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SEMICLASSICAL STATES FOR THE PSEUDO-RELATIVISTIC SCHRÖDINGER EQUATION WITH COMPETING POTENTIALS*

WEN ZHANG†, JIAN ZHANG‡, AND VICENŢIU D. RĂDULESCU§

Abstract. In this paper, we establish concentration and multiplicity properties of positive ground state solutions to the following perturbed pseudo-relativistic Schrödinger equation with competing potentials

$$\begin{cases} (-\epsilon^2\Delta+m^2)^s u+V(x)u=K(x)f(u) & \text{in } \mathbb{R}^N, \\ u\in H^s(\mathbb{R}^N),\ u>0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $N>2s$, ϵ is a small positive parameter, and $(-\Delta+m^2)^s$ is the pseudo-relativistic Schrödinger operator with $s\in(0,1)$ and mass $m>0$. We assume that the potentials V, K and the nonlinearity f are continuous but are not necessarily of class C^1. Under natural hypotheses, combining the extension method, Nehari analysis and the Ljusternik-Schnirelmann category theory, we first study the existence and concentration phenomena of positive solutions for $\epsilon>0$ sufficiently small, as well as multiplicity properties depending on the topology of the set where V attains its global minimum and K attains its global maximum. Moreover, we establish the asymptotic convergence and the exponential decay of positive solutions. In the final part of this paper, we provide a sufficient condition for the non-existence of ground state solutions.

Keywords. Pseudo-relativistic Schrödinger equation; ground states; concentration; multiplicity.

AMS subject classifications. 35R11; 35B09; 35J10; 35J20; 58E05.

1. Introduction

1.1. Historical background.

The Schrödinger equation is central in quantum mechanics and it plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future behaviour of a dynamical system. It is striking to point out that talking about his celebrating equation, Erwin Schrödinger said: "I don't like it, and I'm sorry I ever had anything to do with it". The linear Schrödinger equation is a central tool of quantum mechanics, which provides a thorough description of a particle in a non-relativistic setting. Schrödinger's linear equation is

$$\Delta\psi+\frac{8\pi^2 m}{\hbar^2}\,(E-V(x))\,\psi=0,$$

where ψ is the Schrödinger wave function, m is the mass of the particle, $\hbar$ denotes Planck's renormalized constant, E is the energy, and V stands for the potential energy.

Schrödinger also established the classical derivation of his equation, based upon the analogy between mechanics and optics, and closer to de Broglie's ideas. He developed a perturbation method, inspired by the work of Lord Rayleigh in acoustics, proved the

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†College of Science, Hunan University of Technology and Business, 410205 Changsha, Hunan, China (zwmath2011@163.com).

‡Corresponding author. College of Science, Hunan University of Technology and Business, 410205 Changsha, Hunan, China (zhangjian@hutb.edu.cn).

§Faculty of Applied Mathematics, AGH University of Kraków, 30-059 Kraków, Poland; Brno University of Technology, Faculty of Electrical Engineering and Communication, Technická 3058/10, Brno 61600, Czech Republic; Department of Mathematics, University of Craiova, 200585 Craiova, Romania; "Simion Stoilow" Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania; School of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China (radulescu@inf.ucv.ro).

equivalence between his wave mechanics and Heisenberg's matrix, and introduced the time dependent Schrödinger's equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi,\ x\in\mathbb{R}^N\ (N\geq 2), \tag{1.1}$$

where $p<2N/(N-2)$ if $N\geq 3$ and $p<+\infty$ if $N=2$.

In physical problems, a cubic nonlinearity corresponding to $p=3$ in Equation (1.1) is common; in this case problem (1.1) is called the Gross-Pitaevskii equation. In the study of equation (1.1), Floer and Weinstein [22] and Oh [35] supposed that the potential V is bounded and possesses a non-degenerate critical point at $x=0$. More precisely, it is assumed that V belongs to the class (V_a) (for some real number a) introduced in Kato [27]. Taking $\gamma>0$ and $\hbar>0$ sufficiently small and using a Lyapunov-Schmidt type reduction, Oh [35] proved the existence of *bound state* solutions of problem (1.1), that is, a solution of the form

$$\psi(x,t) = e^{-iEt/\hbar}u(x). \tag{1.2}$$

Using the Ansatz (1.2), we can reduce the nonlinear Schrödinger Equation (1.1) to the semilinear elliptic equation

$$-\frac{\hbar^2}{2m}\Delta u + (V(x)-E)u = |u|^{p-1}u.$$

The change of variable $y=\hbar^{-1}x$ (and replacing y by x) yields

$$-\Delta u + 2m(V_\hbar(x)-E)u = |u|^{p-1}u,\ x\in\mathbb{R}^N, \tag{1.3}$$

where $V_\hbar(x)=V(\hbar x)$.

Let us also recall that in his 1928 pioneering paper, G. Gamow [23] proved the *tunneling effect*, which led to the construction of the electronic microscope and the correct study of the alpha radioactivity. The notion of "solution" used by him was not explicitly mentioned in the paper but it is coherent with the notion of weak solution introduced several years later by other authors such as J. Leray, L. Sobolev and L. Schwartz. Most of the study developed by Gamow was concerned with the bound states $\psi(x,t)$ defined in (1.2), where u solves the stationary equation

$$-\Delta u + V(x)u = \lambda u \quad \text{in } \mathbb{R}^N,$$

for a given potential $V(x)$. Gamow was particularly interested in the Coulomb potential but he also proposed to replace the resulting potential by a simple potential that keeps the main properties of the original one. In this way, if Ω is a subdomain of $\mathbb{R}^N$, Gamow proposed to use the *finite well potential*

$$V_{q,\Omega}(x) = \begin{cases} V(x) & \text{if } x\in\Omega \\ q & \text{if } x\in\mathbb{R}^N\setminus\Omega \end{cases} \quad \text{for some } q\in\mathbb{R}.$$

It seems that the first reference dealing with the limit case, the so-called *infinite well potential*,

$$V_\infty(x;R,V_0) = \begin{cases} V_0 & \text{if } x\in\Omega \\ +\infty & \text{if } x\in\mathbb{R}^N\setminus\Omega \end{cases} \quad \text{for some } V_0\in\mathbb{R},$$

was the book by the 1977 Nobel Prize Mott [34]. The more singular case in which V_0 is the Dirac mass δ_0 is related with the so-called *Quantum Dots*, see Joglekar [26]. In contrast with classical mechanics, in quantum mechanics the incertitude appears (the Heisenberg principle). For instance, for a free particle (i.e. with $V(x)\equiv 0$), in nonrelativistic quantum mechanics, if the wave function $\psi(\cdot,t)$ at time $t=0$ vanishes outside some compact region $\overline{\Omega}$ then at an arbitrarily short time later the wave function is nonzero arbitrarily far away from the original region $\overline{\Omega}$. Thus, the wave function instantaneously spreads to infinity and the probability of finding the particle arbitrarily far away from the initial region is nonzero for all $t>0$.

1.2. Statement of the problem, features and main results. In this paper we consider the following singularly perturbed pseudo-relativistic Schrödinger equation with competing potentials

$$\begin{cases} (-\epsilon^2\Delta+m^2)^s u+V(x)u=K(x)f(u) & \text{in } \mathbb{R}^N, \\ u\in H^s(\mathbb{R}^N),\ u>0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $N>2s$, $\epsilon>0$ is small parameter, $(-\Delta+m^2)^s$ is pseudo-relativistic Schrödinger operator with $s\in(0,1)$ and mass $m>0$, V and K are potential functions and f is the reaction term with subcritical growth. We are interested in the qualitative and asymptotic analysis of solutions to problem (1.4) and we are mainly concerned with existence and multiplicity properties of solutions, as well as with concentration phenomena as $\epsilon\to 0$.

The features of this paper are the following:

(1) the pseudo-relativistic Schrödinger operator generates the nonlocal nature of the problem;

(2) the problem combines the multiple effects generated by two variable potentials;

(3) there exists an interesting competition effect between the external potential and the reaction potential, which implies more complex phenomena to locate the concentration positions;

(4) the main concentration phenomenon creates a bridge between the global maximum point of the solution versus the global minimum of the external potential and the global maximum of the reaction potential;

(5) due to the unboundedness of the domain, the Palais-Smale sequences do not have the compactness property;

(6) the proofs combine some refined estimates and some analysis techniques including extension, topological and variational tools.

Problem (1.4) arises when one is looking for the standing waves of the following time-dependent pseudo-relativistic Schrödinger equations:

$$i\hbar\frac{\partial\Psi}{\partial t}=(-\Delta+m^2)^s\Psi+V(x)\Psi-f(x,\Psi),\quad (x,t)\in\mathbb{R}^N\times\mathbb{R},$$

where Ψ represents the wave function, V is an external potential, m is the mass of free relativistic particle and the nonlinear coupling f describes a self-interaction among many particles. In physics this equation has been successfully used to describe the behavior of bosons, spin-0 particles in relativistic fields.

We observe that the pseudo-relativistic Schrödinger operator in (1.4) can be characterized as

$$(-\Delta+m^2)^s u(x)=\mathscr{F}^{-1}((|\xi|^2+m^2)^s\mathscr{F}u(\xi))(x),\quad x\in\mathbb{R}^N \tag{1.5}$$

for any rapidly decaying function u belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^N)$, where $\mathscr{F}$ denotes the usual Fourier transform and $\mathscr{F}^{-1}$ denotes its inverse transform. Besides, according to [29] (see also [2, 20]), the operator (1.5) also has the following expression with singular integral

$$\begin{aligned}&(-\Delta+m^2)^s u(x)\\ =&C_{N,s} m^{\frac{N+2s}{2}} \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}} W_{\frac{N+2s}{2}}(m|x-y|)\mathrm{d}y + m^{2s}u(x),\end{aligned} \tag{1.6}$$

where P.V. stands for the Cauchy principal value, W_ι is the modified Bessel function of the second kind of order ι, (the asymptotic properties of W_ι can be found in [7, 20]) and $C_{N,s}$ is a positive constant whose exact value is given by

$$C_{N,s} = 2^{-\frac{N+2s}{2}+1} \pi^{-\frac{N}{2}} 2^{2s} \frac{s(1-s)}{\Gamma(2-s)}.$$

Clearly, as $m \to 0$, the operator $(-\Delta+m^2)^s$ reduces to the well-known fractional Laplacian $(-\Delta)^s$ which has the expression with singular integral

$$(-\Delta)^s u(x) = \widehat{C}_{N,s} \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y \tag{1.7}$$

and

$$\widehat{C}_{N,s} = \pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)} s(1-s).$$

From (1.6) and (1.7) we can easily see that the most important difference between the operators $(-\Delta)^s$ and $(-\Delta+m^2)^s$ is that the first one is homogeneous in scaling whereas the second one is inhomogeneous as should be clear from the presence of the Bessel function W_ι in (1.6).

During the last two decades, some fractional problems involving (1.7) have been widely investigated due to many applications in different fields, such as quantum mechanics, phase transitions, anomalous diffusions, chemical reaction in liquids and so on. In particular, a great interest has been devoted to the existence and multiplicity and asymptotic behaviors of solutions for fractional Schrödinger equation

$$\epsilon^{2s}(-\Delta)^s u + V(x)u = f(u) \text{ in } \mathbb{R}^N. \tag{1.8}$$

Here, since we cannot introduce the huge bibliography on this subject, we refer the readers to [1, 17, 19, 21, 25, 32, 36, 37, 44, 45] for the existence, multiplicity, concentration and regularity of positive solutions. We also refer to the monograph [6] in which the author studies several nonlinear fractional Schrödinger equations using suitable variational and topological methods, and the monograph by Molica Bisci-Rădulescu-Servadei [33] for a very comprehensive introduction for the nonlocal fractional problems.

On the other hand, there have been many works concerning with the study of existence and properties of solutions for the fractional equations driven by $(-\Delta+m^2)^s$ with $m>0$. For the case $s=\frac{1}{2}$, Lieb and Yau [30] first studied the following pseudo-relativistic Hartree equation

$$\sqrt{-\Delta+m^2}u + V(x)u = \left(\frac{1}{|x|} * |u|^2\right) u \ \text{ in } \mathbb{R}^N, \tag{1.9}$$

the radially symmetric ground state solution was proved via minimization argument. Lenzmann [28] proved that this ground state solution is unique up to translations and phase change, and the non-degeneracy result of the ground state solution was also obtained. Later, Coti Zelati-Nolasco [15] investigated the existence of positive radially symmetric ground state solution for (1.9) with more general radially symmetric convolution kernel. Under the radially symmetric condition for potential V, Melgaard-Zongo [31] proved the existence of the radially symmetric solutions with high energy. Without the symmetric condition for the external potential V, the positive ground state solution was constructed by Cingolani-Secchi [13], and some asymptotic decay estimates of solutions were also proved. For the general case $s\in(0,1)$, Ambrosio [4] proved the existence and symmetry of ground state solutions for problem (1.4) without external potential V. We also refer to [5, 9] for more results about the regularity and decay of solutions.

When $\epsilon>0$ sufficiently small, the solutions of (1.4) are often referred to as semi-classical states, which have very rich dynamic behaviors, such as concentration, convergence and decay etc. Especially, the concentration phenomenon of semiclassical states, as $\epsilon\to 0$, reflects the transition from quantum mechanics to classical mechanics and it gives rise to significant physical insights. As far as we know there exist relatively few papers treating the existence and concentration of semiclassical states to (1.4). Let us now briefly recall some related results in this direction.

Regarding the study of semiclassical analysis for problem (1.4) we would like to mention the papers [2, 3, 14, 24]. More precisely, under the local hypothesis introduced by del Pino-Felmer [16]: there exists a bounded domain Ω such that

$$\inf_{x\in\Omega} V(x) < \inf_{x\in\partial\Omega} V(x), \tag{1.10}$$

Cingolani-Secchi [14] studied the semi-classical limit for the pseudo-relativistic Hartree equation

$$\sqrt{-\epsilon^2\Delta+m^2}u+V(x)u=(I_\alpha*|u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Using the extension method developed by Caffarelli-Silvestre [11] and the penalization technique introduced by Byeon-Jeanjean [10], they established the existence of a single-spike solution which concentrates at the local minimum points of V. Taking advantage of the same method as in [14], Gao-Rădulescu-Yang-Zheng [24] proved the existence of multi-peak solutions when the nonlinearity satisfies general hypotheses of Berestycki-Lions type.

Also under the condition (1.10), Ambrosio [2] investigated the following general pseudo-relativistic Schrödinger equation with linear external potential and subcritical growth

$$(-\epsilon^2\Delta+m^2)^s u+V(x)u=f(u) \quad \text{in } \mathbb{R}^N, \tag{1.11}$$

where $s\in(0,1)$. Using the penalization method [16] combined with the Ljusternik-Schnirelmann category theory, the author obtained the multiplicity result and concentration properties of semiclassical positive solutions to problem (1.11). We also mention the recent paper [3] in which some similar results for the case of critical growth were established.

We would like to point out that, in all the works mentioned above, the authors only considered the effect of the linear potential V on the existence, multiplicity and concentration phenomena of solutions for problem (1.4) or similar to (1.4). For such case, the problem is autonomous in the reaction, in the sense that the nonlinearities on the right-hand side of the equation do not depend on the variable x. That is why it is quite natural to ask how the appearance of nonlinear potential and linear potential will affect the existence, multiplicity and concentration of solutions to problem (1.4)? This is the main motivation of the present paper and we will give an affirmative answer, which also complements and extends the results before.

Inspired by the above facts, in this paper we will investigate the existence, multiplicity and concentration phenomena of positive solutions to problem (1.4) under the effects of both linear potential V and nonlinear potential K. Following [41] (see also [43]), the combination of linear potential and nonlinear potential is called the competing potentials, which creates difficulties in determining the concentration positions of solutions. This happens because V has the tendency to attract solutions to its minimum points, while the potential K tends to attract solutions to its maximum points. Therefore, the study of the concentration phenomena of semiclassical states to problem (1.4) becomes more delicate and involved under the effects of competing potentials.

More precisely, the main ingredients of this paper are the following four aspects. We first prove the existence of positive ground state solutions for small ϵ. Secondly, we determine two concrete sets related to the potentials V and K as the concentration positions to study the concentration phenomena of these solutions as $\epsilon \to 0$. Thirdly, we analyze the asymptotic convergence of ground state solutions under scaling and translation and the exponential decay estimate. Finally, we investigate the relation between the number of positive solutions and the topology of the set where V attains its global minimum and K attains its global maximum. To the best of our knowledge, the present paper is the first work dealing with multiplicity and concentration properties for the general pseudo-relativistic Schrödinger equations in the presence of two competing potentials.

Concerning the potentials V and K, we use the following notations:

$$V_{\min} = \min V,\ \mathscr{V} = \{x \in \mathbb{R}^N : V(x) = V_{\min}\} \text{ and } V_\infty = \liminf_{|x| \to \infty} V(x)$$

and

$$K_{\max} = \max K,\ \mathscr{K} = \{x \in \mathbb{R}^N : K(x) = K_{\max}\} \text{ and } K_\infty = \limsup_{|x| \to \infty} K(x).$$

Let us now introduce the following assumptions on V and K in the spirit of [18].

- (A_0) $V, K \in C(\mathbb{R}^N, \mathbb{R})$ are bounded, $V_{\min} \in (-m^{2s}, 0)$ and $K_{\min} := \inf K > 0$;
- (A_1) $V_{\min} < V_\infty$ and there is $x_v \in \mathscr{V}$ such that $K(x_v) \geq K(x)$ for all $|x| \geq R$ and some large $R > 0$;
- (A_2) $K_{\max} > K_\infty$ and there is $x_k \in \mathscr{K}$ such that $V(x_k) \leq V(x)$ for all $|x| \geq R$ and some large $R > 0$;
- (A_3) $V, K \in C(\mathbb{R}^N, \mathbb{R})$ are bounded functions such that $0 < V^\infty := \lim_{|x| \to \infty} V(x) \leq V(x)$ and $0 < K(x) \leq K^\infty := \lim_{|x| \to \infty} K(x)$, and $|\mathcal{V}| > 0$ or $|\mathcal{K}| > 0$, where

$$\mathcal{V} = \{x \in \mathbb{R}^N : V^\infty < V(x)\} \text{ and } \mathcal{K} = \{x \in \mathbb{R}^N : K^\infty > K(x)\}.$$

Note that, for case (A_1), we can assume $K(x_v) = \max_{x \in \mathscr{V}} K(x)$, and for case (A_2), we can assume $V(x_k) = \min_{x \in \mathscr{K}} V(x)$. In order to characterize the concentration phenomena of positive ground state solutions, we consider the following sets:

$$\mathscr{A}_v := \{x \in \mathscr{V} : K(x) = K(x_v)\} \cup \{x \notin \mathscr{V} : K(x) > K(x_v)\},$$

and

$$\mathscr{A}_k := \{x \in \mathscr{K} : V(x) = V(x_k)\} \cup \{x \notin \mathscr{K} : V(x) < V(x_k)\}.$$

We also observe that $x_v \in \mathscr{A}_v$ and $x_k \in \mathscr{A}_k$, which implies that $\mathscr{A}_v$ and $\mathscr{A}_k$ are non-empty and bounded sets. Furthermore, if (A_1) holds and $\mathscr{V} \cap \mathscr{K} \neq \emptyset$, we can set $K(x_v) = \max_{x \in \mathscr{V} \cap \mathscr{K}} K(x)$ and

$$\mathscr{A}_v := \{x \in \mathscr{V} \cap \mathscr{K} : K(x) = K(x_v)\},$$

then $\mathscr{A}_v = \mathscr{V} \cap \mathscr{K}$. Similarly, if (A_2) holds and $\mathscr{V} \cap \mathscr{K} \neq \emptyset$, then $\mathscr{A}_k = \mathscr{V} \cap \mathscr{K}$.

Meanwhile, we assume that the nonlinearity f satisfies the following conditions:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(t) = 0$ for all $t < 0$;

(f_2) $f(t) = o(|t|)$ as $t \to 0$;

(f_3) there are $c_0 > 0$ and $p \in (2, 2_s^*)$ with $2_s^* = \frac{2N}{N-2s}$ such that $f(t) \leq c_0(1 + |t|^{p-1})$ for all t;

(f_4) there exists $\theta \in (2, 2_s^*)$ such that

$$0 < \theta F(t) = \theta \int_0^t f(\tau) \mathrm{d}\tau \leq t f(t) \text{ for all } t > 0;$$

(f_5) $\frac{f(t)}{t}$ is increasing for all $t \in (0, \infty)$.

The main results of this paper can be stated as follows:

THEOREM 1.1. *Suppose that (A_0), (A_1) and (f_1)-(f_5) are satisfied, then for all small $\epsilon > 0$*

(i) *problem (1.4) has at least a positive ground state solution u_ϵ;*

(ii) *$\mathscr{L}_\epsilon$ is compact, where $\mathscr{L}_\epsilon$ denotes the set of all ground state solutions;*

(iii) *$u_\epsilon(x)$ possesses a maximum point x_ϵ such that, up to a subsequence, $x_\epsilon \to x_0$ as $\epsilon \to 0$, and $\lim_{\epsilon \to 0} dist(x_\epsilon, \mathscr{A}_v) = 0$, and $v_\epsilon(x) := u_\epsilon(\epsilon x + x_\epsilon)$ converges to a ground state solution of*

$$(-\Delta + m^2)^s u + V(x_0) u = K(x_0) f(u) \text{ in } \mathbb{R}^N.$$

In particular, if $\mathscr{V} \cap \mathscr{K} \neq \emptyset$, then $\lim_{\epsilon \to 0} dist(x_\epsilon, \mathscr{V} \cap \mathscr{K}) = 0$, and up to a subsequence, v_ϵ converges to a ground state solution of

$$(-\Delta + m^2)^s u + V_{\min} u = K_{\max} f(u) \text{ in } \mathbb{R}^N.$$

(iv) *There exist positive constants c, C such that*

$$u_\epsilon(x) \leq C \exp\left(-\frac{c}{\epsilon}|x - x_\epsilon|\right).$$

THEOREM 1.2. *Suppose that (A_0), (A_2) and (f_1)-(f_5) are satisfied, then all the conclusions of Theorem 1.1 remain true with $\mathscr{A}_v$ replaced by $\mathscr{A}_k$.*

In order to study the multiplicity result of positive solutions for problem (1.4), we first recall the definition of Ljusternik-Schnirelmann category. If Y is a given closed subset of a topological space X, the Ljusternik-Schnirelmann category $\mathrm{cat}_X(Y)$ is the least number of closed and contractible sets in X which cover Y.

To obtain the multiplicity result, in the following we assume $\mathscr{V}\cap\mathscr{K}\neq\emptyset$. Let us denote by

$$\Lambda:=\mathscr{V}\cap\mathscr{K} \text{ and } \Lambda_\delta=\{x\in\mathbb{R}^N:\operatorname{dist}(x,\Lambda)\leq\delta\} \text{ for } \delta>0.$$

Evidently, from (A_0), (A_1) and (A_2) we can see that the set Λ is compact. The multiplicity result is the following theorem.

THEOREM 1.3. *Suppose that (A_0), (A_1) (or (A_2)) and (f_1)-(f_5) are satisfied and $\Lambda\neq\emptyset$. Then for any $\delta>0$ there exists $\epsilon_\delta>0$ such that, for any $\epsilon\in(0,\epsilon_\delta)$, problem (1.4) has at least $cat_{\Lambda_\delta}(\Lambda)$ positive solutions.*

Finally, we give the non-existence result of ground state solutions as follows.

THEOREM 1.4. *Suppose that (A_3) and (f_1)-(f_5) are satisfied, then for each $\epsilon>0$, problem (1.4) has no positive ground state solutions.*

Let us now outline the strategies and approaches to establish Theorems 1.1-1.4. Our arguments are based on appropriate topological and variational arguments inspired by [2, 40].

Firstly, we observe that the operator $(-\Delta+m^2)^s$ involved in problem (1.4) has the nonlocal feature and that does not scale like the fractional Laplacian operator $(-\Delta)^s$. More precisely, the operator $(-\Delta+m^2)^s$ is not compatible with the semigroup $\mathbb{R}_+$ acting on functions as $t*u\mapsto u(t^{-1}x)$ for $t>0$. This feature means that some arguments used to handle (1.8) do not work in our problem. To surmount these difficulties, we will use a variant of the s-harmonic extension method from Caffarelli-Silvestre [11] (see also [20]) which allows us to investigate (1.4) by studying a local problem via suitable variational methods.

Secondly, due to the fact that the nonlinear term f is only continuous, the Nehari manifold is not differentiable and some well-known arguments for C^1-Nehari manifold are not applicable in our situation. To overcome this difficulty created by the non-differentiability, we will use the method developed by Szulkin-Weth [39] to handle the present problem. The main idea of this method is to find a homeomorphism mapping between the Nehari manifold and the unit sphere of working space. Then, one can construct a reduction functional on the unit sphere such that critical points of reduction functional are in one-to-one correspondence with critical points of the original functional.

Thirdly, the combined effects of lack of compactness and competition of two potentials bring some difficulties to our analysis, it is difficult to prove that the energy functional has the Palais-Smale compactness property. This goal will be achieved by doing a finer analysis and using the energy comparison method to establish some comparison relationships of the ground state energy value between the original problem and certain auxiliary problems. Furthermore, these comparison relationships we established are also very beneficial for proving the concentration phenomena and nonexistence of solutions, which play a fundamental role in the study. Arguing as in [2], we prove the regularity and exponential decay of solutions. These properties contribute to determining the concentration location of solutions. To obtain the multiplicity result of positive solutions, we use the Ljusternik-Schnirelmann category theory and the techniques due to Benci-Cerami [8] based on precise comparisons between the category of some sublevel sets of the energy functional and the category of the set Λ.

Finally, it should be pointed out that the main results presented in this paper are new and have not been established previously for the general pseudo-relativistic Schrödinger equations. And all of conclusions are new even for the special case $s=\frac{1}{2}$.

Moreover, we would like to point out that our arguments are rather flexible and we believe that the ideas contained here can be applied in other fractional problems with competing potentials.

The remaining part of the paper is organized as follows. In Section 2, we establish a suitable variational framework of problem (1.4) and introduce the Nehari manifold method. In Section 3, we present some results for the autonomous problem. In Section 4, we analyze the Palais-Smale compactness condition. In Section 5, we prove the existence and concentration of positive ground state solutions and we complete the proofs of Theorem 1.1 and Theorem 1.2. In Section 6, we are devoted to the multiplicity result and we give the proof of Theorem 1.3. Finally, we prove the non-existence result of ground state solutions and finish the proof of Theorem 1.4 in Section 7.

2. Variational framework and Nehari manifold

Throughout the paper, we introduce following notations which will be used later.

- The symbol $\mathbb{R}^{N+1}_+$ denotes the half space $\{(x,y): x\in\mathbb{R}^N, y>0\}$;
- $\|\cdot\|_p$ denotes the usual norm of the space $L^p(\mathbb{R}^N)$, $1\le p\le\infty$;
- c, c, C, C_i denote some different positive constants;
- For $x\in\mathbb{R}^N$, $r>0$, we will denote by $B_r(x)$ the ball in $\mathbb{R}^N$ centered at x with radius r;
- For $x\in\mathbb{R}^{N+1}_+$ and $r>0$, $B_r^+(x)$ will be the ball in $\mathbb{R}^{N+1}_+$ centered at x with radius r;
- $u^+=\max\{u,0\}$ and $u^-=\min\{u,0\}$.

In what follows, we introduce some definitions and basic results of the Lebesgue spaces with weight. Let $\Omega\subset\mathbb{R}^{N+1}_+$ be an open set. $L^p(\Omega,y^{1-2s})$ denotes the weighted Lebesgue space of all measurable functions $u:\Omega\to\mathbb{R}$ such that

$$\|u\|_{L^p(\Omega,y^{1-2s})}=\left(\iint_\Omega y^{1-2s}|u|^p\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}<\infty.$$

We define the weighted Sobolev space $H^1(\Omega,y^{1-2s})$ with the norm

$$\|u\|_{H^1(\Omega,y^{1-2s})}=\left(\iint_\Omega y^{1-2s}(|\nabla u|^2+|u|^2)\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}$$

and the inner product

$$(u,v)=\iint_\Omega y^{1-2s}(\nabla u\nabla v+uv)\mathrm{d}x\mathrm{d}y.$$

Let $H^s(\mathbb{R}^N)$ be the usual fractional Sobolev space defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{H^s}=\left(\int_{\mathbb{R}^N}(|\xi|^2+m^2)^s|\mathscr{F}u(\xi)|^2\mathrm{d}\xi\right)^{\frac{1}{2}},$$

where $\mathscr{F}$ denotes the usual Fourier transform and $m>0$.

We define $X^s(\mathbb{R}^{N+1}_+):=H^1(\mathbb{R}^{N+1}_+,y^{1-2s})$, which is the completion of $C_0^\infty(\mathbb{R}^{N+1}_+)$ with respect to the norm

$$\|u\|_{X^s}=\left(\iint_{\mathbb{R}^{N+1}_+}y^{1-2s}(|\nabla u|^2+m^2u^2)\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.$$

According to [20, Lemma 3.1], we can know that $X^s(\mathbb{R}_+^{N+1})$ is continuously embedded in $L^{2q_0}(\mathbb{R}_+^{N+1},y^{1-2s})$, and there holds

$$\|u\|_{L^{2q_0}(\mathbb{R}_+^{N+1},y^{1-2s})}\leq c_1\|u\|_{X^s} \text{ for all } u\in X^s(\mathbb{R}_+^{N+1}), \tag{2.1}$$

where $q_0=1+\frac{2}{N-2s}$. Moreover, we also know that $X^s(\mathbb{R}_+^{N+1})$ is compactly embedded in the space $L^2(B_R^+,y^{1-2s})$ for all $R>0$.

Following [20, Proposition 5], there is a linear trace operator $\mathrm{Tr}: X^s(\mathbb{R}_+^{N+1})\to H^s(\mathbb{R}^N)$ such that

$$\sigma_s\|\mathrm{Tr}(u)\|_{H^s}^2\leq\|u\|_{X^s}^2 \text{ for all } u\in X^s(\mathbb{R}_+^{N+1}),$$

where $\sigma_s=2^{1-2s}\Gamma(1-s)/\Gamma(s)$ is a normalization constant. Moreover, according to the definition of H^s-norm we can get

$$\sigma_s m^{2s}\|\mathrm{Tr}(u)\|_2^2\leq\sigma_s\|\mathrm{Tr}(u)\|_{H^s}^2\leq\|u\|_{X^s}^2 \text{ for all } u\in X^s(\mathbb{R}_+^{N+1}). \tag{2.2}$$

In the sequel, in order to simplify the notation, we denote $\mathrm{Tr}(u)$ by $u(x,0)$.

We observe that $\mathrm{Tr}(X^s(\mathbb{R}_+^{N+1}))\subseteq H^s(\mathbb{R}^N)$, and together with the fact that the embedding $H^s(\mathbb{R}^N)\hookrightarrow L^q(\mathbb{R}^N)$ is continuous for all $q\in[2,2_s^*]$, and locally compact for all $q\in[1,2_s^*)$, we have the following embedding result.

LEMMA 2.1. *$Tr(X^s(\mathbb{R}_+^{N+1}))$ is embedded continuously into $L^q(\mathbb{R}^N)$ for any $q\in[2,2_s^*]$ and compactly into $L^q_{loc}(\mathbb{R}^N)$ for any $q\in[1,2_s^*)$.*

We recall the following Lions compactness lemma, see Lemma 3.3 in [2].

LEMMA 2.2. *Let $p\in[2,2_s^*)$. If $\{u_n\}\subset X^s(\mathbb{R}_+^{N+1})$ is a bounded sequence and if*

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_R(z)}|u_n(x,0)|^p\mathrm{d}x=0,$$

where $R>0$, then $u_n(x,0)\to 0$ in $L^q(\mathbb{R}^N)$ for all $q\in(2,2_s^*)$.

We introduce the extension method for the pseudo-differential operator $(-\Delta+m^2)^s$. Precisely speaking, for any $u\in H^s(\mathbb{R}^N)$, there exists a unique function $w\in X^s(\mathbb{R}_+^{N+1})$ solving the following problem

$$\begin{cases}-\mathrm{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & \text{in } \mathbb{R}_+^{N+1},\\ w(x,0)=u, & \text{on } \mathbb{R}^N.\end{cases}$$

The function w is called the extension of u and has the following properties:

(1) $\frac{\partial w}{\partial\nu^{1-2s}}:=-\lim\limits_{y\to 0}y^{1-2s}\frac{\partial w}{\partial y}(x,y)=\sigma_s(-\Delta+m^2)^s u(x)$ in distribution sense;

(2) $\sigma_s\|u\|_{H^s}^2=\|w\|_{X^s}^2$;

(3) if $u\in\mathcal{S}(\mathbb{R}^N)$, then $w\in C^\infty(\mathbb{R}_+^{N+1})\cap C(\overline{\mathbb{R}_+^{N+1}})$ and it can be expressed as

$$w(x,y)=\int_{\mathbb{R}^N}P_{s,m}(x-z,y)u(z)\mathrm{d}z$$

with

$$P_{s,m}(x,y):=C(N,s)y^{2s}m^{\frac{N+2s}{2}}|(x,y)|^{-\frac{N+2s}{2}}W_{\frac{N+2s}{2}}(m|(x,y)|)$$

and

$$C(N,s)=p_{N,s}2^{\frac{N+2s}{2}-1}/\Gamma(\frac{N+2s}{2}),$$

where $p_{N,s}$ is the constant for the Poisson kernel with $m=0$, see [38].

In order to investigate some properties of solutions, we need to apply some conclusions about local Schauder estimates for degenerate elliptic equations involving the operator

$$-\mathrm{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w \text{ with } m>0.$$

Let $\Omega\subset\mathbb{R}_+^{N+1}$ be a bounded domain with $\partial\Omega\neq\emptyset$, and let $h\in L_{loc}^{\frac{2N}{N+2s}}(\partial\Omega)$ and $g\in L_{loc}^1(\partial\Omega)$. We consider the following problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & \text{in } \Omega,\\ \frac{\partial w}{\partial\nu^{1-2s}}=h(x)w+g(x), & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here we say $w\in H^1(\Omega,y^{1-2s})$ is a weak supersolution (resp. subsolution) to (2.3) in Ω if for any nonnegative $\varphi\in C_0^1(\Omega\cup\partial\Omega)$,

$$\iint_\Omega y^{1-2s}(\nabla w\nabla\varphi+m^2w\varphi)\mathrm{d}x\mathrm{d}y\geq(\leq)\int_{\partial\Omega}[h(x)w(x,0)+g(x)]\varphi(x,0)\mathrm{d}x.$$

We say that $w\in H^1(\Omega,y^{1-2s})$ is a weak solution to (2.3) in Ω if it is both a weak supersolution and a weak subsolution.

For $R>0$, let $\Omega_R:=B_R\times(0,R)$. We introduce the following results proved in [20].

Lemma 2.3. *Let $f,g\in L^q(B_1)$ for some $q>\frac{N}{2s}$.*

(a) *Let $u\in H^1(\Omega_1,y^{1-2s})$ be a weak subsolution to (2.3) in Ω_1, then*

$$\sup_{\Omega_{1/2}}u^+\leq c\left(\|u^+\|_{L^2(\Omega_1,y^{1-2s})}+|g^+|_{L^q(B_1)}\right),$$

where $c>0$ depends only on m,N,s,q and $|f^+|_{L^q(B_1)}$.

(b) *Let $u\in H^1(\Omega_1,y^{1-2s})$ be a nonnegative weak supersolution to (2.3) in Ω_1, then for some $p_0>0$ and any $0<r_1<r_2<1$ we get*

$$\inf_{\bar\Omega_{r_1}}u+|g^-|_{L^q(B_1)}\geq c\|u\|_{L^{p_0}(\Omega_{r_2},y^{1-2s})},$$

where $c>0$ depends only on m,N,s,q,r_1,r_2 and $|f^-|_{L^q(B_1)}$.

(c) *Let $u\in H^1(\Omega_1,y^{1-2s})$ be a nonnegative weak solution to (2.3) in Ω_1, then $u\in C^{0,\alpha}(\bar\Omega_{1/2})$ and*

$$\|u\|_{C^{0,\alpha}(\bar\Omega_{1/2})}\leq c(\|u\|_{L^2(\Omega_1)}+|g|_{L^q(B_1)})$$

where $\alpha\in(0,1)$, $c>0$ depends only on m,N,s,q and $|f|_{L^q(B_1)}$.

Observe that, making the change of variable $x\mapsto\epsilon x$, then problem (1.4) is equivalent to the following problem

$$\begin{cases} (-\Delta+m^2)^su+V(\epsilon x)u=K(\epsilon x)f(u), & \text{in } \mathbb{R}^N,\\ u\in H^s(\mathbb{R}^N),u>0, & \text{in } \mathbb{R}^N. \end{cases} \tag{2.4}$$

Clearly, if u is a solution of problem (2.4), then $v(x):=u(x/\epsilon)$ is a solution of problem (1.4). Thus, to study the original problem (1.4), it suffices to study the equivalent problem (2.4).

Furthermore, according to the previous discussion and using the extension method, we are able to transform problem (2.4) into the following local problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w}{\partial \nu^{1-2s}}=\sigma_s(-V(\epsilon x)w(x,0)+K(\epsilon x)f(w(x,0))), & \text{on } \mathbb{R}^N. \end{cases} \tag{2.5}$$

If w is a solution of problem (2.5), then the trace $u(x)=\mathrm{Tr}(w)=w(x,0)$ is a solution of problem (2.4), and the converse is also true. Therefore, both formulations are equivalent. For the sake of convenience, we set the constant $\sigma_s=1$ for the second equation in (2.5).

Now we establish the variational framework of problem (2.5). For any fixed $\epsilon>0$, we define the working space of problem (2.5)

$$E_\epsilon=\left\{u\in X^s(\mathbb{R}^{N+1}_+):\int_{\mathbb{R}^N}V(\epsilon x)u^2(x,0)\mathrm{d}x<\infty\right\}$$

endowed with the norm

$$\|u\|_\epsilon=\left[\|u\|^2_{X^s}+\int_{\mathbb{R}^N}V(\epsilon x)u^2(x,0)\mathrm{d}x\right]^{\frac{1}{2}}$$

and the inner product

$$(u,v)_\epsilon=\iint_{\mathbb{R}^{N+1}_+}y^{1-2s}(\nabla u\nabla v+m^2uv)\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^N}V(\epsilon x)u(x,0)v(x,0)\mathrm{d}x$$

for all $u,v\in E_\epsilon$. From condition (A_0) we know that the potential V is sign-changing. Still, we can also check that $\|\cdot\|_\epsilon$ is actually a norm. In fact, as observed in [2], we have that

$$\|u\|^2_\epsilon=\left[\|u\|^2_{X^s}+V_{\min}\int_{\mathbb{R}^N}u^2(x,0)\mathrm{d}x\right]+\int_{\mathbb{R}^N}[V(\epsilon x)-V_{\min}]u^2(x,0)\mathrm{d}x.$$

Using (2.2) and the fact $V_{\min}\in(-m^{2s},0)$ we have

$$\left[1+\frac{V_{\min}}{m^{2s}}\right]\|u\|^2_{X^s}\le\left[\|u\|^2_{X^s}+V_{\min}\int_{\mathbb{R}^N}u^2(x,0)\mathrm{d}x\right]\le\|u\|^2_{X^s},$$

which implies that

$$\|u\|^2_{X^s}+V_{\min}\int_{\mathbb{R}^N}u^2(x,0)\mathrm{d}x$$

and $\|\cdot\|_{X^s}$ are equivalent. Therefore, we can see that $\|\cdot\|_\epsilon$ is actually a norm and $E_\epsilon\subset X^s(\mathbb{R}^{N+1}_+)$. Using (A_0) and (2.2) again, we can conclude that

$$\|u\|^2_{X^s}\le\left[\frac{m^{2s}}{m^{2s}+V_{\min}}\right]\|u\|^2_\epsilon. \tag{2.6}$$

We define the energy functional associated to problem (2.5) on E_ϵ

$$\begin{aligned}\Phi_\epsilon(u)&=\frac{1}{2}\|u\|^2_{X^s}+\frac{1}{2}\int_{\mathbb{R}^N}V(\epsilon x)u^2(x,0)\mathrm{d}x-\int_{\mathbb{R}^N}K(\epsilon x)F(u(x,0))\mathrm{d}x\\&=\frac{1}{2}\|u\|^2_\epsilon-\int_{\mathbb{R}^N}K(\epsilon x)F(u(x,0))\mathrm{d}x.\end{aligned}$$

Using Lemma 2.1 and some standard arguments, we can check that $\Phi_\epsilon \in C^1(E_\epsilon,\mathbb{R})$, and critical points of Φ_ϵ correspond to weak solutions of problem (2.5). Moreover, for any $u,v\in E_\epsilon$, we have

$$\langle \Phi'_\epsilon(u),v\rangle = (u,v)_\epsilon - \int_{\mathbb{R}^N} K(\epsilon x) f(u(x,0)) v(x,0)\mathrm{d}x.$$

From (f_1), (f_2) and (f_3) we can deduce that for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|f(t)|\le \varepsilon|t| + C_\varepsilon |t|^{p-1} \text{ and } |F(t)|\le \varepsilon |t|^2 + C_\varepsilon |t|^p \text{ for any } t\in\mathbb{R}. \tag{2.7}$$

Moreover, (f_5) implies that

$$F(t)>0 \text{ and } \frac{1}{2}f(t)t - F(t)>0,\ \forall t>0. \tag{2.8}$$

To prove the positive ground state solutions of problem (2.5), we will use the method of Nehari manifold developed by Szulkin and Weth [39]. Define the Nehari manifold associated to Φ_ϵ by

$$\mathscr{N}_\epsilon := \{u\in E_\epsilon \setminus \{0\} : \langle \Phi'_\epsilon(u),u\rangle = 0\},$$

and the ground state energy value

$$c_\epsilon := \inf_{\mathscr{N}_\epsilon} \Phi_\epsilon.$$

Evidently, we can see that if c_ϵ is achieved by $u_\epsilon\in\mathscr{N}_\epsilon$, then u_ϵ is a critical point of Φ_ϵ. Since c_ϵ is the lowest level for Φ_ϵ, then u_ϵ is called a ground state solution of problem (2.5).

Applying Lemma 2.1 and some standard arguments, it is easy to check that the functional Φ_ϵ satisfies some elementary properties.

LEMMA 2.4. *Suppose that(f_1)-(f_5) are satisfied, then Φ_ϵ satisfies the following properties:*

(1) *Φ'_ϵ maps bounded sets of E_ϵ into bounded sets of E_ϵ;*

(2) *Φ'_ϵ is weakly sequentially continuous in E_ϵ.*

LEMMA 2.5. *Suppose that (A_0) and (f_1)-(f_5) are satisfied, then*

(1) *there exist α, $\varrho>0$ such that $\Phi_\epsilon(u)\ge\alpha$ with $\|u\|_\epsilon=\varrho$;*

(2) *there exist $u\in E_\epsilon$ and $R>0$ with $\|u\|_\epsilon>R$ such that $\Phi_\epsilon(u)<0$.*

Proof.
(1) From (2.2), (2.6), (2.7) and Lemma 2.1 we get

$$\begin{aligned}\Phi_\epsilon(u) &\ge \frac{1}{2}\|u\|_\epsilon^2 - \varepsilon\|u(x,0)\|_2^2 - C_\epsilon K_{\max}\|u(x,0)\|_p^p\\ &\ge \frac{1}{2}\|u\|_\epsilon^2 - \frac{\varepsilon}{m^{2s}}\|u\|_{X^s}^2 - c_2C_\epsilon\|u\|_{X^s}^p\\ &\ge \left[\frac{1}{2} - \frac{\varepsilon}{m^{2s}+V_{\min}}\right]\|u\|_\epsilon^2 - c_3C_\epsilon\|u\|_\epsilon^p.\end{aligned}$$

Using (A_0), $p>2$ and the arbitrariness of ε, then there exist $\alpha,\varrho>0$ such that $\Phi_\epsilon(u)\ge\alpha$ for $\|u\|_\epsilon=\varrho$.

(2) Let $e \in E_\epsilon \backslash \{0\}$, from (f_4) we have

$$\Phi_\epsilon(te) = \frac{1}{2}\|e\|_\epsilon^2 - \int_{\mathbb{R}^N} K(\epsilon x)\frac{F(te)}{t^2}\mathrm{d}x \to -\infty \text{ as } t \to \infty.$$

Evidently, the conclusion (2) holds. □

Lemma 2.5 shows that Φ_ϵ satisfies the usual mountain pass geometry, then we can use a version of mountain pass theorem without the Palais-Smale condition [42] to yield the existence of a Palais-Smale sequence $\{u_n\}$ at level $\tilde{c}_\epsilon$, namely

$$\Phi_\epsilon(u_n) \to \tilde{c}_\epsilon \text{ and } \Phi_\epsilon'(u_n) \to 0,$$

where $\tilde{c}_\epsilon$ is the mountain pass level of Φ_ϵ defined as

$$\tilde{c}_\epsilon = \inf_{\ell \in \Gamma} \max_{t \in [0,1]} \Phi_\epsilon(\ell(t)),$$

and

$$\Gamma = \{\ell \in C([0,1], E_\epsilon) : \ell(0) = 0, \Phi_\epsilon(\ell(1)) < 0\}.$$

Since $\mathscr{N}_\epsilon$ is not differentiable under our conditions, we collect some properties of $\mathscr{N}_\epsilon$ in order to use the Nehari manifold method.

Lemma 2.6. *$\mathscr{N}_\epsilon$ is bounded away from* 0*, and is closed in E_ϵ.*

Proof. Let $u \in \mathscr{N}_\epsilon$, from Lemma 2.1, (2.2), (2.6) and (2.7) we have

$$\begin{aligned}\|u\|_\epsilon^2 &= \int_{\mathbb{R}^N} K(\epsilon x) f(u(x,0))u(x,0)\mathrm{d}x \\ &\le \varepsilon K_{\max}\|u(x,0)\|_2^2 + C_\epsilon K_{\max}\|u(x,0)\|_p^p \\ &\le \varepsilon c_4\|u\|_\epsilon^2 + c_5\|u\|_\epsilon^p.\end{aligned}$$

So, it is easy to see that there exists $\alpha_0 > 0$ such that $\|u\|_\epsilon \ge \alpha_0$.

Let $\{u_n\} \subset \mathscr{N}_\epsilon$ be a sequence such that $u_n \to u$ in E_ϵ. From Lemma 2.4, we know $\Phi_\epsilon'(u_n)$ is bounded, and

$$\langle \Phi_\epsilon'(u_n), u_n\rangle - \langle \Phi_\epsilon'(u), u\rangle = \langle \Phi_\epsilon'(u_n) - \Phi_\epsilon'(u_n), u\rangle + \langle \Phi_\epsilon'(u_n), u_n - u\rangle \to 0,$$

this shows that $\langle \Phi_\epsilon'(u), u\rangle = 0$. Together with the above conclusion we deduce that $\|u\|_\epsilon \ge \alpha_0$ and $u \in \mathscr{N}_\epsilon$. This completes the proof. □

Lemma 2.7. *Let $u \in E_\epsilon \backslash \{0\}$, then there exists a unique $t_u > 0$ such that $t_u u \in \mathscr{N}_\epsilon$. Moreover, $\widehat{m}_\epsilon(u) = t_u u$ is the unique global maximum of Φ_ϵ on $\mathbb{R}^+ u$. In particular, if $u \in \mathscr{N}_\epsilon$, then*

$$\Phi_\epsilon(u) = \max_{t \ge 0} \Phi_\epsilon(tu) \ge \Phi_\epsilon(tu) \text{ for all } t \ge 0.$$

Proof. Let $u \in E_\epsilon \backslash \{0\}$, we define the function $g(t) = \Phi_\epsilon(tu)$ for $t > 0$. According to the proof of Lemma 2.5, we know that $g(0) = 0$, $g(t) > 0$ for t sufficiently small and $g(t) < 0$ for t sufficiently large. Hence, there is $t = t_u$ such that $\max_{t>0} g(t)$ is attained at t_u, so $g'(t_u) = 0$ and $t_u u \in \mathscr{N}_\epsilon$.

Now we prove that t_u is the unique critical point of g. Suppose by contradiction that there exist t_1 and t_2 with $0<t_1<t_2$ such that $t_1u,t_2u\in\mathcal{N}_\epsilon$, then it follows that

$$\|u\|_\epsilon^2=\int_{\mathbb{R}^N}K(\epsilon x)\frac{f(t_1u(x,0))}{t_1u(x,0)}u^2(x,0)\mathrm{d}x$$

and

$$\|u\|_\epsilon^2=\int_{\mathbb{R}^N}K(\epsilon x)\frac{f(t_2u(x,0))}{t_2u(x,0)}u^2(x,0)\mathrm{d}x.$$

From (f_5) we have

$$0=\int_{\mathbb{R}^N}K(\epsilon x)\left[\frac{f(t_1u(x,0))}{t_1u(x,0)}-\frac{f(t_2u(x,0))}{t_2u(x,0)}\right]u^2(x,0)\mathrm{d}x<0,$$

this is impossible, a contradiction. So the proof is now complete. □

Lemma 2.8. *There exists $\lambda>0$ such that $t_u\geq\lambda$ for each $u\in S_\epsilon$, and for each compact subset $\mathcal{W}\subset S_\epsilon$, there exists $C_{\mathcal{W}}>0$ such that $t_u\leq C_{\mathcal{W}}$ for all $u\in\mathcal{W}$, where $S_\epsilon=\{u\in E_\epsilon:\|u\|_\epsilon=1\}$.*

Proof. For each $u\in S_\epsilon$, by Lemma 2.6 and Lemma 2.7, there exists $t_u>0$ such that $t_uu\in\mathcal{N}_\epsilon$, and $t_u=\|t_uu\|_\epsilon\geq\alpha_0$. Next we show that $t_u\leq C_{\mathcal{W}}$ for all $u\in\mathcal{W}\subset S_\epsilon$. Arguing by contradiction we assume that there exist a sequence $\{u_n\}\subset\mathcal{W}\subset S_\epsilon$ and $\{t_n\}$ such that $t_n\to\infty$. Since $\mathcal{W}$ is compact, there exists $u\in\mathcal{W}$ such that $u_n\to u$ in E_ϵ. From the proof of Lemma 2.5, it is clear that $\Phi_\epsilon(t_nu_n)\to-\infty$. However, for any $u\in\mathcal{N}_\epsilon$, we deduce from (2.8) that

$$\begin{aligned}\Phi_\epsilon(u)&=\Phi_\epsilon(u)-\frac{1}{2}\langle\Phi_\epsilon'(u),u\rangle\\&=\int_{\mathbb{R}^N}K(\epsilon x)\left[\frac{1}{2}f(u(x,0))u(x,0)-F(u(x,0))\right]\mathrm{d}x>0.\end{aligned}$$

So, the conclusion $\Phi_\epsilon(t_nu_n)\to-\infty$ is impossible, a contradiction. This ends the proof. □

According to Lemmas 2.5-2.8, the ground state energy value c_ϵ has a minimax characterization given by

$$c_\epsilon=\tilde{c}_\epsilon=\inf_{u\in E_\epsilon\setminus\{0\}}\max_{t\geq0}\Phi_\epsilon(tu)=\inf_{u\in S_\epsilon}\max_{t\geq0}\Phi_\epsilon(tu). \tag{2.9}$$

We refer to see [39] and [42] for the detailed proof .

Lemma 2.9.

(1) *There is $\alpha>0$ independent of ϵ such that $c_\epsilon\geq\alpha>0$.*

(2) *Φ_ϵ is coercive on $\mathcal{N}_\epsilon$, i.e., $\Phi_\epsilon(u)\to\infty$ as $\|u\|_\epsilon\to\infty$, $u\in\mathcal{N}_\epsilon$.*

Proof.

(1) For any $u\in\mathcal{N}_\epsilon$, from Lemma 2.5 we have $\Phi_\epsilon(tu)\geq\alpha>0$ for $t>0$ small. Moreover, by (2.9) we get $c_\epsilon\geq\alpha>0$. So, the conclusion (1) holds.

(2) Let $u\in\mathcal{N}_\epsilon$, then using (f_4) we have

$$\begin{aligned}\Phi_\epsilon(u)=&\Phi_\epsilon(u)-\frac{1}{\theta}\langle\Phi_\epsilon'(u),u\rangle\\ =&\left[\frac{1}{2}-\frac{1}{\theta}\right]\|u\|_\epsilon^2+\int_{\mathbb{R}^N}K(\epsilon x)\left[\frac{1}{\theta}f(u(x,0))u(x,0)-F(u(x,0))\right]\mathrm{d}x\\ \geq&\left[\frac{1}{2}-\frac{1}{\theta}\right]\|u\|_\epsilon^2.\end{aligned}$$

Obviously, this shows that the conclusion (2) holds. ∎

LEMMA 2.10. *The mapping $\widehat{m}_\epsilon:E_\epsilon\backslash\{0\}\to\mathcal{N}_\epsilon$ is continuous, and the map $m_\epsilon:=\widehat{m}_\epsilon|_{S_\epsilon}:S_\epsilon\to\mathcal{N}_\epsilon$ is a homeomorphism between S_ϵ and $\mathcal{N}_\epsilon$ with inverse given by*

$$m_\epsilon^{-1}:\mathcal{N}_\epsilon\to S_\epsilon,\ m_\epsilon^{-1}(u)=u/\|u\|_\epsilon.$$

Proof. First we assume that $u_n\to u\neq0$. Since $\widehat{m}_\epsilon(su)=\widehat{m}_\epsilon(u)$ for each $s>0$, we may assume $u_n\in S_\epsilon$ for all n and it suffices to show that $\widehat{m}_\epsilon(u_n)\to\widehat{m}_\epsilon(u)$ after passing to a subsequence. According to Lemma 2.7, $\widehat{m}_\epsilon(u_n)=t_{u_n}u_n$. It follows from Lemma 2.8 that $\{t_{u_n}\}$ is bounded and bounded away from 0, hence, taking a subsequence, $t_{u_n}\to t_0>0$. By Lemma 2.6, $\mathcal{N}_\epsilon$ is closed and $\widehat{m}_\epsilon(u_n)\to t_0u$ and $t_0u\in\mathcal{N}_\epsilon$. Hence $t_0u=t_uu=\widehat{m}_\epsilon(u)$. From the above proof, the second conclusion is an immediate consequence. ∎

Based on Lemma 2.10, we define the functional $\widehat{I}_\epsilon:E_\epsilon\backslash\{0\}\to\mathbb{R}$ and the restriction $I_\epsilon:S_\epsilon\to\mathbb{R}$

$$\widehat{I}_\epsilon(u)=\Phi_\epsilon(\widehat{m}_\epsilon(u))\text{ and }I_\epsilon=\widehat{I}_\epsilon|_{S_\epsilon}.$$

According to the above preliminary results, we have the following important results. The details can be found in relevant material from Corollary 3.3 in [39].

LEMMA 2.11. *The following conclusions hold:*

(a) *$\widehat{I}_\epsilon\in C^1(E_\epsilon\backslash\{0\},\mathbb{R})$ and for $u,v\in E_\epsilon$ and $u\neq0$,*

$$\langle\widehat{I}_\epsilon'(u),v\rangle=\frac{\|\widehat{m}_\epsilon(u)\|_\epsilon}{\|u\|_\epsilon}\langle\Phi_\epsilon'(\widehat{m}_\epsilon(u)),v\rangle.$$

(b) *$I_\epsilon\in C^1(S_\epsilon,\mathbb{R})$ and $\langle I_\epsilon'(u),v\rangle=\|\widehat{m}_\epsilon(u)\|_\epsilon\langle\Phi_\epsilon'(\widehat{m}_\epsilon(u)),v\rangle$ for $v\in T_u(S_\epsilon)$, where $T_u(S_\epsilon)$ is the tangent space of S_ϵ at u.*

(c) *$\{u_n\}$ is a Palais-Smale sequence for I_ϵ if and only if $\{\widehat{m}_\epsilon(u_n)\}$ is a Palais-Smale sequence for Φ_ϵ.*

(d) *u is a critical point of I_ϵ if and only if $\widehat{m}_\epsilon(u)$ is a critical point of Φ_ϵ. Moreover, the corresponding critical values coincide and and*

$$\inf_{S_\epsilon}I_\epsilon=\inf_{\mathcal{N}_\epsilon}\Phi_\epsilon=c_\epsilon.$$

3. The autonomous problem

We will make use of the limit problem of problem (2.5) to help us to prove the main results. So, in this section we start by considering the autonomous problem

$$\begin{cases}-\mathrm{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & \text{in }\mathbb{R}_+^{N+1},\\ \frac{\partial w}{\partial\nu^{1-2s}}=-\mu w(x,0)+\kappa f(w(x,0)), & \text{on }\mathbb{R}^N.\end{cases}\tag{3.1}$$

where $\mu > -m^{2s}$ and $\kappa > 0$. Define the corresponding working space

$$E_\mu = \left\{ u \in X^s(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^N} \mu u^2(x,0)\mathrm{d}x < \infty \right\}$$

with the norm

$$\|u\|_\mu = \left[\|u\|^2_{X^s} + \int_{\mathbb{R}^N} \mu u^2(x,0)\mathrm{d}x \right]^{\frac{1}{2}}$$

and the inner product

$$(u,v)_\mu = \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(\nabla u \nabla v + uv)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \mu u(x,0)v(x,0)\mathrm{d}x, \ u,v \in E_\mu.$$

It is well known that the solutions of problem (3.1) are precisely critical points of the functional

$$\begin{aligned} \mathcal{J}_{\mu\kappa}(u) &= \frac{1}{2}\|u\|^2_{X^s} + \frac{1}{2}\int_{\mathbb{R}^N} \mu u^2(x,0)\mathrm{d}x - \kappa \int_{\mathbb{R}^N} F(u(x,0))\mathrm{d}x \\ &= \frac{1}{2}\|u\|^2_\mu - \kappa \int_{\mathbb{R}^N} F(u(x,0))\mathrm{d}x. \end{aligned}$$

Clearly, $\mathcal{J}_{\mu\kappa} \in C^1(E_\mu, \mathbb{R})$ and its differential is given by

$$\langle \mathcal{J}'_{\mu\kappa}(u), v\rangle = (u,v)_\mu - \kappa \int_{\mathbb{R}^N} f(u(x,0))v(x,0)\mathrm{d}x$$

for any $u,v \in E_\mu$. The Nehari manifold and ground state energy associated to $\mathcal{J}_{\mu\kappa}$ are defined by

$$\mathcal{N}_{\mu\kappa} := \{u \in E_\mu \setminus \{0\} : \langle \mathcal{J}'_{\mu\kappa}(u), u\rangle = 0\} \text{ and } c_{\mu\kappa} = \inf_{\mathcal{N}_{\mu\kappa}} \mathcal{J}_{\mu\kappa}.$$

As before, we define the mappings

$$\widehat{m}_{\mu\kappa} : E_\mu \setminus \{0\} \to \mathcal{N}_{\mu\kappa} \text{ and } m_{\mu\kappa} = \widehat{m}_{\mu\kappa}|_S : S \to \mathcal{N}_{\mu\kappa},$$

and the inverse of $m_{\mu\kappa}$ is given by

$$m^{-1}_{\mu\kappa} : \mathcal{N}_{\mu\kappa} \to S, \ m^{-1}_{\mu\kappa}(u) = u/\|u\|_\mu.$$

From the previous arguments in Section 2, we see that $\mathcal{J}_{\mu\kappa}$, $\mathcal{N}_{\mu\kappa}$ and $c_{\mu\kappa}$ have similar properties with Φ_ϵ, $\mathcal{N}_\epsilon$ and c_ϵ. Moreover, all related Lemmas in Section 2 hold for the autonomous problem (3.1). So we give these results for problem (3.1).

Lemma 3.1. *The following conclusions hold:*

(a) *$\mathcal{N}_{\mu\kappa}$ is bounded away from 0. Moreover, $\mathcal{N}_{\mu\kappa}$ is closed in E_μ.*

(b) *For $u \in E_\mu \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_{\mu\kappa}$. Moreover, $\widehat{m}_{\mu\kappa}(u) = t_u u$ is the unique global maximum of $\mathcal{J}_{\mu\kappa}$ on $\mathbb{R}^+ u$.*

(c) *There exists $\lambda > 0$ such that $t_u \geq \lambda$ for each $u \in S$, and for each compact subset $\mathcal{W} \subset S$, there exists $C_{\mathcal{W}} > 0$ such that $t_u \leq C_{\mathcal{W}}$ for all $u \in \mathcal{W}$.*

(d) *$c_{\mu\kappa} > 0$ and $\mathcal{J}_{\mu\kappa}$ is positive bounded below on $\mathcal{N}_{\mu\kappa}$.*

(e) *$\mathcal{J}_{\mu\kappa}$ is coercive on $\mathcal{N}_{\mu\kappa}$, i.e., $\mathcal{J}_{\mu\kappa}(u)\to\infty$ as $\|u\|_\mu\to\infty$, $u\in\mathcal{N}_{\mu\kappa}$.*

Let us define the functional $\widehat{I}_{\mu\kappa}:E_\mu\backslash\{0\}\to\mathbb{R}$ and the restriction $I_{\mu\kappa}:S\to\mathbb{R}$

$$\widehat{I}_{\mu\kappa}(u)=\mathcal{J}_{\mu\kappa}(\widehat{m}_{\mu\kappa}(u)) \text{ and } I_{\mu\kappa}=\widehat{I}_{\mu\kappa}|_S.$$

Similarly, we have the following results.

LEMMA 3.2. *The following conclusions hold:*

(a) *$I_{\mu\kappa}\in C^1(S,\mathbb{R})$ and $\langle I'_{\mu\kappa}(u),v\rangle=\|\widehat{m}_{\mu\kappa}(u)\|\langle\mathcal{J}'_{\mu\kappa}(\widehat{m}_{\mu\kappa}(u)),v\rangle$ for $v\in T_u(S)$, where $T_u(S)$ is the tangent space of S at u.*

(b) *$\{u_n\}$ is a Palais-Smale sequence for $I_{\mu\kappa}$ if and only if $\{\widehat{m}_{\mu\kappa}(u_n)\}$ is a Palais-Smale sequence for $\mathcal{J}_{\mu\kappa}$.*

(c) *u is a critical point of $I_{\mu\kappa}$ if and only if $\widehat{m}_{\mu\kappa}(u)$ is a critical point of $\mathcal{J}_{\mu\kappa}$. Moreover, the corresponding values of $I_{\mu\kappa}$ and $\mathcal{J}_{\mu\kappa}$ coincide and*

$$c_{\mu\kappa}=\inf_{\mathcal{N}_{\mu\kappa}}\mathcal{J}_{\mu\kappa}=\inf_S I_{\mu\kappa}.$$

Now we use the Nehari manifold method to establish the existence result of positive ground state solution of problem (3.1), this result is very useful in later arguments.

LEMMA 3.3. *Assume that $\mu>-m^{2s}$, $\kappa>0$ and (f_1)-(f_5) are satisfied. Then problem (3.1) has at least one positive ground state solution $\tilde{u}$ such that $\mathcal{J}_{\mu\kappa}(\tilde{u})=c_{\mu\kappa}$.*

Proof. Firstly, it follows from Lemma 3.1-(d) that $c_{\mu\kappa}>0$. We observe that if $u\in\mathcal{N}_{\mu\kappa}$ satisfies $\mathcal{J}_{\mu\kappa}(u)=c_{\mu\kappa}$, then $m^{-1}_{\mu\kappa}(u)\in S$ is a minimizer of $I_{\mu\kappa}$, and hence a critical point of $I_{\mu\kappa}$. Lemma 3.2 shows that u is a critical point of $\mathcal{J}_{\mu\kappa}$. Hence, we need to prove that there exists a minimizer $\tilde{u}\in\mathcal{N}_{\mu\kappa}$ such that $\mathcal{J}_{\mu\kappa}(\tilde{u})=c_{\mu\kappa}$. Indeed, using Ekeland's variational principle [42], there exists a sequence $\{v_n\}\subset S$ such that $I_{\mu\kappa}(v_n)\to c_{\mu\kappa}$ and $I'_{\mu\kappa}(v_n)\to 0$ as $n\to\infty$. Let $u_n=\widehat{m}_{\mu\kappa}(v_n)\in\mathcal{N}_{\mu\kappa}$ for all $n\in\mathbb{N}$. Then using Lemma 3.2 again, we can get $\mathcal{J}_{\mu\kappa}(u_n)\to c_{\mu\kappa}$ and $\mathcal{J}'_{\mu\kappa}(u_n)\to 0$. According to Lemma 3.1-(e), we can see that $\{u_n\}$ is bounded in E_μ, and $\|u_n\|_\mu\geq\alpha_0$ for some $\alpha_0>0$ by Lemma 3.1-(a).

We claim that

$$\varliminf_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_R(z)}u_n^2(x,0)\mathrm{d}x>0.$$

Otherwise, Lemma 2.2 yields that $u_n(x,0)\to 0$ in $L^q(\mathbb{R}^N)$ for any $q\in(2,2_s^*)$. From the fact that $u_n\in\mathcal{N}_{\mu\kappa}$ and (2.7) we can deduce that

$$0=\langle\mathcal{J}'_{\mu\kappa}(u_n),u_n\rangle=\|u_n\|_\mu^2-\kappa\int_{\mathbb{R}^N}f(u_n(x,0))u_n(x,0)\mathrm{d}x=\|u_n\|_\mu^2+o(1),$$

this shows that $\|u_n\|_\mu\to 0$, which contradicts with $\|u_n\|_\mu\geq\alpha_0$.

Thus, there are $\delta>0$ and $\{k_n\}\subset\mathbb{Z}^N$ such that

$$\int_{B_R(k_n)}u_n^2(x,0)\mathrm{d}x\geq\delta.$$

Setting $\tilde{u}_n(x,y)=u_n(x+k_n,y)$, then we have

$$\int_{B_R(0)}\tilde{u}_n^2(x,0)\mathrm{d}x\geq\delta. \tag{3.2}$$

Since problem (3.1) is autonomous, then $\mathcal{J}_{\mu\kappa}$ possesses the invariance under $\mathbb{Z}^N$-translation, and we have $\|\tilde{u}_n\|_\mu = \|u_n\|_\mu$ and

$$\mathcal{J}_{\mu\kappa}(\tilde{u}_n) \to c_{\mu\kappa} \quad \text{and} \quad \mathcal{J}'_{\mu\kappa}(\tilde{u}_n) \to 0. \tag{3.3}$$

Up to a subsequence, we assume that $\tilde{u}_n \rightharpoonup \tilde{u}$ in E_μ, $\tilde{u}_n(x,0) \to \tilde{u}(x,0)$ in $L^q_{loc}(\mathbb{R}^N)$ for $q \in (2, 2_s^*)$, and $\tilde{u}_n(x,y) \to \tilde{u}(x,y)$ a.e. in $\mathbb{R}^{N+1}_+$, $\tilde{u}_n(x,0) \to \tilde{u}(x,0)$ a.e. in $\mathbb{R}^N$. Therefore, from Lemma 2.4, (3.2) and (3.3) we infer that $\tilde{u} \neq 0$ and $\mathcal{J}'_{\mu\kappa}(\tilde{u}) = 0$. This shows that $\tilde{u} \in \mathcal{N}_{\mu\kappa}$ and $\mathcal{J}_{\mu\kappa}(\tilde{u}) \geq c_{\mu\kappa}$.

On the other hand, applying Fatou's lemma and (2.8), we can get

$$\begin{aligned}
c_{\mu\kappa} &= \lim_{n\to\infty} \left[\mathcal{J}_{\mu\kappa}(\tilde{u}_n) - \frac{1}{2}\langle \mathcal{J}'_{\mu\kappa}(\tilde{u}_n), \tilde{u}_n \rangle \right] \\
&= \lim_{n\to\infty} \left[\kappa \int_{\mathbb{R}^N} \frac{1}{2} f(\tilde{u}_n(x,0))\tilde{u}_n(x,0) - F(\tilde{u}_n(x,0)) \mathrm{d}x \right] \\
&\geq \kappa \int_{\mathbb{R}^N} \left[\frac{1}{2} f(\tilde{u}(x,0))\tilde{u}(x,0) - F(\tilde{u}(x,0)) \right] \mathrm{d}x \\
&= \mathcal{J}_{\mu\kappa}(\tilde{u}) - \frac{1}{2}\langle \mathcal{J}'_{\mu\kappa}(\tilde{u}), \tilde{u} \rangle = \mathcal{J}_{\mu\kappa}(\tilde{u}),
\end{aligned}$$

this shows that $\mathcal{J}_{\mu\kappa}(\tilde{u}) \leq c_{\mu\kappa}$. Then we have $\mathcal{J}_{\mu\kappa}(\tilde{u}) = c_{\mu\kappa}$ and $\tilde{u}$ is critical point of $\mathcal{J}_{\mu\kappa}$. So, $\tilde{u}$ is a ground state solution of problem (3.1).

Next, we show that the ground state solution $\tilde{u}$ is positive. Indeed, taking $\tilde{u}^- = \min\{\tilde{u}, 0\}$ as test function in problem (3.1), we can obtain

$$\|\tilde{u}^-\|_\mu^2 = \kappa \int_{\mathbb{R}^N} f(\tilde{u}^-(x,0))\tilde{u}^-(x,0)\mathrm{d}x = 0.$$

This shows that $\tilde{u}^- = 0$, that is $\tilde{u} \geq 0$ in $\mathbb{R}^{N+1}_+$. By the regularity results (see Lemma 5.5 below), we know that $\tilde{u}(x,0) \in L^q(\mathbb{R}^N)$ for any $q \in [2, \infty]$, and that $\tilde{u} \in L^\infty(\mathbb{R}^{N+1}_+)$. According to Lemma 2.3-(c), we have $\tilde{u} \in C^{0,\alpha}(\mathbb{R}^{N+1}_+)$ for some $\alpha \in (0,1)$. Finally we can apply Harnack's inequality (see Lemma 2.3-(b)) to conclude that $\tilde{u} > 0$ in $\mathbb{R}^{N+1}_+$.

□

The following lemma describes a comparison between the ground state energy values for different parameters μ and κ, which is crucially important in our analysis.

LEMMA 3.4. *Let $\mu_i > -m^{2s}$ and $\kappa_i > 0$ for $i = 1, 2$, with $\min\{\mu_2 - \mu_1, \kappa_1 - \kappa_2\} \geq 0$. Then $c_{\mu_1\kappa_1} \leq c_{\mu_2\kappa_2}$. Additionally, if $\max\{\mu_2 - \mu_1, \kappa_1 - \kappa_2\} > 0$, then $c_{\mu_1\kappa_1} < c_{\mu_2\kappa_2}$. In particular, we have $c_{\mu_1\kappa_i} < c_{\mu_2\kappa_i}$ if $\mu_1 < \mu_2$, and $c_{\mu_i\kappa_1} > c_{\mu_i\kappa_2}$ if $\kappa_1 < \kappa_2$.*

Proof. Let $u \in \mathcal{N}_{\mu_2\kappa_2}$ with $\mathcal{J}_{\mu_2\kappa_2}(u) = c_{\mu_2\kappa_2}$, then, Lemma 3.1-(b) implies that

$$c_{\mu_2\kappa_2} = \mathcal{J}_{\mu_2\kappa_2}(u) = \max_{t \geq 0} \mathcal{J}_{\mu_2\kappa_2}(tu).$$

Using Lemma 3.1-(b) again, there exists $t_0 > 0$ such that $u_0 = t_0 u \in \mathcal{N}_{\mu_1\kappa_1}$ and

$$\mathcal{J}_{\mu_1\kappa_1}(u_0) = \max_{t \geq 0} \mathcal{J}_{\mu_1\kappa_1}(tu_0).$$

According to the above facts we have

$$\begin{aligned}c_{\mu_2\kappa_2}&=\mathcal{J}_{\mu_2\kappa_2}(u)\geq\mathcal{J}_{\mu_2\kappa_2}(u_0)\\&=\mathcal{J}_{\mu_1\kappa_1}(u_0)+\frac{1}{2}(\mu_2-\mu_1)\int_{\mathbb{R}^N}u_0^2(x,0)\mathrm{d}x+(\kappa_1-\kappa_2)\int_{\mathbb{R}^N}F(u_0(x,0))\mathrm{d}x\\&\geq c_{\mu_1\kappa_1}+\frac{1}{2}(\mu_2-\mu_1)\int_{\mathbb{R}^N}u_0^2(x,0)\mathrm{d}x+(\kappa_1-\kappa_2)\int_{\mathbb{R}^N}F(u_0(x,0))\mathrm{d}x.\end{aligned}$$

Evidently, $c_{\mu_2\kappa_2}\geq c_{\mu_1\kappa_1}$. If $\max\{\mu_2-\mu_1,\kappa_1-\kappa_2\}>0$, it is easy to see that $c_{\mu_2\kappa_2}>c_{\mu_1\kappa_1}$ by the above formula. This finishes the proof. ∎

4. Palais-Smale compactness condition

In this section we will analyze the behaviors of Palais-Smale sequence to overcome the lack of compactness. First, combining the nonlocal version of Brezis-Lieb lemma and some standard arguments, we have the following splitting lemma.

LEMMA 4.1. *Let $\{u_n\}$ be a sequence such that $u_n\rightharpoonup u$ in E_ϵ, and set $v_n=u_n-u$. Then we have*

$$\Phi_\epsilon(v_n)=\Phi_\epsilon(u_n)-\Phi_\epsilon(u)+o_n(1),$$

and

$$\langle\Phi_\epsilon'(v_n),\varphi\rangle=\langle\Phi_\epsilon'(u_n),\varphi\rangle-\langle\Phi_\epsilon'(u),\varphi\rangle+o_n(1)$$

uniformly in $\varphi\in E_\epsilon$.

Consider the limit problem

$$\begin{cases}-\mathrm{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & \text{in } \mathbb{R}_+^{N+1},\\ \frac{\partial w}{\partial\nu^{1-2s}}=-V_\infty w(x,0)+K_\infty f(w(x,0)), & \text{on } \mathbb{R}^N.\end{cases}\tag{4.1}$$

As before, we use the notations $\mathcal{J}_{V_\infty K_\infty}$, $\mathcal{N}_{V_\infty K_\infty}$ and $c_{V_\infty K_\infty}$ to denote the associated energy functional, Nehari manifold and ground state energy value of limit problem (4.1), respectively.

LEMMA 4.2. *Let $\{u_n\}$ be a Palais-Smale sequence at level $c>0$ for Φ_ϵ with $u_n\rightharpoonup u$ in E_ϵ. Then the following alternative holds: either $u_n\to u$ in E_ϵ along a subsequence, or $c-\Phi_\epsilon(u)\geq c_{V_\infty K_\infty}$.*

Proof. Define $v_n=u_n-u$ and assume that $v_n\not\to 0$ in E_ϵ. We infer from Lemma 2.7 that for each v_n there exists a unique $\{t_n\}\subset(0,\infty)$ such that $\{t_nv_n\}\subset\mathcal{N}_{V_\infty K_\infty}$. We divide the proof into three steps.

Step 1. The sequence $\{t_n\}$ satisfies

$$\limsup_{n\to\infty}t_n\leq 1.$$

Indeed, assume by contradiction that the above conclusion does not hold. Then, there exist $\tau>0$ and a subsequence of $\{t_n\}$, still denoted by itself, such that

$$t_n\geq 1+\tau \text{ for all } n\in\mathbb{N}.$$

On account of Lemma 4.1, we see that $\langle\Phi_\epsilon'(v_n),v_n\rangle=o_n(1)$ and

$$\|v_n\|_{X^s}^2+\int_{\mathbb{R}^N}V(\epsilon x)v_n^2(x,0)\mathrm{d}x-\int_{\mathbb{R}^N}K(\epsilon x)f(v_n(x,0))v_n(x,0)\mathrm{d}x=o_n(1).$$

By the fact that $\{t_nv_n\}\subset \mathcal{N}_{V_\infty K_\infty}$, we have

$$\|v_n\|^2_{X^s}+\int_{\mathbb{R}^N}V_\infty v_n^2(x,0)\mathrm{d}x-\int_{\mathbb{R}^N}\frac{K_\infty f(t_nv_n(x,0))}{t_nv_n(x,0)}v_n^2(x,0)\mathrm{d}x=0.$$

Consequently,

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[\frac{K_\infty f(t_nv_n(x,0))}{t_nv_n(x,0)}-\frac{K_\infty f(v_n(x,0))}{v_n(x,0)}\right]v_n^2(x,0)\mathrm{d}x\\ =&\int_{\mathbb{R}^N}\left[\frac{K(\epsilon x)f(v_n(x,0))}{v_n(x,0)}-\frac{K_\infty f(v_n(x,0))}{v_n(x,0)}\right]v_n^2(x,0)\mathrm{d}x\\ &+\int_{\mathbb{R}^N}[V_\infty-V(\epsilon x)]v_n^2(x,0)\mathrm{d}x+o_n(1).\end{aligned}\tag{4.2}$$

By the definition of V_∞ and K_∞, for any $\varepsilon>0$, there exists $R=R(\varepsilon)>0$ such that

$$V(\epsilon x)\geq V_\infty-\varepsilon \text{ and } K(\epsilon x)\leq K_\infty+\varepsilon \text{ for any } |x|\geq R.\tag{4.3}$$

Since $v_n\rightharpoonup 0$ in E_ϵ, Lemma 2.2 implies that $v_n(x,0)\to 0$ in $L^q_{loc}(\mathbb{R}^N)$ for $q\in[2,2_s^*)$. Using (2.7) and (4.2) we get

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[\frac{K_\infty f(t_nv_n(x,0))}{t_nv_n(x,0)}-\frac{K_\infty f(v_n(x,0))}{v_n(x,0)}\right]v_n^2(x,0)\mathrm{d}x\\ \leq&\int_{\mathbb{R}^N}[K(\epsilon x)-K_\infty]f(v_n(x,0))v_n(x,0)\mathrm{d}x+\int_{\mathbb{R}^N}\left[(V_\infty-V(\epsilon x))v_n^2(x,0)\right]\mathrm{d}x+o_n(1)\\ \leq&\varepsilon\int_{|x|\geq R}f(v_n(x,0))v_n(x,0)\mathrm{d}x+\varepsilon\int_{|x|\geq R}v_n^2(x,0)\mathrm{d}x\\ &+2K_{\max}\int_{|x|\leq R}f(v_n(x,0))v_n(x,0)\mathrm{d}x+2V_{\max}\int_{|x|\leq R}v_n^2(x,0)\mathrm{d}x++o_n(1)\\ =&c_6\varepsilon+o_n(1).\end{aligned}\tag{4.4}$$

Since $v_n\nrightarrow 0$ in E_ϵ and $\langle\Phi'_\epsilon(v_n),v_n\rangle\to 0$, then there exist $\bar{R}$, $\delta>0$ and $z_n\in\mathbb{R}^N$ such that

$$\int_{B_{\bar{R}}(z_n)}v_n^2(x,0)\mathrm{d}x\geq\delta.\tag{4.5}$$

Note that the above claim is true. Otherwise, using Lemma 2.2, we have $v_n(x,0)\to 0$ in $L^q(\mathbb{R}^N)$ for $q\in(2,2_s^*)$. According to (2.2), (2.6), (2.7) and $\langle\Phi'_\epsilon(v_n),v_n\rangle=o_n(1)$ we can deduce that

$$\begin{aligned}o_n(1)=&\langle\Phi'_\epsilon(v_n),v_n\rangle\\ =&\|v_n\|^2_\epsilon-\int_{\mathbb{R}^N}K(\epsilon x)f(v_n(x,0))v_n(x,0)\mathrm{d}x\\ \geq&\|v_n\|^2_\epsilon-\varepsilon K_{\max}\|v_n(x,0)\|_2^2-C_\varepsilon K_{\max}\|v_n(x,0)\|_p^p\\ \geq&(1-\varepsilon c_7)\|v_n\|^2_\epsilon-c_8\|v_n(x,0)\|_p^p.\end{aligned}$$

Then, $v_n\to 0$ in E_ϵ, which is a contradiction. Setting $\tilde{v}_n=v_n(x+z_n,y)$, we may assume that, up to a subsequence, $\tilde{v}_n\rightharpoonup\tilde{v}$ in E_ϵ and $\tilde{v}_n(x,0)\to\tilde{v}(x,0)$ a.e. in $\mathbb{R}^N$. Thus,

$$\int_{B_{\bar{R}}(0)}\tilde{v}_n^2(x,0)\mathrm{d}x\geq\delta,$$

which implies that $\tilde{v} \neq 0$. Moreover, using the fact that $v_n \geq 0$ for all $n \in \mathbb{N}$, we have that $\tilde{v}(x,0) \geq 0$ a.e. in $\mathbb{R}^N$. Hence, there exists a subset $\Omega_1 \subset \mathbb{R}^N$ with positive measure such that $\tilde{v}(x,0) > 0$ for all $x \in \Omega_1$. Consequently, according to (f_5) and (4.4) we get

$$0 < \int_{\Omega_1} \left[\frac{K_\infty f((1+\tau)v_n(x,0))}{(1+\tau)v_n(x,0)} - \frac{K_\infty f(v_n(x,0))}{v_n(x,0)} \right] v_n^2(x,0)\mathrm{d}x \leq c_6 \varepsilon + o_n(1).$$

Letting $n \to \infty$ in the last inequality and applying Fatou's lemma, it follows that

$$0 < \int_{\Omega_1} \left[\frac{K_\infty f((1+\tau)\tilde{v}(x,0))}{(1+\tau)\tilde{v}(x,0)} - \frac{K_\infty f(\tilde{v}(x,0))}{\tilde{v}(x,0)} \right] \tilde{v}^2(x,0)\mathrm{d}x \leq c_6 \varepsilon,$$

this is a contradiction since the arbitrariness of ε.

From Step 1, we derive that

$$\limsup_{n\to\infty} t_n = 1 \text{ or } \limsup_{n\to\infty} t_n = t_0 < 1.$$

Next we will study each one of these possibilities.

Step 2. The sequence $\{t_n\}$ satisfies

$$\limsup_{n\to\infty} t_n = 1.$$

In this case, there exists a subsequence, such that $t_n \to 1$. Using Lemma 4.1 and the fact that $\mathcal{J}_{V_\infty K_\infty}(t_n v_n) \geq c_{V_\infty K_\infty}$ we have

$$\begin{aligned} c - \Phi_\epsilon(u) + o_n(1) &= \Phi_\epsilon(v_n) \\ &= \Phi_\epsilon(v_n) - \mathcal{J}_{V_\infty K_\infty}(t_n v_n) + \mathcal{J}_{V_\infty K_\infty}(t_n v_n) \\ &\geq \Phi_\epsilon(v_n) - \mathcal{J}_{V_\infty K_\infty}(t_n v_n) + c_{V_\infty K_\infty}. \end{aligned} \tag{4.6}$$

Note that,

$$\begin{aligned} &\Phi_\epsilon(v_n) - \mathcal{J}_{V_\infty K_\infty}(t_n v_n) \\ =&\frac{(1-t_n^2)}{2}\|v_n\|_{X^s}^2 + \frac{1}{2}\int_{\mathbb{R}^N} [V(\epsilon x) - t_n^2 V_\infty] v_n^2(x,0)\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} [K_\infty F(t_n v_n(x,0)) - K(\epsilon x) F(v_n(x,0))]\mathrm{d}x. \end{aligned} \tag{4.7}$$

It follows from (4.3) that

$$V(\epsilon x) - t_n^2 V_\infty = (V(\epsilon x) - V_\infty) + (1-t_n^2)V_\infty \geq -\varepsilon + (1-t_n^2)V_\infty \text{ for any } |x| \geq R,$$

then by $v_n(x,0) \to 0$ in $L^2_{loc}(\mathbb{R}^N)$ and $t_n \to 1$ we have

$$\begin{aligned} &\int_{\mathbb{R}^N} [V(\epsilon x) - t_n^2 V_\infty] v_n^2(x,0)\mathrm{d}x \\ =& \int_{|x|\leq R} [V(\epsilon x) - t_n^2 V_\infty] v_n^2(x,0)\mathrm{d}x + \int_{|x|\geq R} [V(\epsilon x) - t_n^2 V_\infty] v_n^2(x,0)\mathrm{d}x \\ \geq& (V_{\min} - t_n^2 V_\infty) \int_{|x|\leq R} v_n^2(x,0)\mathrm{d}x - \varepsilon \int_{|x|\geq R} v_n^2(x,0)\mathrm{d}x + V_\infty(1-t_n^2) \int_{|x|\geq R} v_n^2(x,0)\mathrm{d}x \\ \geq& o_n(1) - c_7 \varepsilon. \end{aligned} \tag{4.8}$$

Now we prove that

$$\int_{\mathbb{R}^N}[K_\infty F(t_n v_n(x,0)) - K(\epsilon x)F(v_n(x,0))]\mathrm{d}x \geq o_n(1) - c\varepsilon. \tag{4.9}$$

In fact, observe that

$$\begin{aligned}
&\int_{\mathbb{R}^N}[K_\infty F(t_n v_n(x,0)) - K(\epsilon x)F(v_n(x,0))]\mathrm{d}x \\
&= \int_{\mathbb{R}^N}[K_\infty F(t_n v_n(x,0)) - K_\infty F(v_n(x,0))]\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}[K_\infty F(v_n(x,0)) - K(\epsilon x)F(v_n(x,0))]\mathrm{d}x \\
&:= D_1 + D_2.
\end{aligned}$$

On the one hand, using the mean value theorem, $t_n \to 1$ and the boundedness of $\{v_n\}$ we get

$$\begin{aligned}
D_1 &\leq K_\infty \int_{\mathbb{R}^N}|F(t_n v_n(x,0)) - F(v_n(x,0))|\mathrm{d}x \\
&\leq c_8|t_n - 1|\int_{\mathbb{R}^N}|v_n(x,0)|^2\mathrm{d}x + c_9|t_n - 1|\int_{\mathbb{R}^N}|v_n(x,0)|^p\mathrm{d}x \\
&= o_n(1).
\end{aligned}$$

On the other hand, applying the fact $v_n(x,0) \to 0$ in $L^q_{loc}(\mathbb{R}^N)$ for $q \in [2, 2^*_s)$ and (4.3) we obtain

$$\begin{aligned}
D_2 &= \int_{|x|\leq R}[K_\infty - K(\epsilon x)]F(v_n(x,0))\mathrm{d}x + \int_{|x|\geq R}[K_\infty - K(\epsilon x)]F(v_n(x,0))\mathrm{d}x \\
&\geq o_n(1) - c_{10}\varepsilon.
\end{aligned}$$

Combining the above two estimates for D_1 and D_2, we can see that (4.9) holds.

Finally, from (2.6), the boundedness of $\{v_n\}$ and $t_n \to 1$ we can deduce that

$$\frac{(1-t_n^2)}{2}\|v_n\|^2_{X^s} = o_n(1). \tag{4.10}$$

Using (4.6), (4.7) (4.8), (4.9) and (4.10) we have

$$c - \Phi_\epsilon(u) \geq o_n(1) - c_{11}\varepsilon + c_{V_\infty K_\infty},$$

and taking the limit as $\varepsilon \to 0$ we get

$$c - \Phi_\epsilon(u) \geq c_{V_\infty K_\infty}.$$

Step 3. The sequence $\{t_n\}$ satisfies

$$\limsup_{n\to\infty} t_n = t_0 < 1.$$

We assume that there is a subsequence, still denoted by $\{t_n\}$, such that $t_n \to t_0 < 1$. Similarly, from the above arguments, we can get

$$\int_{\mathbb{R}^N}[K_\infty - K(\epsilon x)]\left[\frac{1}{2}f(v_n(x,0))v_n(x,0) - F(v_n(x,0))\right]\mathrm{d}x = o_n(1). \tag{4.11}$$

Since $\langle \Phi'_\epsilon(v_n), v_n\rangle = o_n(1)$, then we have

$$\begin{aligned} c-\Phi_\epsilon(u)+o_n(1) &= \Phi_\epsilon(v_n)-\frac{1}{2}\langle \Phi'_\epsilon(v_n),v_n\rangle \\ &= \int_{\mathbb{R}^N} K(\epsilon x)\left[\frac{1}{2}f(v_n(x,0))v_n(x,0)-F(v_n(x,0))\right]\mathrm{d}x. \end{aligned} \tag{4.12}$$

Recalling that $t_n v_n \in \mathcal{N}_{V_\infty K_\infty}$, and using (f_5), (4.11) and (4.12) we have

$$\begin{aligned} c_{V_\infty K_\infty} &\le \mathcal{J}_{V_\infty K_\infty}(t_n v_n) \\ &= \mathcal{J}_{V_\infty K_\infty}(t_n v_n)-\frac{1}{2}\langle \mathcal{J}'_{V_\infty K_\infty}(t_n v_n), t_n v_n\rangle \\ &= \int_{\mathbb{R}^N} K_\infty\left[\frac{1}{2}f(t_n v_n(x,0))t_n v_n(x,0)-F(t_n v_n(x,0))\right]\mathrm{d}x. \\ &\le \int_{\mathbb{R}^N} K_\infty\left[\frac{1}{2}f(v_n(x,0))v_n(x,0)-F(v_n(x,0))\right]\mathrm{d}x \\ &= \int_{\mathbb{R}^N} K(\epsilon x)\left[\frac{1}{2}f(v_n(x,0))v_n(x,0)-F(v_n(x,0))\right]\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} [K_\infty - K(\epsilon x)]\left[\frac{1}{2}f(v_n(x,0))v_n(x,0)-F(v_n(x,0))\right]\mathrm{d}x \\ &= \Phi_\epsilon(v_n)-\frac{1}{2}\langle \Phi'_\epsilon(v_n),v_n\rangle + o_n(1) \\ &= c-\Phi_\epsilon(u)+o_n(1). \end{aligned}$$

Taking the limit as $n\to\infty$, we get

$$c-\Phi_\epsilon(u)\ge c_{V_\infty K_\infty}.$$

This ends the proof. □

Combining Lemma 4.1 and Lemma 4.2, we have the following compactness result.

LEMMA 4.3. *Let $\{u_n\}$ be a bounded Palais-Smale sequence at level $c<c_{V_\infty K_\infty}$ for Φ_ϵ. Then $\{u_n\}$ has a convergent subsequence in E_ϵ.*

Proof. Let $\{u_n\}$ be a bounded Palais-Smale sequence, up to a subsequence, we may assume that $u_n \rightharpoonup u$ in E_ϵ. Using Lemma 2.4, we can see that $\Phi'_\epsilon(u)=0$. Moreover, it follows from (2.8) that

$$\begin{aligned} \Phi_\epsilon(u) &= \Phi_\epsilon(u)-\frac{1}{2}\langle \Phi'_\epsilon(u),u\rangle \\ &= \int_{\mathbb{R}^N} K(\epsilon x)\left[\frac{1}{2}f(u(x,0))u(x,0)-F(u(x,0))\right]\mathrm{d}x \ge 0. \end{aligned} \tag{4.13}$$

Hence, we have $c-\Phi_\epsilon(u)\le c< c_{V_\infty K_\infty}$. Evidently, we can deduce from Lemma 4.2 that $u_n \to u$ in E_ϵ. This finishes the proof. □

5. Existence and concentration of positive ground states

In this section, we are going to prove the existence and concentration phenomena of positive ground state solutions to problem (2.5). Moreover, we complete the proofs of Theorems 1.1 and 1.2.

We first consider the case that (A_0) and (A_1) are satisfied. For any $x_v \in \mathscr{V}$, we set $\tilde{V}(\epsilon x) = V(\epsilon x + \epsilon x_v)$ and $\tilde{K}(\epsilon x) = K(\epsilon x + \epsilon x_v)$. It is clear that if $\tilde{u}$ is a solution of

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) + m^2 y^{1-2s} w = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w}{\partial \nu^{1-2s}} = -\tilde{V}(\epsilon x)w(x,0) + \tilde{K}(\epsilon x) f(w(x,0)), & \text{on } \mathbb{R}^N. \end{cases}$$

then $u(x) = \tilde{u}(x - \epsilon x_v)$ solves problem (2.5). From (A_0) and (A_1), without loss of generality, we may assume that $x_v = 0 \in \mathscr{V}$ or $x_v = 0 \in \mathscr{V} \cap \mathscr{K}$ if $\mathscr{V} \cap \mathscr{K} \neq \emptyset$. So

$$V(0) = V_{\min} \text{ and } \kappa := K(0) \geq K(x) \text{ for all } |x| \geq R. \tag{5.1}$$

Consider the following problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) + m^2 y^{1-2s} w = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w}{\partial \nu^{1-2s}} = -V_{\min} w(x,0) + \kappa f(w(x,0)), & \text{on } \mathbb{R}^N. \end{cases} \tag{5.2}$$

In the sequel, we also use the associated notations $\mathcal{J}_{V_{\min}\kappa}$, $\mathscr{N}_{V_{\min}\kappa}$ and $c_{V_{\min}\kappa}$ as before, which denote the energy functional, Nehari manifold and ground state energy value of problem (5.2), respectively. Moreover, according to Lemma 3.3 we know that problem (5.2) possesses at least one positive ground state solution.

Next, we give the comparison relationship of the ground state energy value between problem (2.5) and problem (5.2), which will play a crucial role in our arguments.

Lemma 5.1. *We have* $\limsup_{\epsilon\to 0} c_\epsilon \leq c_{V_{\min}\kappa}$. *In particular, if* $\mathscr{V} \cap \mathscr{K} \neq \emptyset$, *then* $\lim_{\epsilon\to 0} c_\epsilon = c_{V_{\min}K_{\max}}$.

Proof. Let $u \in \mathscr{N}_{V_{\min}\kappa}$ be a positive ground state solution of problem (5.2), then we have

$$c_{V_{\min}\kappa} = \mathcal{J}_{V_{\min}\kappa}(u) = \max_{t\geq 0} \mathcal{J}_{V_{\min}\kappa}(tu). \tag{5.3}$$

Moreover, by Lemma 2.7 that there exists $t_\epsilon > 0$ such that $t_\epsilon u \in \mathscr{N}_\epsilon$, and

$$c_\epsilon \leq \Phi_\epsilon(t_\epsilon u) = \max_{t\geq 0} \Phi_\epsilon(tu). \tag{5.4}$$

It is clear to see that $\{t_\epsilon\}$ is bounded. Then, passing to a subsequence, we assume that $t_\epsilon \to t_0$. Observe that

$$\begin{aligned} \Phi_\epsilon(t_\epsilon u) =& \mathcal{J}_{V_{\min}\kappa}(t_\epsilon u) + \frac{t_\epsilon^2}{2}\int_{\mathbb{R}^N} [V(\epsilon x) - V_{\min}]u^2(x,0)\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} [\kappa - K(\epsilon x)]F(t_\epsilon u(x,0))\mathrm{d}x. \end{aligned} \tag{5.5}$$

Using the boundedness of t_ϵ, $K(\epsilon x) \to \kappa$ in a bounded domain and the decay of u, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} [\kappa - K(\epsilon x)]F(t_\epsilon u(x,0))\mathrm{d}x \\ =& \int_{|x|\leq R} [\kappa - K(\epsilon x)]F(t_\epsilon u(x,0))\mathrm{d}x + \int_{|x|\geq R} [\kappa - K(\epsilon x)]F(t_\epsilon u(x,0))\mathrm{d}x \\ =& o_\epsilon(1). \end{aligned} \tag{5.6}$$

Similarly, we can get

$$\frac{t_\epsilon^2}{2}\int_{\mathbb{R}^N}[V(\epsilon x)-V_{\min}]u^2(x,0)\mathrm{d}x=o_\epsilon(1). \tag{5.7}$$

From (5.5), (5.6), and (5.7) we infer that

$$\Phi_\epsilon(t_\epsilon u)=\mathcal{J}_{V_{\min}\kappa}(t_0u)+o_\epsilon(1).$$

Together with (5.3) and (5.4), as $\epsilon\to 0$, we have

$$c_\epsilon\le\Phi_\epsilon(t_\epsilon u)\to\mathcal{J}_{V_{\min}\kappa}(t_0u)\le\max_{t\ge 0}\mathcal{J}_{V_{\min}\kappa}(tu)=\mathcal{J}_{V_{\min}\kappa}(u)=c_{V_{\min}\kappa}.$$

Thus,

$$\limsup_{\epsilon\to 0}c_\epsilon\le c_{V_{\min}\kappa}. \tag{5.8}$$

Now we show that the second conclusion holds. Note that

$$\begin{aligned}\Phi_\epsilon(u)=&\mathcal{J}_{V_{\min}K_{\max}}(u)+\int_{\mathbb{R}^N}[K_{\max}-K(\epsilon x)]F(u(x,0))\mathrm{d}x\\&+\frac{1}{2}\int_{\mathbb{R}^N}[V(\epsilon x)-V_{\min}]u^2(x,0)\mathrm{d}x.\end{aligned}$$

It follows that

$$c_{V_{\min}K_{\max}}\le c_\epsilon.$$

On the other hand, since $\mathscr{V}\cap\mathscr{K}\neq\emptyset$, then $\kappa=K_{\max}$. So, according to (5.8) we can get

$$\lim_{\epsilon\to 0}c_\epsilon=c_{V_{\min}K_{\max}}.$$

The proof is now complete. ∎

Now we give the existence result of positive ground state solutions of problem (2.5).

LEMMA 5.2. *Suppose that* (A_0), (A_1) *and* (f_1)-(f_5) *are satisfied. Then for any* $\epsilon>0$ *small enough, problem* (2.5) *has a positive ground state solution.*

Proof. Note that if $u_\epsilon\in\mathscr{N}_\epsilon$ satisfies $\Phi_\epsilon(u_\epsilon)=c_\epsilon$, by Lemma 2.11 we have

$$I_\epsilon(m_\epsilon^{-1}(u_\epsilon))=\Phi_\epsilon(\widehat{m}_\epsilon(m_\epsilon^{-1}(u_\epsilon)))=\Phi_\epsilon(u_\epsilon)=c_\epsilon=\inf_{S_\epsilon}I_\epsilon.$$

This shows that $m_\epsilon^{-1}(u_\epsilon)\in S_\epsilon$ is a minimizer of I_ϵ, and hence a critical point of I_ϵ. Lemma 2.11 shows that u_ϵ is a critical point of Φ_ϵ. So, it suffices to verify that there exists a minimizer $u_\epsilon\in\mathscr{N}_\epsilon$ such that $\Phi_\epsilon(u_\epsilon)=c_\epsilon$. Indeed, applying the Ekeland variational principle [42], there exists a sequence $\{v_n\}\subset S_\epsilon$ such that $I_\epsilon(v_n)\to c_\epsilon$ and $I_\epsilon'(v_n)\to 0$ as $n\to\infty$. Let $u_n=\widehat{m}_\epsilon(v_n)\in\mathscr{N}_\epsilon$ for all $n\in\mathbb{N}$. Then using Lemma 2.11 again, we get $\Phi_\epsilon(u_n)\to c_\epsilon$ and $\Phi_\epsilon'(u_n)\to 0$. Moreover, Lemma 2.9 shows that $\{u_n\}$ is bounded in E_ϵ. We can assume that, up to a subsequence, $u_n\rightharpoonup u_\epsilon$ in E_ϵ. From (5.1), we have $V_{\min}<V_\infty$ and $\kappa\ge K_\infty$. By Lemma 3.4, we get $c_{V_{\min}\kappa}<c_{V_\infty K_\infty}$, moreover, using Lemma 5.1 we deduce that $c_\epsilon\le c_{V_{\min}\kappa}<c_{V_\infty K_\infty}$ for $\epsilon>0$ small enough. Then, Lemma 4.3 shows that Φ_ϵ satisfies the Palais-Smale compactness condition for $\epsilon>0$ small enough. Using Lemma

2.4 and continuity of Φ_ϵ, we have $\Phi_\epsilon'(u_\epsilon)=0$ and $\Phi_\epsilon(u_\epsilon)=c_\epsilon$. Hence, problem (2.5) possesses a ground state solution u_ϵ. According to Lemma 2.3 and using some similar arguments as in the proof of Lemma 3.3, we can prove the positivity of the ground state solution. ◻

Let $\mathscr{L}_\epsilon$ be the set of all positive ground state solutions of problem (2.5). Then we have the following compactness result for the set $\mathscr{L}_\epsilon$.

Lemma 5.3. *$\mathscr{L}_\epsilon$ is compact in E_ϵ for all small $\epsilon>0$.*

Proof. Suppose by contradiction that, for some $\epsilon_j\to 0$, $\mathscr{L}_{\epsilon_j}$ is not compact in E_ϵ. Thus, for each j, there exists a sequence $\{u_n^j\}\subset\mathscr{L}_{\epsilon_j}$ such that it has no convergent subsequence. However, we observe that $\{u_n^j\}$ is bounded in E_ϵ. So, we may assume without loss of generality that $u_n^j\rightharpoonup u$ in E_ϵ as $n\to\infty$. Arguing as in the proof of Lemma 5.2, we can easily get a contradiction. ◻

Next, we study the concentration phenomena of positive ground state solution u_ϵ obtained in Lemma 5.2 as $\epsilon\to 0$. First we prove the following result, which plays a fundamental role in the study of the behaviors of ground state solutions.

Lemma 5.4. *Let $u_\epsilon\in\mathscr{L}_\epsilon$, then there is a sequence $\{z_\epsilon\}$, up to a subsequence, such that $\epsilon z_\epsilon\to x_0$ as $\epsilon\to 0$, $\lim\limits_{\epsilon\to 0} dist(\epsilon z_\epsilon,\mathscr{A}_v)=0$ and $v_\epsilon(x,y):=u_\epsilon(x+z_\epsilon,y)$ converges strongly to a positive ground state solution of*

$$\begin{cases} -div(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & in\ \mathbb{R}_+^{N+1},\\ \frac{\partial w}{\partial\nu^{1-2s}}=-V(x_0)w(x,0)+K(x_0)f(w(x,0)), & on\ \mathbb{R}^N.\end{cases}$$

In particular, if $\mathscr{V}\cap\mathscr{K}\neq\emptyset$, then $\lim\limits_{\epsilon\to 0} dist(\epsilon z_\epsilon,\mathscr{V}\cap\mathscr{K})=0$, and up to a subsequence, v_ϵ converges strongly to a positive ground state solution of

$$\begin{cases} -div(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & in\ \mathbb{R}_+^{N+1},\\ \frac{\partial w}{\partial\nu^{1-2s}}=-V_{\min}w(x,0)+K_{\max}f(w(x,0)), & on\ \mathbb{R}^N.\end{cases}$$

Proof. Let $\epsilon_j\to 0$ as $j\to\infty$, $u_j:=u_{\epsilon_j}\in\mathscr{L}_{\epsilon_j}$, we first show that there exist $\{z_j\}\subset\mathbb{R}^N$, $R>0$ and $\sigma_0>0$ such that

$$\int_{B_{R_0}(z_j)} u_j^2(x,0)\mathrm{d}x\geq\sigma_0. \tag{5.9}$$

Arguing by contradiction we assume that

$$\lim_{j\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_R(z)} u_j^2(x,0)\mathrm{d}x=0,\ \forall R>0.$$

Then Lemma 2.2 yields that $u_j(x,0)\to 0$ in $L^q(\mathbb{R}^N)$ for $q\in(2,2_s^*)$. On account of (2.2), (2.6), (2.7) and Lemma 2.1, we see that

$$\begin{aligned}0&=\langle\Phi_{\epsilon_j}'(u_j),u_j\rangle\\ &=\|u_j\|_{\epsilon_j}^2-\int_{\mathbb{R}^N}K(\epsilon_jx)f(u_j(x,0))u_j(x,0)\mathrm{d}x\\ &\geq\|u_j\|_{\epsilon_j}^2-\varepsilon K_{\max}\|u_j(x,0)\|_2^2-C_\varepsilon K_{\max}\|u_j(x,0)\|_p^p\\ &\geq(1-\varepsilon c_{12})\|u_j\|_{\epsilon_j}^2-c_{13}\|u_j(x,0)\|_p^p.\end{aligned}$$

Since $p\in(2,2_s^*)$, then $u_j\to 0$ in E_ϵ. However, from Lemma 2.6 we can see that $\{u_j\}$ is positive bounded from below. Clearly this is a contradiction, and so (5.9) holds.

Setting $v_j(x,y)=u_j(x+z_j,y)$, we may assume that, up to a subsequence, $v_j\rightharpoonup v$ in E_ϵ and $v_j(x,0)\to v(x,0)$ in $L^q_{loc}(\mathbb{R}^N)$ for $q\in[2,2_s^*)$, and $v\neq 0$ by (5.9). Moreover, we also observe that v_j is a solution of the problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & \text{in } \mathbb{R}^{N+1}_+,\\ \frac{\partial w}{\partial \nu^{1-2s}}=-V(\epsilon_j x+\epsilon_j z_j)w(x,0)+K(\epsilon_j x+\epsilon_j z_j)f(w(x,0)), & \text{on } \mathbb{R}^N. \end{cases} \tag{5.10}$$

Then, the corresponding energy

$$\begin{aligned} \mathcal{T}_{\epsilon_j}(v_j)=&\frac{1}{2}\|v_j\|^2_{X^s}+\frac{1}{2}\int_{\mathbb{R}^N}V(\epsilon_j x+\epsilon_j z_j)v_j^2(x,0)\mathrm{d}x-\int_{\mathbb{R}^N}K(\epsilon_j x+\epsilon_j z_j)F(v_j(x,0))\mathrm{d}x\\ =&\mathcal{T}_{\epsilon_j}(v_j)-\frac{1}{2}\langle \mathcal{T}'_{\epsilon_j}(v_j),v_j\rangle\\ =&\int_{\mathbb{R}^N}K(\epsilon_j x+\epsilon_j z_j)\left[\frac{1}{2}f(v_j(x,0))v_j(x,0)-F(v_j(x,0))\right]\mathrm{d}x\\ =&\Phi_{\epsilon_j}(u_j)-\frac{1}{2}\langle \Phi'_{\epsilon_j}(u_j),u_j\rangle\\ =&\Phi_{\epsilon_j}(u_j)=c_{\epsilon_j}, \end{aligned} \tag{5.11}$$

and for any $\varphi\in C_0^\infty(\mathbb{R}^N)$ we have

$$\begin{aligned} \langle \mathcal{T}'_{\epsilon_j}(v_j),\varphi\rangle=&\iint_{\mathbb{R}^{N+1}_+}y^{1-2s}(\nabla v_j\nabla\varphi+v_j\varphi)\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^N}V(\epsilon_j x+\epsilon_j z_j)v_j(x,0)\varphi(x,0)\mathrm{d}x\\ &-\int_{\mathbb{R}^N}K(\epsilon_j x+\epsilon_j z_j)f(v_j(x,0))\varphi(x,0)\mathrm{d}x=0. \end{aligned} \tag{5.12}$$

From (A_0), we can assume without loss of generality that $V(\epsilon_j z_j)\to V_0$ and $K(\epsilon_j z_j)\to K_0$ as $j\to\infty$.

Next we will take several steps to complete the proof.

Step 1. We show that v is a positive ground state solution of the limit problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & \text{in } \mathbb{R}^{N+1}_+,\\ \frac{\partial w}{\partial \nu^{1-2s}}=-V_0w(x,0)+K_0f(w(x,0)), & \text{on } \mathbb{R}^N. \end{cases} \tag{5.13}$$

In fact, since $v_j\rightharpoonup v$ in E_ϵ, using some standard arguments we can easily check that

$$\begin{aligned} \int_{\mathbb{R}^N}V(\epsilon_j x+\epsilon_j z_j)v_j(x,0)\varphi(x,0)\mathrm{d}x=&\int_{supp\varphi}V(\epsilon_j x+\epsilon_j z_j)v_j(x,0)\varphi(x,0)\mathrm{d}x\\ &\to\int_{\mathbb{R}^N}V_0v(x,0)\varphi(x,0)\mathrm{d}x. \end{aligned}$$

Similarly, we get

$$\int_{\mathbb{R}^N}K(\epsilon_j x+\epsilon_j z_j)f(v_j(x,0))\varphi(x,0)\mathrm{d}x\to\int_{\mathbb{R}^N}K_0f(v(x,0))\varphi(x,0)\mathrm{d}x.$$

On account of (5.12), we see that v is a solution of problem (5.13). Moreover,

$$\begin{aligned} \mathcal{J}_{V_0K_0}(v)=&\mathcal{J}_{V_0K_0}(v)-\frac{1}{2}\langle \mathcal{J}'_{V_0K_0}(v),v\rangle\\ =&K_0\int_{\mathbb{R}^N}\left[\frac{1}{2}f(v(x,0))v(x,0)-F(v(x,0))\right]\mathrm{d}x\\ \geq& c_{V_0K_0}, \end{aligned}$$

where $c_{V_0K_0}$ is the ground state energy value of $\mathcal{J}_{V_0K_0}$. On the other hand, using Fatou's lemma and Lemma 5.1 we have

$$\begin{aligned} c_{V_0K_0} &\le K_0\int_{\mathbb{R}^N}\left[\frac{1}{2}f(v(x,0))v(x,0)-F(v(x,0))\right]\mathrm{d}x \\ &\le \liminf_{j\to\infty}\left[\int_{\mathbb{R}^N}K(\epsilon_j x+\epsilon_j z_j)\left(\frac{1}{2}f(v_j(x,0))v_j(x,0)-F(v_j(x,0))\right)\mathrm{d}x\right] \\ &= \liminf_{j\to\infty}\mathcal{T}_{\epsilon_j}(v_j)\le \limsup_{j\to\infty}\Phi_{\epsilon_j}(u_j)\le c_{V_0K_0}. \end{aligned}$$

This shows that v is a ground state solution of problem (5.13). Reasoning as in the proof of Lemma 3.3, we show that v is positive. Also, we get

$$\lim_{j\to\infty}\mathcal{T}_{\epsilon_j}(v_j)=\lim_{j\to\infty}c_{\epsilon_j}=\mathcal{J}_{V_0K_0}(v)=c_{V_0K_0}. \tag{5.14}$$

Step 2. We claim that $\{\epsilon_j z_j\}$ is bounded. Arguing by contradiction we assume that, up to a subsequence, $|\epsilon_j z_j|\to\infty$. From (5.1) and (A_1), we find that $V_0>V_{\min}$ and $K_0\le\kappa$. Then we deduce from Lemma 3.4 that $c_{V_0K_0}>c_{V_{\min}\kappa}$. But, according to Lemma 5.1 and (5.14), we can get $c_{\epsilon_j}\to c_{V_0K_0}\le c_{V_{\min}\kappa}$. This is a contradiction. So, $\{\epsilon_j z_j\}$ is bounded.

Consequently, from Step 2, up to subsequence, we can assume that $\epsilon_j z_j\to x_0$ as $j\to\infty$, then $V_0=V(x_0)$ and $K_0=K(x_0)$. Hence, by Step 1 we know that v is a positive ground state solution of the limit problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w}{\partial\nu^{1-2s}}=-V(x_0)w(x,0)+K(x_0)f(w(x,0)), & \text{on } \mathbb{R}^N. \end{cases}$$

Step 3. We prove that $\lim_{j\to\infty}\mathrm{dist}(\epsilon_j z_j,\mathscr{A}_v)=0$. Indeed, we just need to prove $x_0\in\mathscr{A}_v$. We use a contradiction argument to do this. If $x_0\notin\mathscr{A}_v$, then we get $V_0\ge V_{\min}$ and $K_0<\kappa$ by condition (A_1) and the definition of $\mathscr{A}_v$. Moreover, by Lemma 3.4 we have $c_{V_0K_0}>c_{V_{\min}\kappa}$. Thus, we deduce from (5.14) and Lemma 5.1 that

$$\lim_{j\to\infty}c_{\epsilon_j}=c_{V_0K_0}>c_{V_{\min}\kappa}\ge\lim_{j\to\infty}c_{\epsilon_j},$$

which is a contradiction.

Step 4. We verify that $v_j\to v$ in E_ϵ. We adapt the ideas in [18], and let $\eta:[0,\infty)\to[0,1]$ be a smooth function such that $\eta(t)=1$ if $t\le1$, $\eta(t)=0$ if $t\ge2$. Define $\tilde{v}_j(x)=\eta(2|x|/j)v(x)$. By straightforward computation, we can easily check that

$$\|v-\tilde{v}_j\|_{X^s}\to0 \text{ and } \|v(x,0)-\tilde{v}_j(x,0)\|_q\to0 \text{ as } j\to\infty \tag{5.15}$$

for $q\in[2,2_s^*]$. Setting $w_j=v_j-\tilde{v}_j$, it is easy to verify that up to a subsequence,

$$\lim_{j\to\infty}\left|\int_{\mathbb{R}^N}K(\epsilon_j x+\epsilon_j z_j)[F(v_j(x,0))-F(w_j(x,0))-F(\tilde{v}_j(x,0))]\mathrm{d}x\right|=0 \tag{5.16}$$

and

$$\lim_{j\to\infty}\left|\int_{\mathbb{R}^N}K(\epsilon_j x+\epsilon_j z_j)[f(v_j(x,0))-f(w_j(x,0))-f(\tilde{v}_j(x,0))]\varphi(x,0)\mathrm{d}x\right|=0 \tag{5.17}$$

uniformly in $\varphi\in E_\epsilon$ with $\|\varphi\|_\epsilon\le 1$. Using the decay of v and (5.15) we can easily check that

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} V(\epsilon_j x+\epsilon_j z_j)\tilde{v}_j^2(x,0)\mathrm{d}x\to\int_{\mathbb{R}^N} V_0 v^2(x,0)\mathrm{d}x \tag{5.18}$$

and

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} K(\epsilon_j x+\epsilon_j z_j)F(\tilde{v}_j(x,0))\mathrm{d}x\to\int_{\mathbb{R}^N} K_0 F(v(x,0))\mathrm{d}x. \tag{5.19}$$

From (5.14), (5.15), (5.16), (5.18) and (5.19), we infer that

$$\begin{aligned}\mathcal{T}_{\epsilon_j}(w_j)=&\mathcal{T}_{\epsilon_j}(v_j)-\mathcal{J}_{V_0K_0}(v)\\&+\int_{\mathbb{R}^N} K(\epsilon_j x+\epsilon_j z_j)[F(v_j(x,0))-F(w_j(x,0))-F(\tilde{v}_j(x,0))]\mathrm{d}x+o_j(1)\\=&o_j(1),\end{aligned}$$

which implies that $\mathcal{T}_{\epsilon_j}(w_j)\to 0$. Similarly, from (5.17), we also get $\mathcal{T}'_{\epsilon_j}(w_j)\to 0$. Hence, together with (f_4), we obtain

$$\begin{aligned}o_j(1)=&\mathcal{T}_{\epsilon_j}(w_j)-\frac{1}{\theta}\langle\mathcal{T}'_{\epsilon_j}(w_j),w_j\rangle\\=&\left[\frac{1}{2}-\frac{1}{\theta}\right]\|w_j\|_\epsilon^2+\int_{\mathbb{R}^N} K(\epsilon_j x+\epsilon_j z_j)\left[\frac{1}{\theta}f(w_j(x,0))w_j(x,0)-F(w_j(x,0))\right]\mathrm{d}x\\\ge&\left[\frac{1}{2}-\frac{1}{\theta}\right]\|w_j\|_\epsilon^2,\end{aligned}$$

which shows that $\|w_j\|_\epsilon\to 0$. Then, from (5.15) we see that $v_j\to v$ in E_ϵ.

Finally, if $\mathscr{V}\cap\mathscr{K}\ne\emptyset$, then $\mathscr{A}_v=\mathscr{V}\cap\mathscr{K}$. Following the above arguments, we can prove that

$$\lim_{j\to\infty}\mathrm{dist}(\epsilon_j z_j,\mathscr{V}\cap\mathscr{K})=0.$$

So, $x_0\in\mathscr{V}\cap\mathscr{K}$, $V(x_0)=V_{\min}$ and $K(x_0)=K_{\max}$. Moreover, up to a subsequence, v_j converges to a positive ground state solution v of the limit problem

$$\begin{cases}-\mathrm{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & \text{in }\mathbb{R}^{N+1}_+,\\ \frac{\partial w}{\partial\nu^{1-2s}}=-V_{\min}w(x,0)+K_{\max}f(w(x,0)), & \text{on }\mathbb{R}^N.\end{cases}$$

From the above steps, we finish the proofs of all conclusions of Lemma 5.4. ∎

In what follows, we study the regularity and decay properties of the solutions. First, in view of (A_0) and (2.7), we can argue as in the proof of [2, Lemma 4.1] to obtain the following L^∞-estimate.

LEMMA 5.5. *$v_j(x,0)\in L^\infty(\mathbb{R}^N)$ and there exists $C>0$ such that*

$$\|v_j(x,0)\|_\infty\le C\quad\textit{for all } j\in\mathbb{N}.$$

Moreover, $v_j\in L^\infty(\mathbb{R}^{N+1}_+)$ and there exists $\tilde{C}>0$ such that

$$\|v_j\|_{L^\infty(\mathbb{R}^{N+1}_+)}\le\tilde{C}\quad\textit{for all } j\in\mathbb{N}.$$

We observe that $v_j(x,0)$ is a weak solution of the problem

$$(-\Delta+m^2)^s v_j(x,0)=-V_j(x)v_j(x,0)+K_j(x)f(v_j(x,0)) \text{ in } \mathbb{R}^N.$$

Fix $\delta\in(0,m^{2s}+V_{\min})$, we deduce from (A_0) and (2.7) that $v_j(x,0)$ is a weak subsolution to

$$(-\Delta+m^2)^s v_j(x,0)=(\delta-V_{\min})v_j(x,0)+C_\delta v_j^{p-1}(x,0)=:\phi_j \text{ in } \mathbb{R}^N \tag{5.20}$$

for some $C_\delta>0$. Evidently, we can see that $\phi_j\geq 0$ in $\mathbb{R}^N$. Using Lemma 5.4, Lemma 5.5 and interpolation inequality, we know that for any $q\in[2,\infty)$

$$\phi_j\to\phi:=(\delta-V_{\min})v(x,0)+C_\delta v^{p-1}(x,0) \text{ in } L^q(\mathbb{R}^N) \tag{5.21}$$

and $\|\phi_j\|_\infty\leq C$ for all $j\in\mathbb{N}$.

Let $w_j\in H^s(\mathbb{R}^N)$ be the unique solution of

$$(-\Delta+m^2)^s w_j=\phi_j \text{ in } \mathbb{R}^N. \tag{5.22}$$

Then $w_j=\mathscr{B}_{2s,m}*\phi_j$, where

$$\mathscr{B}_{2s,m}(x)=(2\pi)^{-\frac{N}{2}}\mathscr{F}^{-1}((|\xi|^2+m^2)^{-s})(x)$$

is the Bessel kernel with parameter m. According to the scaling property of the Fourier transform, we have $\mathscr{B}_{2s,m}(x)=m^{N-2s}\mathscr{B}_{2s,1}(mx)$. Using formula (4.1) at page 416 in [7], $\mathscr{B}_{2s,m}(x)$ has the following expression

$$\mathscr{B}_{2s,m}(x)=\frac{1}{2^{\frac{N+2s-2}{2}}\pi^{\frac{N}{2}}\Gamma(s)}m^{\frac{N-2s}{2}}W_{\frac{N-2s}{2}}(m|x|)|x|^{\frac{2s-N}{2}}.$$

Moreover, it satisfies the following properties (see [7, p.416-417]):

- $(\mathscr{B}_1)$ $\mathscr{B}_{2s,m}$ is positive, radially symmetric and smooth in $\mathbb{R}^N\setminus\{0\}$;
- $(\mathscr{B}_2)$ $\mathscr{B}_{2s,m}(x)\leq C(\chi_{B_2}(x)|x|^{2s-N}+\chi_{B_2^c}(x)e^{-c|x|})$ for all $x\in\mathbb{R}^N$ and some $C,c>0$;
- $(\mathscr{B}_3)$ $\mathscr{B}_{2s,m}\in L^r(\mathbb{R}^N)$ for all $r\in[1,\frac{N}{N-2s})$.

Based on these information, we are able to prove the following result.

LEMMA 5.6. *$v_j(x,0)\to 0$ as $|x|\to\infty$ uniformly in $j\in\mathbb{N}$.*

Proof. The proof of lemma can be found in [3], for the convenience of the readers, we give the proof here. We fist verify that $w_j(x)\to 0$ as $|x|\to\infty$ uniformly in $j\in\mathbb{N}$. For any $\sigma\in(0,\frac{1}{2})$, we have

$$w_j(x)=(\mathscr{B}_{2s,m}*\phi_j)(x)=\left(\int_{B_{\frac{1}{\sigma}}(x)}+\int_{B^c_{\frac{1}{\sigma}}(x)}\right)\mathscr{B}_{2s,m}(x-y)\phi_j(y)\mathrm{d}y.$$

On the one hand, according to $(\mathscr{B}_1)$ and $(\mathscr{B}_2)$ we get

$$\begin{aligned}0&\leq\int_{B^c_{\frac{1}{\sigma}}(x)}\mathscr{B}_{2s,m}(x-y)\phi_j(y)\mathrm{d}y\leq\|\phi_j\|_\infty\int_{B^c_{\frac{1}{\sigma}}(x)}e^{-c|x-y|}\mathrm{d}y\\&\leq C\int_{\frac{1}{\sigma}}^{\infty}e^{-cr}r^{N-1}\mathrm{d}r=:C\nu(\sigma)\to 0 \text{ as } \sigma\to 0.\end{aligned} \tag{5.23}$$

On the other hand, it follows that

$$0 \le \int_{B_{\frac{1}{\sigma}}(x)} \mathscr{B}_{2s,m}(x-y)\phi_j(y)\mathrm{d}y$$
$$= \int_{B_{\frac{1}{\sigma}}(x)} \mathscr{B}_{2s,m}(x-y)(\phi_j(y)-\phi(y))\mathrm{d}y + \int_{B_{\frac{1}{\sigma}}(x)} \mathscr{B}_{2s,m}(x-y)\phi(y)\mathrm{d}y.$$

Choosing $q \in (1, \min\{\frac{N}{N-2s}, 2\})$ such that $q' = \frac{q}{q-1} > 2$, and using $(\mathscr{B}_3)$ and Hölder inequality we have

$$\int_{B_{\frac{1}{\sigma}}(x)} \mathscr{B}_{2s,m}(x-y)\phi_j(y)\mathrm{d}y \le \|\mathscr{B}_{2s,m}\|_q \|\phi_j - \phi\|_{q'} + \|\mathscr{B}_{2s,m}\|_q \|\phi\|_{L^{q'}(B_{\frac{1}{\sigma}}(x))}.$$

From (5.21) we know that $\|\phi_j - \phi\|_{q'} \to 0$ as $j \to \infty$ and $\|\phi\|_{L^{q'}(B_{\frac{1}{\sigma}}(x))} \to 0$ as $|x| \to \infty$. So, there exist $R_0 > 0$ and $j_0 \in \mathbb{N}$ such that for any $j \ge j_0$ and $|x| \ge R_0$

$$\int_{B_{\frac{1}{\sigma}}(x)} \mathscr{B}_{2s,m}(x-y)\phi_j(y)\mathrm{d}y \le C\sigma. \tag{5.24}$$

We deduce from (5.23) and (5.24) that

$$\int_{\mathbb{R}^N} \mathscr{B}_{2s,m}(x-y)\phi_j(y)\mathrm{d}y \le C(\nu(\sigma)+\sigma). \tag{5.25}$$

for any $j \ge j_0$ and $|x| \ge R_0$.

For each $j \in \{1, \cdots, j_0 - 1\}$, it is clear that there exists a constant $R_j > 0$ such that $\|\phi_j\|_{L^{q'}(B_{\frac{1}{\sigma}}(x))} < \sigma$ for all $|x| \ge R_j$. Hence, for all $|x| \ge R_j$ we have

$$\int_{\mathbb{R}^N} \mathscr{B}_{2s,m}(x-y)\phi_j(y)\mathrm{d}y \le C\nu(\sigma) + \int_{B_{\frac{1}{\sigma}}(x)} \mathscr{B}_{2s,m}(x-y)\phi_j(y)\mathrm{d}y$$
$$\le C\nu(\sigma) + \|\mathscr{B}_{2s,m}\|_q \|\phi_j\|_{L^{q'}(B_{\frac{1}{\sigma}}(x))} \le C(\nu(\sigma)+\sigma). \tag{5.26}$$

Taking $R = \max\{R_0, R_1, \cdots, R_{j_0-1}\}$ and using (5.25) and (5.26) we have

$$\int_{\mathbb{R}^N} \mathscr{B}_{2s,m}(x-y)\phi_j(y)\mathrm{d}y \le C(\nu(\sigma)+\sigma)$$

for $|x| \ge R$ uniformly in $j \in \mathbb{N}$. Letting $\sigma \to 0$, we obtain the desired conclusion for w_j.

Finally, using (5.20), (5.22) and the comparison principle (see [2, Theorem 4.3]) we know that $0 \le v_j(x,0) \le w_j(x)$ in $\mathbb{R}^N$ and we finish the proof of this lemma. □

Lemma 5.7. *There exists $\nu_0 > 0$ such that $\|v_j(x,0)\|_\infty \ge \nu_0$ for all $j \in \mathbb{N}$.*

Proof. First, according to (5.9) we can see that

$$\int_{B_R(0)} v_j^2(x,0)\mathrm{d}x \ge \sigma_0 > 0$$

for some $R > 0$ and $j \ge j_0$. Assume by contradiction that $\|v_j(x,0)\|_\infty \to 0$ as $j \to +\infty$, then

$$0 < \sigma_0 \le \int_{B_R(0)} v_j^2(x,0)\mathrm{d}x \le |B_R| \|v_j(x,0)\|_\infty^2 \to 0 \text{ as } j \to \infty,$$

which is impossible. ∎

We now proceed as in the proof of Theorem 1.1 in [2] (see also [3]) to derive the following exponential estimate.

LEMMA 5.8. *There exist c, $C>0$ such that for all $j\in\mathbb{N}$*

$$u_j(x,0)\le C\exp(-c|x-z_j|).$$

Proof. By Lemma 5.6, (A_0) and (f_1), there exists $R_1>0$ such that

$$K_j(x)f(v_j(x,0))\le \delta v_j(x,0) \tag{5.27}$$

for some $\delta\in(0,m^{2s}+V_{\min})$ and $|x|\ge R_1$. Pick a smooth cut-off function ϕ defined in $\mathbb{R}^N$ such that $0\le\phi\le 1$, $\phi(x)=0$ for $|x|\ge 1$ and $\phi\not\equiv 0$. By using the Riesz representation theorem, there exists a unique function $\widehat{v}\in H^s(\mathbb{R}^N)$ such that

$$(-\Delta+m^2)^s\widehat{v}-(\delta-V_{\min})\widehat{v}=\phi \text{ in } \mathbb{R}^N.$$

Moreover, applying the extension method and Lemma 2.3, we can verify that $\widehat{v}$ is continuous and positive. Evidently, there exists $R_2>1$ we have

$$(-\Delta+m^2)^s\widehat{v}-(\delta-V_{\min})\widehat{v}=0 \text{ in } \bar{B}^c_{R_2}. \tag{5.28}$$

By constructing a suitable comparison function which has the exponential decay at infinity and using the comparison principle, we can obtain the exponential decay property of $\widehat{v}$, that is,

$$0<\widehat{v}\le Ce^{-c|x|}, \quad \text{for all } x\in\mathbb{R}^N \tag{5.29}$$

for some $C,c>0$, the details of the proof are similar to the proof of (57) in [2].

Taking $R=\max\{R_1,R_2\}$, and using (A_0) and (5.27) we have

$$(-\Delta+m^2)^s v_j(x,0)-(\delta-V_{\min})v_j(x,0)\le 0 \text{ in } \bar{B}^c_R. \tag{5.30}$$

Set $a=\min_{\bar{B}_R}\widehat{v}>0$, $b=\sup_{j\in\mathbb{N}}\|v_j(x,0)\|_\infty<\infty$ and $w_j=(b+1)\widehat{v}-av_j(x,0)$. Clearly, $w_j\ge 0$ in $\bar{B}_R$. From (5.28) and (5.30), we infer that

$$(-\Delta+m^2)^s w_j-(\delta-V_{\min})w_j\ge 0 \text{ in } \bar{B}^c_R.$$

Since $\delta-V_{\min}<m^{2s}$, using the comparison principle (see [2, Theorem 4.3]) we can infer that $w_j\ge 0$ in $\mathbb{R}^N$. Then $v_j(x,0)\le\frac{b+1}{a}\widehat{v}$, this, together with (5.29) gives

$$0<v_j(x,0)\le C_0e^{-c|x|} \quad \text{for all } x\in\mathbb{R}^N,$$

for some $C_0,c>0$. Recall that $u_j(x,0)=v_j(x-z_j,0)$, then we have

$$0<u_j(x,0)=v_j(x-z_j,0)\le C_0e^{-c|x-z_j|} \quad \text{for all } x\in\mathbb{R}^N.$$

The proof is completed. ∎

Now we are in a position to complete the proofs of Theorems 1.1 and 1.2.

Proof. **(Proof of Theorem 1.1.)** Firstly, from Lemma 5.2 we can see that problem (2.5) has a positive ground state solution u_ϵ for $\epsilon>0$ small enough. Then $u_\epsilon(x,0)$ is a positive ground state solution of problem (2.4), and $\widehat{u}_\epsilon(x):=u_\epsilon(\frac{x}{\epsilon},0)$ is a positive ground state solution of problem (1.4). So, the conclusion (i) holds. Lemma 5.3 shows that the conclusion (ii) holds.

Next we show the concentration of the maximum points of ground state solutions as $\epsilon\to 0$. From Lemma 5.4, there exists a sequence $\{z_\epsilon\}\subset\mathbb{R}^N$ such that $v_\epsilon(x,y):=u_\epsilon(x+z_\epsilon,y)\to v$ in E_ϵ. If p_ϵ is a global maximum point of $v_\epsilon(x,0)$, we deduce from Lemma 5.6 and Lemma 5.7 that there exists $R_0>0$ such that $p_\epsilon\in B_{R_0}(0)$. Thus, $y_\epsilon=p_\epsilon+z_\epsilon$ is a global maximum point of $u_\epsilon(x,0)$, and $x_\epsilon=\epsilon y_\epsilon$ is a global maximum point of $\widehat{u}_\epsilon(x)$. According to Lemma 5.4, we get

$$x_\epsilon\to x_0 \text{ and } \lim_{\epsilon\to 0}\operatorname{dist}(x_\epsilon,\mathscr{A}_v)=0,$$

Moreover, $\widehat{u}_\epsilon(\epsilon x+x_\epsilon)$ converges to a positive ground state solution $\widehat{u}$ of

$$(-\Delta+m^2)^s u+V(x_0)u=K(x_0)f(u) \text{ in } \mathbb{R}^N.$$

In particular, if $\mathscr{V}\cap\mathscr{K}\neq\emptyset$, then $\mathscr{A}_v=\mathscr{V}\cap\mathscr{K}$ and

$$\lim_{\epsilon\to 0}\operatorname{dist}(x_\epsilon,\mathscr{V}\cap\mathscr{K})=0,$$

and $\widehat{u}_\epsilon(\epsilon x+x_\epsilon)$ converges to a positive ground state solution $\widehat{u}$ of

$$(-\Delta+m^2)^s u+V_{\min}u=K_{\max}f(u) \text{ in } \mathbb{R}^N.$$

So, conclusion (iii) holds.

Finally, from Lemma 5.8 we have

$$\begin{aligned}|\widehat{u}_\epsilon(x)|&=|u_\epsilon(\frac{x}{\epsilon},0)|=|v_\epsilon(\frac{x}{\epsilon}-z_\epsilon,0)|\le C\exp\Big(-c|\frac{x}{\epsilon}-z_\epsilon|\Big)\\&\le C\exp\Big(-c|\frac{x}{\epsilon}-y_\epsilon|\Big)=C\exp\Big(-\frac{c}{\epsilon}|x-x_\epsilon|\Big)\end{aligned}$$

for some $c,C>0$. Consequently, the proof of Theorem 1.1 is completed. ∎

Proof. **(Proof of Theorem 1.2.)** For the case that (A_0) and (A_2) are satisfied, we can assume without loss of generality that $x_k=0\in\mathscr{K}$ or $x_k=0\in\mathscr{V}\cap\mathscr{K}$ if $\mathscr{V}\cap\mathscr{K}\neq\emptyset$. Consequently,

$$K(0)=K_{\max} \text{ and } \tau:=V(0)\le V(x) \text{ for all } |x|\ge R.$$

Arguing as in the proof of Lemma 5.1, we also prove that

$$\limsup_{\epsilon\to 0} c_\epsilon\le c_{\tau K_{\max}}.$$

The remaining proofs are similar to the proof of Theorem 1.1 with suitable modification. Here we omit the details. ∎

6. Multiplicity of positive solutions

In this section we will prove the multiplicity result of positive solutions and complete the proof of Theorem 1.3. For this purpose, we always assume that $\Lambda:=\mathscr{V}\cap\mathscr{K}\neq\emptyset$.

Let u be a positive ground state solution of problem

$$\begin{cases}-\operatorname{div}(y^{1-2s}\nabla w)+m^2y^{1-2s}w=0, & \text{in } \mathbb{R}^{N+1}_+,\\ \frac{\partial w}{\partial\nu^{1-2s}}=-V_{\min}w(x,0)+K_{\max}f(w(x,0)), & \text{on } \mathbb{R}^N,\end{cases}\tag{6.1}$$

and ζ be a smooth nonincreasing cut-off function in $[0,+\infty)$ such that $\zeta(s)=1$ if $0\le s\le \frac{1}{2}$ and $\zeta(s)=0$ if $s\ge 1$. For any $z\in\Lambda$, we define

$$\Psi_{\epsilon,z}(x,y)=\zeta(|(\epsilon x-z,y)|)u(\frac{\epsilon x-z}{\epsilon},y).$$

Then, according to Lemma 2.7, there exists $t_\epsilon>0$ such that

$$\max_{t\ge 0}\Phi_\epsilon(t\Psi_{\epsilon,z})=\Phi_\epsilon(t_\epsilon\Psi_{\epsilon,z}).$$

We define $\gamma_\epsilon:\Lambda\to\mathcal{N}_\epsilon$ by $\gamma_\epsilon(z)=t_\epsilon\Psi_{\epsilon,z}$. By the construction, we can see that $\gamma_\epsilon(z)$ has compact support for any $z\in\Lambda$.

LEMMA 6.1. *The function γ_ϵ satisfies*

$$\lim_{\epsilon\to 0}\Phi_\epsilon(\gamma_\epsilon(z))=c_{V_{\min}K_{\max}} \text{ uniformly in } z\in\Lambda.$$

Proof. Assume by contradiction that there exist $\varepsilon_0>0$, $\{z_n\}\subset\Lambda$ and $\epsilon_n\to 0$ such that

$$|\Phi_{\epsilon_n}(\gamma_{\epsilon_n}(z))-c_{V_{\min}K_{\max}}|\ge\varepsilon_0. \tag{6.2}$$

Using the Lebesgue's dominated convergence theorem, we can easily check that

$$\|\Psi_{\epsilon_n,z_n}\|^2_{X^s}+\int_{\mathbb{R}^N}V(\epsilon_n x)\Psi^2_{\epsilon_n,z_n}(x,0)\mathrm{d}x\to\|u\|^2_{X^s}+\int_{\mathbb{R}^N}V_{\min}u^2(x,0)\mathrm{d}x, \tag{6.3}$$

and

$$\int_{\mathbb{R}^N}K(\epsilon_n x)F(\Psi_{\epsilon_n,z_n}(x,0))\mathrm{d}x\to K_{\max}\int_{\mathbb{R}^N}F(u(x,0))\mathrm{d}x. \tag{6.4}$$

Since $\langle\Phi'_{\epsilon_n}(t_{\epsilon_n}\Psi_{\epsilon_n,z_n}),t_{\epsilon_n}\Psi_{\epsilon_n,z_n}\rangle=0$, making the change of variable $\tilde{x}=\frac{\epsilon_n x-z_n}{\epsilon_n}$ we have

$$\begin{aligned}
&t^2_{\epsilon_n}\|\Psi_{\epsilon_n,z_n}\|^2_{X^s}+t^2_{\epsilon_n}\int_{\mathbb{R}^N}V(\epsilon_n x)\Psi^2_{\epsilon_n,z_n}(x,0)\mathrm{d}x\\
&=\int_{\mathbb{R}^N}K(\epsilon_n x)f(t_{\epsilon_n}\Psi_{\epsilon_n,z_n}(x,0))t_{\epsilon_n}\Psi_{\epsilon_n,z_n}(x,0)\mathrm{d}x\\
&=\int_{\mathbb{R}^N}K(\epsilon_n\tilde{x}+z_n)f(t_{\epsilon_n}\zeta(|(\epsilon_n\tilde{x},0)|)u(\tilde{x},0))t_{\epsilon_n}\zeta(|(\epsilon_n\tilde{x},0)|)u(\tilde{x},0)\mathrm{d}\tilde{x}.
\end{aligned} \tag{6.5}$$

We claim that $t_{\epsilon_n}\to 1$. We first prove that $\{t_{\epsilon_n}\}$ is bounded. Indeed, suppose by contradiction that $t_{\epsilon_n}\to\infty$. Using (6.5) and (f_5) we have

$$\begin{aligned}
&\|\Psi_{\epsilon_n,z_n}\|^2_{X^s}+\int_{\mathbb{R}^N}V(\epsilon_n x)\Psi^2_{\epsilon_n,z_n}(x,0)\mathrm{d}x\\
&=\int_{\mathbb{R}^N}K(\epsilon_n\tilde{x}+z_n)f(t_{\epsilon_n}\zeta(|(\epsilon_n\tilde{x},0)|)u(\tilde{x},0))t_{\epsilon_n}\zeta(|(\epsilon_n\tilde{x},0)|)u(\tilde{x},0)t^{-2}_{\epsilon_n}\mathrm{d}\tilde{x}\\
&\ge K_{\min}\int_{B_{\frac{1}{2}}(0)}\frac{f(t_{\epsilon_n}u(\tilde{x},0))}{t_{\epsilon_n}u(\tilde{x},0)}u^2(\tilde{x},0)\mathrm{d}\tilde{x}\\
&\ge K_{\min}\frac{f(t_{\epsilon_n}u(\tilde{x}_0,0))}{t_{\epsilon_n}u(\tilde{x}_0,0)}\int_{B_{\frac{1}{2}}(0)}u^2(\tilde{x},0)\mathrm{d}\tilde{x},
\end{aligned} \tag{6.6}$$

where $u(\tilde{x}_0,0)=\min\{u(\tilde{x},0):|\tilde{x}|\le\frac{1}{2}\}>0$. From (f_4), (6.3) and (6.6), we get a contradiction. Hence, $\{t_{\epsilon_n}\}$ is bounded. Passing to a subsequence, we may assume that $t_{\epsilon_n}\to t_0\ge 0$. If $t_0=0$, we deduce from (f_2) and (6.5) that

$$\|\Psi_{\epsilon_n,z_n}\|_{X^s}^2+\int_{\mathbb{R}^N}V(\epsilon_n x)\Psi_{\epsilon_n,z_n}^2(x,0)\mathrm{d}x\to 0,$$

this contradicts relation (6.3). Consequently, $t_0>0$.

Next, we prove that $t_0=1$. Letting $n\to\infty$ in (6.5), we have

$$\|u\|_{X^s}^2+\int_{\mathbb{R}^N}V_{\min}u^2(x,0)\mathrm{d}x=K_{\max}\int_{\mathbb{R}^N}\frac{f(t_0u(x,0))}{t_0u(x,0)}u^2(x,0)\mathrm{d}x. \tag{6.7}$$

Moreover, since u is a positive ground state solution of problem (6.1), then we have

$$\|u\|_{X^s}^2+\int_{\mathbb{R}^N}V_{\min}u^2(x,0)\mathrm{d}x=K_{\max}\int_{\mathbb{R}^N}f(u(x,0))u(x,0)\mathrm{d}x. \tag{6.8}$$

It follows from (6.7) and (6.8) that

$$\int_{\mathbb{R}^N}\left[\frac{f(t_0u(x,0))}{t_0u(x,0)}-\frac{f(u(x,0))}{u(x,0)}\right]u^2(x,0)\mathrm{d}x=0.$$

Then, we infer from (f_5) that $t_0=1$. Therefore, using (6.3) and (6.4) we have

$$\begin{aligned}\Phi_{\epsilon_n}(\gamma_{\epsilon_n}(z_n))=&\frac{t_{\epsilon_n}^2}{2}\left[\|\Psi_{\epsilon_n,z_n}\|_{X^s}^2+\int_{\mathbb{R}^N}V(\epsilon_n x)\Psi_{\epsilon_n,z_n}^2(x,0)\mathrm{d}x\right]\\&-\int_{\mathbb{R}^N}K(\epsilon_n x)F(t_{\epsilon_n}\Psi_{\epsilon_n,z_n}(x,0))\mathrm{d}x\\\to&\frac{1}{2}\left[\|u\|_{X^s}^2+\int_{\mathbb{R}^N}V_{\min}u^2(x,0)\mathrm{d}x\right]-K_{\max}\int_{\mathbb{R}^N}F(u(x,0))\mathrm{d}x\\=&\mathcal{J}_{V_{\min}K_{\max}}(u)=c_{V_{\min}K_{\max}}.\end{aligned}$$

Obviously, from (6.2) we can see that this is impossible. This ends the proof. □

Now, we are in the position to introduce the barycenter map. For any $\delta>0$, let $\rho=\rho(\delta)>0$ be such that $\Lambda_\delta\subset B_\rho(0)$. We define $\eta:\mathbb{R}^N\to\mathbb{R}^N$ as follows

$$\eta(x)=x \text{ for } |x|\le\rho \ \text{ and } \ \eta(x)=\frac{\rho x}{|x|} \text{ for } |x|\ge\rho.$$

Let us consider $\beta_\epsilon:\mathcal{N}_\epsilon\to\mathbb{R}^N$ given by

$$\beta_\epsilon(u)=\frac{\int_{\mathbb{R}^N}\eta(\epsilon x)u^2(x,0)\mathrm{d}x}{\int_{\mathbb{R}^N}u^2(x,0)\mathrm{d}x}.$$

Lemma 6.2. *We have the following limit*

$$\lim_{\epsilon\to 0}\beta_\epsilon(\gamma_\epsilon(z))=z \textit{ uniformly in } z\in\Lambda.$$

Proof. If it is not true, then there exist $\sigma_0>0$, $\{z_n\}\subset\Lambda$ and $\epsilon_n\to 0$ such that

$$|\beta_{\epsilon_n}(\gamma_{\epsilon_n}(z_n))-z_n|\ge\sigma_0>0. \tag{6.9}$$

Using the definitions of γ_{ϵ_n} and β_{ϵ_n}, and making the change of variable $\tilde{x}=\frac{\epsilon_n x - z_n}{\epsilon_n}$ we get

$$\beta_{\epsilon_n}(\gamma_{\epsilon_n}(z_n)) = z_n + \frac{\int_{\mathbb{R}^N}[\eta(\epsilon_n\tilde{x}+z_n)-z_n](\zeta(|(\epsilon_n\tilde{x},0)|)u^2(\tilde{x},0)\mathrm{d}\tilde{x}}{\int_{\mathbb{R}^N}(\zeta(|(\epsilon_n\tilde{x},0)|)u^2(\tilde{x},0)\mathrm{d}\tilde{x}}.$$

Taking into account $\{z_n\}\subset\Lambda\subset B_\rho(0)$ and using the Lebesgue dominated convergence theorem, we have

$$|\beta_{\epsilon_n}(\gamma_{\epsilon_n}(z_n))-z_n|\to 0,$$

which contradicts relation (6.9). □

LEMMA 6.3. *Let $\epsilon_n\to 0$ and $\{u_n\}\subset\mathcal{N}_{\epsilon_n}$ be a sequence satisfying $\Phi_{\epsilon_n}(u_n)\to c_{V_{\min}K_{\max}}$. Then there exists $\{\tilde{z}_n\}\subset\mathbb{R}^N$ such that $v_n(x,y)=u_n(x+\tilde{z}_n,y)$ has a convergent subsequence. Moreover, up to a subsequence, $z_n\to z\in\Lambda$, where $z_n=\epsilon_n\tilde{z}_n$.*

Proof. Since $u_n\in\mathcal{N}_{\epsilon_n}$ and $\Phi_{\epsilon_n}(u_n)\to c_{V_{\min}K_{\max}}$, we have that $\{u_n\}$ is bounded. We claim that there are $R_0,\delta>0$ and $\tilde{z}_n\in\mathbb{R}^N$ such that

$$\liminf_{n\to\infty}\int_{B_{R_0}(\tilde{z}_n)} u_n^2(x,0)\mathrm{d}x\geq\delta. \tag{6.10}$$

Indeed, if relation (6.10) does not hold, Lemma 2.2 implies that $u_n(x,0)\to 0$ in $L^q(\mathbb{R}^N)$ for $q\in(2,2_s^*)$. From (2.7) and the fact $u_n\in\mathcal{N}_{\epsilon_n}$, it is easy to verify that $u_n\to 0$ in E_ϵ, which is a contradiction, because $\Phi_{\epsilon_n}(u_n)\to c_{V_{\min}K_{\max}}>0$. So, (6.10) holds.

Setting $v_n(x,y)=u_n(x+\tilde{z}_n,y)$, up to a subsequence, we can assume that $v_n\rightharpoonup v\neq 0$. According to Lemma 2.7, there exists $t_n>0$ such that $\tilde{v}_n=t_nv_n\in\mathcal{N}_{V_{\min}K_{\max}}$. Then we have

$$c_{V_{\min}K_{\max}}\leq\mathcal{J}_{V_{\min}K_{\max}}(\tilde{v}_n)=\mathcal{J}_{V_{\min}K_{\max}}(t_nu_n)\leq\Phi_{\epsilon_n}(t_nu_n)\leq\Phi_{\epsilon_n}(u_n)\to c_{V_{\min}K_{\max}},$$

which shows that $\mathcal{J}_{V_{\min}K_{\max}}(\tilde{v}_n)\to c_{V_{\min}K_{\max}}$. By virtue of Lemma 3.1-(e), we know that $\{\tilde{v}_n\}$ is bounded. Thus, for some subsequence, $\tilde{v}_n\rightharpoonup\tilde{v}$ with $\tilde{v}\neq 0$. Moreover, $\mathcal{J}'_{V_{\min}K_{\max}}(\tilde{v})=0$. Using Lemma 4.1 we have

$$\mathcal{J}_{V_{\min}K_{\max}}(\tilde{v}_n-\tilde{v})\to c_{V_{\min}K_{\max}}-\mathcal{J}_{V_{\min}K_{\max}}(\tilde{v}) \text{ and } \mathcal{J}'_{V_{\min}K_{\max}}(\tilde{v}_n-\tilde{v})\to 0.$$

Observe that, using (2.8) and Fatou's lemma we have

$$\begin{aligned}
c_{V_{\min}K_{\max}} &= \lim_{n\to\infty}\left[\mathcal{J}_{V_{\min}K_{\max}}(\tilde{v}_n)-\frac{1}{2}\langle\mathcal{J}'_{V_{\min}K_{\max}}(\tilde{v}_n),\tilde{v}_n\rangle\right]\\
&=\lim_{n\to\infty}\left[K_{\max}\int_{\mathbb{R}^N}\left(\frac{1}{2}f(\tilde{v}_n(x,0))\tilde{v}_n(x,0)-F(\tilde{v}_n(x,0))\right)\mathrm{d}x\right]\\
&\geq K_{\max}\int_{\mathbb{R}^N}\left[\frac{1}{2}f(\tilde{v}(x,0))\tilde{v}(x,0)-F(\tilde{v}(x,0))\right]\mathrm{d}x\\
&=\mathcal{J}_{V_{\min}K_{\max}}(\tilde{v})-\frac{1}{2}\langle\mathcal{J}'_{V_{\min}K_{\max}}(\tilde{v}),\tilde{v}\rangle\\
&=\mathcal{J}_{V_{\min}K_{\max}}(\tilde{v})\\
&\geq c_{V_{\min}K_{\max}}.
\end{aligned}$$

It follows that

$$\mathcal{J}_{V_{\min}K_{\max}}(\tilde{v}_n-\tilde{v})\to 0 \text{ and } \mathcal{J}'_{V_{\min}K_{\max}}(\tilde{v}_n-\tilde{v})\to 0. \tag{6.11}$$

Moreover, using (f_4) and (6.11) we have

$$\begin{aligned} o_n(1) =& \mathcal{J}_{V_{\min}K_{\max}}(\tilde{v}_n-\tilde{v}) - \frac{1}{\theta}\langle \mathcal{J}'_{V_{\min}K_{\max}}(\tilde{v}_n-\tilde{v}), \tilde{v}_n-\tilde{v}\rangle \\ =& \left(\frac{1}{2}-\frac{1}{\theta}\right)\left[\|\tilde{v}_n-\tilde{v}\|^2_{X^s} + \int_{\mathbb{R}^N} V_{\min}[\tilde{v}_n(x,0)-\tilde{v}(x,0)]^2 \mathrm{d}x\right] \\ &+ K_{\max}\int_{\mathbb{R}^N}\left[\frac{1}{\theta}f(\tilde{v}_n(x,0)-\tilde{v}(x,0))(\tilde{v}_n(x,0)-\tilde{v}(x,0)) - F(\tilde{v}_n(x,0)-\tilde{v}(x,0))\right]\mathrm{d}x \\ \geq& \left(\frac{1}{2}-\frac{1}{\theta}\right)\left[\|\tilde{v}_n-\tilde{v}\|^2_{X^s} + \int_{\mathbb{R}^N} V_{\min}[\tilde{v}_n(x,0)-\tilde{v}(x,0)]^2 \mathrm{d}x\right], \end{aligned}$$

which implies that $\tilde{v}_n \to \tilde{v}$ in $E_{V_{\min}}$. Since $\{t_n\}$ is bounded, we can assume that $t_n \to t_0 > 0$, and so, $v_n \to v$ in $E_{V_{\min}}$.

Next, we will prove that $\{z_n\} = \{\epsilon_n \tilde{z}_n\}$ has a subsequence satisfying $z_n \to z \in \Lambda$. We first show that $\{z_n\}$ is bounded. Indeed, assume by contradiction that $\{z_n\}$ is not bounded. Then, there exists a subsequence, still denoted by $\{z_n\}$, such that $|z_n| \to \infty$. From $\tilde{v}_n \to \tilde{v}$ in $E_{V_{\min}}$, $V_{\min} < V_\infty$ and $K_{\max} > K_\infty$, we can infer that

$$\begin{aligned} c_{V_{\min}K_{\max}} =& \frac{1}{2}\left[\|\tilde{v}\|^2_{X^s} + \int_{\mathbb{R}^N} V_{\min}\tilde{v}^2(x,0)\mathrm{d}x\right] - K_{\max}\int_{\mathbb{R}^N} F(\tilde{v}(x,0))\mathrm{d}x \\ <& \frac{1}{2}\left[\|\tilde{v}\|^2_{X^s} + \int_{\mathbb{R}^N} V_\infty \tilde{v}^2(x,0)\mathrm{d}x\right] - K_\infty \int_{\mathbb{R}^N} F(\tilde{v}(x,0))\mathrm{d}x \\ \leq& \liminf_{n\to\infty}\left[\frac{1}{2}\left[\|\tilde{v}_n\|^2_{X^s} + \int_{\mathbb{R}^N} V(\epsilon_n x + z_n)\tilde{v}_n^2(x,0)\mathrm{d}x\right] - \int_{\mathbb{R}^N} K(\epsilon_n x + z_n)F(\tilde{v}_n(x,0))\mathrm{d}x\right] \\ \leq& \liminf_{n\to\infty}\left[\frac{1}{2}\left[\|t_n u_n\|^2_{X^s} + \int_{\mathbb{R}^N} V(\epsilon_n x)t_n^2 u_n^2(x,0)\mathrm{d}x\right] - \int_{\mathbb{R}^N} K(\epsilon_n x)F(t_n u_n(x,0))\mathrm{d}x\right] \\ =& \liminf_{n\to\infty}\Phi_{\epsilon_n}(t_n u_n) \\ \leq& \liminf_{n\to\infty}\Phi_{\epsilon_n}(u_n) \\ =& c_{V_{\min}K_{\max}}, \end{aligned}$$

which is a contradiction. Thus, $\{z_n\}$ is bounded and, up to a subsequence, we may assume that $z_n \to z$. If $z \notin \Lambda$, then $V_{\min} < V(z)$ and $K_{\max} > K(z)$, and according to the above steps we get a contradiction. Therefore, we conclude that $z \in \Lambda$. □

Let $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive function defined by

$$\vartheta(\epsilon) = \max_{z\in\Lambda} |\Phi_\epsilon(\gamma_\epsilon(z)) - c_{V_{\min}K_{\max}}|.$$

It follows from Lemma 6.1 that $\vartheta(\epsilon) \to 0$ as $\epsilon \to 0$. We introduce a subset $\tilde{\mathcal{N}_\epsilon}$ of $\mathcal{N}_\epsilon$. Setting

$$\tilde{\mathcal{N}_\epsilon} := \{u \in \mathcal{N}_\epsilon : \Phi_\epsilon(u) \leq c_{V_{\min}K_{\max}} + \vartheta(\epsilon)\}.$$

Since $\gamma_\epsilon(z) \in \tilde{\mathcal{N}_\epsilon}$ for all $z \in \Lambda$, then we can deduce that $\tilde{\mathcal{N}_\epsilon} \neq \emptyset$. Moreover, we have the following result.

LEMMA 6.4. *For any $\delta > 0$, then the following holds*

$$\lim_{\epsilon\to 0} \sup_{u\in\tilde{\mathcal{N}_\epsilon}} \inf_{z\in\Lambda_\delta} |\beta_\epsilon(u) - z| = 0.$$

Proof. Let $\epsilon_n \to 0$ as $n\to\infty$. For each $n\in\mathbb{N}$, there exists $\{u_n\}\subset\tilde{\mathscr{N}}_{\epsilon_n}$, such that

$$\inf_{z\in\Lambda_\delta}|\beta_{\epsilon_n}(u_n)-z|=\sup_{u\in\tilde{\mathscr{N}}_{\epsilon_n}}\inf_{z\in\Lambda_\delta}|\beta_{\epsilon_n}(u)-z|+o_n(1).$$

Hence, it is sufficient to prove that there exists $\{z_n\}\subset\Lambda_\delta$ such that

$$\lim_{n\to\infty}|\beta_{\epsilon_n}(u_n)-z_n|=0.$$

Indeed, since $\{u_n\}\subset\tilde{\mathscr{N}}_{\epsilon_n}$, then we have

$$c_{V_{\min}K_{\max}}\le c_{\epsilon_n}\le\Phi_{\epsilon_n}(u_n)\le c_{V_{\min}K_{\max}}+\vartheta(\epsilon_n),$$

which implies that

$$\Phi_{\epsilon_n}(u_n)\to c_{V_{\min}K_{\max}} \text{ and } \{u_n\}\subset\mathscr{N}_{\epsilon_n}.$$

According to Lemma 6.3, there exists $\{\tilde{z}_n\}\subset\mathbb{R}^N$ such that $v_n(x)=u_n(x+\tilde{z}_n)$ has a convergent subsequence. Moreover, up to a subsequence, $z_n=\epsilon_n\tilde{z}_n\to z\in\Lambda$. Therefore, we get

$$\begin{aligned}\beta_{\epsilon_n}(u_n)&=\frac{\int_{\mathbb{R}^N}\eta(\epsilon_n x)u_n^2(x,0)\mathrm{d}x}{\int_{\mathbb{R}^N}u_n^2(x,0)\mathrm{d}x}\\&=\frac{\int_{\mathbb{R}^N}\eta(\epsilon_n\tilde{x}+z_n)u_n^2(\epsilon_n\tilde{x}+z_n,0)\mathrm{d}\tilde{x}}{\int_{\mathbb{R}^N}u_n^2(\epsilon_n\tilde{x}+z_n,0)\mathrm{d}\tilde{x}}\\&=z_n+\frac{\int_{\mathbb{R}^N}[\eta(\epsilon_n\tilde{x}+z_n)-z_n]v_n^2(\tilde{x},0)\mathrm{d}\tilde{x}}{\int_{\mathbb{R}^N}v_n^2(\tilde{x},0)\mathrm{d}\tilde{x}}\\&\to z\in\Lambda.\end{aligned}$$

Consequently, there exists $\{z_n\}\subset\Lambda_\delta$ such that

$$\lim_{n\to\infty}|\beta_{\epsilon_n}(u_n)-z_n|=0.$$

The proof is now complete. □

We shall use the Ljusternik-Schnirelmann category theory and the techniques developed by Benci-Cerami [8] to prove the multiplicity result of positive solutions. Observe that, since $\mathscr{N}_\epsilon$ is not a C^1-submanifold of E_ϵ, we cannot directly apply the Ljusternik-Schnirelmann category theory. Fortunately, from Lemma 2.10, we can know that the mapping m_ϵ is a homeomorphism between $\mathscr{N}_\epsilon$ and S_ϵ, and S_ϵ is a C^1-submanifold of E_ϵ. So we can apply the Ljusternik-Schnirelmann category theory to the functional $I_\epsilon(u)=\Phi_\epsilon(\widehat{m}_\epsilon(u))|_{S_\epsilon}=\Phi_\epsilon(m_\epsilon(u))$. Based on the above facts, we give the completed proof of Theorem 1.3.

Proof. **(Proof of Theorem 1.3.)** For any $\epsilon>0$, we define $\omega_\epsilon:\Lambda\to S_\epsilon$ as follows

$$\omega_\epsilon(z)=m_\epsilon^{-1}(t_\epsilon\Psi_{\epsilon,z})=m_\epsilon^{-1}(\gamma_\epsilon(z)) \text{ for all } z\in\Lambda.$$

Using Lemma 6.1 we get

$$\lim_{\epsilon\to0}I_\epsilon(\omega_\epsilon(z))=\lim_{\epsilon\to0}\Phi_\epsilon(\gamma_\epsilon(z))=c_{V_{\min}K_{\max}} \text{ uniformly in } z\in\Lambda.$$

Moreover, we set

$$\tilde{S}_\epsilon = \{u \in S_\epsilon : I_\epsilon(u) \le c_{V_{\min}K_{\max}} + \vartheta(\epsilon)\},$$

with $\vartheta(\epsilon) = \sup_{z\in\Lambda} |I_\epsilon(\omega_\epsilon(z)) - c_{V_{\min}K_{\max}}| \to 0$ as $\epsilon \to 0$. Hence, $\omega_\epsilon(z) \in \tilde{S}_\epsilon$ for all $z \in \Lambda$, and this shows that $\tilde{S}_\epsilon \ne \emptyset$ for all $\epsilon > 0$.

Combining Lemma 2.10, Lemma 2.11, Lemma 6.1 and Lemma 6.4, we can see that there exists $\epsilon_\delta > 0$ such that the diagram

$$\Lambda \xrightarrow{\gamma_\epsilon} \tilde{\mathcal{N}_\epsilon} \xrightarrow{m_\epsilon^{-1}} \tilde{S}_\epsilon \xrightarrow{m_\epsilon} \tilde{\mathcal{N}_\epsilon} \xrightarrow{\beta_\epsilon} \Lambda_\delta$$

is well defined for any $\epsilon \in (0, \epsilon_\delta)$. By Lemma 6.2, there exists a function $l(\epsilon, z)$ with $|l(\epsilon, z)| < \frac{\delta}{2}$ uniformly in $z \in \Lambda$ for all $\epsilon \in (0, \epsilon_\delta)$, such that $\beta_\epsilon(\gamma_\epsilon(z)) = z + l(\epsilon, z)$ for all $z \in \Lambda$. We define the function $H(t,z) = z + (1-t)l(\epsilon, z)$. Then, $H : [0,1] \times \Lambda \to \Lambda_\delta$ is continuous. Evidently, $H(0,z) = \beta_\epsilon(\gamma_\epsilon(z))$ and $H(1,z) = z$ for all $z \in \Lambda$, and $\beta_\epsilon \circ \gamma_\epsilon = (\beta_\epsilon \circ m_\epsilon) \circ \omega_\epsilon$ is homotopic to the inclusion mapping $id : \Lambda \to \Lambda_\delta$. So, making use of Lemma 2.2 of [12] (see also [8]), we have

$$\operatorname{cat}_{\tilde{S}_\epsilon}(\tilde{S}_\epsilon) \ge \operatorname{cat}_{\Lambda_\delta}(\Lambda).$$

On the other hand, we choose a function $\vartheta(\epsilon) > 0$ such that $\vartheta(\epsilon) \to 0$ as $\epsilon \to 0$ and such that $c_{V_{\min}K_{\max}} + \vartheta(\epsilon)$ is not a critical level for Φ_ϵ. For $\epsilon > 0$ small enough, Lemma 4.3 shows that Φ_ϵ satisfies the Palais-Smale condition in $\tilde{\mathcal{N}_\epsilon}$. Then, using Lemma 2.11, we know that I_ϵ satisfies the Palais-Smale condition in $\tilde{S}_\epsilon$. Therefore, applying the Ljusternik-Schnirelmann category theory (see [12, Theorem 2.1]), we obtain that I_ϵ has at least $\operatorname{cat}_{\tilde{S}_\epsilon}(\tilde{S}_\epsilon)$ critical points on $\tilde{S}_\epsilon$. Then, using Lemma 2.11 again, we can deduce that Φ_ϵ has at least $\operatorname{cat}_{\Lambda_\delta}(\Lambda)$ critical points. This finishes the proof of Theorem 1.3. □

7. Nonexistence of positive ground state solutions

In this section we are going to prove the nonexistence result of positive ground state solutions. We first consider the following auxiliary problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) + m^2 y^{1-2s} w = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w}{\partial \nu^{1-2s}} = -V^\infty w(x,0) + K^\infty f(w(x,0)), & \text{on } \mathbb{R}^N, \end{cases} \tag{7.1}$$

where V^∞ and K^∞ are given in condition (A_3). Clearly, from the discussion in Section 3, we can know that problem (7.1) has a positive ground state solution. In what follows we are ready to give the proof of Theorem 1.4.

Proof. **(Proof of Theorem 1.4.)** First we need to claim that $c_\epsilon = c_{V^\infty K^\infty}$ for each $\epsilon > 0$. On the one hand, from (A_3), we see that $V^\infty \le V(x)$ and $K(x) \le K^\infty$ for all $x \in \mathbb{R}^N$, then $c_\epsilon \ge c_{V^\infty K^\infty}$ by Lemma 3.4.

On the other hand, we show that $c_\epsilon \le c_{V^\infty K^\infty}$ for any fixed $\epsilon > 0$. Let u^∞ be a positive ground state solution of problem (7.1), by Lemma 3.1-(b), we know that u^∞ is the unique global maximum of $\mathcal{J}_{V^\infty K^\infty}(tu^\infty)$. Set $u_n = u^\infty(\cdot - z_n)$, where $\{z_n\} \subset \mathbb{R}^N$ is a sequence satisfying $|z_n| \to \infty$ as $n \to \infty$. According to Lemma 2.7, there exists $t_n > 0$ such that $\widehat{m}_\epsilon(u_n) = t_n u_n \in \mathcal{N}_\epsilon$ is the unique global maximum of $\Phi_\epsilon(tu_n)$ for each n. Moreover, the sequence $\{t_n\}$ is bounded.

Computing directly, we have

$$\begin{aligned} c_\epsilon \le& \Phi_\epsilon(t_n u_n) \\ =& \mathcal{J}_{V^\infty K^\infty}(t_n u_n) + \frac{t_n^2}{2}\int_{\mathbb{R}^N} [V(\epsilon x) - V^\infty] u_n^2(x,0)\mathrm{d}x + \int_{\mathbb{R}^N} [K^\infty - K(\epsilon x)] F(t_n u_n(x,0))\mathrm{d}x \end{aligned}$$

$$\begin{aligned}
&= \mathcal{J}_{V^\infty K^\infty}(t_n u^\infty) + \frac{t_n^2}{2}\int_{\mathbb{R}^N}[V(\epsilon x+\epsilon z_n)-V^\infty]|u^\infty(x,0)|^2 \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}[K^\infty - K(\epsilon x+\epsilon z_n)]F(t_n u^\infty(x,0))\mathrm{d}x \\
&\le c_{V^\infty K^\infty} + \frac{t_n^2}{2}\int_{\mathbb{R}^N}[V(\epsilon x+\epsilon z_n)-V^\infty]|u^\infty(x,0)|^2 \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}[K^\infty - K(\epsilon x+\epsilon z_n)]F(t_n u^\infty(x,0))\mathrm{d}x. \qquad (7.2)
\end{aligned}$$

Using the decay of u^∞, it follows that for any $\varepsilon>0$, there exists $R>0$ such that

$$\int_{|x|\ge R}[V(\epsilon x+\epsilon z_n)-V^\infty]|u^\infty(x,0)|^2 \mathrm{d}x \le c\varepsilon.$$

Moreover, using (A_3) and Lebesgue's dominated convergence theorem we have

$$\lim_{n\to\infty}\int_{|x|\le R}[V(\epsilon x+\epsilon z_n)-V^\infty]|u^\infty(x,0)|^2 \mathrm{d}x = 0.$$

Thus, we have proved that

$$\int_{\mathbb{R}^N}[V(\epsilon x+\epsilon z_n)-V^\infty]|u^\infty(x,0)|^2 \mathrm{d}x = o_n(1). \qquad (7.3)$$

Similarly, using the above arguments and (2.7) we can easily check that

$$\int_{\mathbb{R}^N}[K^\infty - K(\epsilon x+\epsilon z_n)]F(t_n u^\infty(x,0))\mathrm{d}x = o_n(1). \qquad (7.4)$$

So, we deduce from (7.2), (7.3) and (7.4) that $c_\epsilon = c_{V^\infty K^\infty}$ for each $\epsilon>0$.

We complete the proof by using a contradiction argument. Suppose that for some $\epsilon_0>0$ there exists a positive function u_0 such that $u_0\in\mathcal{N}_{\epsilon_0}$ and $c_{\epsilon_0}=\Phi_{\epsilon_0}(u_0)$. Lemma 3.1-(b) shows that u_0 is the unique global maximum of $\Phi_{\epsilon_0}(tu_0)$. Using Lemma 3.1-(b) again, there exists $t^\infty>0$ such that $t^\infty u_0\in\mathcal{N}_{V^\infty K^\infty}$, hence

$$c_{V^\infty K^\infty} \le \mathcal{J}_{V^\infty K^\infty}(t^\infty u_0) = \max_{t\ge 0}\mathcal{J}_{V^\infty K^\infty}(tu_0). \qquad (7.5)$$

On the other hand, using (A_3) we have $\mathcal{J}_{V^\infty K^\infty}(u)\le\Phi_{\epsilon_0}(u)$ for any u. By (7.5) we have

$$c_{V^\infty K^\infty} \le \mathcal{J}_{V^\infty K^\infty}(t^\infty u_0) \le \Phi_{\epsilon_0}(t^\infty u_0) \le \Phi_{\epsilon_0}(u_0) = c_{\epsilon_0} = c_{V^\infty K^\infty}.$$

This shows that

$$c_{V^\infty K^\infty} = \mathcal{J}_{V^\infty K^\infty}(t^\infty u_0) = \Phi_{\epsilon_0}(t^\infty u_0). \qquad (7.6)$$

Observe that

$$\begin{aligned}
\mathcal{J}_{V^\infty K^\infty}(t^\infty u_0) = &\Phi_{\epsilon_0}(t^\infty u_0) + \frac{1}{2}\int_{\mathbb{R}^N}[V^\infty - V(\epsilon_0 x)]|t^\infty u_0(x,0)|^2 \mathrm{d}x \\
&+ \int_{\mathbb{R}^N}[K(\epsilon_0 x) - K^\infty]F(t^\infty u_0(x,0))\mathrm{d}x. \qquad (7.7)
\end{aligned}$$

We deduce from (A_3) that

$$\int_{\mathbb{R}^N} [V^\infty - V(\epsilon_0 x)]|t^\infty u_0(x,0)|^2 \mathrm{d}x < 0, \tag{7.8}$$

and

$$\int_{\mathbb{R}^N} [K(\epsilon_0 x) - K^\infty] F(t^\infty u_0(x,0)) \mathrm{d}x < 0. \tag{7.9}$$

Finally, from (7.7), (7.8) and (7.9), we obtain $\mathcal{J}_{V^\infty K^\infty}(t^\infty u_0) < \Phi_{\epsilon_0}(t^\infty u_0)$, which contradicts relation (7.6). This completes our proof. □

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