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Weak solutions of quasilinear problems with nonlinear boundary condition

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1. Introduction

The growing attention for the study of the p-Laplacian operator Δ_p in the last few decades is motivated by the fact that it arises in various applications. For instance, in Fluid Mechanics, the shear stress $\vec{\tau}$ and the velocity gradient $\nabla_p u$ of certain fluids obey a relation of the form $\vec{\tau}(x) = a(x)\nabla_p u(x)$, where $\nabla_p u = |\nabla u|^{p-2}\nabla u$. Here $p > 1$ is an arbitrary real number and the case $p = 2$ (respectively $p < 2,\ p > 2$) corresponds to a Newtonian (respectively pseudoplastic, dilatant) fluid. The resulting equations of motion then involve $\operatorname{div}(a\nabla_p u)$, which reduces to $a\Delta_p u = a \operatorname{div} \nabla_p u$, provided that a is constant. The p-Laplacian also appears in the study of torsional creep (elastic for $p = 2$, plastic as $p \to \infty$, see [7]), flow through porous media ($p = \frac{3}{2}$, see [12]) or glacial sliding ($p \in (1, \frac{4}{3}]$, see [9]).

Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with (possible noncompact) smooth boundary Γ and n is the unit outward normal on Γ. We consider the nonlinear elliptic boundary value problem:

$$-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda(1 + |x|)^{\alpha_1}|u|^{p-2}u + (1 + |x|)^{\alpha_2}|u|^{q-2}u \quad \text{in } \Omega,$$

$$a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x) \cdot |u|^{p-2}u = g(x,u) \quad \text{on } \Gamma. \tag{A}$$

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We assume throughout that $1<p<N$, $p<q<p^*=Np/(N-p)$, $-N<\alpha_1<-p$, $-N<\alpha_2<q\cdot(N-p)/p-N$, $0<a_0\leq a\in L^\infty(\Omega)$ and $b:\Gamma\to\mathbb{R}$ is a continuous function satisfying

$$\frac{c}{(1+|x|)^{p-1}}\leq b(x)\leq\frac{C}{(1+|x|)^{p-1}},$$

for constants $0<c\leq C$.

Let $g:\Gamma\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that

(A1) $|g(x,s)|\leq g_0(x)+g_1(x)|s|^{m-1}$; $q\leq m<p\cdot\frac{N-1}{N-p}$,

where $g_i:\Gamma\to\mathbb{R}$ (i=0,1) are measurable functions satisfying $g_0\in L^{m/(m-1)}(\Gamma;w_3^{1/(1-m)})$,

$$0\leq g_i\leq C_g w_3 \quad \text{a.e. in } \Gamma,$$

for a constant $C_g>0$, with $w_3(x)=(1+|x|)^{\alpha_3}$, $x\in\Gamma$, and $-N<\alpha_3<m\cdot(N-p)/p-N+1$.

We also assume

(A2) $\lim_{s\to 0}\frac{g(x,s)}{b(x)|s|^{p-1}}=0$ uniformly in x.

(A3) There exists $\mu\in(p,q]$ such that

$$\mu G(x,s)\leq sg(x,s) \quad \text{for a.e. } x\in\Gamma \text{ and every } s\in\mathbb{R}.$$

(A4) There is a non-empty open set $U\subset\Gamma$ with $G(x,s)>0$ for $(x,s)\in U\times(0,\infty)$, where G is the primitive function of g with respect to the second variable, i.e., $G(x,s)=\int_0^s g(x,t)\,\mathrm{d}t$.

Our first result asserts that, under the above hypotheses, problem (A) has at least a solution in an appropriate space.

Eigenvalue problems involving the p-Laplacian have been the subject of much recent interest (we refer only to [1,3,4,6]). Our purpose is to prove the existence of an eigensolution for the following eigenvalue problem:

$$\begin{aligned}&-\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)=\lambda[(1+|x|)^{\alpha_1}|u|^{p-2}u+(1+|x|)^{\alpha_2}|u|^{q-2}u] \quad \text{in } \Omega,\\ &a(x)|\nabla u|^{p-2}\nabla u\cdot n+b(x)|u|^{p-2}u=\lambda g(x,u) \quad \text{on } \Gamma.\end{aligned} \tag{B}$$

In the study of this problem we drop assumptions (A2) and (A4).

2. Preliminaries and the main results

Let $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(\mathbb{R}^N)$ — functions restricted on Ω. We define the weighted Sobolev space E as the completion of $C_\delta^\infty(\Omega)$ in the norm

$$\|u\|_E=\left(\int_\Omega\left(|\nabla u(x)|^p+\frac{1}{(1+|x|)^p}|u(x)|^p\right)\mathrm{d}x\right)^{1/p}.$$

Denote by $L^p(\Omega;w_1),L^q(\Omega;w_2)$ and $L^m(\Gamma;w_3)$ the weighted Lebesgue spaces with weight functions

$$w_i(x)=(1+|x|)^{\alpha_i}, \quad i=1,2,3$$

and the norms defined by

$$\|u\|_{p,w_1}^p = \int_\Omega w_1|u(x)|^p\,\mathrm{d}x,$$

$$\|u\|_{q,w_2}^q = \int_\Omega w_2|u(x)|^q\,\mathrm{d}x \quad \text{and} \quad \|u\|_{m,w_3}^m = \int_\Gamma w_3|u(x)|^m\,\mathrm{d}\Gamma.$$

Then we have the following embedding and trace theorem.

Theorem 1. *If*

$$p \le r \le \frac{pN}{N-p} \quad \text{and} \quad -N < \alpha \le r\cdot\frac{N-p}{p} - N, \tag{1}$$

then the embedding $E \subset L^r(\Omega; w)$ *is continuous, where* $w(x) = (1+|x|)^\alpha$. *If the upper bounds for* r *in* (1) *are strict, then the embedding is compact. If*

$$p \le m \le p\cdot\frac{N-1}{N-p} \quad \text{and} \quad -N < \alpha_3 \le m\cdot\frac{N-p}{p} - N + 1, \tag{2}$$

then the trace operator $E \to L^m(\Gamma; w_3)$ *is continuous. If the upper bounds for* m *in* (2) *are strict, then the trace is compact.*

This theorem is a consequence of Theorem 2 and Corollary 6 of Pflüger [11].

Lemma 1. *The quantity*

$$\|u\|_b^p = \int_\Omega a(x)|\nabla u|^p\,\mathrm{d}x + \int_\Gamma b(x)|u|^p\,\mathrm{d}\Gamma$$

defines an equivalent norm on E.

For the proof of this result we refer to [10], Lemma 2.
We denote by N_g, N_G the corresponding Nemytskii operators.

Lemma 2. *The operators*

$$N_g : L^m(\Gamma; w_3) \to L^{m/(m-1)}(\Gamma; w_3^{1/(1-m)}), \qquad N_G : L^m(\Gamma; w_3) \to L^1(\Gamma)$$

are bounded and continuous.

Proof. Let $m' = m/(m-1)$ and $u \in L^m(\Gamma; w_3)$. Then, by (A1) we have

$$\int_\Gamma |N_g(u)|^{m'}\cdot w_3^{1/(1-m)}\,\mathrm{d}\Gamma \le 2^{m'-1}\left(\int_\Gamma g_0^{m'}\cdot w_3^{1/(1-m)}\,\mathrm{d}\Gamma + \int_\Gamma g_1^{m'}|u|^m\cdot w_3^{1/(1-m)}\,\mathrm{d}\Gamma\right)$$

$$\le 2^{m'-1}\left(C + C_g\cdot\int_\Gamma |u|^m\cdot w_3\,\mathrm{d}\Gamma\right),$$

which shows that N_g is bounded. In a similar way, we obtain

$$\int_\Gamma |N_G(u)|\,\mathrm{d}\Gamma \le \int_\Gamma g_0|u|\,\mathrm{d}\Gamma + \int_\Gamma g_1|u|^m\,\mathrm{d}\Gamma$$

$$\le \left(\int_\Gamma g_0^{m'} \cdot w_3^{1/(1-m)}\,\mathrm{d}\Gamma\right)^{1/m'} \cdot \left(\int_\Gamma |u|^m \cdot w_3\,\mathrm{d}\Gamma\right)^{1/m} + C_1 \cdot \int_\Gamma |u|^m \cdot w_3\,\mathrm{d}\Gamma$$

and we claim that N_G is bounded.

Now, from the usual properties of Nemytskii operators we deduce the continuity of these operators. □

By weak solution of problem (A) we mean a function $u \in E$ such that

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v\,\mathrm{d}x + \int_\Gamma b(x)|u|^{p-2}uv\,\mathrm{d}\Gamma$$

$$= \lambda \int_\Omega w_1|u|^{p-2}uv\,\mathrm{d}x + \int_\Omega w_2|u|^{q-2}uv\,\mathrm{d}x + \int_\Gamma g(x,u)v\,\mathrm{d}\Gamma, \quad \forall v \in E.$$

Define

$$\tilde{\lambda} := \inf_{u\in E;\, u\neq 0} \left(\frac{\int_\Omega a(x)|\nabla u|^p\,\mathrm{d}x + \int_\Gamma b(x)|u|^p\,\mathrm{d}\Gamma}{\int_\Omega |u|^p \cdot w_1\,\mathrm{d}x}\right).$$

Our first result is

Theorem 2. *Assume that conditions* (A1)–(A4) *hold. Then, for every* $\lambda < \tilde{\lambda}$, *problem* (A) *has a nontrivial weak solution.*

We stress that for the following result of the paper we drop assumptions (A2) and (A4).

By weak solution of problem (B) we mean a function $u \in E$ such that

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v\,\mathrm{d}x + \int_\Gamma b(x)|u|^{p-2}uv\,\mathrm{d}\Gamma$$

$$= \lambda \left[\int_\Omega w_1|u|^{p-2}uv\,\mathrm{d}x + \int_\Omega w_2|u|^{q-2}uv\,\mathrm{d}x + \int_\Gamma g(x,u)v\,\mathrm{d}\Gamma\right], \quad \forall v \in E.$$

We now state the main result of solving problem (B).

Theorem 3. *Assume that hypotheses* (A1) *and* (A3) *hold. Let d be an arbitrary real number such that $1/d$ is not an eigenvalue λ in problem* (B), *and satisfying*

$$d > \frac{1}{\tilde{\lambda}}. \tag{3}$$

Then there exists $\bar{\rho} > 0$ such that for all $r > \rho \ge \bar{\rho}$, eigenvalue problem (B) *has an eigensolution $(u,\lambda) = (u_d, \lambda_d) \in E \times \mathbb{R}$ for which one has*

$$\lambda_d \in \left[\frac{1}{d + r^2\|u_d\|_b^{m-p}}, \frac{1}{d + \rho^2\|u_d\|_b^{m-p}}\right].$$

3. Proof of Theorem 2

The key argument in the proof is the Mountain-Pass Theorem in the following variant (see [2]):

Ambrosetti–Rabinowitz Theorem. *Let X be a real Banach space and $F : X \to \mathbb{R}$ be a C^1-functional. Suppose that F satisfies the Palais–Smale condition and the following geometric assumptions*:

$$\text{there exist positive constants } R \text{ and } c_0 \text{ such that } F(u) \geq c_0,$$

$$\text{for all } u \in X \text{ with } \|u\| = R; \tag{4}$$

$$F(0) < c_0 \text{ and there exists } v \in X \text{ such that}$$

$$\|v\| > R \text{ and } F(v) < c_0. \tag{5}$$

Then the functional F possesses at least a critical point.

Throughout this section we use the same notations as was previously done in the case of problem (A) excepting that $h(x,s) = w_2(x)|s|^{q-2}s,\ \forall x \in \Omega,\ s \in \mathbb{R}$.

The energy functional corresponding to (A) is defined as $F : E \to \mathbb{R}$

$$F(u) = \frac{1}{p}\int_\Omega a(x)\cdot|\nabla u|^p\,\mathrm{d}x + \frac{1}{p}\int_\Gamma b(x)\cdot|u|^p\,\mathrm{d}\Gamma$$

$$-\frac{\lambda}{p}\int_\Omega w_1\cdot|u|^p\,\mathrm{d}x - \int_\Gamma G(x,u)\,\mathrm{d}\Gamma - \int_\Omega H(x,u)\,\mathrm{d}x,$$

where H denotes the primitive function of h with respect to the second variable.

By Lemma 1 we have $\|\cdot\|_b \simeq \|\cdot\|_E$. We may write

$$F(u) = \frac{1}{p}\cdot\|u\|_b^p - \frac{\lambda}{p}\int_\Omega w_1\cdot|u|^p\,\mathrm{d}x - \int_\Gamma G(x,u)\,\mathrm{d}\Gamma - \int_\Omega H(x,u)\,\mathrm{d}x.$$

We observe that

$$|H(x,u)| = \frac{1}{q}w_2(x)|u|^q. \tag{6}$$

Since $p < q < p^*, -N < \alpha_1 < -p$ and $-N < \alpha_2 < q\cdot(N-p)/p - N$ we can apply Theorem 1 and we obtain that the embeddings $E \subset L^p(\Omega; w_1)$ and $E \subset L^q(\Omega; w_2)$ are compact. This and (6) imply that F is well defined.

Our hypothesis

$$\lambda < \tilde{\lambda} := \inf_{u \in E;\, u \neq 0} \frac{\|u\|_b^p}{\|u\|_{p,w_1}^p}$$

implies the existence of some $C_0 > 0$ such that, for every $v \in E$

$$\|v\|_b^p - \lambda\|v\|_{p,w_1}^p \geq C_0\|v\|_b^p.$$

We shall prove in what follows that F satisfies the hypotheses of the Mountain-Pass Theorem.

Lemma 3. *Under assumptions* (A1)–(A4), *the functional* F *is Fréchet-differentiable on* E *and satisfies the Palais–Smale condition.*

Proof. Denote $I(u) = (1/p)\|u\|_b^p$, $K_G(u) = \int_\Gamma G(x,u)\,\mathrm{d}\Gamma$, $K_H(u) = \int_\Omega H(x,u)\,\mathrm{d}x$ and $K_\Phi(u) = \int_\Omega (1/p) w_1 |u|^p\,\mathrm{d}x$, where $\Phi(x,u) = (1/p)w_1(x)|u|^p$. Then the directional derivative of F in the direction $v \in E$ is

$$\langle F'(u), v\rangle = \langle I'(u), v\rangle - \lambda\langle K'_\Phi(u), v\rangle - \langle K'_G(u), v\rangle - \langle K'_H(u), v\rangle,$$

where

$$\langle I'(u), v\rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v\,\mathrm{d}x + \int_\Gamma b(x)|u|^{p-2}uv\,\mathrm{d}\Gamma;$$

$$\langle K'_G(u), v\rangle = \int_\Gamma g(x,u)v\,\mathrm{d}\Gamma;$$

$$\langle K'_H(u), v\rangle = \int_\Omega h(x,u)v\,\mathrm{d}x; \qquad \langle K'_\Phi(u), v\rangle = \int_\Omega w_1|u|^{p-2}uv\,\mathrm{d}x.$$

Clearly, $I' : E \to E^\star$ is continuous. The operator K'_G is a composition of the operators

$$K'_G : E \to L^m(\Gamma; w_3) \xrightarrow{N_g} L^{m/(m-1)}(\Gamma; w_3^{1/(1-m)}) \xrightarrow{l} E^\star,$$

where $\langle l(u), v\rangle = \int_\Gamma uv\,\mathrm{d}\Gamma$. Since

$$\int_\Gamma |uv|\,\mathrm{d}\Gamma \le \left(\int_\Gamma |u|^{m'} w_3^{1/(1-m)}\,\mathrm{d}\Gamma\right)^{1/m'} \cdot \left(\int_\Gamma |v|^m w_3\,\mathrm{d}\Gamma\right)^{1/m},$$

then l is continuous, by Theorem 1. As a composition of continuous operators, K'_G is continuous, too. Moreover, by our assumptions on w_3, the trace operator $E \to L^m(\Gamma; w_3)$ is compact and therefore, K'_G is also compact.

Set $\phi(u) = w_1|u|^{p-2}u$. By the proof of Lemma 2 we deduce that the Nemytskii operator corresponding to any function which satisfies (A1) is bounded and continuous. Hence N_h and N_ϕ are bounded and continuous. We note that

$$K'_\Phi : E \subset L^p(\Omega; w_1) \xrightarrow{N_\phi} L^{p/(p-1)}(\Omega; w_1^{1/(1-p)}) \xrightarrow{\eta} E^\star,$$

where $\langle \eta(u), v\rangle = \int_\Omega uv\,\mathrm{d}x$. Since

$$\int_\Omega |uv|\,\mathrm{d}x \le \left(\int_\Omega |u|^{p/(p-1)} w_1^{1/(1-p)}\,\mathrm{d}x\right)^{(p-1)/p} \cdot \left(\int_\Omega |v|^p w_1\,\mathrm{d}x\right)^{1/p},$$

it follows that η is continuous. But K'_Φ is the composition of three continuous operators and by the assumptions on w_1, the embedding $E \subset L^p(\Omega; w_1)$ is compact. This implies that K'_Φ is compact.

In a similar way, we obtain that K'_H is compact and the continuous Fréchet-differentiability of F follows.

Now, let $u_n \in E$ be a Palais–Smale sequence, i.e.,

$$|F(u_n)| \le C \quad \text{for all } n \tag{7}$$

and

$$\|F'(u_n)\|_{E^\star} \to 0 \quad \text{as } n\to\infty. \tag{8}$$

We first prove that (u_n) is bounded in E. Note that (8) implies:

$$|\langle F'(u_n), u_n\rangle| \le \mu \cdot \|u_n\|_b \quad \text{for } n \text{ large enough.}$$

This and (7) imply that

$$C + \|u_n\|_b \ge F(u_n) - \frac{1}{\mu} \cdot \langle F'(u_n), u_n\rangle. \tag{9}$$

But

$$\begin{aligned}\langle F'(u_n), u_n\rangle &= \int_\Omega a(x)|\nabla u_n|^p \, \mathrm{d}x + \int_\Gamma b(x)|u_n|^p \, \mathrm{d}\Gamma - \lambda \cdot \int_\Omega w_1|u_n|^p \, \mathrm{d}x \\ &\quad - \int_\Omega h(x,u_n)u_n \, \mathrm{d}x - \int_\Gamma g(x,u_n)u_n \, \mathrm{d}\Gamma = \|u_n\|_b^p - \lambda \cdot \|u_n\|_{p,w_1}^p \\ &\quad - \int_\Omega h(x,u_n)u_n \, \mathrm{d}x - \int_\Gamma g(x,u_n)u_n \, \mathrm{d}\Gamma\end{aligned}$$

and

$$F(u_n) = \frac{1}{p}(\|u_n\|_b^p - \lambda \cdot \|u_n\|_{p,w_1}^p) - \int_\Omega H(x,u_n) \, \mathrm{d}x - \int_\Gamma G(x,u_n) \, \mathrm{d}\Gamma.$$

We have

$$\begin{aligned}F(u_n) - \frac{1}{\mu} \cdot \langle F'(u_n), u_n\rangle &= \left(\frac{1}{p} - \frac{1}{\mu}\right)\left(\|u_n\|_b^p - \lambda \cdot \|u_n\|_{p,w_1}^p\right) \\ &\quad - \left(\int_\Omega H(x,u_n) \, \mathrm{d}x - \frac{1}{\mu}\int_\Omega h(x,u_n)u_n \, \mathrm{d}x\right) \\ &\quad - \left(\int_\Gamma G(x,u_n) \, \mathrm{d}\Gamma - \frac{1}{\mu}\int_\Gamma g(x,u_n)u_n \, \mathrm{d}\Gamma\right).\end{aligned}$$

By (A3) we deduce that

$$\int_\Gamma G(x,u_n) \, \mathrm{d}\Gamma \le \frac{1}{\mu}\int_\Gamma g(x,u_n)u_n \, \mathrm{d}\Gamma. \tag{10}$$

A simple computation yields

$$\int_\Omega H(x,u_n) \, \mathrm{d}x = \frac{1}{q}\int_\Omega h(x,u_n)u_n \, \mathrm{d}x \le \frac{1}{\mu}\int_\Omega h(x,u_n)u_n \, \mathrm{d}x. \tag{11}$$

By (10) and (11) we obtain that

$$F(u_n) - \frac{1}{\mu} \cdot \langle F'(u_n), u_n\rangle \ge \left(\frac{1}{p} - \frac{1}{\mu}\right) C_0\|u_n\|_b^p. \tag{12}$$

Relations (9) and (12) imply that

$$C + \|u_n\|_b \ge \left(\frac{1}{p} - \frac{1}{\mu}\right) C_0\|u_n\|_b^p.$$

This shows that (u_n) is bounded in E.

To prove that (u_n) contains a Cauchy sequence we use the following inequalities for $\xi, \zeta \in \mathbb{R}^N$ (see [5, Lemma 4.10]):

$$|\xi - \zeta|^p \le C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta) \quad \text{for } p \ge 2, \tag{13}$$

$$|\xi - \zeta|^2 \le C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p} \quad \text{for } 1 < p < 2. \tag{14}$$

Then we obtain in the case $p \ge 2$:

$$\begin{aligned}
\|u_n - u_k\|_b^p &= \int_\Omega a(x)|\nabla u_n - \nabla u_k|^p \,\mathrm{d}x + \int_\Gamma b(x)|u_n - u_k|^p \,\mathrm{d}\Gamma \\
&\le C(\langle I'(u_n), u_n - u_k\rangle - \langle I'(u_k), u_n - u_k\rangle) \\
&= C(\langle F'(u_n), u_n - u_k\rangle - \langle F'(u_k), u_n - u_k\rangle \\
&\quad + \lambda\langle K'_\Phi(u_n), u_n - u_k\rangle - \lambda\langle K'_\Phi(u_k), u_n - u_k\rangle \\
&\quad + \langle K'_G(u_n), u_n - u_k\rangle - \langle K'_G(u_k), u_n - u_k\rangle \\
&\quad + \langle K'_H(u_n), u_n - u_k\rangle - \langle K'_H(u_k), u_n - u_k\rangle) \\
&\le C(\|F'(u_n)\|_{E^\star} + \|F'(u_k)\|_{E^\star} \\
&\quad + |\lambda| \cdot \|K'_\Phi(u_n) - K'_\Phi(u_k)\|_{E^\star} + \|K'_G(u_n) - K'_G(u_k)\|_{E^\star} \\
&\quad + \|K'_H(u_n) - K'_H(u_k)\|_{E^\star})\|u_n - u_k\|_b.
\end{aligned}$$

Since $F'(u_n) \to 0$ and K'_Φ, K'_G, K'_H are compact, we can assume, passing eventually to a subsequence, that (u_n) converges in E.

If $1 < p < 2$, then we use the estimate

$$\|u_n - u_k\|_b^2 \le C'|\langle I'(u_n), u_n - u_k\rangle - \langle I'(u_k), u_n - u_k\rangle|(\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}). \tag{15}$$

Since $\|u_n\|_b$ is bounded, the same arguments lead to a convergent subsequence. In order to prove estimate (15) we recall the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \le C_s(x^s + y^s) \quad \text{for any } x, y \in (0, \infty). \tag{16}$$

Then, we obtain

$$\begin{aligned}
\|u_n - u_k\|_b^2 &= \left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p \,\mathrm{d}x + \int_\Gamma b(x)|u_n - u_k|^p \,\mathrm{d}\Gamma\right)^{2/p} \\
&\le C_p\left[\left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p \,\mathrm{d}x\right)^{2/p} + \left(\int_\Gamma b(x)|u_n - u_k|^p \,\mathrm{d}\Gamma\right)^{2/p}\right].
\end{aligned} \tag{17}$$

Using (14) and (16) and the Hölder inequality we find

$$\begin{aligned}
&\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p \,\mathrm{d}x \\
&\quad = \int_\Omega a(x)\,(|\nabla u_n - \nabla u_k|^2)^{p/2}\,\mathrm{d}x \\
&\quad \le C\int_\Omega a(x)\,((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k) \\
&\qquad (\nabla u_n - \nabla u_k))^{p/2}(|\nabla u_n| + |\nabla u_k|)^{p(2-p)/2}\,\mathrm{d}x \\
&\quad = C\int_\Omega (a(x)\,(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)\,(\nabla u_n - \nabla u_k))^{p/2} \\
&\qquad \times (a(x)\,(|\nabla u_n| + |\nabla u_k|)^p)^{(2-p)/2}\,\mathrm{d}x \\
&\quad \le C\left(\int_\Omega a(x)\,(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)\,(\nabla u_n - \nabla u_k)\,\mathrm{d}x\right)^{p/2} \\
&\qquad \left(\int_\Omega a(x)\,(|\nabla u_n| + |\nabla u_k|)^p\,\mathrm{d}x\right)^{(2-p)/2} \\
&\quad \le \tilde{C}_p\left(\int_\Omega a(x)|\nabla u_n|^p\,\mathrm{d}x + \int_\Omega a(x)|\nabla u_k|^p\,\mathrm{d}x\right)^{(2-p)/2} \\
&\qquad \left(\int_\Omega a(x)\,(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)\,(\nabla u_n - \nabla u_k)\,\mathrm{d}x\right)^{p/2} \\
&\quad \le \bar{C}_p\left[\left(\int_\Omega a(x)|\nabla u_n|^p\,\mathrm{d}x\right)^{(2-p)/2} + \left(\int_\Omega a(x)|\nabla u_k|^p\,\mathrm{d}x\right)^{(2-p)/2}\right] \\
&\qquad \left(\int_\Omega a(x)\,(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)\,(\nabla u_n - \nabla u_k)\,\mathrm{d}x\right)^{p/2} \\
&\quad \le \bar{C}_p\left(\int_\Omega a(x)\,(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)\,(\nabla u_n - \nabla u_k)\,\mathrm{d}x\right)^{p/2} \\
&\qquad \times(\|u_n\|_b^{((2-p)p)/2} + \|u_k\|_b^{((2-p)p)/2}).
\end{aligned}$$

Using the last inequality and (16) we have the estimate

$$\begin{aligned}
&\left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\,\mathrm{d}x\right)^{2/p} \\
&\quad \le C_p'\left(\int_\Omega a(x)\,(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)\cdot(\nabla u_n - \nabla u_k)\,\mathrm{d}x\right) \\
&\qquad \times(\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}). \qquad (18)
\end{aligned}$$

In a similar way, we can obtain the estimate

$$\left(\int_\Gamma b(x)|u_n-u_k|^p\,\mathrm{d}\Gamma\right)^{2/p}$$
$$\le C_p'\left(\int_\Gamma b(x)(|u_n|^{p-2}u_n-|u_k|^{p-2}u_k)(u_n-u_k)\,\mathrm{d}x\right)$$
$$\times(\|u_n\|_b^{2-p}+\|u_k\|_b^{2-p}). \tag{19}$$

It is now easy to observe that inequalities (17)–(19) imply estimate (15). The proof of Lemma 3 is complete.

Verification of (4): Using (6) we have

$$\left|\int_\Omega H(x,u)\,\mathrm{d}x\right|\le\int_\Omega|H(x,u)|\,\mathrm{d}x\le\frac{1}{q}\|u\|_{q,w_2}^q$$

and by Theorem 1 we have that there exists $A>0$ such that

$$\|u\|_{q,w_2}^q\le A\|u\|_b^q\quad\text{for all } u\in E.$$

This fact implies that

$$F(u)=\frac{1}{p}(\|u\|_b^p-\lambda\|u\|_{p,w_1}^p)-\int_\Omega H(x,u)\,\mathrm{d}x-\int_\Gamma G(x,u)\,\mathrm{d}\Gamma$$
$$\ge\frac{C_0}{p}\|u\|_b^p-\frac{A}{q}\|u\|_b^q-\int_\Gamma G(x,u)\,\mathrm{d}\Gamma.$$

By (A1) and (A2) we deduce that for every $\epsilon>0$ there exists $C_\epsilon>0$ such that

$$|G(x,u)|\le\epsilon b(x)|u|^p+C_\epsilon w_3(x)|u|^m.$$

Consequently,

$$F(u)\ge\frac{C_0}{p}\|u\|_b^p-\frac{A}{q}\|u\|_b^q-\int_\Gamma(\epsilon b(x)|u|^p+C_\epsilon w_3(x)|u|^m)\,\mathrm{d}\Gamma$$
$$\ge\frac{C_0}{p}\|u\|_b^p-\frac{A}{q}\|u\|_b^q-\epsilon c_1\|u\|_b^p-C_\epsilon C_2\|u\|_b^m.$$

For $\epsilon>0$ and $R>0$ small enough, we deduce that, for every $u\in E$ with $\|u\|_b=R$, $F(u)\ge c_0>0$.

Verification of (5): We choose a nonnegative function $\psi\in C_\delta^\infty(\Omega)$ such that $\emptyset\ne\operatorname{supp}\psi\cap\Gamma\subset U$. From $G(x,s)\ge c_3s^\mu-c_4$ on $U\times(0,\infty)$ and (A1) we claim that

$$F(t\psi)=\frac{t^p}{p}(\|\psi\|_b^p-\lambda\|\psi\|_{p,w_1}^p)-\int_\Omega H(x,t\psi)\,\mathrm{d}x-\int_\Gamma G(x,t\psi)\,\mathrm{d}\Gamma$$
$$\le\frac{t^p}{p}(\|\psi\|_b^p-\lambda\|\psi\|_{p,w_1}^p)-c_3t^\mu\int_U\psi^\mu\,\mathrm{d}\Gamma+c_4|U|-\frac{t^q}{q}\int_\Omega w_2\psi^q\,\mathrm{d}x.$$

Since $q \geq \mu > p$, we obtain $F(t\psi) \to -\infty$ as $t \to \infty$. It follows that if $t > 0$ is large enough, $F(t\psi) < 0$. By Ambrosetti–Rabinowitz Theorem, problem (A) has a nontrivial weak solution. □

4. Proof of Theorem 3

We start with the following auxiliary result.

Lemma 4. *Under assumption* (A1), *if* $q \leq m$, *there exists a number* $\bar{\rho} > 0$ *such that for each* $\rho \geq \bar{\rho}$ *the function*

$$v \mapsto \frac{\rho^2}{m}\|v\|_b^m - \frac{1}{p}\|v\|_{p,w_1}^p - \int_\Omega H(x,v)\,\mathrm{d}x - \int_\Gamma G(x,v)\,\mathrm{d}\Gamma, \quad \forall v \in E,$$

is bounded from below on E.

Proof. The growth condition for g implies that

$$\left|\int_\Gamma G(x,v)\,\mathrm{d}\Gamma\right| \leq \int_\Gamma \left(g_0(x)|v| + \frac{1}{m}g_1(x)|v|^m\right)\mathrm{d}\Gamma$$

$$\leq \left(\int_\Gamma g_0^{m/(m-1)} w_3^{1/(1-m)}\,\mathrm{d}\Gamma\right)^{(m-1)/m} \|v\|_{L^m(\Gamma;w_3)} + C_g\|v\|_{L^m(\Gamma;w_3)}^m$$

$$\leq C_0 + C\|v\|_b^m, \quad \forall v \in E,$$

with constants $C_0 > 0,\ C > 0$. One also obtains that

$$\left|\int_\Omega H(x,v)\,\mathrm{d}x\right| = \frac{1}{q}\|v\|_{L^q(\Gamma;w_3)}^q \leq C_2\|v\|_b^q \leq \overline{C_0} + \bar{C}\|v\|_b^m, \quad \forall v \in E,$$

with constants $\bar{C}_0 > 0$, $\bar{C} > 0$. Clearly, we can choose now the positive number $\bar{\rho}$ as desired. □

In view of Lemma 4, one can find numbers $b_0 > 0$ and $\alpha > 0$ such that

$$\frac{\bar{\rho}^2}{m}\|v\|_b^m + \frac{2}{m}b_0 - \frac{1}{p}\|v\|_{p,w_1}^p - \int_\Omega H(x,v)\,\mathrm{d}x$$

$$-\int_\Gamma G(x,v)\,\mathrm{d}\Gamma \geq \alpha > 0, \quad \forall v \in E. \tag{20}$$

With $b_0 > 0$ and $\bar{\rho} > 0$ as above we consider numbers $r > \rho \geq \bar{\rho}$ and a function $\beta \in C^1(\mathbb{R})$ such that

$$\beta(0) = \beta(r) = 0, \quad \beta(\rho) = b_0, \tag{21}$$

$$\beta'(t) < 0 \iff t < 0 \quad \text{or} \quad \rho < t < r, \tag{22}$$

$$\lim_{|t| \to +\infty} \beta(t) = +\infty. \tag{23}$$

Lemma 5. *Assume that conditions* (A1) *and* (A3) *are fulfilled. Then, for any* $d>0$ *satisfying* (3), *the functional* $J : E \times \mathbb{R} \to \mathbb{R}$ *is defined by*

$$J(v,t)=\frac{t^2}{m}\|v\|_b^m+\frac{2}{m}\beta(t)-\frac{1}{p}\|v\|_{p,w_1}^p-\int_\Omega H(x,v)\,\mathrm{d}x$$
$$-\int_\Gamma G(x,v)\,\mathrm{d}x+\frac{d}{p}\|v\|_b^p,\quad \forall (v,t)\in E\times\mathbb{R} \tag{24}$$

is of class C^1 *and satisfies the Palais–Smale condition.*

Proof. The property of J which is continuously differentiable has been already justified in the proof of Theorem 2.

In order to check the Palais–Smale condition let the sequences $\{v_n\}\subset E$ and $\{t_n\}\subset\mathbb{R}$ satisfy

$$|J(v_n,t_n)|\le M,\quad \forall n\ge 1 \tag{25}$$

$$J'_v(v_n,t_n)=t_n^2\,\|v_n\|_b^{m-p}I'(v_n)$$
$$-K'_\Phi(v_n)-K'_H(v_n)-K'_G(v_n)+\mathrm{d}I'(v_n)\to 0\quad \text{as } n\to\infty, \tag{26}$$

$$J'_t(v_n,t_n)=\frac{2}{m}\,(t_n\|v_n\|_b^m+\beta'(t_n))\to 0, \tag{27}$$

where I, K_Φ, K_H, K_G have been introduced in the proof of Lemma 3.

From (20), (21), (24), and (25) we infer that

$$M\ge\frac{t_n^2}{m}\,\|v_n\|_b^m+\frac{2}{m}\beta(t_n)-\frac{1}{p}\|v_n\|_{p,w_1}^p$$
$$-\int_\Omega H(x,v_n)\,\mathrm{d}x-\int_\Gamma G(x,v_n)\,\mathrm{d}x+\frac{d}{p}\,\|v_n\|_b^p$$
$$\ge\frac{t_n^2-\rho^2}{m}\,\|v_n\|_b^m+\frac{2}{m}(\beta(t_n)-\beta(\rho))+\frac{d}{p}\,\|v_n\|_b^p.$$

Condition (23) in conjunction with the inequality above yields the boundedness of $\{t_n\}$.

Let us check the boundedness of $\{v_n\}$ along a subsequence. Without loss of generality, we may admit that $\{v_n\}$ is bounded away from 0. From (22) we deduce that the sequence $\{t_n\|v_n\|_b^m\}$ is bounded. Therefore, it is sufficient to argue in the case where $t_n\to 0$. From (24) it turns out that

$$\frac{1}{p}\|v_n\|_{p,w_1}^p+\int_\Omega H(x,v_n)\,\mathrm{d}x+\int_\Gamma G(x,v_n)\,\mathrm{d}x-\frac{d}{p}\|v_n\|_b^p$$

is bounded. By (26) it is seen that

$$\frac{1}{\|v_n\|_b}(-\langle K'_\Phi(v_n),v_n\rangle-\langle K'_H(v_n),v_n\rangle-\langle K'_G(v_n),v_n\rangle+d\|v_n\|_b^p)\to 0\quad \text{as } n\to\infty.$$

Then, for n sufficiently large, assumption (A3) allows to write

$$\begin{aligned} M+1+\|v_n\|_b &\geq d\left(\frac{1}{p}-\frac{1}{\mu}\right)\|v_n\|_b^p+\left(\frac{1}{\mu}-\frac{1}{q}\right)\|v_n\|^q_{L^q(\Omega,w_2)} \\ &\quad+\int_\Gamma\left(\frac{1}{\mu}g(x,v_n)v_n-G(x,v_n)\right)\,\mathrm{d}\Gamma+\left(\frac{1}{\mu}-\frac{1}{p}\right)\|v_n\|^p_{p,w_1} \\ &\geq\left(\frac{1}{p}-\frac{1}{\mu}\right)(d\|v_n\|_b^p-\|v_n\|^p_{p,w_1})\geq\left(\frac{1}{p}-\frac{1}{\mu}\right)\left(d-\frac{1}{\tilde{\lambda}}\right)\|v_n\|_b^p. \end{aligned}$$

By (3), this establishes the boundedness of $\{v_n\}$ in E.

In view of the compactness of the mappings K'_Φ, K'_H, K'_G (see the proof of Lemma 3), by (26) we get that

$$(d+t_n^2\,\|v_n\|_b^{m-p})\,I'(v_n)$$

converges in E as $n\to\infty$. The boundedness of $\{t_n\}$ and $\{v_n\}$ ensures that $\{I'(v_n)\}$ is convergent in E^* along a subsequence. Assume that $p\geq 2$. Inequality (13) shows that

$$\begin{aligned} \|u_n-u_k\|_b^p &\leq C\left[\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_k|^{p-2}\nabla u_k)\cdot(\nabla u_n-\nabla u_k)\,\mathrm{d}x\right. \\ &\quad\left.+\int_\Gamma b(x)(|u_n|^{p-2}u_n-|u_k|^{p-2}u_k)(u_n-u_k)\,\mathrm{d}\Gamma\right] \\ &=C\langle I'(u_n)-I'(u_k),u_n-u_k\rangle \\ &\leq C\|I'(u_n)-I'(u_k)\|_b^*\|u_n-u_k\|_b \quad \text{if } p\geq 2. \end{aligned}$$

Consequently, if $p\geq 2$, $\{v_n\}$ possesses a convergent subsequence. Proceeding in the same way with inequality (14) in place of (13) we obtain the result for $1<p<2$. □

In the proof of Theorem 3 we shall make use of the following variant of the Mountain-Pass Theorem (see [8]):

Lemma 6. *Let E be a Banach space and let $J: E\times\mathbb{R}\to\mathbb{R}$ be a C^1 functional verifying the hypotheses*

(a) *there exist constants $\rho>0$ and $\alpha>0$ such that $J(v,\rho)\geq\alpha$, for every $v\in E$;*

(b) *there is some $r>\rho$ with $J(0,0)=J(0,r)=0$.*

Then the number

$$c:=\inf_{g\in\mathscr{P}}\max_{0\leq\tau\leq 1}J(g(\tau))$$

is a critical value of J, where

$$\mathscr{P}:=\{g\in C([0,1]),E\times\mathbb{R});\ g(0)=(0,0),\ g(1)=(0,r)\}.$$

Proof of Theorem 3. We apply Lemma 6 to the function J defined in (24). It is clear that assertion (a) is verified with $\rho > 0$ and $\alpha > 0$ described in Lemma 4 and (20). Due to relation (21), condition (b) in Lemma 6 holds. Lemma 5 ensures that the functional J satisfies the Palais–Smale condition. Therefore, Lemma 6 yields a nonzero element $(u,t) \in E \times \mathbb{R}$ such that

$$J'_v(u,t) = (d + t^2/\|u\|_b^{m-p})I'(u) - K'_\Phi(u) - K'_H(u) - K'_G(u) = 0, \tag{28}$$

$$J'_t(u,t) = \frac{2}{m}(t\|u\|_b^m + \beta'(t)) = 0. \tag{29}$$

From (29) it follows that

$$t\beta'(t) \le 0. \tag{30}$$

Combining (30) and (22) we derive that if $t \neq 0$, then $u \neq 0$ and

$$\rho \le t \le r. \tag{31}$$

Therefore, for each d in (3) such that $1/d$ is not an eigenvalue in (B) and each $r > \rho \ge \bar{\rho}$ we deduce that there exists a critical point $(u,t) = (u_d, t_d) \in E \times \mathbb{R}_+$ of J, where $t = t_d$ verifies (31). Consequently, relation (28) establishes that $u_d \in E$ is an eigenfunction in problem (B) where the corresponding eigenvalue is

$$\lambda_d = \frac{1}{d + t_d^2\,\|u_d\|_b^{m-p}},$$

with $t = t_d$ satisfying (31). This completes the proof. □

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