13) and recall that $\|y_n\| = 1$).

We compare relations (15) and (18) and arrive at a contradiction.

Next, we assume that $y = 0$. Let $\mu > 0$ and set $v_n = (p\mu)^{1/p} y_n$ for all $n \in \mathbb{N}$. Evidently, we have

$$\begin{aligned} &v_n \to 0 \text{ in } L^r(\Omega) \text{ (see (14))}, \\ \Rightarrow &\int_\Omega F(z,v_n)dz \to 0 \text{ as } n \to \infty. \end{aligned} \tag{19}$$

Consider the functional $\psi : W^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\,\gamma_p(u) - \int_\Omega F(z,u)dz \text{ for all } u \in W^{1,\theta}(\Omega).$$

Clearly, $\psi \in C^1(W^{1,\theta}(\Omega), \mathbb{R})$ and

$$\psi \leqslant \varphi. \tag{20}$$

We can find $t_n \in [0,1]$ such that

$$\psi(t_n u_n) = \min\{\psi(tu_n) : \ 0 \leqslant t \leqslant 1\} \text{ for all } n \in \mathbb{N}. \tag{21}$$

Because of (13), we can find $n_0 \in \mathbb{N}$ such that

$$0 < \frac{(p\mu)^{1/p}}{\|u_n\|} \leqslant 1 \text{ for all } n \geqslant n_0. \tag{22}$$

Therefore

$$\begin{aligned}
\psi(t_n u_n) &\geqslant \psi(v_n) \text{ (see (21), (22))} \\
&\geqslant \mu\gamma_p(y_n) - \int_\Omega F(z, v_n)dz \\
&\geqslant \mu c_1 - \int_\Omega F(z, v_n)dz \text{ (see (2) and recall that } \|y_n\| = 1) \\
&\geqslant \frac{\mu}{2} c_1 \text{ for all } n \geqslant n_1 \geqslant n_0 \text{ (see (19))}.
\end{aligned}$$

Since $\mu > 0$ is arbitrary, it follows that

$$\psi(t_n u_n) \to +\infty \text{ as } n \to \infty. \tag{23}$$

Note that

$$\psi(0) = 0 \text{ and } \psi(u_n) \leqslant c_4 \text{ for all } n \in \mathbb{N} \text{ (see (7), (20))}. \tag{24}$$

By (23) and (24), we can infer that

$$t_n \in (0,1) \text{ for all } n \geqslant n_2. \tag{25}$$

From (21) and (25), we can see that for all $n \geqslant n_2$ we have

$$\begin{aligned}
0 &= t_n \frac{d}{dt}\psi(tu_n)|_{t=t_n} \\
&= \langle \psi'(t_n u_n), t_n u_n \rangle \text{ (by the chain rule)} \\
&= \gamma_p(t_n u_n) - \int_\Omega f(z, t_n u_n)(t_n u_n)dz.
\end{aligned} \tag{26}$$

It follows that

$$0 \leqslant t_n u_n^+ \leqslant u_n^+ \text{ and } \ -u_n^- \leqslant -t_n u_n^- \leqslant 0 \text{ for all } n \in \mathbb{N}$$

(recall that $u_n^+ = \max\{u_n, 0\}$ and $u_n^- = \max\{-u_n, 0\}$).

By hypothesis H_1(iii), we have

$$\eta(z, t_n u_n^+) \leqslant \eta(z, u_n^+) + e(z) \text{ for a.a. } z \in \Omega \text{ and all } n \in \mathbb{N},$$
$$\eta(z, -t_n u_n^-) \leqslant \eta(z, -u_n^-) + e(z) \text{ for a.a. } z \in \Omega \text{ and all } n \in \mathbb{N}.$$

From these two inequalities and since $u_n = u_n^+ - u_n^-$, we obtain

$$\begin{aligned}
&\eta(z, t_n u_n) \leqslant \eta(z, u_n) + e(z) \text{ for a.a. } z \in \Omega \text{ and all } n \in \mathbb{N}, \\
\Rightarrow\ & f(z, t_n u_n)(t_n u_n) \leqslant \eta(z, u_n) + e(z) + pF(z, t_n u_n) \text{ for a.a. } z \in \Omega \text{ and all } n \in \mathbb{N}.
\end{aligned} \tag{27}$$

We return to (26) and apply (27). Then

$$\begin{aligned}
&\gamma_p(t_n u_n) - p\int_\Omega F(z, t_n u_n)dz \leqslant \int_\Omega \eta(z, u_n)dz + \|e\|_1 \text{ for all } n \in \mathbb{N}, \\
\Rightarrow\ & p\psi(t_n u_n) \leqslant c_8 \text{ for some } c_8 > 0 \text{ and all } n \in \mathbb{N} \text{ (see (12)}.
\end{aligned} \tag{28}$$

We compare (23) and (28) and arrive at a contradiction.

This proves the Claim.

On account of this claim, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,\theta}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega) \tag{29}$$

(see hypotheses H_0).

From (29), we have

$$Du_n \to Du \text{ in } L^p_{a_0}(\Omega, \mathbb{R}^N) \quad \text{and} \quad Du_n(z) \to Du(z) \text{ a.a. } z \in \Omega. \tag{30}$$

In (9), we choose $h = u_n - u \in W^{1,\theta}(\Omega)$, pass to the limit as $n \to \infty$ and use (30) and the monotonicity of $A_p(\cdot)^{a_0}$. We obtain

$$\begin{aligned} &\limsup_{n\to\infty} \langle A_p^{a_0}(u_n), u_n - u\rangle \leqslant 0, \\ \Rightarrow &\limsup_{n\to\infty} \|Du_n\|_{L^p_{a_0}(\Omega,\mathbb{R}^N)} \leqslant \|Du\|_{L^p_{a_0}(\Omega,\mathbb{R}^N)}. \end{aligned}$$

On the other hand, from (30), we have

$$\liminf_{n\to\infty} \|Du_n\|_{L^p_{a_0}(\Omega,\mathbb{R}^N)} \geqslant \|Du\|_{L^p_{a_0}(\Omega,\mathbb{R}^N)}.$$

Therefore, we conclude that

$$\|Du_n\|_{L^p_{a_0}(\Omega,\mathbb{R}^N)} \to \|Du\|_{L^p_{a_0}(\Omega,\mathbb{R}^N)}. \tag{31}$$

The space $L^p_{a_0}(\Omega, \mathbb{R}^N)$ is uniformly convex, hence it has the Kadec–Klee property (see s [**18**, Remark 2.7.30, p. 127]). So, it follows from (30) and (31) that

$$\begin{aligned} &Du_n \to Du \text{ in } L^p_{a_0}(\Omega, \mathbb{R}^N), \\ \Rightarrow\ &Du_n \to Du \text{ in } L^q(\Omega, \mathbb{R}^N) \text{ since } L^p_{a_0}(\Omega, \mathbb{R}^N) \hookrightarrow L^q(\Omega, \mathbb{R}^N) \text{ continuously}, \\ \Rightarrow\ &\rho_\theta(|Du_n - Du|) \to 0 \text{ (see Proposition 2)}, \\ \Rightarrow\ &\|u_n - u\| \to 0 \text{ (see (29) and Proposition 2)}, \\ \Rightarrow\ &\varphi \text{ satisfies the C-condition.} \end{aligned}$$

The proof is now complete. ☐

PROPOSITION 5. *If hypotheses H_0, H_1 hold, then the functional $\varphi(\cdot)$ has a local (1,1)-linking at 0.*

Proof. Since the critical points of φ are solutions of problem (1), we may assume that K_φ is finite or otherwise we already have infinitely many nontrivial solutions of (1) and so we are done.

Choose $\rho \in (0, 1)$ so small that $K_\varphi \cap \bar{B}_\rho = \{0\}$ (here, $B_\rho = \{u \in W^{1,\theta}(\Omega) : \ \|u\| < \rho\}$). Let $V = \mathbb{R}$ and let $\delta > 0$ as postulated by hypothesis H_1(iv). Recall that on a finite-dimensional normed space all norms are equivalent. So, by taking $\rho \in (0, 1)$ even smaller as necessary, we have

$$\|u\| \leqslant \rho \Rightarrow |u| \leqslant \delta \text{ for all } u \in V = \mathbb{R}. \tag{32}$$

Then for $u \in V \cap \bar{B}_\rho$, we have

$$\begin{aligned} \varphi(u) &\leqslant \frac{1}{p}\,\gamma_p(u) - \frac{|u|^q}{q} \int_\Omega \theta(z)dz \text{ (see (32) and Hypothesis } H_1(iv)) \\ &= \frac{|u|^p}{p}\left(\int_\Omega \xi(z)dz + \int_{\partial\Omega} \beta(z)d\sigma\right) - \frac{|u|^q}{q}\int_\Omega \theta(z)dz \\ &\leqslant c_9\|u\|^p - c_{10}\|u\|^q \text{ for some } c_9, c_{10} > 0 \text{ (see hypotheses } H_0 \text{ and } H_1\text{(iv))}. \end{aligned}$$

Since $q < p$, choosing $\rho \in (0,1)$ small, we conclude that

$$\varphi|_{V \cap \bar{B}_\rho} \leqslant 0. \tag{33}$$

Let

$$D = \{u \in W^{1,\theta}(\Omega) : \ \|Du\|_q^q \geqslant \hat{\lambda}_2(q)\|u\|_q^q\}.$$

For all $u \in D$, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\,\gamma_p(u) + \frac{1}{q}\,\|Du\|_q^q - \int_{\{|u|\leqslant\delta\}} F(z,u)dz - \int_{\{|u|>\delta\}} F(z,u)dz \\
&\geqslant \frac{1}{p}\,\gamma_p(u) + \frac{1}{q}\left(\|Du\|_q^q - \int_\Omega \hat{\lambda}|u|^q dz\right) - \int_\Omega F(z,u)dz \\
&\qquad \text{(see hypotheses } H_1(iv)) \\
&\geqslant \frac{1}{p}\,\gamma_p(u) + \frac{1}{q}\int_\Omega (\hat{\lambda}_2(q) - \hat{\lambda})|u|^q dz - c_{11}\|u\|^r \\
&\qquad \text{for some } c_{11} > 0 \text{ (since } u \in D \text{ and see hypothesis } H_1(iv)) \\
&\geqslant \frac{c_{11}}{p}\,\|u\|^p - c_{11}\,\|u\|^r \text{ (see (22))}.
\end{aligned}$$

Since $p < r$, for sufficiently small $\rho \in (0,1)$, we have

$$\varphi|_{D \cap \bar{B}_\rho \setminus \{0\}} > 0. \tag{34}$$

Let $U = \bar{B}_\rho$, $E_0 = V \cap \partial B_\rho$, $E = V \cap \bar{B}_\rho$ and D as above. We have $0 \notin E_0$, $E_0 \subseteq E \subseteq U = \bar{B}_\rho$ and $E_0 \cap D = \emptyset$ (see Definition 3).

Let Y be the topological complement of V. We have that

$$W^{1,\theta}(\Omega) = V \oplus Y \text{ (see [18, pp. 73, 74])}.$$

So, every $u \in W^{1,\theta}(\Omega)$ can be written in a unique way as

$$u = v + y \text{ with } v \in V,\ y \in Y.$$

We consider the deformation $h : [0,1] \times (W^{1,\theta}(\Omega) \setminus D) \to W^{1,\theta}(\Omega) \setminus D$ defined by

$$h(t,u) = (1-t)u + t\rho\,\frac{v}{\|v\|} \text{ for all } t \in [0,1]\ ,\ u \in W^{1,\theta}(\Omega) \setminus D.$$

We have

$$h(0,u) = u \text{ and } h(1,u) = \rho\,\frac{v}{\|v\|} \in V \cap \partial B_\rho = E_0.$$

It follows that E_0 is a deformation retract of $W^{1,\theta}(\Omega) \setminus D$ (see [**17**, Definition 5.3.10, p. 385]). Hence,

$$i_* : H_0(E_0) \to H_0(W^{1,\theta}(\Omega) \setminus \{0\})$$

is an isomorphism (see [**8**, Theorem 11.5, p.30] and [**18**, Remark 6.1.6, p. 460]).

The set $E = V \cap B_\rho$ is contractible (it is an interval). Hence $H_0(E, E_0) = 0$ (see [**8**, Theorem 11.5, p. 30]). Therefore, if $j_* : H_0(E_0) \to H_0(E)$, then $\dim \operatorname{im} j_* = 1$ (see [**8**, Remark 6.1.26, p. 468]). So, finally we have

$$\begin{aligned}
&\dim \operatorname{im} i_* - \dim \operatorname{im} j_* = 2 - 1 = 1, \\
\Rightarrow\ &\varphi(\cdot) \text{ has a local (1,1)-linking at 0, see Definition 3.}
\end{aligned}$$

The proof is now complete. ∎

By Proposition 5 and Theorem 6.6.17 of Papageorgiou, Rădulescu and Repovš [18, p. 538], we have

$$\dim C_1(\varphi, 0) \geqslant 1. \tag{35}$$

Moreover, Proposition 3.9 of Papageorgiou, Rădulescu and Repovš [17] leads to the following result.

PROPOSITION 6. *If hypotheses H_0, H_1 hold, then $C_k(\varphi, \infty) = 0$ for all $k \in \mathbb{N}_0$.*

We are now ready for the existence theorem concerning the superlinear case.

THEOREM 7. *If hypotheses H_0, H_1 hold, then problem (1) has a nontrivial solution $u_0 \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega)$.*

Proof. On account of (35) and Proposition 6, we can apply Proposition 6.2.42 of Papageorgiou, Rădulescu and Repovš [18, p. 499]. So, we can find $u_0 \in W^{1,\theta}(\Omega)$ such that

$$\begin{aligned} &u_0 \in K_\varphi \setminus \{0\}, \\ \Rightarrow\ &u_0 \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega) \text{ is a solution of problem (1), see [18, Section 3.2].} \end{aligned}$$

The proof is now complete. □

4. *The resonant case*

In this section, we are concerned with the resonant case (p-linear case). Our hypotheses allow resonance at $\pm\infty$ with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$.

The new conditions on the reaction $f(z, x)$ are the following.

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leqslant \hat{a}(z)(1 + |x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\hat{a} \in L^\infty(\Omega)$, $p < r < q^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x\to\pm\infty} pF(z, x)/|x|^p \leqslant \hat{\lambda}_1(p)$ uniformly for a.a. $z \in \Omega$;

(iii) we have

$$f(z, x)x - pF(z, x) \to +\infty \text{ uniformly for a.a. } z \in \Omega, \text{ as } x \to \pm\infty;$$

(iv) there exist $\delta > 0$, $\theta \in L^\infty(\Omega)$ and $\hat{\lambda} > 0$ such that

$$0 \leqslant \theta(z) \text{ for a.a. } z \in \Omega,\ \theta \not\equiv 0,\ \hat{\lambda} \leqslant \hat{\lambda}_2(q),$$

$$\theta(z)|x|^q \leqslant qF(z, x) \leqslant \hat{\lambda}|x|^q \text{ for a.a. } z \in \Omega \text{ and all } |x| \leqslant \delta.$$

REMARK 4. Hypothesis H_2(ii) implies that at $\pm\infty$, we can have resonance with respect to the principal eigenvalue of the operator $u \mapsto -\mathrm{div}\,(a_0(z)|Du|^{p-2}Du)$ with Robin boundary condition.

PROPOSITION 8. *If hypotheses H_0, H_2 hold, then the energy functional $\varphi(\cdot)$ is coercive.*

Proof. We have

$$\begin{aligned} \frac{d}{dx}\left(\frac{F(z, x)}{|x|^p}\right) &= \frac{f(z, x)|x|^p - p|x|^{p-2}xF(z, x)}{|x|^{2p}} \\ &= \frac{|x|^{p-2}x[f(z, x)x - pF(z, x)]}{|x|^{2p}}. \end{aligned}$$

On account of hypothesis H_2(iii), given any $\gamma > 0$, we can find $M_1 = M_1(\gamma) > 0$ such that

$$f(z,x)x - pF(z,x) \geqslant \gamma \text{ for a.a. } z \in \Omega \text{ and all } |x| \geqslant M_1.$$

Hence, we obtain

$$\frac{d}{dx}\left(\frac{F(z,x)}{|x|^p}\right)\begin{cases} \geqslant \dfrac{\gamma}{x^{p+1}} & \text{if } x \geqslant M_1 \\ \leqslant -\dfrac{\gamma}{|x|^{p+1}} & \text{if } x \leqslant -M_1. \end{cases}$$

Integrating, we obtain

$$\frac{F(z,x)}{|x|^p} - \frac{F(z,x)}{|u|^p} \geqslant -\frac{\gamma}{p}\left(\frac{1}{|x|^p} - \frac{1}{|u|^p}\right) \text{ for a.a. } z \in \Omega \text{ and all } |x| \geqslant |u| \geqslant M_1. \tag{36}$$

On account of hypothesis $H_2(ii)$, given $\varepsilon > 0$, we can find $M_2 = M_2(\varepsilon) > 0$ such that

$$F(z,x) \leqslant \frac{1}{p}(\hat{\lambda}_1(p) + \varepsilon)|x|^p \text{ for a.a. } z \in \Omega \text{ and all } |x| \geqslant M_2.$$

Using this inequality in (36) and letting $|x| \to \infty$, we obtain

$$\begin{aligned} &\frac{1}{p}(\hat{\lambda}_1(p) + \varepsilon) - \frac{F(z,u)}{|u|^p} \geqslant \frac{\gamma}{p}\frac{1}{|u|^p} \text{ for a.a. } z \in \Omega \text{ and all } |u| \geqslant M = \max\{M_1, M_2\}, \\ &\Rightarrow (\hat{\lambda}_1(p) + \varepsilon)|u|^p - pF(z,u) \geqslant \gamma \text{ for a.a. } z \in \Omega \text{ and all } |u| \geqslant M. \end{aligned} \tag{37}$$

Arguing by contradiction, suppose that $\varphi(\cdot)$ is not coercive. Then we can find $\{u_n\}_{n\geqslant 1} \subseteq W^{1,\theta}(\Omega)$ such that

$$\|u_n\| \to \infty \text{ and } \varphi(u_n) \leqslant M_0 \text{ for some } M_0 > 0 \text{ and all } n \in \mathbb{N}. \tag{38}$$

Let $y_n = u_n/\|u_n\|$ for all $n \in \mathbb{N}$. Then $\|y_n\| = 1$, hence we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,\theta}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{39}$$

From (38), we have

$$\begin{aligned} &\frac{1}{p}\gamma_p(y_n) + \frac{1}{q}\frac{1}{\|u_n\|^{p-q}}\int_\Omega |Dy_n|^q dz - \int_\Omega \frac{F(z,u_n)}{\|u_n\|^p}dz \leqslant \frac{M_0}{\|u_n\|^p}, \\ &\Rightarrow \gamma_p(y_n) + \frac{p}{q}\frac{1}{\|u_n\|^{p-q}}\int_\Omega |Dy_n|^q dz \leqslant \tau_n + (\hat{\lambda}_1(p) + \varepsilon)\|y_n\|_p^p \text{ with } \tau_n \to 0, \text{ see (37)}, \\ &\Rightarrow \gamma_p(y) \leqslant (\hat{\lambda}_1(p) + \varepsilon)\|y\|^p \text{ (see (39))}, \\ &\Rightarrow \gamma_p(y) \leqslant \hat{\lambda}_1(p)\|y\|_p^p \text{ (since } \varepsilon > 0 \text{ is arbitrary)}, \\ &\Rightarrow y = \mu\hat{u}_1(p) \text{ for some } \mu \in \mathbb{R} \text{ (see (4))}. \end{aligned}$$

If $\mu = 0$, then $y = 0$ and so $\gamma_p(y_n) \to 0$. Hence, as in the proof of Proposition 4, we have $y_n \to 0$ in $W^{1,\theta}(\Omega)$, contradicting the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

So, $\mu \neq 0$ and since $\hat{u}_1(p)(z) > 0$ for a.a. $z \in \Omega$, we have $|u_n(z)| \to +\infty$ for a.a. $z \in \Omega$. By (38) and (4), we have

$$\int_\Omega \left[\frac{1}{p}\hat{\lambda}_1(p)|u_n|^p - F(z,u_n)\right]dz \leqslant M_0 \text{ for all } n \in \mathbb{N}. \tag{40}$$

However, from (37) and since $\gamma > 0$ is arbitrary, we can infer that

$$\begin{aligned} &\frac{1}{p}\hat{\lambda}_1(p)|u_n|^p - F(z,u_n) \to +\infty \text{ for a.a. } z \in \Omega, \text{ as } n \to \infty, \\ &\Rightarrow \int_\Omega \left[\frac{1}{p}\hat{\lambda}_1(p)|u_n|^p - F(z,u_n)\right]dz \to +\infty \text{ by Fatou's lemma.} \end{aligned} \tag{41}$$

Comparing (40) and (41), we arrive at a contradiction. Therefore, we can conclude that $\varphi(\cdot)$ is coercive. □

Using Proposition 8 and Proposition 5.1.15 of Papageorgiou, Rădulescu and Repovš [**18**, p. 369], we obtain the following result.

COROLLARY 9. *If hypotheses* H_0, H_2 *hold, then the energy functional* $\varphi(\cdot)$ *is bounded below and satisfies the C-condition.*

Now we are ready for the multiplicity theorem in the resonant case.

THEOREM 10. *If hypotheses* H_0, H_2 *hold, then problem* (1) *has at least two nontrivial solutions* u_0, $\hat{u} \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega)$.

Proof. By Proposition 5, we know that $\varphi(\cdot)$ has a local (1,1)-linking at the origin. Note that for that result mattered only the behavior of $f(z,\cdot)$ near zero and this is common in hypotheses H_1 and H_2. Also, we know that $\varphi(\cdot)$ is sequentially weakly lower semicontinuous. This fact in conjunction with Proposition 8, permit the use of the Weierstrass–Tonelli theorem. So, we can find $u_0 \in W^{1,\theta}(\Omega)$ such that

$$\varphi(u_0) = \min\{\varphi(u) : \ u \in W^{1,\theta}(\Omega)\}. \tag{42}$$

On account of hypothesis $H_2(iv)$ and since $q < p$, we have

$$\begin{aligned} &\varphi(u_0) < 0 = \varphi(0), \\ \Rightarrow\ & u_0 \neq 0 \text{ and } u_0 \in K_\varphi, \\ \Rightarrow\ & u_0 \in K_\varphi \cap L^\infty(\Omega) \text{ is a nontrivial solution of (1).} \end{aligned}$$

Moreover, by Corollary 6.7.10 of Papageorgiou, Rădulescu and Repovš [**18**, p. 552], we can find $\hat{u} \in K_\varphi$, $\hat{u} \notin \{0, u_0\}$. Then $\hat{u} \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega)$ is the second nontrivial solution of problem (1). □

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