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Partial Differential Equations

Asymptotics for the blow-up boundary solution of the logistic equation with absorption

Comportement asymptotique de la solution explosant au bord de l'équation logistique avec absorption

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Abstract

Let Ω be a smooth bounded domain in $\mathbf{R}^N$. Assume that $f \geqslant 0$ is a C^1-function on $[0, \infty)$ such that $f(u)/u$ is increasing on $(0, +\infty)$. Let a be a real number and let $b \geqslant 0$, $b \not\equiv 0$ be a continuous function such that $b \equiv 0$ on $\partial\Omega$. The purpose of this Note is to establish the asymptotic behaviour of the unique positive solution of the logistic problem $\Delta u + au = b(x)f(u)$ in Ω, subject to the singular boundary condition $u(x) \to +\infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$. Our analysis is based on the Karamata regular variation theory. ***To cite this article: F.-C. Cîrstea, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Soit Ω un domaine borné et régulier de $\mathbf{R}^N$. On suppose que $f \in C^1[0, \infty)$ est $\geqslant 0$ et telle que $f(u)/u$ soit strictement croissante sur $(0, +\infty)$. Soit a un réel et $b \geqslant 0$, $b \not\equiv 0$, une fonction continue sur $\overline{\Omega}$ telle que $b \equiv 0$ sur $\partial\Omega$. Dans cette Note on établit le comportement asymptotique de l'unique solution positive du problème logistique $\Delta u + au = b(x)f(u)$ sur Ω avec la donnée au bord singulière $u(x) \to +\infty$ si $\operatorname{dist}(x, \partial\Omega) \to 0$. Notre analyse porte sur la théorie de la variation régulière de Karamata. ***Pour citer cet article : F.-C. Cîrstea, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit $\Omega \subset \mathbf{R}^N$ $(N \geqslant 3)$ un domaine borné et régulier, a un paramètre réel et $0 \not\equiv b \in C^{0,\mu}(\overline{\Omega})$, $b \geqslant 0$ dans Ω. On considère le problème logistique avec explosion au bord

$$\Delta u + au = b(x)f(u) \quad \text{dans } \Omega, \qquad u(x) \to +\infty \quad \text{si } d(x) := \operatorname{dist}(x, \partial\Omega) \to 0, \tag{1}$$

où $f \in C^1[0, \infty)$ est $\geqslant 0$ et satisfait la condition de Keller–Osserman (voir [6,7]) et telle que $f(u)/u$ soit strictement croissante sur $(0, +\infty)$. Soit $\Omega_0 := \operatorname{int}\{x \in \Omega\colon\ b(x) = 0\}$. On suppose que $\partial\Omega_0$ est régulier

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(éventuellement vide), $\overline{\Omega}_0 \subset \Omega$ et $b > 0$ sur $\Omega \setminus \overline{\Omega}_0$. On désigne par $\lambda_{\infty,1}$ la première valeur propre de l'opérateur $(-\Delta)$ dans $H_0^1(\Omega_0)$, avec la convention $\lambda_{\infty,1} = +\infty$ si $\Omega_0 = \emptyset$. Dans [2] on montre que le problème (1) admet une solution positive u_a si et seulement si $a < \lambda_{\infty,1}$. L'unicité de la solution u_a est établie dans [1]. Soit $\mathcal{K}$ l'ensemble des fonctions $k:(0,\nu) \to (0,\infty)$ (pour un certain ν), de classe C^1, croissantes, telles que $\lim_{t\searrow 0}(\int_0^t k(s)\,\mathrm{d}s/k(t))^{(i)} := \ell_i$, pour $i = \overline{0,1}$.

Soit RV_q ($q \in \mathbf{R}$) l'ensemble des fonctions positives et mesurables $Z:[A,\infty) \to \mathbf{R}$ (avec $A > 0$) telles que $\lim_{u\to\infty} Z(\xi u)/Z(u) = \xi^q$, pour tout $\xi > 0$. On désigne par NRV_q la classe des fonctions f définies par $f(u) = Cu^q \exp\{\int_B^u \phi(t)/t\,\mathrm{d}t\}$, pour tout $u \geqslant B > 0$, où $C > 0$ et $\phi \in C[B,\infty)$ satisfait $\lim_{t\to\infty}\phi(t) = 0$. Supposons que $0 \leqslant f \in C^1[0,\infty) \cap NRV_{\rho+1}$ ($\rho > 0$) est telle que $f(u)/u$ soit strictement croissante sur $(0,\infty)$ et que $b \equiv 0$ sur $\partial\Omega$ vérifie $b(x) = k^2(d)(1 + \mathrm{o}(1))$ si $d(x) \to 0$, avec $k \in \mathcal{K}$. Alors, pour chaque $a < \lambda_{\infty,1}$, le problème (1) admet une unique solution positive u_a (voir [1]). Le but de cette Note est d'établir la vitesse d'explosion au bord de la solution u_a.

Pour chaque $\zeta > 0$, soit

$$\mathcal{R}_{0,\zeta} = \left\{ \begin{matrix} k: & k(u^{-1}) = d_0 u[\Lambda(u)]^{-1}\exp\left[-\int_{d_1}^u (s\Lambda(s))^{-1}\,\mathrm{d}s\right]\ (u \geqslant d_1),\ 0 < \Lambda \in C^1[d_1,\infty), \\ & \lim_{u\to\infty}\Lambda(u) = \lim_{u\to\infty} u\Lambda'(u) = 0,\ \lim_{u\to\infty} u^{\zeta+1}\Lambda'(u) = \ell_\star \in \mathbf{R},\ d_0,\ d_1 > 0 \end{matrix} \right\}.$$

On a $\mathcal{R}_{0,\zeta} \subset \mathcal{K}$. De plus, si $k \in \mathcal{R}_{0,\zeta}$ alors $\ell_1 = 0$ et $\lim_{t\to 0} k(t) = 0$.

On définit les classes $\mathcal{F}_{\rho\eta} = \{f \in NRV_{\rho+1}(\rho > 0)$: $\phi \in RV_\eta$ ou $-\phi \in RV_\eta\}$, si $\eta \in (-\rho-2, 0]$ et $\mathcal{F}_{\rho 0,\tau} = \{f \in \mathcal{F}_{\rho 0}$: $\lim_{u\to\infty}(\ln u)^\tau \phi(u) = \ell^\star \in \mathbf{R}\}$, pour $\tau \in (0,\infty)$.

On démontre le résultat suivant.

Théorème 1. *On suppose que* $b(x) = k^2(d)(1 + \tilde{c}d^\theta + \mathrm{o}(d^\theta))$ *si* $d(x) \to 0$ (*avec* $\theta > 0$, $\tilde{c} \in \mathbf{R}$), *où* $k \in \mathcal{R}_{0,\zeta}$. *Soit* $f \in C^1[0,\infty)$ *telle que* $f \geqslant 0$ *et* $f(u)/u$ *soit strictement croissante sur* $(0,\infty)$. *De plus, on suppose que* f *satisfait l'une des conditions suivantes de croissance à l'infini* : (i) $f(u) = Cu^{\rho+1}$ *dans un voisinage de l'infini* ; (ii) $f \in \mathcal{F}_{\rho\eta}$ *avec* $\eta \neq 0$; (iii) $f \in \mathcal{F}_{\rho 0,\tau_1}$ *avec* $\tau_1 = \varpi/\zeta$, *où* $\varpi = \min\{\theta,\zeta\}$.

Alors, pour tout $a \in (-\infty, \lambda_{\infty,1})$, *l'unique solution positive* u_a *du problème* (1) *satisfait*

$$u_a(x) = \xi_0 h(d)\big(1 + \chi d^\varpi + \mathrm{o}(d^\varpi)\big) \quad \text{si } d(x) \to 0,$$

où $\xi_0 = [2(2+\rho)^{-1}]^{1/\rho}$ *et* h *est définie par* $\int_{h(t)}^\infty [2F(s)]^{-1/2}\,\mathrm{d}s = \int_0^t k(s)\,\mathrm{d}s$, *pour* $t > 0$ *suffisament petit. L'expression de* χ *est donnée par*

$$\chi = \begin{cases} -(1+\zeta)\ell_\star(2\zeta)^{-1}\mathrm{Heaviside}(\theta-\zeta) - \tilde{c}\rho^{-1}\mathrm{Heaviside}(\zeta-\theta) = \chi_1 & \text{dans les cas (i) et (ii)}, \\ \chi_1 - \ell^\star\rho^{-1}(-\rho\ell_\star/2)^{\tau_1}(1/(\rho+2) + \ln\xi_0) & \text{pour le cas (iii)}. \end{cases}$$

Notons que le seul cas lié à ce résultat et correspondant à la situation particulière $\Omega_0 = \emptyset$, $f(u) = u^{\rho+1}$, $k(t) = ct^\alpha \in \mathcal{K}$ (avec $c, \alpha > 0$), $\theta = 1$, a été étudié dans [4]. Dans ce travail, les deux premiers termes du développement asymptotique de u_a autour de $\partial\Omega$ tiennent compte de $d(x)$ ainsi que de la courbure moyenne H de $\partial\Omega$. Dans notre approche, on n'a pas besoin de la restriction $b > 0$ dans Ω et on garde la condition $b \equiv 0$ sur $\partial\Omega$, comme restriction naturelle héritée du problème logistique (voir [4]). De plus, on améliore la vitesse d'explosion de u_a pour une large classe de potentiels b, avec $\theta > 0$ quelconque et k appartenant à un riche ensemble de fonctions.

Let $\Omega \subset \mathbf{R}^N$ ($N \geqslant 3$) be a smooth bounded domain. Consider the blow-up logistic problem

$$\Delta u + au = b(x)f(u) \quad \text{in } \Omega, \qquad u(x) \to +\infty \quad \text{as } d(x) := \mathrm{dist}(x, \partial\Omega) \to 0, \tag{1}$$

where $f \in C^1[0,\infty)$, a is a real parameter and $0 \not\equiv b \in C^{0,\mu}(\overline{\Omega})$ (for some $\mu \in (0,1)$) satisfies $b \geqslant 0$ in Ω. Suppose that the absorption term f fulfills both

(A) $f \geqslant 0$ and $f(u)/u$ is increasing on $(0,\infty)$

and the Keller–Osserman condition (see [6,7]) $\int_1^\infty [F(t)]^{-1/2}\,\mathrm{d}t < \infty$, where $F(t) = \int_0^t f(s)\,\mathrm{d}s$.

Assume throughout that $\Omega_0 \Subset \Omega$ satisfies the exterior cone condition (possibly, $\Omega_0 = \emptyset$) and $b > 0$ on $\Omega \setminus \overline{\Omega}_0$, where $\Omega_0 := \operatorname{int}\{x \in \Omega\colon b(x) = 0\}$. Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of $(-\Delta)$ in $H_0^1(\Omega_0)$. Set $\lambda_{\infty,1} = +\infty$ if $\Omega_0 = \emptyset$. Under the above assumptions, we have proved in [2] that (1) has a positive solution u_a if and only if $a < \lambda_{\infty,1}$. Moreover, the uniqueness of u_a is studied in [1]. Denote by $\mathcal{K}$ the set of all positive increasing C^1-functions k defined on $(0, \nu)$, for some $\nu > 0$, which satisfy $\lim_{t\searrow 0}(\int_0^t k(s)\,ds/k(t))^{(i)} := \ell_i$, $i \in \overline{0,1}$. We have $\ell_0 = 0$ and $\ell_1 \in [0, 1]$.

Let us now recall some basic definitions related to the Karamata regular variation theory (see [5,8]). Let RV_q ($q \in \mathbf{R}$) be the set of all positive measurable functions $Z\colon [A,\infty) \to \mathbf{R}$ (for some $A > 0$) satisfying $\lim_{u\to\infty} Z(\xi u)/Z(u) = \xi^q$, $\forall \xi > 0$. Define by NRV_q the class of functions f in the form $f(u) = Cu^q \exp\{\int_B^u \phi(t)/t\,dt\}$, $\forall u \geqslant B > 0$, where $C > 0$ is a constant and $\phi \in C[B,\infty)$ satisfies $\lim_{t\to\infty}\phi(t) = 0$. The Karamata Representation Theorem shows that $NRV_q \subset RV_q$.

If $f \in NRV_{\rho+1}$ ($\rho > 0$) satisfies (A) and $b \equiv 0$ on $\partial\Omega$ such that $b(x) = k^2(d)(1 + o(1))$ as $d(x) \to 0$, for some $k \in \mathcal{K}$, then for any $a \in (-\infty, \lambda_{\infty,1})$, there is a unique positive solution u_a of Eq. (1). Note that the Keller–Osserman condition is automatically fulfilled. Moreover, we have $\lim_{u\to\infty} \Xi(u) = \lim_{u\to\infty}[F(u)]^{1/2}[f(u) \times \int_u^\infty (F(s))^{-1/2}\,ds]^{-1} = \rho[2(\rho+2)]^{-1}$ (see [1]).

We have seen in [1] that the uniqueness of u_a is essentially based on the same boundary behaviour shown by any positive solution of (1). The purpose of this Note is to refine the blow-up rate of u_a near $\partial\Omega$ by giving the second term in the expansion of u_a near $\partial\Omega$. This is a more subtle question which represents the goal of more recent literature (see [4] and the references therein). The approach we give is very general and, as a novelty, it relies on the theory of regular variation instituted in the 1930s by Karamata and subsequently developed by himself and many others (see [5,8]). For any $\zeta > 0$, set $\mathcal{K}_{0,\zeta}$ the subset of $\mathcal{K}$ with $\ell_1 = 0$ and $\lim_{t\searrow 0} t^{-\zeta}(\int_0^t k(s)\,ds/k(t))' := L_\star \in \mathbf{R}$. It can be proven that $\mathcal{K}_{0,\zeta} \equiv \mathcal{R}_{0,\zeta}$, where

$$\mathcal{R}_{0,\zeta} = \left\{\begin{array}{ll} k\colon & k(u^{-1}) = d_0 u[\Lambda(u)]^{-1}\exp\left[-\int_{d_1}^u (s\Lambda(s))^{-1}\,ds\right]\ (u \geqslant d_1),\ 0 < \Lambda \in C^1[d_1,\infty), \\ & \lim_{u\to\infty}\Lambda(u) = \lim_{u\to\infty} u\Lambda'(u) = 0,\ \lim_{u\to\infty} u^{\zeta+1}\Lambda'(u) = \ell_\star \in \mathbf{R},\ d_0,\ d_1 > 0 \end{array}\right\}.$$

Moreover, $\ell_\star$ and $L_\star$ are connected by $L_\star = -(1+\zeta)\ell_\star/\zeta$ (see [3] for details). Define

$$\mathcal{F}_{\rho\eta} = \left\{f \in NRV_{\rho+1}\ (\rho > 0)\colon\ \phi \in RV_\eta \text{ or } -\phi \in RV_\eta\right\}, \quad \eta \in (-\rho-2, 0];$$

$$\mathcal{F}_{\rho 0,\tau} = \left\{f \in \mathcal{F}_{\rho 0}\colon \lim_{u\to\infty}(\ln u)^\tau \phi(u) = \ell^\star \in \mathbf{R}\right\}, \quad \tau \in (0,\infty).$$

Our main result establishes the following asymptotic estimate.

Theorem 1. *Assume that*

$$b(x) = k^2(d)\big(1 + \tilde{c}d^\theta + o\big(d^\theta\big)\big) \quad \text{if } d(x) \to 0, \text{ where } k \in \mathcal{R}_{0,\zeta},\ \theta > 0,\ \tilde{c} \in \mathbf{R}. \tag{2}$$

Suppose that f fulfills (A) *and one of the following growth conditions at infinity*: (i) $f(u) = Cu^{\rho+1}$ *in a neighbourhood of infinity*; (ii) $f \in \mathcal{F}_{\rho\eta}$ *with* $\eta \neq 0$; (iii) $f \in \mathcal{F}_{\rho 0,\tau_1}$ *with $\tau_1 = \varpi/\zeta$, where $\varpi = \min\{\theta, \zeta\}$.*

Then, for any $a \in (-\infty, \lambda_{\infty,1})$, the unique positive solution u_a of (1) *satisfies*

$$u_a(x) = \xi_0 h(d)\big(1 + \chi d^\varpi + o\big(d^\varpi\big)\big) \quad \text{if } d(x) \to 0, \text{ where } \xi_0 = \left[2(2+\rho)^{-1}\right]^{1/\rho} \tag{3}$$

and h is defined by $\int_{h(t)}^\infty [2F(s)]^{-1/2}\,ds = \int_0^t k(s)\,ds$, for $t > 0$ small enough. The expression of χ is

$$\chi = \begin{cases} -(1+\zeta)\ell_\star(2\zeta)^{-1}\,\text{Heaviside}(\theta-\zeta) - \tilde{c}\rho^{-1}\,\text{Heaviside}(\zeta-\theta) := \chi_1 & \text{if (i) or (ii) holds,} \\ \chi_1 - \ell^\star\rho^{-1}(-\rho\ell_\star/2)^{\tau_1}[1/(\rho+2) + \ln\xi_0] & \text{if } f \text{ obeys (iii).} \end{cases}$$

Note that the only case related, in same way, to our Theorem 1 corresponds to $\Omega_0 = \emptyset$, $f(u) = u^{\rho+1}$ on $[0,\infty)$, $k(t) = ct^\alpha \in \mathcal{K}$ (where $c, \alpha > 0$), $\theta = 1$ in (2), being studied in [4]. There, the two-term asymptotic expansion of u_a near $\partial\Omega$ ($a \in \mathbf{R}$ since $\lambda_{\infty,1} = \infty$) involves both the distance function $d(x)$ and the mean curvature H of $\partial\Omega$.

However, the blow-up rate of u_a we present in Theorem 1 is of a different nature since the class $\mathcal{R}_{0,\zeta}$ does not include $k(t) = ct^\alpha$.

Our main result contributes to the knowledge in some new directions. More precisely, the blow-up rate of the unique positive solution u_a of (1) (found in [1]) is here refined

(a) on the maximal interval $(-\infty, \lambda_{\infty,1})$ for the parameter a, which is in connection with an appropriate semilinear eigenvalue problem; thus, the condition $b > 0$ in Ω (which appears in [4]) is removed by defining the set Ω_0, but we maintain $b \equiv 0$ on $\partial\Omega$ since this is a *natural* restriction inherited from the logistic problem (see [4] for details);

(b) when b satisfies (2), where θ is *any* positive number and k belongs to a very rich class of functions, namely $\mathcal{R}_{0,\zeta}$. The equivalence $\mathcal{R}_{0,\zeta} \equiv \mathcal{K}_{0,\zeta}$ shows the connection to the larger class $\mathcal{K}$ (introduced in [1]) for which the uniqueness of u_a holds. In addition, the explicit form of $k \in \mathcal{R}_{0,\zeta}$ shows us how to built $k \in \mathcal{K}_{0,\zeta}$;

(c) for a wide class of functions $f \in NRV_{\rho+1}$ where either $\phi \equiv 0$ (case (i)) or ϕ (resp., $-\phi$) belongs to RV_η with $\eta \in (-\rho-2, 0]$ (cases (ii) and (iii)). Therefore, the theory of regular variation plays a key role in understanding the general framework and the approach as well.

Proof of Theorem 1. We first state two auxiliary results (see [3] for their proofs).

Lemma 1. *Assume* (2) *and* $f \in NRV_{\rho+1}$ *satisfies* (A). *Then h has the following properties*:

(i) $h \in C^2(0,\nu)$, $\lim_{t\searrow 0} h(t) = \infty$ *and* $\lim_{t\searrow 0} h'(t) = -\infty$;
(ii) $\lim_{t\searrow 0} h''(t)/[k^2(t) f(h(t)\xi)] = (2+\rho\ell_1)/[\xi^{\rho+1}(2+\rho)]$, $\forall \xi > 0$;
(iii) $\lim_{t\searrow 0} h(t)/h''(t) = \lim_{t\searrow 0} h'(t)/h''(t) = \lim_{t\searrow 0} h(t)/h'(t) = 0$;
(iv) $\lim_{t\searrow 0} h'(t)/[th''(t)] = -\rho\ell_1/(2+\rho\ell_1)$ *and* $\lim_{t\searrow 0} h(t)/[t^2 h''(t)] = \rho^2\ell_1^2/[2(2+\rho\ell_1)]$;
(v) $\lim_{t\searrow 0} h(t)/[th'(t)] = \lim_{t\searrow 0}[\ln t]/[\ln h(t)] = -\rho\ell_1/2$;
(vi) *If* $\ell_1 = 0$, *then* $\lim_{t\searrow 0} t^j h(t) = \infty$, *for all* $j > 0$;
(vii) $\lim_{t\searrow 0} 1/[t^\zeta \ln h(t)] = -\rho\ell_\star/2$ *and* $\lim_{t\searrow 0} h'(t)/[t^{\zeta+1}h''(t)] = \rho\ell_\star/(2\zeta)$, $\forall k \in \mathcal{R}_{0,\zeta}$.

Let $\tau > 0$ be arbitrary. For any $u > 0$, define $T_{1,\tau}(u) = \{\rho/[2(\rho+2)] - \Xi(u)\}(\ln u)^\tau$ and $T_{2,\tau}(u) = \{f(\xi_0 u)/[\xi_0 f(u)] - \xi_0^\rho\}(\ln u)^\tau$. Note that if $f(u) = Cu^{\rho+1}$, for u in a neighbourhood V_∞ of infinity, then $T_{1,\tau}(u) = T_{2,\tau}(u) = 0$ for each $u \in V_\infty$.

Lemma 2. *Assume* (A) *and* $f \in \mathcal{F}_{\rho\eta}$. *The following hold*: (i) *If* $f \in \mathcal{F}_{\rho 0,\tau}$, *then* $\lim_{u\to\infty} T_{1,\tau}(u) = -\ell^\star/(\rho+2)^2$ *and* $\lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \ell^\star \ln \xi_0$. (ii) *If* $f \in \mathcal{F}_{\rho\eta}$ *with* $\eta \neq 0$, *then* $\lim_{u\to\infty} T_{1,\tau}(u) = \lim_{u\to\infty} T_{2,\tau}(u) = 0$.

Fix $\varepsilon \in (0, 1/2)$. We can find $\delta > 0$ such that $d(x)$ is of class C^2 on $\{x \in \mathbf{R}^N\colon d(x) < \delta\}$, k is nondecreasing on $(0,\delta)$, and $h'(t) < 0 < h''(t)$ for all $t \in (0,\delta)$ (see [1] for details). A straightforward computation shows that $\lim_{t\searrow 0} t^{1-\theta}k'(t)/k(t) = \infty$, for every $\theta > 0$. Using now (2), it follows that we can diminish $\delta > 0$ such that $k^2(t)[1 + (\tilde{c} - \varepsilon)t^\theta]$ is increasing on $(0,\delta)$ and

$$1 + (\tilde{c} - \varepsilon)d^\theta < b(x)/k^2(d) < 1 + (\tilde{c} + \varepsilon)d^\theta, \quad \forall x \in \Omega \text{ with } d \in (0,\delta). \tag{4}$$

We define $u^\pm(x) = \xi_0 h(d)(1 + \chi_\varepsilon^\pm d^\varpi)$, with $d \in (0,\delta)$, where $\chi_\varepsilon^\pm = \chi \pm \varepsilon[1 + \text{Heaviside}(\zeta - \theta)]/\rho$. Take $\delta > 0$ small enough such that $u^\pm(x) > 0$, for each $x \in \Omega$ with $d \in (0,\delta)$. By the Lagrange mean value theorem, we obtain $f(u^\pm(x)) = f(\xi_0 h(d)) + \xi_0 \chi_\varepsilon^\pm d^\varpi h(d) f'(\Upsilon^\pm(d))$, where $\Upsilon^\pm(d) = \xi_0 h(d)(1 + \lambda^\pm(d)\chi_\varepsilon^\pm d^\varpi)$, for some $\lambda^\pm(d) \in [0,1]$. We claim that

$$\lim_{d\searrow 0} f\big(\Upsilon^\pm(d)\big)/f\big(\xi_0 h(d)\big) = 1. \tag{5}$$

Fix $\sigma \in (0,1)$ and $M > 0$ such that $|\chi_\varepsilon^\pm| < M$. Choose $\mu^\star > 0$ so that $|(1 \pm Mt)^{\rho+1} - 1| < \sigma/2$, for all $t \in (0, 2\mu^\star)$. Let $\mu_\star \in (0, (\mu^\star)^{1/\varpi})$ be such that, for every $x \in \Omega$ with $d \in (0,\mu_\star)$

$$\big|f\big(\xi_0 h(d)(1 \pm M\mu^\star)\big)/f\big(\xi_0 h(d)\big) - (1 \pm M\mu^\star)^{\rho+1}\big| < \sigma/2.$$

Hence, $1-\sigma < (1-M\mu^\star)^{\rho+1}-\sigma/2 < f(\Upsilon^\pm(d))/f(\xi_0 h(d)) < (1+M\mu^\star)^{\rho+1}+\sigma/2 < 1+\sigma$, for every $x\in\Omega$ with $d\in(0,\mu_\star)$. This proves (5).

Step 1. There exists $\delta_1\in(0,\delta)$ so that $\Delta u^+ + au^+ - k^2(d)[1+(\tilde c-\varepsilon)d^\theta]f(u^+)\leqslant 0$, $\forall x\in\Omega$ with $d\in(0,\delta_1)$ and $\Delta u^- + au^- - k^2(d)[1+(\tilde c+\varepsilon)d^\theta]f(u^-)\geqslant 0$, $\forall x\in\Omega$ with $d\in(0,\delta_1)$.

Indeed, for every $x\in\Omega$ with $d\in(0,\delta)$, we have

$$\Delta u^\pm + au^\pm - k^2(d)\big[1+(\tilde c\mp\varepsilon)d^\theta\big]f(u^\pm) = \xi_0 d^\varpi h''(d)\bigg[a\chi_\varepsilon^\pm\frac{h(d)}{h''(d)} + \chi_\varepsilon^\pm\Delta d\,\frac{h'(d)}{h''(d)} + 2\varpi\chi_\varepsilon^\pm\frac{h'(d)}{dh''(d)}$$
$$+\varpi\chi_\varepsilon^\pm\Delta d\,\frac{h(d)}{dh''(d)} + \varpi(\varpi-1)\chi_\varepsilon^\pm\frac{h(d)}{d^2h''(d)} + \Delta d\,\frac{h'(d)}{d^\varpi h''(d)} + \frac{a\,h(d)}{d^\varpi h''(d)} + \sum_{j=1}^4\mathcal S_j^\pm(d)\bigg] \tag{6}$$

where, for any $t\in(0,\delta)$, we denote

$$\mathcal S_1^\pm(t) = (-\tilde c\pm\varepsilon)t^{\theta-\varpi}k^2(t)f\big(\xi_0 h(t)\big)\big/\big[\xi_0 h''(t)\big],\quad \mathcal S_2^\pm(t) = \chi_\varepsilon^\pm\big(1-k^2(t)h(t)f'\big(\Upsilon^\pm(t)\big)\big/h''(t)\big),$$
$$\mathcal S_3^\pm(t) = (-\tilde c\pm\varepsilon)\chi_\varepsilon^\pm t^\theta k^2(t)h(t)f'\big(\Upsilon^\pm(t)\big)\big/h''(t),\quad \mathcal S_4^\pm(t) = t^{-\varpi}\big(1-k^2(t)f\big(\xi_0 h(t)\big)\big/\big[\xi_0 h''(t)\big]\big).$$

By Lemma 1(ii), we find $\lim_{t\searrow 0}k^2(t)f(\xi_0 h(t))[\xi_0 h''(t)]^{-1}=1$, which yields $\lim_{t\searrow 0}\mathcal S_1^\pm(t) = (-\tilde c\pm\varepsilon)\times$ Heaviside$(\zeta-\theta)$. Using [1, Lemma 1] and (5), we obtain $\lim_{t\searrow 0}k^2(t)h(t)f'(\Upsilon^\pm(t))/h''(t)=\rho+1$. Hence, $\lim_{t\searrow 0}\mathcal S_2^\pm(t) = -\rho\chi_\varepsilon^\pm$ and $\lim_{t\searrow 0}\mathcal S_3^\pm(t)=0$.

Using the expression of h'', we derive $\mathcal S_4^\pm(t) = \frac{k^2(t)f(h(t))}{h''(t)}\sum_{i=1}^3\mathcal S_{4,i}(t)$, $\forall t\in(0,\delta)$, where we denote $\mathcal S_{4,1}(t) = 2\frac{\Xi(h(t))}{t^\varpi}(\int_0^t k(s)\,\mathrm ds/k(t))'$, $\mathcal S_{4,2}(t) = 2\frac{T_{1,\tau_1}(h(t))}{[t^\zeta\ln h(t)]^{\tau_1}}$ and $\mathcal S_{4,3}(t) = -\frac{T_{2,\tau_1}(h(t))}{[t^\zeta\ln h(t)]^{\tau_1}}$.

Since $\mathcal R_{0,\zeta}\equiv\mathcal K_{0,\zeta}$, we find $\lim_{t\searrow 0}\mathcal S_{4,1}(t) = -(1+\zeta)\rho\ell_\star\zeta^{-1}(\rho+2)^{-1}$ Heaviside$(\theta-\zeta)$.

Cases (i), (ii). By Lemmas 1(vii) and 2(ii), we find $\lim_{t\searrow 0}S_{4,2}(t) = \lim_{t\searrow 0}S_{4,3}(t)=0$. In view of Lemma 1(ii), we derive that $\lim_{t\searrow 0}S_4^\pm(t) = -(1+\zeta)\rho\ell_\star(2\zeta)^{-1}$ Heaviside$(\theta-\zeta)$.

Case (iii). By Lemmas 1(vii) and 2(i), $\lim_{t\searrow 0}S_{4,2}(t) = -2\ell^\star(\rho+2)^{-2}(-\rho\ell_\star/2)^{\tau_1}$ and $\lim_{t\searrow 0}S_{4,3}(t) = -2\ell^\star(\rho+2)^{-1}(-\rho\ell_\star/2)^{\tau_1}\ln\xi_0$. Using Lemma 1(ii) once more, we arrive at $\lim_{t\searrow 0}S_4^\pm(t) = -(1+\zeta)\rho\ell_\star(2\zeta)^{-1}\times$ Heaviside$(\theta-\zeta) - \ell^\star(-\rho\ell_\star/2)^{\tau_1}[1/(\rho+2)+\ln\xi_0]$.

Note that in each of the cases (i)–(iii), the definition of $\chi_\varepsilon^\pm$ yields $\lim_{t\searrow 0}\sum_{j=1}^4\mathcal S_j^+(t) = -\varepsilon<0$ and $\lim_{t\searrow 0}\sum_{j=1}^4\mathcal S_j^-(t)=\varepsilon>0$. By Lemma 1(vii), $\lim_{t\searrow 0}\frac{h'(t)}{t^\varpi h''(t)}=0$. But $\lim_{t\searrow 0}\frac{h(t)}{h'(t)}=0$, so $\lim_{t\searrow 0}\frac{h(t)}{t^\varpi h''(t)}=0$. Thus, using Lemma 1(iii) and (iv), relation (6) concludes our Step 1.

Step 2. There exists $M^+,\delta^+>0$ such that $u_a(x)\leqslant u^+(x)+M^+$, for all $x\in\Omega$ with $0<d<\delta^+$.

Define $(0,\infty)\ni u\mapsto\Psi_x(u) = au - b(x)f(u)$, $\forall x$ with $d\in(0,\delta_1)$. Clearly, $\Psi_x(u)$ is decreasing when $a\leqslant 0$. Suppose $a\in(0,\lambda_{\infty,1})$. Obviously, $f(t)/t:(0,\infty)\to(f'(0),\infty)$ is bijective. Let $\delta_2\in(0,\delta_1)$ be such that $b(x)<1$, $\forall x$ with $d\in(0,\delta_2)$. Let u_x define the unique positive solution of $b(x)f(u)/u = a+f'(0)$, $\forall x$ with $d\in(0,\delta_2)$. Hence, for any x with $d\in(0,\delta_2)$, $u\to\Psi_x(u)$ is decreasing on (u_x,∞). But $\lim_{d(x)\searrow 0}\frac{b(x)f(u^+(x))}{u^+(x)}=+\infty$ (use $\lim_{d(x)\searrow 0}u^+(x)/h(d)=\xi_0$, (A) and Lemma 1(ii) and (iii)). So, for δ_2 small enough, $u^+(x)>u_x$, $\forall x$ with $d\in(0,\delta_2)$.

Fix $\sigma\in(0,\delta_2/4)$ and set $\mathcal N_\sigma := \{x\in\Omega\colon\ \sigma<d(x)<\delta_2/2\}$. We define $u_\sigma^*(x) = u^+(d-\sigma,s)+M^+$, where (d,s) are the local coordinates of $x\in\mathcal N_\sigma$. We choose $M^+>0$ large enough to have $u_\sigma^*(\delta_2/2,s)\geqslant u_a(\delta_2/2,s)$, $\forall\sigma\in(0,\delta_2/4)$ and $\forall s\in\partial\Omega$. Using (4) and Step 1, we find

$$\begin{aligned}-\Delta u_\sigma^*(x) &\geqslant au^+(d-\sigma,s) - \big[1+(\tilde c-\varepsilon)(d-\sigma)^\theta\big]k^2(d-\sigma)f\big(u^+(d-\sigma,s)\big)\\ &\geqslant au^+(d-\sigma,s) - \big[1+(\tilde c-\varepsilon)d^\theta\big]k^2(d)f\big(u^+(d-\sigma,s)\big)\geqslant\Psi_x\big(u^+(d-\sigma,s)\big)\\ &\geqslant\Psi_x(u_\sigma^*) = au_\sigma^*(x) - b(x)f\big(u_\sigma^*(x)\big)\quad\text{in }\mathcal N_\sigma.\end{aligned}$$

Thus, by [2, Lemma 1], $u_a\leqslant u_\sigma^*$ in $\mathcal N_\sigma$, $\forall\sigma\in(0,\delta_2/4)$. Letting $\sigma\to 0$, we have proved Step 2.

Step 3. There exists $M^-, \delta^- > 0$ such that $u_a(x) \geqslant u^-(x) - M^-$, for all $x \in \Omega$ with $0 < d < \delta^-$.

For every $r \in (0, \delta)$, define $\Omega_r = \{x \in \Omega\colon\ 0 < d(x) < r\}$. We will prove that for $\lambda > 0$ sufficiently small, $\lambda u^-(x) \leqslant u_a(x)$, $\forall x \in \Omega_{\delta_2/4}$. Indeed, fix arbitrarily $\sigma \in (0, \delta_2/4)$. Define $v_\sigma^*(x) = \lambda u^-(d+\sigma, s)$, for $x = (d, s) \in \Omega_{\delta_2/2}$. We choose $\lambda \in (0,1)$ small enough such that $v_\sigma^*(\delta_2/4, s) \leqslant u_a(\delta_2/4, s)$, $\forall \sigma \in (0, \delta_2/4)$, $\forall s \in \partial\Omega$. Using (4), Step 1 and (A), we find

$$\begin{aligned}\Delta v_\sigma^*(x) + a v_\sigma^*(x) &\geqslant \lambda k^2(d+\sigma)\bigl[1 + (\tilde{c}+\varepsilon)(d+\sigma)^\theta\bigr] f\bigl(u^-(d+\sigma, s)\bigr)\\ &\geqslant k^2(d)\bigl[1 + (\tilde{c}+\varepsilon)d^\theta\bigr] f\bigl(\lambda u^-(d+\sigma, s)\bigr) \geqslant b f(v_\sigma^*),\end{aligned}$$

for all $x = (d, s) \in \Omega_{\delta_2/4}$, that is v_σ^* is a subsolution of $\Delta u + au = b(x) f(u)$ in $\Omega_{\delta_2/4}$. By [2, Lemma 1], we conclude that $v_\sigma^* \leqslant u_a$ in $\Omega_{\delta_2/4}$. Letting $\sigma \to 0$, we find $\lambda u^-(x) \leqslant u_a(x)$, $\forall x \in \Omega_{\delta_2/4}$.

Since $\lim_{d\searrow 0} u^-(x)/h(d) = \xi_0$, by using (A) and Lemma 1 (ii) and (iii), we can easily obtain $\lim_{d\searrow 0} k^2(d) f(\lambda^2 u^-(x))/u^-(x) = \infty$. So, there exists $\tilde{\delta} \in (0, \delta_2/4)$ such that

$$k^2(d)\bigl[1 + (\tilde{c}+\varepsilon)d^\theta\bigr] f\bigl(\lambda^2 u^-\bigr)/u^- \geqslant \lambda^2 |a|, \quad \forall x \in \Omega \text{ with } 0 < d \leqslant \tilde{\delta}. \tag{7}$$

By Lemma 1(i) and (v), we deduce that $u^-(x)$ decreases with d when $d \in (0, \tilde{\delta})$ (if necessary, $\tilde{\delta} > 0$ is diminished). Choose $\delta_* \in (0, \tilde{\delta})$, close enough to $\tilde{\delta}$, such that

$$h(\delta_*)\bigl(1 + \chi_\varepsilon^- \delta_*^\varpi\bigr) \big/ \bigl[h(\tilde{\delta})\bigl(1 + \chi_\varepsilon^- \tilde{\delta}^\varpi\bigr)\bigr] < 1 + \lambda. \tag{8}$$

For each $\sigma \in (0, \tilde{\delta} - \delta_*)$, we define $z_\sigma(x) = u^-(d+\sigma, s) - (1-\lambda)u^-(\delta_*, s)$. We prove that z_σ is a subsolution of $\Delta u + au = b(x) f(u)$ in Ω_{δ_*}. Using (8), $z_\sigma(x) \geqslant u^-(\tilde{\delta}, s) - (1-\lambda)u^-(\delta_*, s) > 0$ $\forall x = (d, s) \in \Omega_{\delta_*}$. By (4) and Step 1, z_σ is a subsolution of $\Delta u + au = b(x) f(u)$ in Ω_{δ_*} if

$$k^2(d+\sigma)\bigl[1 + (\tilde{c}+\varepsilon)(d+\sigma)^\theta\bigr]\bigl[f\bigl(u^-(d+\sigma, s)\bigr) - f\bigl(z_\sigma(d, s)\bigr)\bigr] \geqslant a(1-\lambda)u^-(\delta_*, s), \tag{9}$$

for all $(d, s) \in \Omega_{\delta_*}$. Applying the Lagrange mean value theorem and (A), we infer that (9) is a consequence of $k^2(d+\sigma)[1 + (\tilde{c}+\varepsilon)(d+\sigma)^\theta]\, f(z_\sigma(d, s))/z_\sigma(d, s) \geqslant |a|$, $\forall (d, s) \in \Omega_{\delta_*}$. This inequality holds by virtue of (7), (8) and the decreasing character of u^- with d.

On the other hand, $z_\sigma(\delta_*, s) \leqslant \lambda u^-(\delta_*, s) \leqslant u_a(x)$, $\forall x = (\delta_*, s) \in \Omega$. Clearly, $\limsup_{d\to 0}(z_\sigma - u_a)(x) = -\infty$ and $b > 0$ in Ω_{δ_*}. Thus, by [2, Lemma 1], $z_\sigma \leqslant u_a$ in Ω_{δ_*}, $\forall \sigma \in (0, \tilde{\delta} - \delta_*)$. Letting $\sigma \to 0$, we conclude the assertion of Step 3.

By Steps 2 and 3, $\chi_\varepsilon^+ \geqslant \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} - M^+/[\xi_0 d^\varpi h(d)]$ $\forall x \in \Omega$ with $d \in (0, \delta^+)$ and $\chi_\varepsilon^- \leqslant \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} + M^-/[\xi_0 d^\varpi h(d)]$ $\forall x \in \Omega$ with $d \in (0, \delta^-)$. Passing to the limit as $d \to 0$ and using Lemma 1(vi), we obtain $\chi_\varepsilon^- \leqslant \liminf_{d\to 0}\{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi}$ and $\limsup_{d\to 0}\{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} \leqslant \chi_\varepsilon^+$. Letting $\varepsilon \to 0$, we conclude our proof. □

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