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EXISTENCE RESULTS FOR HEMIVARIATIONAL INEQUALITIES INVOLVING RELAXED $\eta - \alpha$ MONOTONE MAPPINGS

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ABSTRACT. We establish some existence results for hemivariational inequalities with relaxed $\eta - \alpha$ monotone mappings on bounded, closed and convex subsets in reflexive Banach spaces. Our proofs rely essentially on a fixed point theorem for set valued mappings which is due to Tarafdar [20]. We also give a sufficient condition for the existence of solutions in the case of unbounded subsets.

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1. INTRODUCTION

Inequality theory plays an important role in many fields, such as mechanics, engineering sciences, economics, optimal control, etc. Because of their wide applicability, inequality problems have become an important area of investigation in the past several decades, an important part of this research focusing on the existence of the solutions. Inequality problems can be divided into two main classes: that of variational inequalities and that of hemivariational inequalities. The study of variational inequality problems began in the early sixties with the pioneering work of G. Fichera [4], J. L. Lions and G. Stampacchia [8]. The most basic result is due to Hartman and Stampacchia [5], which states that if X is a finite dimensional Banach space, $K \subset X$ is compact and convex, and A is a continuous operator, then the variational inequality problem of finding $u \in K$ such that

$$\langle Au, v-u \rangle \geq 0, \quad \forall v \in K \tag{1.1}$$

has a solution. When K is not compact, or X is infinite dimensional, certain monotonicity properties are required to prove the existence of solution. In the last years, a

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number of authors have introduced several important generalizations of monotonicity, such as pseudomonotonicity, quasimonotonicity, semimonotonicity, relaxed monotonicity, relaxed $\eta - \alpha$ monotonicity, etc; see for example [1, 3, 6, 7, 21] and the references therein.

In 1999, Panagiotopoulos, Fundo and Rădulescu [17] extended the classical results from [5], proving several versions of theorems of Hartman-Stampacchia's type for the case of hemivariational inequalities on compact or on closed and convex subsets, in infinite and finite dimensional Banach spaces.

By replacing the subdifferential of a convex function by the generalized gradient (in the sense of F. H. Clarke) of a locally Lipschitz functional, hemivariational inequalities arise whenever the energetic functional associated to a concrete problem is nonconvex. This new type of inequalities appears as a generalization of the variational inequalities, but hemivariational inequalities are much more general, in the sense that they are not equivalent to minimum problems but, give rise to substationarity problems. The theory of hemivariational inequalities can be viewed as a new field of Nonsmooth Mechanics since the main ingredient used in the study of these inequalities is the notion of Clarke subdifferential of a locally Lipschitz functional. The mathematical theory hemivariational inequalities, as well as their applications in Mechanics, Engineering or Economics were introduced and developed by P. D. Panagiotopoulos [13]–[16] in the case of nonconvex energy functions. For a treatment of this theory and further comments we recommend the monographs by Z. Naniewicz and P. D. Panagiotopoulos [12], D. Motreanu and P. D. Panagiotopoulos [9], D. Motreanu and V. Rădulescu [10, 11], and V. Rădulescu [18, 19].

Inspired and motivated by [17], in this paper we propose an extension of the results of Panagiotopoulos, Fundo and Rădulescu to a more general framework, that of *relaxed* $\eta - \alpha$ *monotone mappings*. Using a fixed point theorem for set valued mappings due to Tarafdar, some existence results are established.

2. THE ABSTRACT FRAMEWORK

Throughout this paper X will denote a real reflexive Banach space with its dual space X^*, $T : X \to L^p(\Omega; \mathbb{R}^k)$ will be a linear and compact operator where $1 < p < \infty$ and Ω is a bounded and open subset of $\mathbb{R}^N$. We shall denote $Tu = \hat{u}$ and by p' the conjugated exponent of p. Let K be a nonempty subset of X (in order to simplify some computations we shall assume that $0 \in K$), $A : K \to X^*$ a nonlinear operator and $j = j(x, y) : \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a Carathéodory function, locally Lipschitz with respect to the second variable which satisfies the following condition:

-there exist $g_1 \in L^{p/(p-1)}(\Omega; \mathbb{R})$ and $g_2 \in L^\infty(\Omega; \mathbb{R})$ such that

$$|z| \le g_1(x) + g_2(x)|y|^{p-1} \tag{2.1}$$

a.e. $x \in \Omega$, for all $y \in \mathbb{R}^k$ and all $z \in \partial j(x, y)$.

We shall use the notation $j^0(x, y; h)$ for the Clarke's generalized directional derivative (see e.g. [2] or [11]) of the locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to the direction $h \in \mathbb{R}^k$, where $x \in \Omega$, that is,

$$j^0(x, y; h) = \limsup_{\substack{w \to y \\ \lambda \downarrow 0}} \frac{j(x, w + \lambda h) - j(x, w)}{\lambda}.$$

Accordingly, Clarke's generalized gradient $\partial j(x, y)$ of the locally Lipschitz map $j(x, \cdot)$ is defined by

$$\partial j(x, y) = \{z \in \mathbb{R}^k : \; z \cdot h \le j^0(x, y; h), \text{ for all } h \in \mathbb{R}^k\},$$

where the symbol "$\cdot$" means the inner product on $\mathbb{R}^k$.

The euclidian norm in $\mathbb{R}^k$, $k \ge 1$, resp. the duality pairing between a Banach space and its dual will be denoted by $|\cdot|$, respectively $\langle \cdot, \cdot \rangle$. We also denote by $\|\cdot\|_p$ the norm in the space $L^p(\Omega; \mathbb{R}^k)$ defined by

$$\|\hat{u}\|_p = \left(\int_\Omega |\hat{u}(x)|^p \, dx \right)^{1/p}, \quad 1 < p < \infty.$$

Definition 2.1. A mapping $A : K \to X^*$ is said to be relaxed $\eta - \alpha$ monotone on K(see [3]) if there exists a mapping $\eta : K \times K \to X$ and a function $\alpha : X \to \mathbb{R}$ with $\alpha(tw) = t^q \alpha(w)$ for all $t > 0$, all $w \in X$ and $q > 1$ a constant, such that

$$\langle Av - Au, \eta(v, u) \rangle \ge \alpha(v - u), \quad \forall u, v \in K. \tag{2.2}$$

Special cases:

1. If $\eta(v, u) = v - u$ for all $u, v \in K$ and $\alpha \equiv 0$ then (2.2) becomes
$$\langle Av - Au, v - u \rangle \ge 0, \quad \forall u, v \in K,$$
and A is said to be *monotone*.
2. If $\eta(v, u) = v - u$ for all $u, v \in K$ and $\alpha(w) = -\mu \|w\|^2$, $\mu > 0$ a constant, then (2.2) becomes
$$\langle Av - Au, v - u \rangle \ge -\mu \|v - u\|^2, \quad \forall u, v \in K,$$
and A is said to be *relaxed monotone*.
3. If $\eta(v, u) = v - u$ for all $u, v \in K$, then (2.2) becomes
$$\langle Av - Au, v - u \rangle \ge \alpha(v - u), \quad \forall u, v \in K,$$
and A is said to be *relaxed α monotone*.

Example 2.2. Let $K = [-1, 1]$, $Au = u^2$ and $\eta(v, u) = u^2 - v^2$. Then A is relaxed $\eta - \alpha$ monotone on K with $\alpha(w) = -4w^2$. It is easy to check that A is not monotone on K.

Example 2.3. Let $K = [0, \infty)$ and $A : K \to \mathbb{R}$ defined by $Au = -u^p$ with $p > 1$. Then A is not monotone on K since

$$\langle Av - Au, v - u\rangle = (-v^p + u^p)(v - u) = -(v-u)^2(v^{p-1} + uv^{p-2} + \ldots + u^{p-2}v + u^{p-1}) \leq 0.$$

On the other hand, A is relaxed $\eta - \alpha$ monotone on K with $\eta(v, u) = u^p - v^p$ and $\alpha \equiv 0$.

Example 2.4. Let $K = \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}$ defined by $Au = -u$. Obviously A is not monotone on $\mathbb{R}$. On the other hand A is simultaneous relaxed monotone ($\mu = 1$), relaxed α monotone (with $\alpha(w) = -cw^2$, where $c \geq 1$ is a constant) and relaxed $\eta-\alpha$ monotone (with $\eta(v, u) = v - u$ and $\alpha(w) = -cw^2,\ c \geq 1$). We point out the fact that A is relaxed $\eta - \alpha$ monotone with

$$\eta(v, u) = \begin{cases} \frac{1}{2}(u - v); & \text{if } u \leq v \\ \frac{1}{2}(v - u); & \text{if } u > v. \end{cases}$$

and

$$\alpha(w) = \frac{1}{2}w|w|.$$

Remark 2.5. It is easy to check, by the above definitions and examples, that the class of relaxed $\eta - \alpha$ monotone mappings is strictly broader than the class of monotone mappings.

Definition 2.6. Let $A : K \to X^*$ and $\eta : K \times K \to X$ be two mappings. A is said to be η-hemicontinuous (see [1]) if, for any fixed $u, v \in K$, the mapping $f : [0, 1] \to (-\infty, +\infty)$ defined by $f(t) = \langle A(u + t(v - u)), \eta(v, u)\rangle$ is continuous at 0^+.

The following lemma will be needed in the proof of the main result.

Lemma 2.7 (Tarafdar [20])**.** *Let $K \neq \emptyset$ be a convex subset of a Hausdorff topological vector space E. Let $F : K \to 2^K$ (by 2^K we understand the family of all the subsets of K) be a set valued map such that*

(T1) *for each $u \in K$, $F(u)$ is a nonempty convex subset of K;*
(T2) *for each $v \in K$, $F^{-1}(v) = \{\, u \in K :\ v \in F(u)\}$ contains an open set O_v which may be empty;*
(T3) $\bigcup_{v\in K} O_v = K$;
(T4) *there exists a nonempty set V_0 contained in a compact convex subset V_1 of K such that $D = \bigcap_{v\in V_0} O_v^c$ is either empty or compact (where O_v^c is the complement of O_v in K).*

Then there exist a point $u_0 \in K$ such that $u_0 \in F(u_0)$.

3. MAIN RESULTS

In this section, we study the existence of the following hemivariational inequality problem:

Find $u \in K$ such that

$$\langle Au, \eta(v,u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0, \quad \forall v \in K. \tag{3.1}$$

Lemma 3.1. *Let K be a nonempty, closed and convex subset of X and $A : K \to X^*$ a relaxed $\eta - \alpha$ monotone and η-hemicontinuous mapping. Assume that:*

(i) $\eta(u,u) = 0, \quad \forall u \in K$;

(ii) *for any fixed $v, w \in K$, the mapping $u \longmapsto \langle Aw, \eta(u,v)\rangle$ is convex.*

Then $u \in K$ is a solution of (3.1) *if and only if it solves the following problem:*

Find $u \in K$ such that

$$\langle Av, \eta(v,u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq \alpha(v-u), \quad \forall v \in K. \tag{3.2}$$

Proof. ("$\Longrightarrow$") Let u be a solution of (3.1). Then

$$\langle Au, \eta(v,u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0 \quad \forall v \in K.$$

On the other hand, by the relaxed $\eta - \alpha$ monotonicity of A we have

$$\langle Av - Au, \eta(v,u)\rangle \geq \alpha(v,u), \quad \forall v \in K.$$

Combining the above estimates we obtain that

$$\langle Av, \eta(v,u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq \alpha(v-u) \quad \forall v \in K.$$

which implies that u is a solution of (3.2).

("$\Longleftarrow$") Conversely, we assume that u is a solution of (3.2) and fix $v \in K$. Letting

$$w = u + t(v-u), \quad t \in (0,1],$$

then $w \in K$. It follows from (3.2) that

$$\begin{aligned} \langle Aw, \eta(w,u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{w}(x) - \hat{u}(x))\, dx \;&\geq\; \alpha(w-u) \\ &=\; \alpha(t(v-u)) \\ &=\; t^q \alpha(v-u). \end{aligned}$$

By conditions (i) and (ii) we have

$$\begin{aligned} \langle Aw, \eta(w,u)\rangle \;&\leq\; t\langle Aw, \eta(v,u)\rangle + (1-t)\langle Aw, \eta(u,u)\rangle \\ &=\; t\langle Aw, \eta(v,u)\rangle. \end{aligned}$$

Combining the above relations with the positive homogeneity of the mapping $\hat{v} \longmapsto j^0(x, \hat{u}; \hat{v})$ we obtain

$$t\langle Aw, \eta(v,u)\rangle + t\int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq t^q \alpha(v-u)$$

which leads to

$$\langle Aw, \eta(v,u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq t^{q-1} \alpha(v-u), \quad \forall v \in K. \tag{3.3}$$

Letting $t \to 0^+$ in (3.3) and using the η-hemicontinuity of A we deduce that u solves (3.1). □

Theorem 3.2. *Let K be a nonempty, bounded, closed and convex subset of X and $A : K \to X^*$ be a relaxed $\eta - \alpha$ monotone and η-hemicontinuous mapping. Assume the following conditions are fulfilled:*

(H1) $\eta(u,v) + \eta(v,u) = 0, \quad \forall u, v \in K$;

(H2) *For any fixed $v, w \in K$ the mapping $u \longmapsto \langle Aw, \eta(u,v)\rangle$ is convex and lower semicontinuous;*

(H3) *For any sequence $\{x_n\} \subset X$, $x_n \rightharpoonup x$ implies*

$$\limsup_{n\to\infty} \alpha(x_n) \geq \alpha(x).$$

Then there exists at least one solution for (3.1).

Proof. Arguing by contradiction suppose that (3.1) has no solutions. Then, for each $u \in K$ there exists $v \in K$ such that

$$\langle Au, \eta(v,u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < 0. \tag{3.4}$$

This implies by Lemma 3.1 that, for each $u \in K$ there exists $v \in K$ such that

$$\langle Av, \eta(v,u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < \alpha(v-u). \tag{3.5}$$

We define the set valued map $F : K \to 2^K$

$$F(u) := \left\{ v \in K : \ \langle Au, \eta(v,u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < 0 \right\}.$$

We shall prove that F satisfies the conditions of Lemma 2.7. Obviously, $F(u)$ is nonempty for each $u \in K$. Let $u \in K$ be arbitrary fixed and $w = (1-t)v_1 + tv_2$, with $v_1, v_2 \in F(u)$, $t \in [0,1]$. Using the assumption (H2) and the convexity of the mapping $\hat{v} \longmapsto j^0(x,\hat{u};\hat{v})$, we have

$$\begin{aligned}\langle Au, \eta(w,u)\rangle + \int_\Omega j^0(x,\hat{u};\hat{w}-\hat{u})\,dx &\le (1-t)\langle Au,\eta(v_1,u)\rangle + t\langle Au,\eta(v_2,u)\rangle \\ &\quad +(1-t)\int_\Omega j^0(x,\hat{u};\hat{v}_1-\hat{u})\,dx \\ &\quad +t\int_\Omega j^0(x,\hat{u};\hat{v}_2-\hat{u})\,dx \\ &< 0.\end{aligned}$$

The above relation shows that $w \in F(u)$, which implies that $F(u)$ is convex for each $u \in K$.

For each $v \in K$,

$$\begin{aligned} F^{-1}(v) &= \{u \in K :\ v \in F(u)\} \\ &= \left\{u \in K :\ \langle Au,\eta(v,u)\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx < 0\right\} \\ &\supseteq \left\{u \in K :\ \langle Av,\eta(v,u)\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx < \alpha(v-u)\right\} \\ &:= O_v.\end{aligned}$$

To prove the above inclusion we will show that $[F^{-1}(v)]^c \subseteq O_v^c$. If $u \in [F^{-1}(v)]^c$, then

$$\langle Au,\eta(v,u)\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx \ge 0$$

Taking into account that A is relaxed $\eta-\alpha$ monotone we have

$$\langle Av - Au, \eta(v,u)\rangle \ge \alpha(v-u)$$

Summing the last two relations we deduce that

$$\langle Av,\eta(v,u)\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx \ge \alpha(v-u)$$

which is equivalent to $u \in O_v^c$.

We claim that O_v^c is weakly closed. Let $\{u_n\} \subset O_v^c$ be a sequence that converges weakly to u as $n \to \infty$. We have to prove that $u \in O_v^c$. Since j satisfies condition (2.1), by Lemma 1 [17] (p. 44) part (b) the application

$$(u,v) \longmapsto \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx$$

is weakly upper semicontinuous. Using (H1)-(H3) we deduce that

$$\begin{aligned}
\alpha(v-u) &\leq \limsup_{n\to\infty} \alpha(v-u_n)\\
&\leq \limsup_{n\to\infty} \left[\langle Av, -\eta(u_n,v)\rangle + \int_\Omega j^0(x,\hat{u}_n(x);\hat{v}(x)-\hat{u}_n(x))\,dx\right]\\
&\leq -\liminf_{n\to\infty} \langle Av, \eta(u_n,v)\rangle + \limsup_{n\to\infty} \int_\Omega j^0(x,\hat{u}_n(x);\hat{v}(x)-\hat{u}_n(x))\,dx\\
&\leq -\langle Av, \eta(u,v)\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx\\
&= \langle Av, \eta(v,u)\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx
\end{aligned}$$

This is equivalent to $u \in O_v^c$.

We have to prove next that $\bigcup_{v\in K} O_v = K$. It suffices to show that $K \subseteq \bigcup_{v\in K} O_v$, the converse inclusion being obvious. For an arbitrary $u \in K$, by (3.5) there exists $v \in K$ such that $u \in O_v$, hence the desired inclusion holds.

The set $D = \bigcap_{v\in K} O_v^c$ is empty or weakly closed as it is the intersection of weakly closed sets O_v^c. Since K is nonempty, bounded, closed and convex and X is reflexive, it follows that K is weakly compact, hence D is also weakly compact. Therefore, the conditions (T1–T4) of Lemma 2.7 are satisfied in the weak topology. It follows that there exists $u_0 \in F(u_0)$ which implies

$$0 = \langle Au_0, \eta(u_0,u_0)\rangle + \int_\Omega j^0(x,\hat{u}_0(x);\hat{u}_0(x)-\hat{u}_0(x))\,dx < 0.$$

Thus, we have reached a contradiction which assures that (3.1) has at least one solution and the proof of Theorem 3.2 is now complete. □

Corollary 3.3. *Let K be a nonempty, bounded, closed and convex subset of X and $A : K \to X^*$ be a monotone and hemicontinuous operator. Then the problem:*

Find $u \in K$ such that

$$\langle Au, v-u\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx \geq 0, \quad \forall v \in K; \tag{3.6}$$

has at least one solution.

Proof. It suffices to define $\eta : K \times K \to X$, $\eta(u,v) = u - v$ and $\alpha : X \to \mathbb{R}$, $\alpha \equiv 0$ and observe that A is relaxed $\eta - \alpha$ monotone and η-hemicontinuous. Applying Theorem 3.2 the conclusion follows. □

Theorem 3.4. *Assume that the same hypotheses as in Theorem* 3.2 *hold without the assumption of boundedness of K. Suppose in addition that there exists $q \geq p$ such that*

$$\frac{\langle Aw, \eta(w,0)\rangle}{\|w\|^q} \to \infty, \qquad \text{as } \|w\| \to \infty.$$

Then the inequality problem (3.1) *admits at leats one solution.*

Proof. Set $K_n = \{v \in K : \ \|v\| \leq n\}$. Using Theorem 3.2, we get the existence of $u_n \in K_n$ such that

$$\langle Au_n, \eta(v, u_n)\rangle + \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}_n(x) - \hat{u}_n(x))\, dx \geq 0, \quad \forall v \in K_n. \tag{3.7}$$

STEP 1. There exists a positive integer n_0 such that $\|u_{n_0}\| < n_0$.

Arguing by contradiction let us suppose that $\|u_n\| = n$ for all $n \geq 1$. Setting $v = 0$ in (3.7), we have

$$\langle Au_n, \eta(u_n, 0)\rangle \leq \int_\Omega j^0(x, \hat{u}_n(x); -\hat{u}_n(x))\, dx \tag{3.8}$$

On the other hand, for each $x \in \Omega$ where holds true the condition (2.1) and for each $y, h \in \mathbb{R}^k$, there exists $z \in \partial j(x, y)$ such that

$$j^0(x, y; h) = z \cdot h = \max\{w \cdot h : \ w \in \partial j(x, y)\},$$

(see [2], Prop 2.1.2). It follows from (2.1) that

$$|j^0(x, y; h)| \leq |z| \cdot |h| \leq (g_1(x) + g_2(x)|y|^{p-1})|h|.$$

Using (3.8), Hölder's inequality and the fact that $T : X \to L^p(\Omega; \mathbb{R}^k)$ is linear and compact we get

$$\begin{aligned}
\langle Au_n, \eta(u_n, 0)\rangle &\leq \int_\Omega g_1(x)|\hat{u}_n(x)| + g_2(x)|\hat{u}_n(x)|^p\, dx \\
&\leq c_1\|\hat{u}_n\|_p + c_2\|\hat{u}_n\|_p^p \\
&\leq c_3\|u_n\| + c_4\|u_n\|^p,
\end{aligned}$$

where c_1, c_2, c_3, c_4 are positive constants.

Thus,

$$\frac{\langle Au_n, \eta(u_n, 0)\rangle}{\|u_n\|^q} \leq c_3\|u_n\|^{1-q} + c_4\|u_n\|^{p-q}$$

and passing to the the limit as $n \to \infty$ we obtain a contradiction, since $1 < p \leq q$.

STEP 2. u_{n_0} solves inequality problem (3.1).

Since $\|u_{n_0}\| < n_0$, for each $v \in K$ we can choose $\varepsilon > 0$ such that $w = u_{n_0} + \varepsilon(v - u_{n_0}) \in K_{n_0}$ (it suffices to take $\varepsilon = 1$ if $v = u_{n_0}$ and $\varepsilon < (n_0 - \|u_{n_0}\|)/\|v - u_{n_0}\|$ if $v \neq u_{n_0}$).

It follows from (3.7) and the positive homogeneity of the map $\hat{v} \mapsto j^0(x,\hat{u};\hat{v})$ that

$$\begin{aligned} 0 &\leq \langle Au_{n_0}, \eta(w,u_{n_0})\rangle + \int_\Omega j^0(x,\hat{u}_{n_0}(x);\hat{w}(x)-\hat{u}_{n_0}(x))\,dx \\ &\leq \varepsilon\langle Au_{n_0}, \eta(v,u_{n_0})\rangle + (1-\varepsilon)\langle Au_{n_0}, \eta(u_{n_0},u_{n_0})\rangle \\ &\quad +\varepsilon \int_\Omega j^0(x,\hat{u}_{n_0}(x);\hat{v}(x)-\hat{u}_{n_0}(x))\,dx \\ &= \varepsilon\left[\langle Au_{n_0}, \eta(v,u_{n_0})\rangle + \int_\Omega j^0(x,\hat{u}_{n_0}(x);\hat{v}(x)-\hat{u}_{n_0}(x))\,dx\right]. \end{aligned}$$

Dividing by ε the conclusion follows. □

Remark 3.5. (a) If instead A is continuous on finite dimensional subspaces, the conclusions of Theorems 3.2, 3.4 and Corollary 3.3 are also true.

(b) Corollary 3.3 states the same result as Theorem 2 in [17] which implies that Theorems 3.2 and 3.4 improve and generalize the known results of Panagiotopoulos-Fundo-Rădulescu [17].

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