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HIGH ENERGY BLOWUP AND BLOWUP TIME FOR A CLASS OF SEMILINEAR PARABOLIC EQUATIONS WITH SINGULAR POTENTIAL ON MANIFOLDS WITH CONICAL SINGULARITIES*

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Abstract. In this paper, we consider a class of semilinear parabolic equations with singular potential on manifolds with conical singularities. At high initial energy level $J(u_0)>d$, we present a new sufficient condition to describe the global existence and nonexistence of solutions for problem (1.1)-(1.3) respectively. Moreover, by applying the Levine's concavity method, we give some affirmative answers to finite time blow up of solutions at arbitrary positive initial energy $J(u_0)>0$, including the upper bound of blowup time. Finally, we show a lower bound of the blowup time and blowup rate for problem (1.1)-(1.3) under arbitrary initial energy level.

Keywords. Finite time blow up; blowup time; parabolic equation; conical singularities; singular potential.

AMS subject classifications. 35K20; 35K55; 35A01; 35D30.

1. Introduction

In this paper, we concern with the following initial boundary value problem for semilinear conical degenerate parabolic equations with singular potential

$$u_t-\Delta_{\mathbb{B}}u+\kappa V(x)u=g(x)|u|^{p-1}u,\ x\in \text{int}\mathbb{B},t>0, \tag{1.1}$$

$$u(x,0)=u_0,\quad x\in \text{int}\mathbb{B}, \tag{1.2}$$

$$u(x,t)=0,\quad x\in\partial\mathbb{B},t\geq 0, \tag{1.3}$$

where $u_0\in\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})$, $1<p<\frac{N+2}{N-2}$ and $N>2$ is the dimension of $\mathbb{B}$, $N\in\mathbb{N}$, $\kappa\in\mathbb{R}$. Here the domain $\mathbb{B}=[0,1)\times X$ is regarded as the local model near the conical point on conical singular manifolds, where $X\subset\mathbb{R}^{N-1}$ is a closed compact C^∞ manifold. Denote by $\text{int}\mathbb{B}$ the interior of $\mathbb{B}$ and the boundary of $\mathbb{B}$ by $\partial\mathbb{B}:=\{0\}\times X$. Near $\partial\mathbb{B}$ we often use the coordinates $x=(x_1,\widetilde{x})=(x_1,x_2,x_3,...,x_N)\in\mathbb{B}$ for $0\leq x_1<1$ and $\widetilde{x}\in X$. The Fuchsian type Laplacian is defined as

$$\Delta_{\mathbb{B}}=\nabla^2_{\mathbb{B}}=(x_1\partial_{x_1})^2+\partial^2_{x_2}+...+\partial^2_{x_N},$$

which is a special case of totally characteristic degeneracy operators on a stretched conical manifold and $\nabla_{\mathbb{B}}=(x_1\partial_{x_1},\partial_{x_2},...,\partial_{x_N})$ denotes the corresponding gradient operator with conical degeneracy on the boundary $\partial\mathbb{B}$.

In Equation (1.1), we assume that $g=g(x_1,\widetilde{x})\in L^\infty(\text{int}\mathbb{B})\cap C(\text{int}\mathbb{B})$ is a positive weighted function and $V=V(x_1,\widetilde{x})\in L^\infty(\text{int}\mathbb{B})\cap C(\text{int}\mathbb{B})$ is non-negative potential function which satisfies the cone Hardy's inequality (see Lemma 2.1 for detail). Specifically,

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the potential $V(x_1,\widetilde{x})$ contains two kinds of singular potential functions:

$$V_1=\left(\frac{N-3}{2}\right)^2\frac{1}{|x_1^2+\widetilde{x}^2|}$$

and

$$V_2=\left(\frac{N-1}{2}\right)^2\frac{x_1^{-2}e^{-\frac{1}{x_1^2}}}{e^{-\frac{1}{x_1^2}}+\widetilde{x}^2}.$$

Obviously, original point is the singular point of V_1, meanwhile, V_2 is unbounded on $\partial\mathbb{B}$. Notice that V_1 is the classical Hardy potential for Δ and V_2 is a new Hardy potential induced by $\Delta_{\mathbb{B}}$ [1]. Before further exploring the conical singular manifold, we briefly introduce some of the dynamics of the parabolic equations associated with this paper, including the related progress of the heat equation problem with potential function, and the topics to be dealt with in our paper.

Due to the fact that the study of the classical semilinear heat equation on the bounded domain $\Omega\subset\mathbb{R}^n$

$$u_t-\Delta u=|u|^{p-1}u \tag{1.4}$$

has been tackled with so many different unlinked tools, it is impossible to describe all of these conclusions systematically. Depending on suitable properties of the initial datum in $H_0^1(\Omega)$, we shall focus on the special interests of the dynamical behavior of the solution for different initial energy levels. As reported in [2], these conclusions revealed the relations between the solution and the initial datum by using the variational method, so-called potential well method. Hence the following discussions aim to describe all of the manifolds of the initial data which leads to the global solution or finite time blow up solution as shared and summarized in [3–12].

Before this, we concentrate first on the celebrated potential term. For bounded potentials or potentials with moderate singularities, the existence, uniqueness, and behavior of solutions to the linear parabolic equation $u_t-\Delta u-V(|x|)u=0$ is similar with the corresponding properties of the heat kernel as the potential V is small enough. Interestingly, such situation changes dramatically for very singular potentials, a particularly typical one of singular potentials is the so-called inverse-square potential $V(\xi)\sim\lambda|\xi|^{-2}$ due to many physical models, such as the reaction-diffusion of chemicals [13] and some combustion models [14,15]. In particular, as a consequence of the important character of a lack of regularity in the origin, the inverse-square potential has a number of remarkable mathematical properties which make it useful in the linearized analysis for nonlinear diffusion equations in solid-fuel ignition phenomena [16]. Another prominent application of singular potentials can be found in nonrelativistic quantum mechanics when we consider the Schrödinger equation by referring to [17–20] for a detailed discussion of the backgrounds.

From the purely mathematical theoretical point of view, the singular potential has mathematical interest in itself as it represents a threshold. To survey the singular potential Baras and Goldstein [21] considered the linear heat equation with singular potential

$$u_t-\Delta u=V(x)u+f(x,t)$$

and discovered that existence and non-existence of positive solutions is crucially determined by the value of the parameter c of $V(x)=c|x|^{-2}$. Even an attempt at a brief review of this linear case is beyond the scope of this introduction, here we refer to [13, 22–25] for more details. For the nonlinear case, P. Souplet [26] obtained the stability properties of the parabolic equation with decaying potentials in $\mathbb{R}^n$ as follows

$$u_t - \Delta u + V(x)u = u^p. \tag{1.5}$$

Assuming the initial data $u_0 \in L_k^\infty$ satisfies $\|u_0\|_{\infty,k} \leq \delta$, it was shown that the corresponding solution u of Cauchy problem of (1.5) is global in time under suitable conditions on $V(x)$. When $u_0 \in L_k^\infty \cap H^1(\mathbb{R})$ and $u_0 \geq 0$, they further achieved a uniform priori estimate for global solutions and $E(t) \geq 0$. Besides that, if $V(x)$ and u_0 are radial, there is an additional result that some global solutions have ω-limit sets containing a positive equilibrium. More literature on potential functions can be found in [27, 28] and the references therein.

Different from the classical domain $\Omega \subset \mathbb{R}^n$, in the case of a singular manifold domain, say conical points on the boundary, it is no longer possible to define the derivative in the classical sense. Therefore, its scope and techniques fall out of the traditional analysis when near these singularities. Fortunately, there is a special pseudo-differential operator that can reflect the singular structure of manifold, i.e., Fuchsian type Laplace operator; such mathematical advances were sharpened in the work of Kondrat'ev [29] as well as in related works of Schulze and collaborators [30–33]. In short, with the help of the pseudo-differential operator, the specific differential and integral are properly defined, and the corresponding functional analysis framework is constructed. Through these analysis tools, the dynamic behavior of several different types of solutions for differential equations in the distribution sense is studied, which provides an affirmative method for dealing with problems of partial differential equations that arise in singular or degenerate geometric situations. With this observation a lot of mathematicians began a variety of studies of differential equations associated with the manifold with singularities, see [34–39]. Especially, comparing with the classical case of problem (1.4), Chen [40] first investigated the initial boundary value problem of semilinear conical degenerate parabolic equation

$$u_t - \Delta_{\mathbb{B}} u = |u|^{p-1}u, \tag{1.6}$$

and obtained not only the existence of global solutions with exponential decay, but also the blow up in finite time with low initial energy and critical initial energy $J(u_0) \leq d$. Until recently, M. Alimohammady [41] also focused on the influence of the initial data to solution of differential equation especially the critical value of the singular potential and discussed the existence theorem for a class of semilinear totally characteristic hypoelliptic equations on manifold with singularities

$$\partial_t^k u - \Delta_{\mathbb{B}} u - V(x)u = g(x)|u|^{p-1}u, \tag{1.7}$$

where $k \geq 1$. As a key ingredient, in cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, by utilizing the cone Sobolev embedding inequality, cone Poincaré inequality and potential well method, M. Alimohammady proved the global existence and finite time blow up of solution to problem (1.7) under sub-critical initial energy level, i.e, $J(u_0) < d$. As another main result, he also derived an exponential decay of solution with $J(u_0) < d$ provided the solution exists globally.

An interesting as well as important point is the dynamic behavior of the solution to problem (1.1)-(1.3) at high initial energy level, i.e., $J(u_0) > d$. In fact, as the potential energy functional $J(u)$ lacks the sharp constraints of mountain pass level d, demonstrating the invariant set under arbitrary high initial energy level will be a challenging task. Corresponding to the case of usual Euclidean domain studied in [42,43], a new different obstacle for studying the well-posedness of problem (1.1)-(1.3) at high initial energy levels would be the analysis of the dynamic behavior of solutions in a degenerate measure space, including the use of various forms of inequalities governed by cone differential operator, which would bring additional technical complications caused by the singularity of the domain. In addition, the existence of the undefinite sign potential function is one of the important features of this paper as it covers the applicability of a wider range of problems, but it increases the difficulty of estimating the conditions of both the existence of global solutions and blowup solutions. Noticing the proof in [41], to decide whether or not the resulting solution is global or blows up in finite time, they indicated the sharp criteria stem from initial datum u_0 and exhibited the sufficient condition by freezing the upper bound of initial energy, i.e., $J(u_0)<d$. However, the unexpected restrictions of initial energy occupies our present attention, after releasing the subcritical initial energy to the arbitrary positive case, (from $0<J(u_0)<d$ to $J(u_0)>0$), a corresponding attraction motivates us to explore whether there are still sharp-like conditions similar to the above described in [41]? Or, at least to provide sufficient conditions for the initial datum u_0 to determine whether the solution of the problem (1.1)-(1.3) exists globally or blows up in finite time, respectively. To handle these, the present paper will give the sufficient condition of initial datum u_0 under high initial energy which leads to global existence and nonexistence of solutions, including a sufficient condition for the finite time blow up of solutions at arbitrary positive initial energy. At the same time, we will also complete some estimates about the blow up time and the blow up rate.

For the high initial energy levels case, according to the discussion above, the three basic successive issues we will focus on include *what*, *when* and *how*. Specifically, what kind of initial data will lead to the solution existing globally or blowing up in finite time? Granted that finite time blow up occurs, when does the blow up occur? Can we estimate the blow up time? Last but not least, how does finite time blow up occur? In summary, our paper will give some affirmative answers as the main results of this paper.

(a) **Global existence and finite time blow up for high initial energy. (Theorem 3.1 in the present paper).** When the initial energy is larger than the critical initial energy level, i.e., $J(u_0)>d$, by characterizing the high-energy steady-state set $\mathcal{N}_\alpha$, Theorem 3.1 gives a sharp-like condition, which uses the $L_2^{\frac{N}{2}}$-norm as the radius to divide the phase space $\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$ into three parts, as shown in Figure 1.1. It not only shows the sufficient condition of initial datum u_0 leading to global existence and blow up of solutions respectively, but also clarifies the distribution of these initial data in terms of $L_2^{\frac{N}{2}}$-norm in $\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$.

(b) **Finite time blow up and upper bound of blowup time for positive initial energy I (Theorem 3.2 in the present paper) and positive initial energy II (Theorem 3.3 in the present paper).** The most interesting part of the present paper seems to be these two theorems, i.e., Theorem 3.2 and Theorem 3.3, roughly speaking, which can be regarded as some kind of sacrifice of the wide range of conditions for initial data ensuring the finite time blow up of the solution in order to obtain the estimates of the upper bound of the blowup time corresponding to two different sets of the initial data. Different from the sharp-like condition of the global existence in time and finite time blow up of the solution given in Theorem 3.1 aiming

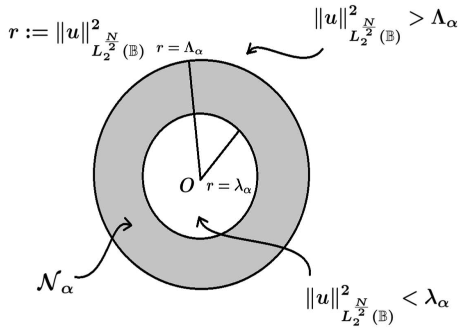

FIG. 1.1. *Schematic diagram of* $\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})$ *space segmentation*

to reveal the relations between the conditions of the initial data for global existence and finite time blow up of solution respectively, Theorem 3.2 and Theorem 3.3 aim to focus on the upper bound of the blowup time by adding some more restrictive conditions on the initial data. Indeed, these restrictions can be given in many possible ways and forms, but our two theorems describe two important aspects of them, by the coefficients of the norm of the initial datum $\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}$ in the conditions of Theorem 3.2 and Theorem 3.3 in the present paper. Generally speaking, such coefficient in Theorem 3.2 is about the parameters related to diffusion term and the potential term on the left side of the equation, while the coefficient in Theorem 3.3 is from the parameters close to the nonlinear source term on the right side of the equation. Based on these two different restrictions about the initial conditions, the finite time blow up of solution is proved, and the corresponding different estimates of upper bounds of blowup time are derived. More detailed analysis will be given in Remarks 5.1 and 6.1.

(c) **Lower bound of blowup time and blowup rate for blowup solution. (Theorem 3.4 in the present paper).** Based on the blowup solution by supposing that the finite time blow up happens to the solution, Theorem 3.4 estimates the lower bound of the blowup time and the lower bound of the blowup rate of the blowup solution. Unlike Theorem 3.2 and Theorem 3.3, the lower bound of the blowup time is independent of the initial energy condition, so the results given by Theorem 3.4 can apply to the finite time blow up solution described in Theorem 3.1-Theorem 3.3 and all the other possible and suitable finite time blow up solutions, as long as the condition of Theorem 3.4 can be satisfied.

Open problems. (Global existence for positive initial energy). It is well known that Laplacian operators in Euclidean space can be used as infinitesimal generators of semigroups. Therefore, using semigroups method to represent the solution, then combined with the potential well method and comparison principles, we can analyze the dynamic behavior of solutions for nonlinear parabolic equations more clearly. For example, to discuss the existence for global solutions and blowup solutions in $\mathcal{N}_+$ under positive initial energy level, see [2,8,44] for details. However, it is still unknown whether conclusions similar to the classical comparison principle can be obtained for the degenerate cone operator case. Thus, it prevents us from discussing further the other results tied to the global well-posedness of the solution with high initial energy case. Another important research topic on problem (1.1)-(1.3) is what kind of sufficient conditions the initial data needs to satisfy under positive initial energy levels to induce the global existence of solutions. However, as an open problem, we would like to point that it is truly difficult to find such a sufficient condition. The reason for this is that the appropriate boundedness estimation is the necessary tool to demonstrate the global solutions, while we nowadays have not found the right ways to control the long-term global dynamic behavior of solutions by virtue of the boundedness of the initial energy. Another brief explanation is the lack of a proof of invariant set. Perhaps there is a deeper level of topological invariance and other profound mechanisms that we have not fully realized yet, which may guide one to explore further.

2. Preliminaries

2.1. Cone Sobolev spaces. Owing to the differential topology defect, the singular domain; which means that the boundary has singularity such as conical, edge, corner, etc., compared with the smooth one, there are more difficulties in analysis including sufficient generality and regularity. For instance, the doubling property[1] as a normal property for smooth domain does not apply to singular domain (see [45] Example 3.1), but the doubling property plays a major role for establishing many basic inequalities such as Poincaré inequality [46]. Moreover, it is rarely the real smooth case. The non-smooth domain is naturally encountered in various practical settings in applications, in which many inconveniences appear in microlocal non-smooth part [47]. Of course, after drawing the singular domain as singular manifold, the most difficult task is to identify the differential operator reasonably on the singular manifold, which should depend on the geometry of the corresponding manifold. Hence, instead of general singular one, we focus on the essential singular space equipped with rich additional structures, which permit one to define differential operators in a natural way. We call it a manifold with conical singularities.

We first give the definition of a manifold with conical singularities.

Definition 2.1. *Let B be a second countable Hausdorff space and $b_1,b_2,...,b_n\in B$ such that $B\backslash\{b_1,b_2,...,b_n\}$ is a smooth manifold (without boundary) of dimension N. To say that B is a manifold with conical singularities $b_1,b_2,...,b_n$ means to require the existence of a system Φ of homeomorphisms $\phi: B\to\mathbb{B}_\phi/\sim$, where*

- (i) *$\mathbb{B}_\phi$ is a smooth compact manifold with boundary $\partial\mathbb{B}_\phi=\cup_{j=1}^n X_\phi(j)$ with $N-1$ dimensional smooth compact manifolds $X_\phi(j)$;*
- (ii) *$\sim$ means shrinking each of the $X_\phi(j)$ to a point;*
- (iii) *$\phi: B\backslash\{b_1,b_2,...,b_n\}\to int\mathbb{B}_\phi$ is a diffeomorphism and $\phi(b_j)=X_\phi/\sim$;*
- (iv) *for each $\phi_1,\phi_2\in\Phi$ the restriction $\phi_1\circ\phi_2^{-1}: int\mathbb{B}_{\phi_2}\to int\mathbb{B}_{\phi_1}$ extends to a diffeomorphism (of manifolds with boundary) $\mathbb{B}_{\phi_2}\to\mathbb{B}_{\phi_1}$.*

As showed in [29, 30], to construct the associated geometric operators on conic manifolds needs resolving the singularity of the manifold, which is reflected in Definition 2.1. Details of this desingularization, for instance, by introducing polar coordinates, can be found in [30]. In order to make the reader understand this process more intuitively, we show in Figure 2.1 a graph formed by stretching and quotient map of a singular manifold B, and indicate the relationship between the singular manifold B, the stretched manifold

[1] For a metric space M with a metric d, a measure f over (M,d) is said to have the doubling property if $f(2B)/f(B)$ is bounded where B and $2B$ are balls with the same center and the radius of $2B$ is twice as that of B.

$\mathbb{B}$ and the projection of the stretched manifold $\mathbb{B}/\sim$.

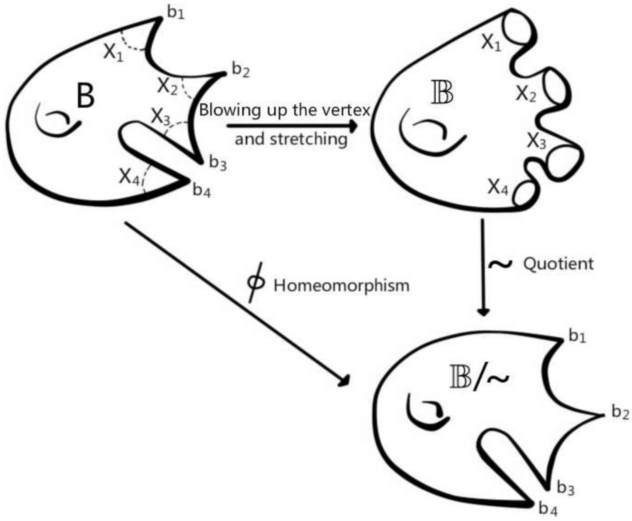

FIG. 2.1. *The relationship between B, $\mathbb{B}$ and $\mathbb{B}/\sim$*

Smooth manifold $\mathbb{B}$ is produced by stretching singular manifold B, which provides facility to construct the differential structures. Naturally, when the operators contained in the differential structure act near the boundary $\partial\mathbb{B}$, the behavior of operators should be restricted so that they can reflect the singular characteristics. In details, this restriction should have two effects: to accurately express the derivative calculation near the boundary and to definitely exhibit the singularity on the boundary as shown by the operator $\mathcal{A}$ which will be defined below. Of course, the "stretching" does not commit us to stray from the subject. The projection of the smooth manifold $\mathbb{B}/\sim$, specified by above differential structure in the quotient topology sense, is exactly homeomorphic to the original singular manifold B. Such viewpoint of B.W. Schulze and other collaborators [30–32] makes our research finally focus on the pair $(\mathbb{B},\mathcal{D}_B)$, rather than the troublesome B, where $\mathcal{D}_B \subset \mathrm{Diff}(\mathbb{B})$[2] consists of the so-called cone-degenerate operators, which provide a smooth continuation to the entire stretched manifold $\mathbb{B}$ but degenerates on the boundary $\partial\mathbb{B}$.

To understand the degenerate behavior of the cone-degenerate operators clearly, we herein mention the $-x_1\partial_{x_1}$-differential form appearing in operator $\Delta_{\mathbb{B}}$ and discuss some details. The typical $-x_1\partial_{x_1}$ differential operators, called Fuchsian type operators, defined in a neighborhood of $x_1=0$ of the following form

$$\mathcal{A}:=x_1^{-s}\sum_{k=0}^{s}a_k(x_1)\left(-x_1\frac{\partial}{\partial x_1}\right)^k,$$

where $(x_1,\widetilde{x})\in X^\wedge:=\mathbb{R}_+\times X$ is a cylinder manifold near the boundary $\partial\mathbb{B}$ (see Figure 2.2), and $a_k(x_1)\in C^\infty(\overline{\mathbb{R}}_+,\mathrm{Diff}^{s-k}(X))$, $s\in\mathbb{R}$ are smooth up to $x_1=0$ [30–33]. Such operators exhibit both degenerate and singular behaviors as follows: the derivative

[2] $\mathrm{Diff}^s(\Omega)$ is a Fréchet space, which denotes all differential operators on Ω of order s with smooth coefficients in local coordinates, where Ω is a C^∞ manifold. In particular, $\mathrm{Diff}^0(\Omega):=\mathrm{Diff}(\Omega)$.

$x_1\partial_{x_1}$ in the direction of cone axis vanishes at $x_1=0$ and the factor x_1^{-s} is singular at $x_1=0$.

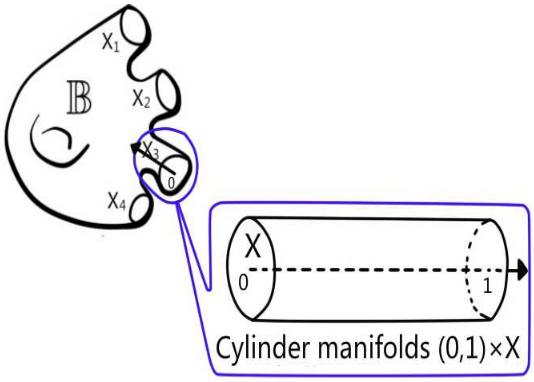

FIG. 2.2. *Stretched manifold* $\mathbb{B}$ *and cylinder manifold* $(0,1)\times X$

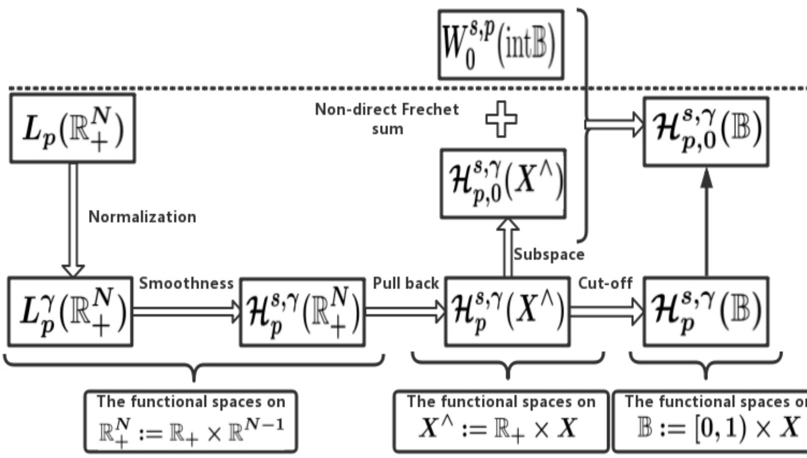

FIG. 2.3. *A mind map of the construction of functional spaces.*

As a widely accepted idea from the works by Kondrat'ev and Schulze [29–31] that the Mellin transform being a Fourier transform in logarithmic variables is particularly adapted to conical degeneration calculus which is well behaved under a $\mathbb{R}_+$ direction. Moreover, the Fuchsian type operator $-x_1\partial_{x_1}$ can be translated by Mellin transform into multiplication with complex covariable, which motivates a better pseudodifferential symbols calculus near the singularities. Cost-effectively, we will skip the calculation related to Mellin transform and give the weighted Sobolev spaces directly [30,31,39,41], details related to Mellin transform can also be found in [48].

The key to constructing the Sobolev space is to reasonably define it on the cylinder manifold, in which the cylinder manifold is in accordance with the stretched manifold near the boundary. However, the metrics are different in the $\mathbb{R}_+$ direction and non-$\mathbb{R}_+$ directions near the boundary, thus, it is necessary to consider their actions in the function spaces respectively. Firstly, introducing the usual Lebesgue space in the sense of polar coordinates may be the best choice as it just meets our needs. Subsequently, we realize the normalization of $L_p(\mathbb{R}_+^N)$ through the weight factor $x_1^{-\gamma}$ and introduce the derivative to get the $\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+^N)$ space. With the aid of the pull back function, we map the $\mathcal{H}_p^{s,\gamma}$ space from the half space $\mathbb{R}_+^N$ onto the cylinder manifold $X^\wedge$. Finally, we give the weighted Sobolev spaces $\mathcal{H}_{p,0}^{s,\gamma}(\mathbb{B})$ space combined by $\mathcal{H}_{p,0}^{s,\gamma}(X^\wedge)$ and $W_0^{s,p}(\mathrm{int}(\mathbb{B}))$ via non-direct Fréchet sum (Refer to [30] in page 6). We present a mind map in Figure 2.3, which will help guide the reader to understand the path we use to construct the weighted Sobolev spaces.

Firstly, we define the weighted Sobolev spaces on $\mathbb{R}_+^N := \mathbb{R}_+ \times \mathbb{R}^{N-1}$.

Definition 2.2 (The space $L_p(\mathbb{R}_+^N)$ and $L_p^\gamma(\mathbb{R}_+^N)$ [31] p. 139, [39]). *For $(x_1,\widetilde{x}) \in \mathbb{R}_+^N$, we say that $u(x_1,\widetilde{x}) \in L_p(\mathbb{R}_+^N, \frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x})$ with $1<p<\infty$, if*

$$\|u\|_{L_p(\mathbb{R}_+^N)} = \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^{N-1}} x_1^N |u(x_1,\widetilde{x})|^p \frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{1}{p}} < \infty.$$

The weighted L_p spaces with weight $\gamma \in \mathbb{R}$ is denoted by $L_p^\gamma(\mathbb{R}_+^N, \frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x})$, i.e., if $u \in L_p^\gamma(\mathbb{R}_+^N, \frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x})$, then $x_1^{-\gamma}u \in L_p(\mathbb{R}_+^N, \frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x})$, which consists of functions $u(x_1,\widetilde{x})$ with

$$\|u\|_{L_p^\gamma(\mathbb{R}_+^N)} = \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^{N-1}} x_1^N |x_1^{-\gamma}u(x_1,\widetilde{x})|^p \frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{1}{p}} < \infty.$$

Definition 2.3 (The space $\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+^N)$ [30] p. 3, [39]). *For $s \in \mathbb{N}, \gamma \in \mathbb{R}$ and $1<p<\infty$, assuming $u(x_1,\widetilde{x}) \in \mathcal{D}'(\mathbb{R}_+^N)$, where the dual $(C_0^\infty(\mathbb{R}_+^N))' =: \mathcal{D}'(\mathbb{R}_+^N)$ is the space of all distributions in $\mathbb{R}_+^N$, we denote the spaces*

$$\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+^N) := \left\{ u \in \mathcal{D}'(\mathbb{R}_+^N) \,\middle|\, (x_1\partial_{x_1})^k \partial_{\widetilde{x}}^\beta u \in L_p^\gamma(\mathbb{R}_+^N, \frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}) \right\}$$

for any $k \in \mathbb{N}$, multi-index $\beta \in \mathbb{N}^{N-1}$ with $k+|\beta| \le s$. Therefore, $\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+^N)$ is a Banach space with norm

$$\|u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+^N)} = \sum_{k+|\beta|\le s} \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^{N-1}} x_1^N |x_1^{-\gamma}(x_1\partial_{x_1})^k \partial_{\widetilde{x}}^\beta u(x_1,\widetilde{x})|^p \frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{1}{p}}.$$

Turning to the weighted Sobolev spaces on cylinder manifold $\mathbb{R}_+ \times X$, we now briefly mention the necessary constructs from [30,31] as follows.

Note that X is a closed compact C^∞ manifold and let $\mathcal{O} = \{O_1,...,O_M\}$ be an open covering of X by coordinate neighborhoods. If we fix a subordinate partition of unity[3] $\{\psi_1,...,\psi_M\}$ and charts $\chi_j : O_j \to \mathbb{R}^{N-1}$, $j=1,...,M$, $M \in \mathbb{N}$, then $\mathcal{H}_p^{s,\gamma}(X^\wedge)$ denotes the closure of $C_0^\infty(X^\wedge)$ with the norm

$$\|u\|_{\mathcal{H}_p^{s,\gamma}(X^\wedge)} = \left(\sum_{j=1}^M \|(1\times\chi_j^*)^{-1}\psi_j u\|^p_{\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+^N)}\right)^{\frac{1}{p}}.$$

[3]Partition of unity: The set of continuous functions from a topological space to the unit interval $[0,1]$ such that for every point x, there is a neighborhood of x where a cofinite number of the functions are 0, and such that the sum of all the function values at x is 1.

Here $1\times\chi_j^*:C_0^\infty(\mathbb{R}_+\times\mathbb{R}^{N-1})\to C_0^\infty(\mathbb{R}_+\times O_j)$ is the pull back function[4] with respect to $1\times\chi_j:\mathbb{R}_+\times O_j\to\mathbb{R}_+\times\mathbb{R}^{N-1}$. Roughly speaking, the $\mathcal{H}_p^{s,\gamma}(X^\wedge)$ consists, by definition, of all functions belonging to $\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+^N)$ in arbitrary local coordinates on X. Denote $\mathcal{H}_{p,0}^{s,\gamma}(X^\wedge)$ as the subspace of $\mathcal{H}_p^{s,\gamma}(X^\wedge)$ which is defined as the closure of $C_0^\infty(X^\wedge)$ with respect to the norm $\|\cdot\|_{\mathcal{H}_p^{s,\gamma}(X^\wedge)}$.

Finally, after all the above preparations, we will give the weighted Sobolev space on the stretched manifold $\mathbb{B}=[0,1)\times X$ with the help of cut-off function. The weighted Sobolev spaces $\mathcal{H}_{p,0}^{s,\gamma}(\mathbb{B})$ on stretched manifold $\mathbb{B}$ consists of two parts. The first part is denoted by $\mathcal{H}_{p,0}^{s,\gamma}(X^\wedge)$ on a collar neighborhood of $\partial\mathbb{B}$; and the another one is denoted by $W_0^{s,p}(\mathrm{int}\mathbb{B})$ on $\mathrm{int}\mathbb{B}$, which are coherent with each other via the non-direct Fréchet sum [30]. Specifically, assume $\omega(x_1)\in C_0^\infty(\overline{\mathbb{R}}_+)$ is a non-negative real-valued cut-off function which equals 1 near $\partial\mathbb{B}$ and equals 0 outside the collar neighborhood of the boundary, i.e.,

$$\omega(x_1)=\begin{cases}1 & \text{near } x_1=0,\\ 0 & \text{for} \quad x_1\geq \text{const}.\end{cases}$$

Then, we have the following definition.

Definition 2.4 (The space $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ and $\mathcal{H}_{p,0}^{s,\gamma}(\mathbb{B})$ [30, 39]). *The space $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ for $s\in\mathbb{N},\gamma\in\mathbb{R}$ denotes the subspace of all $u\in W_{\mathrm{loc}}^{s,p}(\mathrm{int}\mathbb{B})$ such that*

$$\mathcal{H}_p^{s,\gamma}(\mathbb{B})=\left\{u\in W_{\mathrm{loc}}^{s,p}(\mathrm{int}\mathbb{B})\,|\,\omega u\in\mathcal{H}_p^{s,\gamma}(X^\wedge)\right\}$$

for any cut-off function ω, supported by a collar neighborhood of $[0,1)$. Here $W_{\mathrm{loc}}^{s,p}(\mathrm{int}\mathbb{B})$ denotes the subspace of all $u\in\mathcal{D}'(\mathrm{int}\mathbb{B})$ such that $\phi u\in W^{s,p}(\mathrm{int}\mathbb{B})$ for every $\phi\in C_0^\infty(\mathrm{int}\mathbb{B})$.

Moreover, the subspace $\mathcal{H}_{p,0}^{s,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is defined as follows

$$\mathcal{H}_{p,0}^{s,\gamma}(\mathbb{B})=[\omega]\mathcal{H}_{p,0}^{s,\gamma}(X^\wedge)+[1-\omega]W_0^{s,p}(\mathrm{int}\mathbb{B}),$$

where $W_0^{s,p}(\mathrm{int}\mathbb{B})$ denotes the closure of $C_0^\infty(\mathrm{int}\mathbb{B})$ in Sobolev spaces $W^{s,p}(\widetilde{X})$ and $\widetilde{X}$ is a closed compact C^∞ manifold of dimension N that contains $\mathbb{B}$ as a submanifold with boundary.[5]

Proposition 2.1 (Cone Sobolev inequality [39]). *Let $1\leq p<N$, $\frac{1}{p^*}=\frac{1}{p}-\frac{1}{N}$ and $\gamma\in\mathbb{R}$. The following estimate*

$$\|u\|_{L_{p^*}^{\gamma^*}(\mathbb{R}_+^N)}\leq c_1\|(x_1\partial_{x_1})u\|_{L_p^\gamma(\mathbb{R}_+^N)}+(c_1+c_2)\sum_{i=2}^N\|\partial_{x_i}u\|_{L_p^\gamma(\mathbb{R}_+^N)}+\frac{c_2}{c_3}\|u\|_{L_p^\gamma(\mathbb{R}_+^N)}$$

holds for all $u(x_1,\widetilde{x})\in C_0^\infty(\mathbb{R}_+^N)$, where

$$\gamma^*=\gamma-1,$$
$$c_1=\frac{(N-1)p}{N(N-p)},$$

[4] The pull back by χ of a function v on $\mathbb{R}^{N-1}$ is the function $\chi_j^*v=v\circ\chi$ on O, where $v:\mathbb{R}^{N-1}\to O$. In short, pull-back can be interpreted as a composition.

[5] Let A is an algebra, E a Fréchet space which is a left module over A, i.e., the elements $a\in A$ induce by mulitiplication $e\to ae$ linear oprators $a:E\to E$ with the usual algebraic rules, then we set $[a]E:=\{\text{the closure of } \{ae:e\in E\} \text{ in E}\}$ for every $a\in A$.

$$c_2 = \frac{(N-1)p\left|(N-1)-\frac{(\gamma-1)(N-1)p}{N-p}\right|^{\frac{1}{N}}}{N(N-p)},$$
$$c_3 = \frac{(N-1)p}{N-p}.$$

Moreover, if $u(x_1,\widetilde{x}) \in \mathcal{H}^{1,\gamma}_{p,0}(\mathbb{R}^N_+)$, we have

$$\|u\|_{L^{\gamma^*}_{p^*}(\mathbb{R}^N_+)} \leq c\|u\|_{\mathcal{H}^{1,\gamma}_p(\mathbb{R}^N_+)},$$

where the constant $c = c_1 + c_2$.

PROPOSITION 2.2 (Cone Poincaré inequality [39]). *Let $1<p<\infty$, $\gamma \in \mathbb{R}$. If $u(x_1,\widetilde{x}) \in \mathcal{H}^{1,\gamma}_{p,0}(\mathbb{B})$ then*

$$\|u(x_1,\widetilde{x})\|_{L^\gamma_p(\mathbb{B})} \leq C^{\frac{1}{2}}_{poin}\|\nabla_{\mathbb{B}} u(x_1,\widetilde{x})\|_{L^\gamma_p(\mathbb{B})},$$

where the constant C_{poin} depends only on $\mathbb{B}$ and p. In particular, it follows that $\|u\|_{\mathcal{H}^{1,\gamma}_{2,0}(\mathbb{B})}$ and $\|\nabla_{\mathbb{B}} u\|_{L^\gamma_2(\mathbb{B})}$ are the equivalent norms in the bounded domain by using the cone Poincaré inequality.

PROPOSITION 2.3 (Cone Hölder inequality [41]). *If $u \in L^{\frac{N}{p}}_p(\mathbb{B}), v \in L^{\frac{N}{p'}}_{p'}(\mathbb{B})$ with $p,p' \in [1,\infty]$ and $\frac{1}{p}+\frac{1}{p'}=1$, then we have the following cone type Hölder inequality*

$$(u,v)_{\mathbb{B}} \leq \|u\|_{L^{\frac{N}{p}}_p(\mathbb{B})}\|v\|_{L^{\frac{N}{p'}}_{p'}(\mathbb{B})},$$

where we denote $(u,v)_{\mathbb{B}} := \int_{\mathbb{B}} uv\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}$ and use it throughout this paper.

LEMMA 2.1 (Cone Hardy's inequality [1,26]). *For all $u \in \mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})\backslash\{0\}$ and $\kappa \in \mathbb{R}\backslash\{0\}$, the following inequality*

$$\left(\int_{\mathbb{B}} V(x_1,\widetilde{x})|u|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{-\kappa}{2|\kappa|}} \leq C_*^{\frac{-\kappa}{|\kappa|}}\left(\int_{\mathbb{B}}|\nabla_{\mathbb{B}} u|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{-\kappa}{2|\kappa|}}$$

holds, where the best constant $C_*>0$ is selected as follows

$$C_*^2 = \begin{cases} \inf\left\{\dfrac{\|V^{\frac{1}{2}}u\|_{L^{\frac{N}{2}}_2(\mathbb{B})}}{\|\nabla_{\mathbb{B}} u\|_{L^{\frac{N}{2}}_2(\mathbb{B})}} \,\middle|\, u \in \mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})\right\}, & \text{if } \kappa>0; \\ \sup\left\{\dfrac{\|V^{\frac{1}{2}}u\|_{L^{\frac{N}{2}}_2(\mathbb{B})}}{\|\nabla_{\mathbb{B}} u\|_{L^{\frac{N}{2}}_2(\mathbb{B})}} \,\middle|\, u \in \mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})\right\}, & \text{if } \kappa<0. \end{cases}$$

REMARK 2.1. The Hardy's inequality and its extensions and refinements are not only of intrinsic interest but are indispensable tools in many areas of mathematics and mathematical physics, such as to deal with the potential functions like those in our paper. According to different model backgrounds, the potential function in the equation considered by the previous work frequently appears as a positive one or a negative one [1, 26]. Naturally, the Hardy's inequality will also have different manifestations.

As a simplification, the Hardy's inequality given by Lemma 2.1 contains the above two cases, namely

$$\|V^{\frac{1}{2}}u\|^2 \geq C_*^2\|\nabla_{\mathbb{B}}u\|^2 \text{ for } \kappa>0$$

and

$$\|V^{\frac{1}{2}}u\|^2 \leq C_*^2\|\nabla_{\mathbb{B}}u\|^2 \text{ for } \kappa<0,$$

which allow us to deal with both cases simultaneously as shown in the problem (1.1)-(1.3).

2.2. Potential well. To prove our main results, we begin by giving the definition of weak solution of (1.1)-(1.3). Then we introduce the mathematical analysis tool, the potential well method with its structures.

DEFINITION 2.5 (Weak solution). *Function $u(x,t)$ is called a weak solution to problem (1.1)-(1.3) on $[0,T)\times\mathbb{B}$, in which T is the maximum existence time of solution, if it satisfies*

(i) $u\in L^\infty(0,T;\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B}))$ *and* $u_t\in L^2(0,T;L_2^{\frac{N}{2}}(\mathbb{B}))$;

(ii) *for any* $v\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$ *and a.e.* $t\in[0,T)$, *the identity*

$$\int_{\mathbb{B}} u_t(t)v\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}+\int_{\mathbb{B}}\nabla_{\mathbb{B}}u(t)\nabla_{\mathbb{B}}v\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}+\kappa\int_{\mathbb{B}}Vu(t)v\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}$$
$$=\int_{\mathbb{B}}g|u(t)|^{p-1}u(t)v\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}$$

holds.

Now, we introduce the potential energy functional

$$J(u(t))=\frac{1}{2}\|\nabla_{\mathbb{B}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\frac{\kappa}{2}\|V^{\frac{1}{2}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-\frac{1}{p+1}\|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})} \tag{2.1}$$

and the Nehari functional

$$I(u(t))=\|\nabla_{\mathbb{B}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\kappa\|V^{\frac{1}{2}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-\|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}, \tag{2.2}$$

which helps define the Nehari manifold

$$\mathcal{N}=\left\{u\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})\backslash\{0\}\;\middle|\; I(u)=0\right\},$$

which can divide the $\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$ space into two parts, that is the positive part

$$\mathcal{N}_+=\left\{u\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})\;\middle|\; I(u)>0\right\}$$

and the negative part

$$\mathcal{N}_-=\left\{u\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})\;\middle|\; I(u)<0\right\}.$$

In aid of the Nehari manifolds we can also define the depth of potential well

$$d=\inf_{u\in\mathcal{N}}J(u).$$

Furthermore, we consider the levels of $J(u)$ by

$$J^{\alpha}=\left\{u\in\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})\ \middle|\ J(u)<\alpha\right\}.$$

Hence, for all $\alpha>d$, we define

$$\begin{aligned}\mathcal{N}_{\alpha}:=\mathcal{N}\cap J^{\alpha}&\equiv\left\{u\in\mathcal{N}\ \middle|\ \|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}<\frac{2\alpha(p+1)}{(1+\kappa C_*^2)(p-1)}\right\}\neq\emptyset\\&\equiv\left\{u\in\mathcal{N}\ \middle|\ \|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}<\frac{2C_{poin}\alpha(p+1)}{(1+\kappa C_*^2)(p-1)}\right\}\neq\emptyset.\end{aligned}$$

Remark 2.2 (Some explanations on $\mathcal{N}_\alpha$). As the manifolds $\mathcal{N}_\alpha$ is a key tool for us to partition the manifold of the initial data under the high initial energy level, which connects the Nehari manifold $\mathcal{N}$ and the upper bound of the potential functional $J(u)$, also the top norm $\left(\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)$ and the base norm $\left(\|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)$ of the space $\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})$, we need to explain these connections clearly in this remark. In aid of the cone Hardy's inequality, we can derive the relationship between $J(u)$ and $I(u)$ as follows

$$\begin{aligned}J(u)=&\frac{1}{2}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\frac{\kappa}{2}\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-\frac{1}{p+1}\|g^{\frac{1}{p+1}}u\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}\\=&\frac{p-1}{2(p+1)}\left(\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\kappa\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)+\frac{1}{p+1}I(u)\\\geq&\frac{p-1}{2(p+1)}(1+\kappa C_*^2)\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\frac{1}{p+1}I(u),\end{aligned}\tag{2.3}$$

here as we need to consider the two possible opposite signs of $\kappa\in\mathbb{R}$, both the versions of the cone Poincaré inequality will be applied. For any $u\in\mathcal{N}_\alpha$, i.e., $u\in\mathcal{N}$ and $J(u)<\alpha$, we obtain from (2.3) that

$$\alpha>J(u)\geq\frac{(p-1)(1+\kappa C_*^2)}{2(p+1)}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})},$$

which means

$$\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}<\frac{2\alpha(p+1)}{(p-1)(1+\kappa C_*^2)}.$$

i.e., the definition of the first line of $\mathcal{N}_\alpha$.

From the cone Poincaré inequality, we also rewrite the above inequality as

$$\|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}<\frac{2\alpha C_{poin}(p+1)}{(1+\kappa C_*^2)(p-1)},$$

which is the definition of the second line of $\mathcal{N}_\alpha$.

Now it is convenient for us to define

$$\lambda_\alpha:=\inf\left\{\|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\middle|u\in\mathcal{N}_\alpha\right\}$$

and

$$\Lambda_\alpha := \sup\left\{\|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\middle| u\in\mathcal{N}_\alpha\right\}.$$

Clearly we have the following monotonicity properties

$$\alpha \mapsto \lambda_\alpha \text{ is nonincreasing}$$

and

$$\alpha \mapsto \Lambda_\alpha \text{ is nondecreasing.}$$

Finally, let us introduce following two sets,

$$\mathcal{S}_G := \mathcal{N}_+ \cap \left\{u\in\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})\middle| \|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \le \lambda_\alpha, J(u)<\alpha\right\}$$

and

$$\mathcal{S}_B := \mathcal{N}_- \cap \left\{u\in\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})\middle| \|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \ge \Lambda_\alpha, J(u)<\alpha\right\}.$$

But $\mathcal{S}_G$ or $\mathcal{S}_B$ is not sharp.

3. The main results

As shown in (a)-(c) in Section 1, as the descriptions, summaries and explanations of the main conclusions obtained in the present paper, in this section we shall exhibit these main results in the following four theorems. Roughly speaking, Theorem 3.1 provides a very general condition on the initial data to classify them for the global solution and the non-global solution. By further restricting the initial condition for the finite time blow up solution, Theorem 3.2 estimates the upper bound of the blowup time, when the initial condition is related to the parameters associated to the diffusion term and the potential term, comparing to Theorem 3.3, which makes it by the parameters related to the nonlinearity. Finally, Theorem 3.4 estimates the lower bound of the blowup time, and the lower bound of the blowup rate of the blowup solution without any restrictions on the initial data but assuming the finite time blowup to be true.

THEOREM 3.1 (Global existence and finite time blow up for high initial energy). *For any $\alpha\in(d,+\infty)$ and $\kappa\in(-C_*^{-2},+\infty)$, the following conclusions hold*

(i) *If $u_0\in\mathcal{S}_G$, then the weak solution of problem (1.1)-(1.3) exists global in time and $u(t)\to 0$ as $t\to+\infty$;*

(ii) *If $u_0\in\mathcal{S}_B$, then the weak solution of problem (1.1)-(1.3) blows up in finite time.*

THEOREM 3.2 (Finite time blow up and upper bound of blowup time for positive initial energy I). *Let $u_0\in\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})$ and $\kappa\in(-C_*^{-2},+\infty)$, assume that $J(u_0)>0$ and*

$$J(u_0) < \frac{(p-1)(\kappa C_*^2+1)}{2C_{poin}(p+1)}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \tag{3.1}$$

hold, then the weak solution u of problem (1.1)-(1.3) blows up in finite time, where C_{poin} is the optimal constant of cone Poincaré inequality

$$C_{poin}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \ge \|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}.$$

Moreover, there exists a time T_1^* satisfying

$$T_1^* \le \frac{4\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)\left(\frac{(p-1)(1+\kappa C_*^2)}{2C_{poin}(p+1)}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - J(u_0)\right)}$$

such that

$$\lim_{t\to T_1^*}\int_0^t \|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\,\mathrm{d}\tau = +\infty.$$

THEOREM 3.3 (Finite time blow up and upper bound of blowup time for positive initial energy II). *Let $u_0\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$, assume that $J(u_0)>0$ and*

$$J(u_0) < \frac{p-1}{2|\mathbb{B}|^{\frac{(p-1)^2}{4}} C_{g_*}^{\frac{p-1}{2}}(p+1)}\|u_0\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})} \tag{3.2}$$

hold, then the solution u of problem (1.1)-(1.3) blows up in finite time, where C_{g_*} is the essential supremum of the weighted function $g^{-1}\in L^\infty(\mathrm{int}\mathbb{B})\cap C(\mathrm{int}\mathbb{B})$, i.e., $C_{g_*}=$ $\inf\{M\,|\,|g(x)|^{-1}\le M, a.e.\ x\in\mathbb{B}\}$ and $|\mathbb{B}|$ denotes the volume of $\mathbb{B}$. Moreover, there exists a time T_2^* satisfying

$$T_2^* \le \frac{4\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)\left(\frac{p-1}{2C_{g_*}^{\frac{p-1}{2}}(p+1)}\|u_0\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})} - J(u_0)\right)}$$

such that

$$\lim_{t\to T_2^*}\int_0^t \|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\,\mathrm{d}\tau = +\infty.$$

THEOREM 3.4 (Lower bound of blowup time and blowup rate of blowup solution). *Let $u_0\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$ and $\kappa\in(-C_*^{-2},+\infty)$. Suppose that u is the blowup solution of problem (1.1)-(1.3), then a lower bound for the blow up time T_* of the solution is given by*

$$T_* \ge \frac{54(4\epsilon-4+\kappa C_*^2\epsilon^2)^2}{8\epsilon^5\tilde{C}^3C_g^3S^4|\mathbb{B}|^{2(1-\delta)}(2\delta-1)\|u_0\|^{\epsilon(2\delta-1)}_{L_\epsilon^N}}$$

and the lower estimate of the blow up rate is described by

$$\|u(t)\|_{L_\epsilon^N} \ge \left(\frac{8\epsilon^5\tilde{C}^3C_g^3S^4|\mathbb{B}|^{2(1-\delta)}(2\delta-1)}{54(4\epsilon-4+\kappa C_*^2\epsilon^2)^2}\right)^{\frac{1}{\epsilon(1-2\delta)}}(T_*-t)^{\frac{1}{\epsilon(1-2\delta)}},$$

where $|\mathbb{B}|$ denotes the volume of $\mathbb{B}$,

$$\delta = \frac{1}{2}+\frac{3(p-1)}{2\epsilon}$$

and the odd number ϵ satisfies

$$\epsilon > 3(p-1).$$

Furthermore, $C_g>0$ is the essential supremum which depends on the weighted function $g\in L^\infty(\mathrm{int}\mathbb{B})\cap C(\mathrm{int}\mathbb{B})$, i.e., $C_g=\inf\{M\,|\,|g(x)|\le M, a.e.\ x\in\mathbb{B}\}$, $\tilde{C}$ is the optimal imbedding constant for $L_{\frac{N-2}{3N}}^{\frac{3N^2}{N-2}}(\mathbb{B})\hookrightarrow L^\infty(\mathbb{B})$ and S is the optimal constant of the cone Sobolev embedding, i.e.,

$$S^{-1}=\inf_{u\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})\setminus\{0\}}\frac{\|\nabla_{\mathbb{B}}u\|_{L_2^{\frac{N}{2}}(\mathbb{B})}}{\|u\|_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}}.$$

4. Proof of Theorem 3.1

Our goal in this section is to achieve a sharp-like theorem depending tightly on the initial data in $\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$ which can describe what kinds of initial data lead to the global existence or the finite time blow up of the solution at arbitrary high initial energy level, i.e., $J(u_0)>d$. To begin, we provide some of the significant characteristics of the Nehari manifold. Without the special claim, here and in the following sections, the solution $u(t)$ for $t\in[0,T)$ we mentioned in each beginning always represents a local solution.

As some auxiliary results to prove Theorem 3.1, we first state the relationship between the invariant set $\mathcal{N}_+$ and the potential energy functional $J(u)$, and combine the two to give a bounded conclusion. Both results play a key role in proving the global existence of the solution under the high initial energy level. At the same time, we also give a fundamental feature of the invariant manifold $\mathcal{N}_-$ in space $\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$, which is regarded as a powerful point for blow up analysis.

Lemma 4.1. *Suppose that $u\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$, then*

(i) *for every $u\in\mathcal{N}_+$, we infer $J(u)>0$;*

(ii) *for each $\alpha>0$, we assert that $J^\alpha\cap\mathcal{N}_+$ is a bounded set in $\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$;*

(iii) *for any $u\in\mathcal{N}_-$, we conclude*

$$\mathrm{dist}(0,\mathcal{N}_-):=\min_{u\in\mathcal{N}_-}\|\nabla_{\mathbb{B}}u\|_{L_2^{\frac{N}{2}}(\mathbb{B})}\ge\left(\frac{1+\kappa C_*^2}{C_gS^{p+1}}\right)^{\frac{1}{p-1}}>0.$$

Proof.

(i) For $u\in\mathcal{N}_+$, i.e., $I(u)>0$, which implies $\|\nabla_{\mathbb{B}}u\|_{L_2^{\frac{N}{2}}(\mathbb{B})}\neq0$, then by cone Hardy's inequality we have

$$\begin{aligned}J(u)=&\frac{1}{2}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\frac{\kappa}{2}\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-\frac{1}{p+1}\|g^{\frac{1}{p+1}}u\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}\\
=&\frac{p-1}{2(p+1)}\left(\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\kappa\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)+\frac{1}{p+1}I(u)\\
\ge&\frac{p-1}{2(p+1)}(1+\kappa C_*^2)\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\frac{1}{p+1}I(u)\\
>&\frac{(p-1)(1+\kappa C_*^2)}{2(p+1)}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\\
>&0.\end{aligned}$$

(ii) Since $I(u)>0$, from the proof of (i) and $J(u)<\alpha$, we get

$$\alpha>J(u)>\frac{(p-1)(1+\kappa C_*^2)}{2(p+1)}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})},$$

which yields $\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}<\frac{2\alpha(p+1)}{(p-1)(1+\kappa C_*^2)}$.

(iii) For $u\in\mathcal{N}_-$, i.e., $I(u)<0$, we also have $\|\nabla_{\mathbb{B}}u\|_{L_2^{\frac{N}{2}}(\mathbb{B})}\neq 0$. From the definition of $\mathcal{N}_-$, we get

$$\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\kappa\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}<\|g^{\frac{1}{p+1}}u\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}=\int_{\mathbb{B}}|g||u|^{p+1}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}. \tag{4.1}$$

For the last term in (4.1), recalling $g\in L^\infty(\mathrm{int}\mathbb{B})\cap C(\mathrm{int}\mathbb{B})$, we know

$$\int_{\mathbb{B}}|g||u|^{p+1}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\leq\|g\|_{L^\infty(\mathbb{B})}\int_{\mathbb{B}}|u|^{p+1}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}=\|g\|_{L^\infty(\mathbb{B})}\|u\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}. \tag{4.2}$$

Thus, together with (4.1), (4.2) and the cone embedding inequality, we obtain

$$\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\kappa\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\leq C_gS^{p+1}\|\nabla_{\mathbb{B}}u\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})}, \tag{4.3}$$

where $C_g=\inf\{M\,|\,|g(x)|\leq M,\text{a.e. } x\in\mathbb{B}\}$.
On the other hand, using cone Hardy's inequality, we have

$$(1+\kappa C_*^2)\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\leq\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\kappa\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}. \tag{4.4}$$

Combining (4.3) and (4.4) gives $\|\nabla_{\mathbb{B}}u\|_{L_2^{\frac{N}{2}}(\mathbb{B})}>\left(\frac{1+\kappa C_*^2}{C_gS^{p+1}}\right)^{\frac{1}{p-1}}>0$.

∎

To clarify some monotonic properties and conservation law with respect to t for problem (1.1)-(1.3), we need the following lemmas.

LEMMA 4.2. *Suppose that $u\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})$, then we assert*

$$\frac{\mathrm{d}}{\mathrm{d}t}J(u(t))=-\|u_t(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})},\ t\in[0,T), \tag{4.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}=-2I(u(t)),\ t\in[0,T), \tag{4.6}$$

furthermore,

$$J(u(t))+\int_0^t\|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau=J(u_0). \tag{4.7}$$

Proof. Let u be any weak solution of problem (1.1)-(1.3). Multiplying (1.1) by u_t and integrating on $\mathbb{B}$ gives

$$\|u_t\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\|g^{\frac{1}{p+1}}u\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})},$$

that is

$$-\|u_t\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\frac{\kappa}{2}\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-\frac{1}{p+1}\|g^{\frac{1}{p+1}}\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}\right),$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) = -\|u_t(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \leq 0.$$

On the other hand, testing (1.1) by u also gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} = \|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})} - \|\nabla_{\mathbb{B}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - \kappa\|V^{\frac{1}{2}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} = -I(u(t)).$$

Finally, integrating (4.5) with respect to t from 0 to t, one infers (4.7) immediately. □

Next we define $T^*(u_0)$ as the maximum existence time for initial datum u_0 and give the proof of Theorem 3.1.

Proof. **(Proof of Theorem 3.1.)**

(i) In this case, as $u_0 \in \mathcal{S}_G$, namely $u_0 \in \mathcal{N}_+$, $J(u_0) < \alpha$ and $\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \leq \lambda_\alpha$. Recalling that λ_α is non-increasing, one also sees that $\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \leq \lambda_\alpha \leq \lambda_{J(u_0)}$. Then we need the invariance of $\mathcal{N}_+$, which claims that $u(t) \in \mathcal{N}_+$ for any $t \in [0, T^*(u_0))$ provided $u_0 \in \mathcal{N}_+$. Arguing by contradiction, due to the continuity of $I(t)$ in t, we suppose that there exists the first time $t_1 \in (0, T^*(u_0))$ such that $u(t) \in \mathcal{N}_+$ for $0 \leq t < t_1$ and $u(t_1) \in \mathcal{N}$. From $u(t) \in \mathcal{N}_+$ for $0 \leq t < t_1$, $\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \leq \lambda_\alpha \leq \lambda_{J(u_0)}$ and Lemma 4.2 one deduces that

$$\|u(t_1)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} < \|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \leq \lambda_{J(u_0)} \tag{4.8}$$

and

$$J(u(t_1)) < J(u_0). \tag{4.9}$$

Then by $u(t_1) \in \mathcal{N}$ and (4.9), we have $u(t_1) \in \mathcal{N}_{J(u_0)}$. Together with the definition of $\lambda_{J(u_0)}$ and $u(t_1) \in \mathcal{N}_{J(u_0)}$, it follows that $\|u(t_1)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \geq \lambda_{J(u_0)}$, which contradicts (4.8), hence $u(t) \in \mathcal{N}_+$ for any $t \in [0, T^*(u_0))$. In addition, combining (4.9) and (ii) of Lemma 4.1, $J^{J(u_0)} \cap \mathcal{N}_+$ is bounded in $\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})$, i.e., $\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} < \frac{2J(u_0)(p+1)}{(1+\kappa C_*^2)(p-1)}$, which yields $T^*(u_0) = \infty$ and the global existence of the weak solution.

Next we deal with the asymptotic behavior of the solution as $t \to +\infty$. We firstly denote the ω-limits set of $u_0 \in \mathcal{S}_G$ as follows,

$$\omega(u_0) = \bigcap_{t \geq 0} \overline{\{u(\xi) \,|\, \xi \geq t\}}.$$

Clearly, the ω-limits set consists of all the stationary solutions of (1.1)-(1.3) which solve the following elliptic problem

$$\begin{gathered} -\Delta_{\mathbb{B}}u + \kappa V(x)u = g(x)|u|^{p-1}u, x \in \mathbb{B}, \\ u(x) = 0, x \in \partial\mathbb{B}. \end{gathered}$$

According to the fact $u(t) \in \mathcal{N}_+$ for all $t \geq 0$ and Lemma 4.1-(i), we derive from (4.5) that $J(u(t)) > 0$ is bounded below and decreasing with respect to t, which means that there is a constant $c \geq 0$ such that

$$\lim_{t \to +\infty} J(u(t)) = c.$$

Obviously, for any $\widehat{u}_0\in\omega(u_0)$, we can infer that $J(\widehat{u}(t))=c$ for all $t\geq 0$, where $\widehat{u}(t)$ denotes the weak solution of (1.1)-(1.3) with initial data $\widehat{u}_0$. Meanwhile, we know that $\widehat{u}(t)=\widehat{u}_0$ for all $t\geq 0$, which together with (4.6) shows that

$$-2I(\widehat{u}(t))=0, t\in[0,\infty). \tag{4.10}$$

Thus, the formula (4.10) implies that

$$\omega(u_0)\in\mathcal{N}\cup\{0\}. \tag{4.11}$$

However, if $\widehat{u}_0\in\omega(u_0)$, by the fact that $\widehat{u}_0\in\mathcal{S}_G$, we also have

$$J(\widehat{u}_0)<\alpha \text{ and } \|\widehat{u}_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}<\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\leq\lambda_\alpha.$$

Then from the definition of $\mathcal{N}_\alpha$, it follows that $\widehat{u}_0\notin\mathcal{N}_\alpha$ and

$$\omega(u_0)\cap\mathcal{N}=\emptyset. \tag{4.12}$$

Hence, from (4.11) and (4.12), we conclude that $\omega(u_0)=\{0\}$, which means that the solution u of problem (1.1)-(1.3) exists global in time and decays to zero as $t\to+\infty$ provided $u_0\in\mathcal{S}_G$.

(ii) If $u_0\in\mathcal{S}_B$. A similar contradictory argument as that in (i) of this proof indicates that $u(t)\in\mathcal{N}_-$ for all $t\in[0,T^*(u_0))$. In order to prove the finite time blow up we suppose $T^*(u_0)=+\infty$. As (4.5) shows that $J(u(t))$ is non-increasing with respect to t, it has the following two possible cases:

(a) There is a constant C such that $\lim_{t\to+\infty}J(u(t))=C$
or

(b) $\lim_{t\to+\infty}J(u(t))=-\infty$.

Next, we shall prove that neither of the above cases can hold by showing the contradiction to $T^*(u_0)=+\infty$, then we infer that the solution u blows up in finite time.

For case (a), that is to suppose that case (a) happens, by (4.5) and $\lim_{t\to+\infty}J(u(t))=C$, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}J(u(t))=-\|u_t(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\to 0 \text{ as } t\to+\infty,$$

which indicates that $u(t)$ is approaching the stationary solution of problem (1.1)-(1.3) as $t\to+\infty$, i.e., $u(t)\in\mathcal{N}$ or $u(t)=0$ as $t\to+\infty$. However, $u(t)\in\mathcal{N}_-$ for any $t>0$ implies that $u(t)\notin\mathcal{N}$ for any $t>0$, which tells us that $u(t)=0$ as $t\to+\infty$. On the other hand, from $u(t)\in\mathcal{N}_-$ and (iii) in Lemma 4.1, we can conclude that $u(t)\neq 0$ as $t\to+\infty$, which contradicts $u(t)=0$ as $t\to+\infty$. Hence, case (a) does not happen.

Next we test the possibility of case (b), similar to what we did for case (a), we suppose that case (b) happens, then due to the continuity of $J(u)$ with respect to t, there must exist a first $t_1<T^*(u_0)$ such that $J(u(t_1))<0$. Since we have proved that $u(t)\in\mathcal{N}_-$ for all $t\in[0,T^*(u_0))$, then $u(t_1)\in\mathcal{N}_-$. Taking $u(t_1)$ as a new initial data, by Theorem 1.3 in [41] we know that the corresponding solution $U(t)=u(t_1+t)$ blows up in finite time, which contradicts the assumption that u is a global solution. Hence, case (b) does not happen.

Therefore, combining (a) and (b), we assert that $T^*(u_0)<\infty$, i.e., the solution u of problem (1.1)-(1.3) blows up in finite time provided $u_0\in\mathcal{S}_B$. ∎

5. Proof of Theorem 3.2

The well-known lemma that we introduce here originated from [49] and will be used to demonstrate the trend of the auxiliary function to effectively analyze the finite time blow up of solutions. At the same time, this lemma can also estimate the upper bound of the blowup time.

Lemma 5.1 ([49]). *Suppose that a positive, twice-differentiable function $\psi(t)$ satisfies the inequality*

$$\psi''(t)\psi(t)-(1+\theta)(\psi'(t))^2\geq 0,\ t>0,$$

where $\theta>0$ is some positive constant. If $\psi(0)>0$ and $\psi'(0)>0$, then there exists $0<t_*\leq\frac{\psi(0)}{\theta\psi'(0)}$ such that $\psi(t)$ tends to infinity as $t\to t_*$.

Next, we establish the following invariant lemma, which plays a crucial role in the proof of the blowup theorem at arbitrary positive initial energy.

Lemma 5.2. *Assume that $u_0\in\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})$ satisfies (3.1), then $u(t)\in\mathcal{N}_-$ for all $t\in[0,T)$.*

Proof. Noting that

$$J(u_0)=\frac{p-1}{2(p+1)}\left(\|\nabla_{\mathbb{B}}u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\kappa\|V^{\frac{1}{2}}u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)+\frac{1}{p+1}I(u_0),$$

then (3.1) indicates that $u_0\in\mathcal{N}_-$.

Now, we prove $u(t)\in\mathcal{N}_-$ for all $t\in[0,T)$. Arguing by contradiction, by the continuity of $I(u(t))$ in t, we suppose that there exists a first time $\iota\in(0,T)$ such that $u(t)\in\mathcal{N}_-$ for $0\leq t<\iota$ and $u(\iota)\in\mathcal{N}$, then by (4.6) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}=-2I(u(t))>0 \text{ for all } t\in[0,\iota), \tag{5.1}$$

which implies that

$$\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}<\|u(\iota)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}. \tag{5.2}$$

From (4.5) it follows that

$$J(u(\iota))\leq J(u_0). \tag{5.3}$$

By the definition of $J(u)$, $u(\iota)\in\mathcal{N}$ and cone Hardy's inequality, we derive

$$\begin{aligned}J(u(\iota))&=\frac{p-1}{2(p+1)}\left(\|\nabla_{\mathbb{B}}u(\iota)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\kappa\|V^{\frac{1}{2}}u(\iota)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)+\frac{1}{p+1}I(u(\iota))\\&=\frac{p-1}{2(p+1)}\left(\|\nabla_{\mathbb{B}}u(\iota)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\kappa\|V^{\frac{1}{2}}u(\iota)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\\&\geq\frac{p-1}{2(p+1)}\left(1+\kappa C_*^2\right)\|\nabla_{\mathbb{B}}u(\iota)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}.\end{aligned}$$

Combining (5.3) and (3.1), we get

$$\frac{(p-1)(1+\kappa C_*^2)}{2C_{poin}(p+1)}\|u(\iota)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\leq J(u_0)<\frac{(p-1)(1+\kappa C_*^2)}{2C_{poin}(p+1)}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})},$$

that is

$$\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} > \|u(\iota)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})},$$

which contradicts (5.2). Hence, we have proved this lemma. □

Now we prove the high energy blowup and estimate the upper bound of the blowup time of solutions for problem (1.1)-(1.3).

Proof. **(Proof of Theorem 3.2.)** For the sake of clarity, we divide the proof of the theorem into two parts.

Part I. Finite time blow up. Arguing by contradiction, we assume the maximum existence time $T=+\infty$. From the definition of $J(u)$, $I(u)$ and (4.6) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} =& -2I(u(t))\\
=& -2\left(\|\nabla_{\mathbb{B}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + \kappa\|V^{\frac{1}{2}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - \|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}\right)\\
=& -4\left(\frac{1}{2}\|\nabla_{\mathbb{B}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + \frac{\kappa}{2}\|V^{\frac{1}{2}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - \frac{1}{p+1}\|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}\right)\\
&+\left(2-\frac{4}{p+1}\right)\|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}\\
=& -4J(u(t)) + \frac{2(p-1)}{p+1}\|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}.
\end{aligned} \tag{5.4}$$

In the rest of the proof, we consider the following two cases.

Case I. $J(u(t))\geq 0$ for all $t>0$.

From (3.1), we choose ρ satisfying

$$1<\rho<\frac{\xi_1\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{4J(u_0)},$$

where $\xi_1 := \frac{2(p-1)(1+\kappa C_*^2)}{C_{poin}(p+1)}$. Then substituting (4.7) into (5.4) and by virtue of $J(u)\geq 0$, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} =& 4(\rho-1)J(u(t)) - 4\rho J(u(t)) + \frac{2(p-1)}{p+1}\|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}\\
\geq& -4\rho J(u_0) + 4\rho\int_0^t \|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\,\mathrm{d}\tau + \frac{2(p-1)}{p+1}\|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}.
\end{aligned} \tag{5.5}$$

From Lemma 5.2 (i.e., $I(u)<0$) and Lemma 2.1, we know

$$\begin{aligned}
\|g^{\frac{1}{p+1}}u\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})} >& \|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + \kappa\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\\
\geq& \frac{1+\kappa C_*^2}{C_{poin}}\|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})},
\end{aligned}$$

which makes (5.5) to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} > -4\rho J(u_0) + 4\rho \mathbf{\Psi_1}(t) + \xi_1 \|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}, \tag{5.6}$$

where $\mathbf{\Psi_1}(t) := \int_0^t \|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau$. Hence we get the following differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - \xi_1 \|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} > -4\rho J(u_0),$$

which yields

$$\xi_1 \|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} > \xi_1 \|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} e^{\xi_1 t} + 4\rho J(u_0)\left(1 - e^{\xi_1 t}\right). \tag{5.7}$$

Next, we define

$$\mathbf{\Psi_2}(t) := \int_0^t \|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau.$$

As the solution has been already supposed to be global in time, $\mathbf{\Psi_2}(t)$ is bounded for $t \in [0,\infty)$. Here we have

$$\mathbf{\Psi_2}'(t) = \|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \tag{5.8}$$

and

$$\mathbf{\Psi_2}''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}.$$

Taking

$$0 < \varepsilon < \frac{1}{2\rho\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}\left(\xi_1 \|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - 4\rho J(u_0)\right),$$

we derive

$$\xi_1 \|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - 4\rho J(u_0) > 2\rho\varepsilon \|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}. \tag{5.9}$$

Substituting (5.7) into (5.6), by (5.9) and $e^{\xi_1 t} \geq 1$ for $t \geq 0$, we arrive at

$$\begin{aligned}\mathbf{\Psi_2}''(t) &> 4\rho\mathbf{\Psi_1}(t) + \left(\xi_1 \|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - 4\rho J(u_0)\right)e^{\xi_1 t} \\ &> 4\rho\mathbf{\Psi_1}(t) + 2\rho\varepsilon\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}.\end{aligned} \tag{5.10}$$

Let

$$\mathbf{\Psi_3}(t) := \left(\int_0^t \|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau\right)^2 + \varepsilon^{-1}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\int_0^t \|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau + c,$$

where

$$c > \frac{1}{4}\varepsilon^{-2}\|u_0\|^4_{L_2^{\frac{N}{2}}(\mathbb{B})}. \tag{5.11}$$

According to the definition of $\mathbf{\Psi_3}(t)$, we can derive

$$\mathbf{\Psi_3}'(t) = \left(2\mathbf{\Psi_2}(t) + \varepsilon^{-1}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\mathbf{\Psi_2}'(t) \tag{5.12}$$

and

$$\mathbf{\Psi_3}''(t) = \left(2\mathbf{\Psi_2}(t) + \varepsilon^{-1}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\mathbf{\Psi_2}''(t) + 2(\mathbf{\Psi_2}'(t))^2. \tag{5.13}$$

In order to well exhibit the coming results, we define (its positivity can be ensured by (5.11))

$$\delta := 4c - \varepsilon^{-2}\|u_0\|^4_{L_2^{\frac{N}{2}}(\mathbb{B})} > 0,$$

then (5.12) gives

$$\begin{aligned}
(\mathbf{\Psi_3}'(t))^2 &= \left(4\mathbf{\Psi_2}^2(t) + 4\varepsilon^{-1}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathbf{\Psi_2}(t) + \varepsilon^{-2}\|u_0\|^4_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)(\mathbf{\Psi_2}'(t))^2 \\
&= \left(4\mathbf{\Psi_2}^2(t) + 4\varepsilon^{-1}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathbf{\Psi_2}(t) + 4c - \delta\right)(\mathbf{\Psi_2}'(t))^2 \\
&= (4\mathbf{\Psi_3}(t) - \delta)(\mathbf{\Psi_2}'(t))^2,
\end{aligned} \tag{5.14}$$

which says

$$4\mathbf{\Psi_3}(t)(\mathbf{\Psi_2}'(t))^2 = (\mathbf{\Psi_3}'(t))^2 + \delta(\mathbf{\Psi_2}'(t))^2. \tag{5.15}$$

Taking advantage of the inner product in $L_2^{\frac{N}{2}}(\mathbb{B})$, we have the following identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} = (u(t), u_t(t))_{\mathbb{B}}.$$

Integrating the above identity over $[0,t]$ gives

$$\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} = \|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + 2\int_0^t (u(\tau), u_t(\tau))_{\mathbb{B}}\mathrm{d}\tau. \tag{5.16}$$

Combining (5.8) and (5.16), by Hölder and Cauchy's inequalities we get

$$\begin{aligned}
\left(\mathbf{\Psi_2}'(t)\right)^2 &= \|u(t)\|^4_{L_2^{\frac{N}{2}}(\mathbb{B})} = \left(\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + 2\int_0^t (u(\tau), u_t(\tau))_{\mathbb{B}}\mathrm{d}\tau\right)^2 \\
&\le \left(\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + 2(\mathbf{\Psi_2}(t))^{\frac{1}{2}}(\mathbf{\Psi_1}(t))^{\frac{1}{2}}\right)^2 \\
&= \|u_0\|^4_{L_2^{\frac{N}{2}}(\mathbb{B})} + 4\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}(\mathbf{\Psi_2}(t))^{\frac{1}{2}}(\mathbf{\Psi_1}(t))^{\frac{1}{2}} + 4\mathbf{\Psi_2}(t)\mathbf{\Psi_1}(t) \\
&\le \|u_0\|^4_{L_2^{\frac{N}{2}}(\mathbb{B})} + 4\mathbf{\Psi_2}(t)\mathbf{\Psi_1}(t) + 2\varepsilon\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathbf{\Psi_2}(t) + 2\varepsilon^{-1}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathbf{\Psi_1}(t)
\end{aligned}$$

$$:= \mathbf{\Psi_4}(t). \tag{5.17}$$

From (5.13), (5.15) and (5.10), we get

$$\begin{aligned}
&2\mathbf{\Psi_3}(t)\mathbf{\Psi_3}''(t)\\
=&2\left(\left(2\mathbf{\Psi_2}(t)+\varepsilon^{-1}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\mathbf{\Psi_2}''(t)+2(\mathbf{\Psi_2}'(t))^2\right)\mathbf{\Psi_3}(t)\\
=&2\left(2\mathbf{\Psi_2}(t)+\varepsilon^{-1}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\mathbf{\Psi_2}''(t)\mathbf{\Psi_3}(t)+4(\mathbf{\Psi_2}'(t))^2\mathbf{\Psi_3}(t)\\
=&2\left(2\mathbf{\Psi_2}(t)+\varepsilon^{-1}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\mathbf{\Psi_2}''(t)\mathbf{\Psi_3}(t)+(\mathbf{\Psi_3}'(t))^2+\delta(\mathbf{\Psi_2}'(t))^2\\
>&4\rho\mathbf{\Psi_3}(t)\left(2\mathbf{\Psi_2}(t)+\varepsilon^{-1}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\left(2\mathbf{\Psi_1}(t)+\varepsilon\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)+(\mathbf{\Psi_3}'(t))^2+\delta(\mathbf{\Psi_2}'(t))^2\\
=&4\rho\mathbf{\Psi_3}(t)\mathbf{\Psi_4}(t)+(\mathbf{\Psi_3}'(t))^2+\delta(\mathbf{\Psi_2}'(t))^2.
\end{aligned}\tag{5.18}$$

Combining (5.18), (5.14) and (5.17), we obtain

$$\begin{aligned}
&2\mathbf{\Psi_3}(t)\mathbf{\Psi_3}''(t)-(1+\rho)(\mathbf{\Psi_3}'(t))^2\\
>&4\rho\mathbf{\Psi_3}(t)\mathbf{\Psi_4}(t)+\delta(\mathbf{\Psi_2}'(t))^2-\rho(\mathbf{\Psi_3}'(t))^2\\
=&4\rho\mathbf{\Psi_3}(t)\mathbf{\Psi_4}(t)-\rho\left(4\mathbf{\Psi_3}(t)-\delta\right)(\mathbf{\Psi_2}'(t))^2+\delta(\mathbf{\Psi_2}'(t))^2\\
=&4\rho\mathbf{\Psi_3}(t)\mathbf{\Psi_4}(t)-4\rho\mathbf{\Psi_3}(t)(\mathbf{\Psi_2}'(t))^2+\delta(1+\rho)(\mathbf{\Psi_2}'(t))^2\\
\geq&4\rho\mathbf{\Psi_3}(t)\mathbf{\Psi_4}(t)-4\rho\mathbf{\Psi_3}(t)\mathbf{\Psi_4}(t)\\
=&0,
\end{aligned}$$

that is

$$\mathbf{\Psi_3}(t)\mathbf{\Psi_3}''(t)-\frac{1+\rho}{2}(\mathbf{\Psi_3}'(t))^2>0,\quad t\in[0,+\infty),$$

which implies that

$$\left(\mathbf{\Psi_3}^{-\beta}(t)\right)''=-\frac{\beta}{\mathbf{\Psi_3}^{\beta+2}(t)}\left(\mathbf{\Psi_3}''(t)\mathbf{\Psi_3}(t)-(\beta+1)(\mathbf{\Psi_3}'(t))^2\right)<0,\ t\in[0,+\infty),$$

where $\beta=\frac{\rho-1}{2}>0$. Since $\mathbf{\Psi_3}(0)=c>\frac{1}{4}\varepsilon^{-2}\|u_0\|^4_{L_2^{\frac{N}{2}}(\mathbb{B})}>0$ and $\mathbf{\Psi_3}'(0)=\varepsilon^{-1}\|u_0\|^4_{L_2^{\frac{N}{2}}(\mathbb{B})}>0$, by Lemma 5.1, it follows that there exists a

$$0<T^*\leq\frac{2\mathbf{\Psi_3}(0)}{(\rho-1)\mathbf{\Psi_3}'(0)} \tag{5.19}$$

such that

$$\lim_{t\to T^*}\mathbf{\Psi_3}^{-\beta}(t)=0,$$

i.e.,

$$\lim_{t\to T^*}\mathbf{\Psi_3}(t)=+\infty,$$

which contradicts $T=+\infty$. Now, by employing the continuity of both $\mathbf{\Psi_3}(t)$ and $\mathbf{\Psi_2}(t)$ with respect to t, we can conclude that $\mathbf{\Psi_2}(t)$ tends to infinity as t approaches T^*, which contradicts $T^*=+\infty$.

Case II. There exist some $\tilde{t}$ such that $J(u(\tilde{t}))<0$.

Since $J(u_0)>0$, by the continuity of $J(u)$ in t, we can assume that there exists a first time $t_0>0$ such that $J(u(t_0))=0$ and $J(u(\hat{t}))<0$ for some $\hat{t}>t_0$. We take $u(\hat{t})$ as a new initial datum, then from Lemma 5.2, we have $u(t)\in\mathcal{N}_-$ for $t>\hat{t}$. Similarly as the proof of Theorem 1.3 in [41], we can deduce that the solution of problem (1.1)-(1.3) blows up in finite time.

Combining the above two cases, we conclude that u blows up in finite time.

Remark 5.1. Although we have given an estimate of the upper bound of the blow up time in (5.19) in a specific form as follows by simple calculation

$$0<T^*\leq\frac{2c}{(\rho-1)\varepsilon^{-1}\|u_0\|^4_{L_2^{\frac{N}{2}}(\mathbb{B})}},$$

where $c>\frac{\varepsilon^{-2}}{4}\|u_0\|^4_{L_2^{\frac{N}{2}}(\mathbb{B})}$ is a constant, due to the unknown constant c the work of the argument has not reached the end. In order to further clarify the upper bound of the blow up time T^*, and even clarify what factors such an upper bound is related to, we need to carry out in-depth analysis, for example, to estimate the bound of the constant c, so as to reach the conclusion, which is the task of the following Part II.

Part II. Upper bound of blowup time estimate. Next, we estimate the upper bound of blowup time as follows. Let u be the weak solution of problem (1.1)-(1.3), T is the maximum existence time. For $t\in[0,T)$, we denote

$$\mathbf{\Psi_5}(t):=\int_0^t\|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau+(T-t)\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\mu(t+\nu)^2,$$

where $\mu>0$ and $\nu>0$ are two constants, which will be determined later in the argumentation process. Then we have

$$\mathbf{\Psi_5}'(t)=\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+2\mu(t+\nu),\quad t\in[0,T).$$

By (4.6) and the relationship between $J(u)$ and $I(u)$ in (2.3), it follows that

$$\begin{aligned}\mathbf{\Psi_5}''(t)&=-2I(u(t))+2\mu\\&\geq(p-1)(1+\kappa C_*^2)\|\nabla_{\mathbb{B}}u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-2(p+1)J(u(t)),\quad t\in[0,T).\end{aligned}\tag{5.20}$$

Combining the cone Poincaré inequality and (4.7), (5.20) becomes

$$\begin{aligned}\mathbf{\Psi_5}''(t)\geq&\frac{(p-1)(1+\kappa C_*^2)}{C_{poin}}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-2(p+1)J(u(t))\\\geq&\frac{(p-1)(1+\kappa C_*^2)}{C_{poin}}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-2(p+1)J(u_0)+2(p+1)\int_0^t\|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau.\end{aligned}$$

From Lemma 5.2 and Lemma 4.2, we know $\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}$ is strictly increasing on $[0,T)$. Therefore, we obtain

$$\mathbf{\Psi_5}''(t)\geq2(p+1)\left(\frac{\xi_1}{4}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-J(u_0)+\mathbf{\Psi_1}(t)\right)\tag{5.21}$$

and $\mathbf{\Psi_5}'(t)\geq 2\mu(t+\nu)>0$ for $t\in[0,T)$, which implies $\mathbf{\Psi_5}(t)\geq \mathbf{\Psi_5}(0)=T\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\mu\nu^2>0$ for all $t\in[0,T)$.

On the other hand, we derive

$$
\begin{aligned}
-\frac{1}{4}(\mathbf{\Psi_5}'(t))^2=&-\left(\frac{1}{2}\left(\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)+\mu(t+\nu)\right)^2\\
=&\underbrace{\left(\int_0^t\|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau+\mu(t+\nu)^2\right)\left(\int_0^t\|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau+\mu\right)}_{I_1}\\
&\underbrace{-\left(\frac{1}{2}\left(\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)+\mu(t+\nu)\right)^2}_{I_2}\\
&-\left(\mathbf{\Psi_5}(t)-(T-t)\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\left(\int_0^t\|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau+\mu\right).
\end{aligned}
$$

To estimate the above inequality clearly, we show $I_1-I_2\geq 0$ next. Firstly, we have

$$
\begin{aligned}
I_1-I_2=&I_1-\left(\frac{1}{2}\int_0^t\frac{\mathrm{d}}{\mathrm{d}\tau}\|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau+\mu(t+\nu)\right)^2\\
=&I_1-\left(\int_0^t(u(\tau),u_t(\tau))_{\mathbb{B}}\mathrm{d}\tau+\mu(t+\nu)\right)^2.
\end{aligned}
$$

By using the cone Hölder inequality, it follows that

$$
\begin{aligned}
&I_1-I_2\\
\geq&I_1-\left(\int_0^t\|u(\tau)\|_{L_2^{\frac{N}{2}}(\mathbb{B})}\|u_t(\tau)\|_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau+\mu(t+\nu)\right)^2\\
\geq&I_1-\left(\left(\int_0^t\|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t\|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau\right)^{\frac{1}{2}}+\mu(t+\nu)\right)^2\\
=&\left(\mathbf{\Psi_2}(t)+\mu(t+\nu)^2\right)\left(\mathbf{\Psi_1}(t)+\mu\right)-\left((\mathbf{\Psi_2}(t))^{\frac{1}{2}}(\mathbf{\Psi_1}(t))^{\frac{1}{2}}+\mu(t+\nu)\right)^2\\
=&\left(\sqrt{\mu}(\mathbf{\Psi_2}(t))^{\frac{1}{2}}\right)^2-2\sqrt{\mu}(\mathbf{\Psi_2}(t))^{\frac{1}{2}}\sqrt{\mu}(t+\nu)(\mathbf{\Psi_1}(t))^{\frac{1}{2}}+\left(\sqrt{\mu}(t+\nu)(\mathbf{\Psi_1}(t))^{\frac{1}{2}}\right)^2\\
=&\left(\sqrt{\mu}(\mathbf{\Psi_2}(t))^{\frac{1}{2}}-\sqrt{\mu}(t+\nu)(\mathbf{\Psi_1}(t))^{\frac{1}{2}}\right)^2\\
\geq&0.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
-(\mathbf{\Psi_5}'(t))^2\geq&-4\left(\mathbf{\Psi_5}(t)-(T-t)\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\left(\int_0^t\|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau+\mu\right)\\
\geq&-4\mathbf{\Psi_5}(t)\left(\mathbf{\Psi_1}(t)+\mu\right).
\end{aligned}
\tag{5.22}
$$

Then by (5.22), we achieve

$$
\mathbf{\Psi_5}(t)\mathbf{\Psi_5}''(t)-\frac{p+1}{2}(\mathbf{\Psi_5}'(t))^2
$$

$$\geq \boldsymbol{\Psi_5}(t)\left(\boldsymbol{\Psi_5}''(t)-2(p+1)\left(\boldsymbol{\Psi_1}(t)+\mu\right)\right). \tag{5.23}$$

Substituting (5.21) into (5.23), we get

$$\boldsymbol{\Psi_5}(t)\boldsymbol{\Psi_5}''(t)-\frac{p+1}{2}(\boldsymbol{\Psi_5}'(t))^2 \geq 2(p+1)\boldsymbol{\Psi_5}(t)\left(\xi_1\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-J(u_0)-\mu\right).$$

By (3.1), we can choose a small enough $\mu\in(0,\sigma]$ such that

$$\frac{\xi_1}{4}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-J(u_0)-\mu\geq 0,$$

where $\sigma:=\frac{\xi_1}{4}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-J(u_0)$. Then,

$$\boldsymbol{\Psi_5}(t)\boldsymbol{\Psi_5}''(t)-\frac{p+1}{2}(\boldsymbol{\Psi_5}'(t))^2\geq 0,$$

which implies that the conditions of Lemma 5.1 are satisfied. It is easy to verify that $\boldsymbol{\Psi_5}(0)=T\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}+\mu\nu^2>0$, $\boldsymbol{\Psi_5}'(0)\geq 2\mu\nu>0$, then by Lemma 5.1, we derive

$$T^*\leq \frac{2\boldsymbol{\Psi_5}(0)}{(p-1)\boldsymbol{\Psi_5}'(0)}\leq \frac{\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)\mu\nu}T+\frac{\nu}{p-1}. \tag{5.24}$$

Now we choose ν to be large enough such that

$$\nu\in\left(\frac{\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)\mu},+\infty\right), \tag{5.25}$$

then it follows from (5.24) that

$$T^*\leq \frac{\mu\nu^2}{(p-1)\mu\nu-\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}. \tag{5.26}$$

Next, we need to deal with the two parameters μ and ν in (5.26) to specify the upper bound of the blow up time. First we introduce the following set to describe a pair of (μ,ν)

$$\mathfrak{M}:=\left\{(\mu,\nu)\;\middle|\;\nu\in\left(\frac{\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)\sigma},+\infty\right),\mu\in\left(\frac{\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)\nu},\sigma\right]\right\},$$

then the upper bound of the blow up time can be considered as

$$T^*\leq \inf_{(\mu,\nu)\in\mathfrak{M}}\frac{\mu\nu^2}{(p-1)\mu\nu-\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}. \tag{5.27}$$

Letting $f(\mu,\nu):=\frac{\mu\nu^2}{(p-1)\mu\nu-\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}$, and then differentiating $f(\mu,\nu)$ with respect to μ, we deduce

$$f'(\mu,\nu)=\frac{-\nu^2\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{\left((p-1)\mu\nu-\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)^2}<0,$$

which means that $f(\mu,\nu)$ is decreasing with μ, then for any ν, we have

$$\inf_{(\mu,\nu)\in\mathfrak{M}} f(\mu,\nu)=\inf_{\nu} f(\sigma,\nu)=\inf_{\nu}\frac{\sigma\nu^2}{(p-1)\sigma\nu-\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}. \tag{5.28}$$

By differentiating $f(\sigma,\nu)$ with respect to ν and taking $f'(\sigma,\nu)=0$, we can get the minimum point $\nu_{\min}=\frac{2\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)\sigma}$, then

$$\inf_{\nu} f(\sigma,\nu)=f(\sigma,\nu_{\min})=\frac{4\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)\sigma},$$

which together with (5.28) implies that

$$\inf_{(\mu,\nu)\in\mathfrak{M}} f(\mu,\nu)=f(\sigma,\nu_{\min})=\frac{4\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)\sigma}. \tag{5.29}$$

Hence, from (5.27) and (5.29), we obtain

$$\begin{aligned}T^*\le \inf_{(\mu,\nu)\in\mathfrak{M}} f(\mu,\nu)&=\frac{4\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)\sigma}\\ &=\frac{8C_{poin}(p+1)\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}{(p-1)^2(1+\kappa C_*^2)\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}-2C_{poin}(p^2-1)J(u_0)}.\end{aligned}$$

□

Since both the conclusions of Theorem 3.1 and Theorem 3.2 involve the finite time blow up of solutions, next we explain the correlation between them and compare their own characteristics in the following remark.

Remark 5.2. As we said in the introduction, both Theorem 3.1 and Theorem 3.2 give sufficient conditions for finite time blow up, while they place the different concerns on blow up, which naturally require different assumptions on the initial data. Obviously, as the sharp condition for the global existence and finite time blow up of the solution with low initial energy level has been obtained in [41], Theorem 3.1 concerns the key task to classify the initial data under high initial energy ($J(u_0)>d$). We hope to make a complete division of the whole space as far as possible, so as to expect that the sufficient conditions obtained in this way can cover more initial data to help us judge the global existence and finite time blow up of solutions. However, the intention of Theorem 3.2 lies in not only obtaining the sufficient conditions that satisfy the initial data leading to the solution blowing up in finite time, but also estimating the corresponding upper bound of blow up time. Comparing the sufficient conditions in Theorem 3.1 and Theorem 3.2, we find that the sufficient conditions for finite time blow up of solutions have overlapping parts, but the overlapping parts between them are not easy to see. Therefore, we want to explain this important conclusion through the following simple argument, namely, assuming that the initial energy level satisfies $J(u_0)>d$, we assert that if the initial data satisfies (3.1) in Theorem 3.2, then $u_0\in\mathcal{S}_B$.

From the proof of Lemma 5.2, we know that if u_0 satisfies (3.1), then $u_0\in\mathcal{N}_-$. For any $u\in\mathcal{N}_{J(u_0)}$, together with the definition of $\mathcal{N}_{J(u_0)}$, we deduce $\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}<\frac{2J(u_0)(p+1)}{(1+\kappa C_*^2)(p-1)}$. By cone Poincaré inequality, we have

$$\|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\le C_{poin}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}<\frac{2C_{poin}J(u_0)(p+1)}{(1+\kappa C_*^2)(p-1)}. \tag{5.30}$$

Picking $\Lambda_{J(u_0)}=\sup\left\{\|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\middle|u\in\mathcal{N}_{J(u_0)}\right\}$, then (5.30) tells us

$$\Lambda_{J(u_0)}<\frac{2C_{poin}J(u_0)(p+1)}{(1+\kappa C_*^2)(p-1)}. \tag{5.31}$$

Substituting (3.1) into (5.31), we get $\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}>\Lambda_{J(u_0)}$. Hence, we conclude that if u_0 satisfies (3.1), then

$$u_0\in\mathcal{N}_-\cap\left\{u_0\in\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})\middle|\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\ge\Lambda_{J(u_0)},J(u)<J(u_0)\right\}=\mathcal{S}_B.$$

However, rearranging (3.1), we obtain

$$\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}>\frac{2C_{poin}J(u_0)(p+1)}{(\kappa C_*^2+1)(p-1)},$$

together with (5.31), which tells that if $u_0\in\mathcal{S}_B$, it does not necessarily satisfy (3.1).

Therefore, through the above demonstration, we find that the conditions of Theorem 3.2 are more strict, while this "strictness" enables us to obtain a better estimation of the upper bound of blow up time.

6. Proof of Theorem 3.3

In order to prove that the solution to problem (1.1)-(1.3) blows up in finite time at arbitrary positive initial energy, we start with the following invariant manifold related to initial data.

LEMMA 6.1. *Assume that $u_0\in\mathcal{H}^{1,\frac{N}{2}}_{2,0}(\mathbb{B})$ satisfies (3.2), then $u(t)\in\mathcal{N}_-$ for all $t\in[0,T)$.*

Proof. By using the cone Hölder inequality and the fact $0<g\in L^\infty(\mathrm{int}\mathbb{B})\cap C(\mathrm{int}\mathbb{B})$, we have

$$\begin{aligned}\|u\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})}&=\left(\int_{\mathbb{B}}g^{-\frac{2}{p+1}}\cdot g^{\frac{2}{p+1}}|u|^2\frac{dx_1}{x_1}d\tilde{x}\right)^{\frac{p+1}{2}}\\&\le\left(\left(\int_{\mathbb{B}}|g^{-\frac{2}{p+1}}|^{\frac{p+1}{p-1}}\frac{dx_1}{x_1}d\tilde{x}\right)^{\frac{p-1}{p+1}}\left(\int_{\mathbb{B}}|g^{\frac{2}{p+1}}u^2|^{\frac{p+1}{2}}\frac{dx_1}{x_1}d\tilde{x}\right)^{\frac{2}{p+1}}\right)^{\frac{p+1}{2}}\end{aligned}$$

$$\le \left(\left(\int_{\mathbb{B}} |g^{-1}|^{\frac{2}{p-1}} \frac{dx_1}{x_1} d\widetilde{x} \right)^{\frac{p-1}{p+1}} \left(\int_{\mathbb{B}} |g^{\frac{1}{p+1}} u|^{p+1} \frac{dx_1}{x_1} d\widetilde{x} \right)^{\frac{2}{p+1}} \right)^{\frac{p+1}{2}}$$
$$= \|g^{-1}\|^{\frac{p-1}{2}}_{L^{\frac{N(p-1)}{2}}_{\frac{2}{p-1}}(\mathbb{B})} \|g^{\frac{1}{p+1}} u\|^{p+1}_{L^{\frac{N}{p+1}}_{p+1}(\mathbb{B})}. \tag{6.1}$$

On the other hand, due to the imbedding theorem that $L^\infty(\mathbb{B}) \hookrightarrow L^{\frac{N(p-1)}{2}}_{\frac{2}{p-1}}(\mathbb{B})$, it follows that

$$\|g^{-1}\|^{\frac{p-1}{2}}_{L^{\frac{N(p-1)}{2}}_{\frac{2}{p-1}}(\mathbb{B})} \le |\mathbb{B}|^{\frac{(p-1)^2}{4}} \|g^{-1}\|^{\frac{p-1}{2}}_{L^\infty(\mathbb{B})} \le |\mathbb{B}|^{\frac{(p-1)^2}{4}} C_{g_*}^{\frac{p-1}{2}}, \tag{6.2}$$

where $C_{g_*} = \inf\{M \mid |g(x)|^{-1} \le M, \text{a.e. } x \in \mathbb{B}\}$. Therefore, from (6.1) and (6.2), one can deduce

$$\|g^{\frac{1}{p+1}} u\|^{p+1}_{L^{\frac{N}{p+1}}_{p+1}(\mathbb{B})} \ge \frac{\|u\|^{p+1}_{L^{\frac{N}{2}}_{2}(\mathbb{B})}}{|\mathbb{B}|^{\frac{(p-1)^2}{4}} C_{g_*}^{\frac{p-1}{2}}}. \tag{6.3}$$

Substituting (3.2) into (6.3) and using the relationship of $J(u)$ and $I(u)$ in (2.3), we arrive at

$$\|g^{\frac{1}{p+1}} u_0\|^{p+1}_{L^{\frac{N}{p+1}}_{p+1}(\mathbb{B})} > \frac{2(p+1)}{p-1} J(u_0) = \|\nabla_{\mathbb{B}} u_0\|^2_{L^{\frac{N}{2}}_{2}(\mathbb{B})} + \kappa \|V^{\frac{1}{2}} u_0\|^2_{L^{\frac{N}{2}}_{2}(\mathbb{B})} + \frac{2}{p-1} I(u_0). \tag{6.4}$$

Recalling the definition of $I(u)$, (6.4) turns into

$$0 > \|\nabla_{\mathbb{B}} u_0\|^2_{L^{\frac{N}{2}}_{2}(\mathbb{B})} + \kappa \|V^{\frac{1}{2}} u_0\|^2_{L^{\frac{N}{2}}_{2}(\mathbb{B})} - \|g^{\frac{1}{p+1}} u_0\|^{p+1}_{L^{\frac{N}{p+1}}_{p+1}(\mathbb{B})} + \frac{2}{p-1} I(u_0) > \frac{p+1}{p-1} I(u_0),$$

which implies $u_0 \in \mathcal{N}_-$.

Next, we prove $u(t) \in \mathcal{N}_-$ for all $t \in [0,T)$. Arguing by contradiction, by the continuity of $I(u(t))$ in t, we suppose that there exists a first time $\iota \in (0,T)$ such that $u \in \mathcal{N}_-$ for $0 \le t < \iota$ and $u(\iota) \in \mathcal{N}$. Similar as the proof of Lemma 5.2, we can retrieve (5.1), (5.2) and (5.3), and only need to modify the remainder as follows

$$\begin{aligned} J(u(\iota)) &= \frac{p-1}{2(p+1)} \left(\|\nabla_{\mathbb{B}} u(\iota)\|^2_{L^{\frac{N}{2}}_{2}(\mathbb{B})} + \kappa \|V^{\frac{1}{2}} u(\iota)\|^2_{L^{\frac{N}{2}}_{2}(\mathbb{B})} \right) + \frac{1}{p+1} I(u(\iota)) \\ &= \frac{p-1}{2(p+1)} \left(\|\nabla_{\mathbb{B}} u(\iota)\|^2_{L^{\frac{N}{2}}_{2}(\mathbb{B})} + \kappa \|V^{\frac{1}{2}} u(\iota)\|^2_{L^{\frac{N}{2}}_{2}(\mathbb{B})} \right) \\ &= \frac{p-1}{2(p+1)} \|g^{\frac{1}{p+1}} u(\iota)\|^{p+1}_{L^{\frac{N}{p+1}}_{p+1}(\mathbb{B})}, \end{aligned}$$

which together with (6.3) yields

$$J(u(\iota)) \ge \frac{p-1}{2|\mathbb{B}|^{\frac{(p-1)^2}{4}} C_{g_*}^{\frac{p-1}{2}} (p+1)} \|u(\iota)\|^{p+1}_{L^{\frac{N}{2}}_{2}(\mathbb{B})}.$$

Combining (5.3) with (3.2), we get

$$\frac{p-1}{2|\mathbb{B}|^{\frac{(p-1)^2}{4}} C_{g_*}^{\frac{p-1}{2}} (p+1)} \|u(\iota)\|^{p+1}_{L^{\frac{N}{2}}_{2}(\mathbb{B})} \le J(u_0) < \frac{p-1}{2|\mathbb{B}|^{\frac{(p-1)^2}{4}} C_{g_*}^{\frac{p-1}{2}} (p+1)} \|u_0\|^{p+1}_{L^{\frac{N}{2}}_{2}(\mathbb{B})},$$

that is

$$\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} > \|u(\iota)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})},$$

which contradicts (5.2). Hence, we finish the proof. ∎

Next we prove the high energy blowup and estimate the upper bound of the blowup time of solutions to problem (1.1)-(1.3).

Proof. **(Proof of Theorem 3.3.)** For the sake of clarity, we divide the proof of the theorem into two parts.

Part I. Finite time blow up

Similar as the proof of Theorem 3.2, we can attain the blowup results by slightly modifying the corresponding argument in Theorem 3.2 as follows.

Arguing by contradiction, we assume the maximum existence time $T=+\infty$. From the definition of $J(u)$, $I(u)$ and (4.6) we also have (5.4). Similarly, in the rest of the proof, we again consider the following two cases.

Case I. $J(u(t))\geq 0$ for all $t>0$.

From (3.2), we choose ρ' satisfying

$$1<\rho'<\frac{\xi_2\|u_0\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})}}{4J(u_0)}, \tag{6.5}$$

where $\xi_2:=\frac{2(p-1)}{|\mathbb{B}|^{\frac{(p-1)^2}{4}}C_{g_*}^{\frac{p-1}{2}}(p+1)}$. Substituting (4.7) into (5.4) and by virtue of $J(u)\geq 0$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} &= 4(\rho'-1)J(u(t))-4\rho' J(u(t))+\frac{2(p-1)}{p+1}\|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}\\ &\geq -4\rho' J(u_0)+4\rho'\int_0^t\|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\,\mathrm{d}\tau+\frac{2(p-1)}{p+1}\|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}.\end{aligned} \tag{6.6}$$

From (6.3), (4.6) and Lemma 6.1, we know

$$\|g^{\frac{1}{p+1}}u\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}\geq\frac{\|u\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})}}{|\mathbb{B}|^{\frac{(p-1)^2}{4}}C_{g_*}^{\frac{p-1}{2}}}>\frac{\|u_0\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})}}{|\mathbb{B}|^{\frac{(p-1)^2}{4}}C_{g_*}^{\frac{p-1}{2}}},$$

which makes (6.6) to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}>-4\rho' J(u_0)+4\rho'\boldsymbol{\Psi}_{\mathbf{1}}(t)+\xi_2\|u_0\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})}. \tag{6.7}$$

Similar as the proof of Theorem 3.2, thanks to (6.5) and (3.2), we can choose a constant ε' such that

$$0<\varepsilon'<\frac{1}{2\rho'\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}}\left(\xi_2\|u_0\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})}-4\rho' J(u_0)\right),$$

therefore,

$$\xi_2\|u_0\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})} - 4\rho' J(u_0) > 2\rho'\varepsilon'\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}. \tag{6.8}$$

Substituting (6.8) into (6.7), one arrives at

$$\boldsymbol{\Psi_2}''(t) > 4\rho'\boldsymbol{\Psi_1}(t) + 2\rho'\varepsilon'\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}.$$

The remainder proof for Part I is completely similar as the proof of Theorem 3.2, hence, we omit it.

Part II. Estimate of the upper bound of blowup time. Next, we estimate the upper bound of blowup time as follows. Let u be the weak solution of problem (1.1)-(1.3), T be the maximum existence time. For $t\in[0,T)$, we denote

$$\boldsymbol{\Psi_6}(t) := p\int_0^t \|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau + p(T-t)\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + \mu(t+\nu)^2,$$

where $\mu>0$ and $\nu>0$ are two constants, which will be determined later in the argumentation process. Differentiating $\boldsymbol{\Psi_6}(t)$ with respect to t, we have

$$\boldsymbol{\Psi_6}'(t) = p\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - p\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + 2\mu(t+\nu).$$

Note that the relationship between $J(u)$ and $I(u)$ in (2.3) can also be expressed as

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + \frac{\kappa}{2}\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - \frac{1}{p+1}\|g^{\frac{1}{p+1}}u\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})} \\ &= \frac{1}{2}I(u) + \frac{p-1}{2(p+1)}\|g^{\frac{1}{p+1}}u\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})}. \end{aligned} \tag{6.9}$$

From (4.6) and (6.9), it follows that

$$\begin{aligned} \boldsymbol{\Psi_6}''(t) &= -2pI(u(t)) + 2\mu \\ &\geq \frac{2p(p-1)}{p+1}\|g^{\frac{1}{p+1}}u(t)\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})} - 4pJ(u(t)), \end{aligned}$$

which together with (6.3) and (4.7) turns into

$$\begin{aligned} \boldsymbol{\Psi_6}''(t) &\geq \frac{2p(p-1)}{|\mathbb{B}|^{\frac{(p-1)^2}{4}}C_{g_*}^{\frac{p-1}{2}}(p+1)}\|u(t)\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})} - 4pJ(u(t)) \\ &= \frac{2p(p-1)}{|\mathbb{B}|^{\frac{(p-1)^2}{4}}C_{g_*}^{\frac{p-1}{2}}(p+1)}\|u(t)\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})} - 4pJ(u_0) + 4p\int_0^t \|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\mathrm{d}\tau. \end{aligned}$$

From Lemma 6.1 and (4.6) in Lemma 4.2, we know that $\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}$ is strictly increasing on $[0,T)$. Therefore, we obtain

$$\boldsymbol{\Psi_6}''(t) \geq 4p\left(\frac{\xi_2}{4}\|u_0\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})} - J(u_0) + \boldsymbol{\Psi_1}(t)\right) \tag{6.10}$$

and $\mathbf{\Psi_6}'(t) \geq 2\mu(t+\nu) > 0$, which implies $\mathbf{\Psi_6}(t) \geq \mathbf{\Psi_6}(0) = T\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + \mu\nu^2 > 0$ for all $t \in [0,T)$.

On the other hand, we derive

$$\begin{aligned}
-\frac{1}{4}(\mathbf{\Psi_6}'(t))^2 =& -\left(\frac{p}{2}\left(\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - \|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right) + \mu(t+\nu)\right)^2 \\
=& \underbrace{\left(p\int_0^t \|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \mathrm{d}\tau + \mu(t+\nu)^2\right)\left(\int_0^t \|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \mathrm{d}\tau + \mu\right)}_{I_3} \\
& - \underbrace{\left(\frac{p}{2}\left(\|u(t)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - \|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right) + \mu(t+\nu)\right)^2}_{I_4} \\
& - \left(\mathbf{\Psi_6}(t) - p(T-t)\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\left(\int_0^t \|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \mathrm{d}\tau + \mu\right).
\end{aligned}$$

Then we show $I_3 - I_4 \geq 0$ as follows

$$\begin{aligned}
I_3 - I_4 =& I_3 - \left(\frac{p}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \mathrm{d}\tau + \mu(t+\nu)\right)^2 \\
=& I_3 - \left(p\int_0^t (u(\tau), u_t(\tau))_{\mathbb{B}} \mathrm{d}\tau + \mu(t+\nu)\right)^2,
\end{aligned}$$

by using the cone Hölder inequality, which tells

$$\begin{aligned}
& I_3 - I_4 \\
\geq & I_3 - \left(p\int_0^t \|u(\tau)\|_{L_2^{\frac{N}{2}}(\mathbb{B})} \|u_t(\tau)\|_{L_2^{\frac{N}{2}}(\mathbb{B})} \mathrm{d}\tau + \mu(t+\nu)\right)^2 \\
\geq & I_3 - \left(\left(p\int_0^t \|u(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \mathrm{d}\tau\right)^{\frac{1}{2}} \left(p\int_0^t \|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \mathrm{d}\tau\right)^{\frac{1}{2}} + \mu(t+\nu)\right)^2 \\
= & \left(p\mathbf{\Psi_2}(t) + \mu(t+\nu)^2\right)\left(\mathbf{\Psi_1}(t) + \mu\right) - \left((p\mathbf{\Psi_2}(t))^{\frac{1}{2}}(p\mathbf{\Psi_1}(t))^{\frac{1}{2}} + \mu(t+\nu)\right)^2 \\
= & \left(\sqrt{\mu}(p\mathbf{\Psi_2}(t))^{\frac{1}{2}}\right)^2 - 2\sqrt{\mu}(p\mathbf{\Psi_2}(t))^{\frac{1}{2}}\sqrt{\mu}(t+\nu)(\mathbf{\Psi_1}(t))^{\frac{1}{2}} + \left(\sqrt{\mu}(t+\nu)(\mathbf{\Psi_1}(t))^{\frac{1}{2}}\right)^2 \\
= & \left(\sqrt{\mu}(p\mathbf{\Psi_2}(t))^{\frac{1}{2}} - \sqrt{\mu}(t+\nu)(\mathbf{\Psi_1}(t))^{\frac{1}{2}}\right)^2 \\
\geq & 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
-(\mathbf{\Psi_6}'(t))^2 \geq & -4\left(\mathbf{\Psi_6}(t) - p(T-t)\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)\left(\int_0^t \|u_t(\tau)\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \mathrm{d}\tau + \mu\right) \\
\geq & -4\mathbf{\Psi_6}(t)(\mathbf{\Psi_1}(t) + \mu).
\end{aligned} \tag{6.11}$$

Then in view of (6.11) and (6.10), for any $\vartheta > 1$, we achieve

$$\mathbf{\Psi_6}(t)\mathbf{\Psi_6}''(t) - \vartheta(\mathbf{\Psi_6}'(t))^2$$

$$\geq \mathbf{\Psi_6}(t)\left(\mathbf{\Psi_6}''(t) - 4\vartheta\left(\mathbf{\Psi_1}(t) + \mu\right)\right)$$
$$\geq 4\mathbf{\Psi_6}(t)\left(\frac{p\xi_2}{4}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - pJ(u_0) + (p-\vartheta)\mathbf{\Psi_1}(t) - \vartheta\mu\right),$$

taking $\vartheta = p$, we have

$$\mathbf{\Psi_6}(t)\mathbf{\Psi_6}''(t) - \vartheta(\mathbf{\Psi_6}'(t))^2 \geq 4p\mathbf{\Psi_6}(t)\left(\frac{\xi_2}{4}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - J(u_0) - \mu\right).$$

Choose a small enough $\mu' \in (0, \sigma']$ such that

$$\frac{\xi_2}{4}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - J(u_0) - \mu \geq 0,$$

where $\sigma' := \frac{\xi_2}{4}\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} - J(u_0)$. Then we have

$$\mathbf{\Psi_6}(t)\mathbf{\Psi_6}''(t) - \vartheta(\mathbf{\Psi_6}'(t))^2 \geq 0,$$

which implies that the conditions of Lemma 5.1 are satisfied. It is easy to verify that $\mathbf{\Psi_6}(0) = pT\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + \mu\nu^2 > 0$, $\mathbf{\Psi_6}'(0) \geq 2\mu\nu > 0$. The remainder proof is similar as the proof of Theorem 3.2-Part II, hence, we omit it. ∎

As we did in Remark 5.2, Theorem 3.1 and Theorem 3.3 also have the correlations similar to Theorem 3.1 and Theorem 3.2, so we give the following remark.

Remark 6.1. Similar to the phenomenon discussed in Remark 5.2, Theorem 3.1 and Theorem 3.3 also have different emphases on finite time blow up of solutions, and we mainly discuss the overlapping parts of Theorem 3.1 and Theorem 3.3 when considering the finite time blow up of solutions under high initial energy levels. Comparing the sufficient conditions in Theorem 3.1 and Theorem 3.3, we conclude the result through the following simple argument, namely, assuming that the initial energy level satisfies $J(u_0) > d$, we assert that if the initial data satisfies (3.2) in Theorem 3.3, then $u_0 \in \mathcal{S}_B$.

From the proof of Lemma 6.1, we know that if u_0 satisfies (3.2), then $u_0 \in \mathcal{N}_-$. Similar as the proof in Remark 5.2, for any $u \in \mathcal{N}_{J(u_0)}$, together with the definition of $\mathcal{N}_{J(u_0)}$, we get

$$\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + \kappa\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} < \frac{2(p+1)}{p-1}J(u_0).$$

Combining above inequality with (4.5), (6.3), we have

$$\begin{aligned}
\frac{\|u\|^{p+1}_{L_2^{\frac{N}{2}}(\mathbb{B})}}{|\mathbb{B}|^{\frac{(p-1)^2}{4}} C_{g_*}^{\frac{p-1}{2}}} &\leq \|g^{\frac{1}{p+1}}u\|^{p+1}_{L_{p+1}^{\frac{N}{p+1}}(\mathbb{B})} \\
&= \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} + \kappa\|V^{\frac{1}{2}}u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \\
&< \frac{2(p+1)}{p-1}J(u_0).
\end{aligned} \tag{6.12}$$

Picking $\Lambda_{J(u_0)} = \sup\left\{\|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \,\middle|\, u \in \mathcal{N}_{J(u_0)}\right\}$, and (6.12) tells us

$$\Lambda^{\frac{p+1}{2}}_{J(u_0)} = \left(\|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})}\right)^{\frac{p+1}{2}} < \frac{2|\mathbb{B}|^{\frac{(p-1)^2}{4}} C_{g_*}^{\frac{p-1}{2}} J(u_0)(p+1)}{p-1}. \tag{6.13}$$

Substituting (3.2) into (6.13), we get $\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} > \Lambda_{J(u_0)}$. Hence, from above discussion we know that if u_0 satisfies (3.2), then

$$u_0 \in \mathcal{N}_- \cap \left\{ u_0 \in \mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B}) \middle| \|u\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} \geq \Lambda_{J(u_0)}, J(u) < J(u_0) \right\} = \mathcal{S}_B.$$

However, rearranging (3.2), we obtain

$$\|u_0\|^2_{L_2^{\frac{N}{2}}(\mathbb{B})} > \left(\frac{2|\mathbb{B}|^{\frac{(p-1)^2}{4}} C_{g_*}^{\frac{p-1}{2}} J(u_0)(p+1)}{p-1} \right)^{-\frac{p+1}{2}},$$

together with (6.13), we see that if $u_0 \in \mathcal{S}_B$, it does not necessarily satisfy (3.2).

Therefore, compared with (ii) in Theorem 3.1, the sufficient conditions of Theorem 3.3 are stricter, but they also give us a better estimate of the upper bound of blow up time.

7. Proof of Theorem 3.4

In this section we seek the lower bound of blow up time to problem (1.1)-(1.3).

Proof. **(Proof of Theorem 3.4.)** Let

$$\mathbf{\Psi_7}(t) := \int_{\mathbb{B}} |u(t)|^{\epsilon} \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x},$$

where the constant ϵ is an odd number and will be chosen later.

Since u is the solution of (1.1)-(1.3), we have

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{\Psi_7}(t)}{\mathrm{d}t} &= \epsilon \left(|u|^{\epsilon-2}u, u_t \right)_{\mathbb{B}} \\
&= \epsilon \left(|u|^{\epsilon-2}u, \Delta_{\mathbb{B}} u - \kappa V u + g(x)|u|^{p-1}u \right)_{\mathbb{B}} \\
&= \epsilon \Bigg(-(\epsilon-1) \int_{\mathbb{B}} |u|^{\epsilon-2} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x} - \kappa \int_{\mathbb{B}} V|u|^{\epsilon} \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x} \\
&\quad + \int_{\mathbb{B}} g(x)|u|^{\epsilon+p-1} \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x} \Bigg).
\end{aligned}$$

For convenience we now set $v = |u|^{\frac{\epsilon}{2}}$, then

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{\Psi_7}(t)}{\mathrm{d}t} =& -\frac{4(\epsilon-1)}{\epsilon} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} v|^2 \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x} - \kappa\epsilon \int_{\mathbb{B}} V|v|^2 \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x} \\
&+ \epsilon \int_{\mathbb{B}} g(x) v^{2+\frac{2(p-1)}{\epsilon}} \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x}.
\end{aligned} \tag{7.1}$$

By virtue of the cone Hardy's inequality (Lemma 2.1), (7.1) turns into

$$\frac{\mathrm{d}\mathbf{\Psi_7}(t)}{\mathrm{d}t} \leq -\left(\frac{4(\epsilon-1)}{\epsilon} + \kappa C_*^2 \epsilon \right) \int_{\mathbb{B}} |\nabla_{\mathbb{B}} v|^2 \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x} + \epsilon \int_{\mathbb{B}} g(x) v^{2+\frac{2(p-1)}{\epsilon}} \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x}. \tag{7.2}$$

For the second term on the right-hand side of (7.2), we deal with it by using the cone Hölder inequality and imbedding theorem $L^{\infty}(\mathbb{B}) \hookrightarrow L_3^{\frac{N}{3}}(\mathbb{B})$ as follows

$$\int_{\mathbb{B}} g(x) v^{2+\frac{2(p-1)}{\epsilon}} \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x} \leq \left(\int_{\mathbb{B}} g(x)^3 \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x} \right)^{\frac{1}{3}} \left(\int_{\mathbb{B}} v^{3+\frac{3(p-1)}{\epsilon}} \frac{\mathrm{d}x_1}{x_1} \mathrm{d}\widetilde{x} \right)^{\frac{2}{3}}$$

$$\leq \tilde{C}\|g\|_{L^\infty(\mathbb{B})}\int_{\mathbb{B}} v^{2+\frac{2(p-1)}{\epsilon}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x},$$

where $\tilde{C}$ is the optimal imbedding constant of $L^\infty(\mathbb{B})\hookrightarrow L_3^{\frac{N}{3}}(\mathbb{B})$. Together with the fact that $g(x)\in L^\infty(\mathbb{B})\cap C(\mathbb{B})$, it follows that

$$\int_{\mathbb{B}} g(x)v^{2+\frac{2(p-1)}{\epsilon}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\leq \tilde{C}C_g\int_{\mathbb{B}} v^{2+\frac{2(p-1)}{\epsilon}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}, \tag{7.3}$$

where $C_g>0$ is the essential supremum which depends on the weighted function $g(x)$, i.e., $C_g=\inf\{M\,|\,|g(x)|\leq M, \text{a.e. } x\in\mathbb{B}\}$.

Again from the cone Hölder inequality, we know that

$$\int_{\mathbb{B}} v^{2+\frac{2(p-1)}{\epsilon}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\leq \left(\int_{\mathbb{B}} v^4\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{1}{3}}\left(\int_{\mathbb{B}} v^{1+\frac{3(p-1)}{\epsilon}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{2}{3}}, \tag{7.4}$$

then applying the cone Sobolev inequality

$$\int_{\mathbb{B}} v^4\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\leq S^4\int_{\mathbb{B}}|\nabla_{\mathbb{B}}v|^4\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x} \tag{7.5}$$

and substituting (7.4) and (7.5) into (7.3), we see

$$\int_{\mathbb{B}} g(x)v^{2+\frac{2(p-1)}{\epsilon}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\leq \tilde{C}C_gS^{\frac{4}{3}}\left(\int_{\mathbb{B}}|\nabla_{\mathbb{B}}v|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{2}{3}}\left(\int_{\mathbb{B}} v^{1+\frac{3(p-1)}{\epsilon}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{2}{3}}, \tag{7.6}$$

where S is the optimal constant of the cone Sobolev embedding $\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})\hookrightarrow L_4^{\frac{N}{4}}(\mathbb{B})$.

By reusing the cone Hölder inequality and recalling $v=|u|^{\frac{\epsilon}{2}}$, we can easily get

$$\int_{\mathbb{B}} v^{1+\frac{3(p-1)}{\epsilon}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}=\int_{\mathbb{B}}|u|^{\frac{\epsilon+3(p-1)}{2}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\leq|\mathbb{B}|^{1-\delta}(\mathbf{\Psi_7}(t))^{\delta}, \tag{7.7}$$

where $|\mathbb{B}|$ denotes the volume of $\mathbb{B}$ and $\delta:=\frac{1}{2}+\frac{3(p-1)}{2\epsilon}$. In particular, the constant ϵ needs to satisfy $\epsilon>3(p-1)$ in order to ensure $\delta<1$.

Hence, substituting (7.7) into (7.6), we arrive at

$$\begin{aligned}\int_{\mathbb{B}} g(x)v^{2+\frac{2(p-1)}{\epsilon}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}&\leq \tilde{C}C_gS^{\frac{4}{3}}|\mathbb{B}|^{\frac{2}{3}(1-\delta)}\left(\int_{\mathbb{B}}|\nabla_{\mathbb{B}}v|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{2}{3}}(\mathbf{\Psi_7}(t))^{\frac{2\delta}{3}}\\&:=K\left(\int_{\mathbb{B}}|\nabla_{\mathbb{B}}v|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{2}{3}}(\mathbf{\Psi_7}(t))^{\frac{2\delta}{3}}.\end{aligned} \tag{7.8}$$

Using the following Young's inequality

$$X^rY^s\leq rX+sY \text{ for } r+s=1, X,Y\geq 0,$$

we can rewrite (7.8) with a parameter $\theta>0$ as follows

$$\begin{aligned}\int_{\mathbb{B}} g(x)v^{2+\frac{2(p-1)}{\epsilon}}\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}&\leq K\left(\theta\int_{\mathbb{B}}|\nabla_{\mathbb{B}}v|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}\right)^{\frac{2}{3}}\left(\theta^{-2}(\mathbf{\Psi_7}(t))^{2\delta}\right)^{\frac{1}{3}}\\&\leq\frac{2K\theta}{3}\int_{\mathbb{B}}|\nabla_{\mathbb{B}}v|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}+\frac{K\theta^{-2}}{3}(\mathbf{\Psi_7}(t))^{2\delta}.\end{aligned} \tag{7.9}$$

Combining (7.8), (7.9) with (7.2), we obtain

$$\frac{\mathrm{d}\mathbf{\Psi_7}(t)}{\mathrm{d}t} \leq (\theta\eta_2 - \eta_1)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} v(t)|^2 \frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x} + \frac{\theta^{-2}\eta_2}{2}\left(\mathbf{\Psi_7}(t)\right)^{2\delta},$$

where

$$\begin{cases} \eta_1 := \frac{4(\epsilon-1)}{\epsilon} + \kappa C_*^2\epsilon, \\ \eta_2 := \frac{2}{3}\epsilon\tilde{C}C_g S^{\frac{4}{3}}|\mathbb{B}|^{\frac{2}{3}(1-\delta)}. \end{cases}$$

Choosing θ to make the coefficient of $\int_{\mathbb{B}} |\nabla_{\mathbb{B}} v|^2 \frac{\mathrm{d}x_1}{x_1}\mathrm{d}\widetilde{x}$ vanish, we reach to

$$\frac{\mathrm{d}\mathbf{\Psi_7}(t)}{\mathrm{d}t} \leq \gamma(\mathbf{\Psi_7}(t))^{2\delta}, \tag{7.10}$$

where $\gamma := \frac{\theta^{-2}\eta_2}{2}$. Or upon integration we have for $t < T_*$

$$\frac{1}{\mathbf{\Psi_7}(0)^{2\delta-1}} - \frac{1}{\mathbf{\Psi_7}(t)^{2\delta-1}} \leq \gamma(2\delta-1)t.$$

So that letting $t \to T_*$, we conclude that

$$T_* \geq \frac{1}{\gamma(2\delta-1)(\mathbf{\Psi_7}(0))^{2\delta-1}} = \frac{54(4\epsilon-4+\kappa C_*^2\epsilon^2)^2}{8\epsilon^5\tilde{C}^3C_g^3S^4|\mathbb{B}|^{2(1-\delta)}(2\delta-1)\|u_0\|_{L_\epsilon^N}^{\epsilon(2\delta-1)}}.$$

Moreover, integrating (7.10) from t to T_*, we obtain

$$T_* - t \geq \int_{\mathbf{\Psi_7}(t)}^{\infty} \frac{\mathrm{d}\theta}{\gamma\theta^{2\delta}} := F(\mathbf{\Psi_7}(t)). \tag{7.11}$$

Obviously, $F(\mathbf{\Psi_7}(t))$ is a decreasing function of $\mathbf{\Psi_7}(t)$, which means its inverse function F^{-1} exists and it is also a decreasing one. Therefore, we know

$$\mathbf{\Psi_7}(t) \geq F^{-1}(T_* - t),$$

which implies the existence of the lower bound of blowup rate. By calculating the generalized integral in (7.11), we deduce that

$$T_* - t \geq \frac{1}{\gamma(2\delta-1)}\left(\mathbf{\Psi_7}(t)\right)^{1-2\delta},$$

which yields

$$\mathbf{\Psi_7}(t) \geq \left(\gamma(2\delta-1)(T_*-t)\right)^{\frac{1}{1-2\delta}}.$$

That is,

$$\|u(t)\|_{L_\epsilon^N} \geq \left(\frac{8\epsilon^5\tilde{C}^3C_g^3S^4|\mathbb{B}|^{2(1-\delta)}(2\delta-1)}{54(4\epsilon-4+\kappa C_*^2\epsilon^2)^2}\right)^{\frac{1}{\epsilon(1-2\delta)}} (T_*-t)^{\frac{1}{\epsilon(1-2\delta)}}.$$

□

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