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Normalized solutions for (p, q)-Laplacian equations with mass supercritical growth

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Abstract

In this paper, we study the following (p, q)-Laplacian equation with L^p-constraint:

$$\begin{cases} -\Delta_p u - \Delta_q u + \lambda |u|^{p-2} u = f(u), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p \, dx = c^p, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases}$$

where $1 < p < q < N$, $\Delta_i = \operatorname{div}(|\nabla u|^{i-2} \nabla u)$, with $i \in \{p, q\}$, is the i-Laplacian operator, λ is a Lagrange multiplier and $c > 0$ is a constant. The nonlinearity f is assumed to be continuous and satisfying weak mass supercritical conditions. The purpose of this paper is twofold: to establish the existence of ground states, and to reveal the basic behavior of the ground state energy E_c as $c > 0$ varies. Moreover, we introduce a new approach based on the direct minimization of the energy functional on the linear combination of Nehari and Pohozaev constraints intersected with the closed ball of radius c^p in $L^p(\mathbb{R}^N)$. The analysis developed in this paper allows to provide the general growth assumptions imposed to the reaction f.

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1. Introduction

In this paper, we consider the following (p, q)-Laplacian equation with L^p-constraint:

$$\begin{cases} -\Delta_p u - \Delta_q u + \lambda |u|^{p-2}u = f(u), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p \, dx = c^p, \\ u \in E, \end{cases} \tag{1.1}$$

where $1 < p < q < N$, $\Delta_i = \operatorname{div}(|\nabla u|^{i-2}\nabla u)$, with $i \in \{p, q\}$, is the i-Laplacian operator, $\lambda \in \mathbb{R}$ is a Lagrange multiplier, $c > 0$ is a given constant, $f \in C(\mathbb{R}, \mathbb{R})$, $E := W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$.

The features of problem (1.1) are the following:

(i) The presence of two differential operators with different growth, which generates a double phase associated energy.
(ii) The problem combines the effects generated by a general nonlinearity and an unbalanced operator.
(iii) Due to the unboundedness of the domain, the Palais-Smale sequences do not have the compactness property.

Since the content of the paper is closely concerned with unbalanced growth, we briefly introduce in what follows the related background and applications and we recall some pioneering contributions to these fields. Equation (1.1) is driven by a differential operator with unbalanced growth due to the presence of the (p, q)-Laplace operator. This type of problem comes from a general reaction-diffusion system:

$$u_t = \operatorname{div}[A(\nabla u)\nabla u] + c(x, u), \quad \text{and} \quad A(\nabla u) = |\nabla u|^{p-2} + |\nabla u|^{q-2},$$

where the function u is a state variable and describes the density or concentration of multicomponent substances, $\operatorname{div}[A(\nabla u)\nabla u]$ corresponds to the diffusion with coefficient $A(\nabla u)$ and $c(x, u)$ is the reaction and relates to source and loss processes. Originally, the idea to treat such operators comes from Zhikov [49] who introduced such classes to provide models of strongly anisotropic materials, see also the monograph of Zhikov et al. [50]. We refer to the remarkable works initiated by Marcellini [31–33], where the author investigated the regularity and existence of solutions of elliptic equations with unbalanced growth conditions. The (p, q)-Laplacian equation (1.1) is also motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born-Infeld equation [15] that appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell's Lagrangian density:

$$-\operatorname{div}\left(\frac{\nabla u}{(1 - 2|\nabla u|^2)^{\frac{1}{2}}}\right) = h(u) \quad \text{in } \Omega.$$

Indeed, by the Taylor formula, we have

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \frac{3}{2\cdot 2^2}x^2 + \frac{5!!}{3!\cdot 2^3}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)!\cdot 2^{n-1}}x^{n-1} + \cdots \quad \text{for } |x| < 1.$$

Taking $x = 2|\nabla u|^2$ and adopting the first order approximation, we obtain problem (1.1) for $p = 2$ and $q = 4$. Furthermore, the n-th order approximation problem is driven by the multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

We also refer to the following fourth-order relativistic operator

$$u \mapsto \operatorname{div}\left(\frac{|\nabla u|^2}{(1-|\nabla u|^4)^{\frac{3}{4}}}\nabla u\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Again, by Taylor's formula, we have

$$x^2(1-x^4)^{-\frac{3}{4}} = x^2 + \frac{3x^6}{4} + \frac{21x^{10}}{32} + \cdots.$$

This shows that the fourth-order relativistic operator can be approximated by the following operator

$$u \mapsto \Delta_4 u + \frac{3}{4}\Delta_8 u.$$

For more details on the physical backgrounds and other applications, we refer to Bahrouni et al. [6] (for phenomena associated with transonic flows) and to Benci et al. [11] (for models arising in quantum physics).

In the past few decades, equation (1.1) has been the subject of extensive mathematical studies. Using various variational and topological arguments, many authors studied the existence and multiplicity results of nontrivial solutions, ground state solutions, nodal solutions and some qualitative properties of solutions, respectively. We refer to [19,35,38] for the case of bounded domains. In this classical setting, we recall the seminal papers by Ni et al. [36], Li et al. [28], del Pino et al. [16,17] and Ambrosetti et al. [5]. The regularity results, existence and multiplicity of solutions to problem (1.1) on the whole space can be found in [3,22,47].

The study developed in this paper is inspired by the interest of physicists to the existence of normalized solutions. Indeed, prescribed mass appears in nonlinear optics and in the theory of Bose-Einstein condensates, see [18,30] and the reference therein. In particular, when $p = 2$, $q = 0$, $f(u)$ is replaced by $|u|^{l-2}u$, equation (1.1) is reduced to the following semilinear elliptic equation

$$-\Delta u = \lambda u + |u|^{l-2}u, \quad (\lambda, u) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.2}$$

In the L^2-subcritical case, namely $l < 2\left(1+\frac{2}{N}\right)$, the functional on the constraint is coercive. Hence one can obtain the existence of a global minimizer by minimizing on the sphere, cf. [29,42]. In the L^2-subcritical case, that is $l > 2\left(1+\frac{2}{N}\right)$, the functional on the sphere could not be bounded from below. But one of the main difficulties in dealing with normalized solutions as critical points of a functional constrained to a sphere consists in proving the Palais-Smale condition. Jeanjean [23] overcame this problem in the L^2-supercritical case by using a mountain pass structure for an auxiliary functional proving the existence of at least one normalized solution of (1.2). More precisely, Jeanjean studied the following equation

$$-\Delta u + \lambda u = f(u), \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

We recall below the conditions introduced there.

(H_0) $f : \mathbb{R} \to \mathbb{R}$ is continuous and odd.
(H_1) There exist $\alpha, \beta \in \mathbb{R}$ satisfying $2 + \frac{4}{N} < \alpha \le \beta < 2^*$ such that

$$0 < \alpha F(t) \le f(t)t \le \beta F(t) \quad \text{for any } t \in \mathbb{R}\backslash\{0\},$$

where $2^* := \frac{2N}{N-2}$ for $N \ge 3$ and $2^* := +\infty$ when $N = 1, 2$, $F(t) := \int_0^t f(\tau)\, d\tau$.
(H_2) The function $\tilde{F}(t) := f(t)t - 2F(t)$ is of class C^1 and satisfies

$$\tilde{F}'(t)t > \left(2 + \frac{4}{N}\right)\tilde{F}(t) \quad \text{for any } t \neq 0.$$

In [23], under the conditions (H_0) and (H_1), Jeanjean obtained a radial normalized solution at a mountain pass value when $N \ge 2$. Moreover, when (H_2) is also assumed, the existence of normalized ground states was proved in any dimension $N \ge 1$. Recently, Jeanjean and Lu [25] made a more in-depth study of (1.3) in the mass supercritical case. First, they relaxed some of the classical growth assumptions on f. In particular, the first part of (H_1) i.e.

$$\text{there exists } \alpha > 2 + \frac{4}{N} \text{ such that } 0 < \alpha F(t) \le f(t)t \text{ for any } t \neq 0 \tag{1.4}$$

was used in a technical but essential way not only in showing that the problem is mass supercritical but also in obtaining bounded constrained Palais-Smale sequences. They showed that under a weak and more natural mass supercritical condition. Consequently, they managed to extend the previous results on the existence of normalized ground states and the multiplicity of radial normalized solutions. Furthermore, they address new issues, such as the monotonicity of the ground state energy as a function of L^2 constraint constant or the existence of infinitely many nonradial sign-changing solutions. In the last, they stressed that all of their results were obtained only assuming that the nonlinearity f, as any function built on f, is continuous. Similar to the results in [25], Bieganowski and Mederski [14] introduced a new view point to the problem of the existence of ground state on $\bar{S}_m$ by searching a global minimum for the energy functional on

$$\bar{\mathcal{P}} := \left\{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \|u\|^2_{L^2(\mathbb{R}^N)} \le m \text{ and } \bar{P}(u) = 0\right\},$$

where

$$\bar{P}(u) := \int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}(u)\, dx, \quad \bar{S}_m := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2\, dx = m \right\}.$$

This interesting approach relies on stronger regularity assumptions and in particular the function $\tilde{F}$ needs to be of class C^1. So the approach of [14] did not permit to recover the results of [25] in full generality. For other relevant results on the normalized solutions of elliptic equation, see [9,10,26]. When it comes to combined nonlinearities, the following works deserve to be highlighted. Soave in [39] first studied the following nonlinear Schrödinger equation

$$-\Delta u + \lambda u = |u|^{p-2}u + \mu |u|^{q-2}u, \quad \text{in} \quad \mathbb{R}^N. \tag{1.5}$$

The author considered the existence and nonexistence of the normalized solution for equation (1.5) with $\mu \in \mathbb{R}$ and combined power nonlinearities $2 < q \le 2 + \frac{4}{N} \le p < 2^*$ with $2^* := \frac{2N}{N-2}$ if $N \ge 3$ and $2^* := +\infty$ if $N = 1, 2$, where $N \ge 1$ and made pioneering work by using the variational method and Pohozaev constraint. In particular, when $2 < q < 2 + \frac{4}{N} < p < 2^*$, the author obtained the existence of two solutions (local minimizer and Moutain-Pass type) for equation (1.5). Furthermore, the author got the orbital stability of the ground state set when $q = 2 + \frac{4}{N} < p < 2^*$. Later, Soave in [40] further studied the existence and nonexistence of the normalized solution for equation (1.5) with $\mu \in \mathbb{R}$, $p = 2^* = \frac{2N}{N-2}$ and $q \in (2, 2^*)$ where $N \ge 3$ by using the similar technique in [39]. However, when $2 < q < 2 + \frac{4}{N} < p = 2^*$, the author obtained only the existence of local minimizer for equation (1.5). It is worth mentioning that the existence of the second normalized solution (Mountain-Pass type) for equation (1.5) in $N \ge 3$ is given by [24,46].

Nowadays, to our best knowledge, when $p \ne 2$, $q = 0$, $f(u)$ is replaced by $|u|^{l-2}u$, there are few results on the following p-Laplacian equation

$$-\Delta_p u = \lambda |u|^{p-2}u + |u|^{l-2}u. \tag{1.6}$$

In particular, when $|u|^{l-2}u$ is $g(x,t)$ with $g(x,t)$ is L^p-subcritical in the sense that

$$\lim_{|t|\to+\infty} \frac{g(x,t)}{|t|^{\tilde{p}-1}} = 0$$

holds uniformly for $x \in \mathbb{R}^N$, where $\tilde{p} := \frac{p^2}{N} + p$. Li and Yan [27] obtained the existence of normalized ground state solutions. In [21], Gu et al. proved the existence of normalized ground state solutions with a trapping potential for (1.6) in case of $l = \tilde{p}$. Recently, Zhang and Zhang [48] considered the following p-Laplacian equation with a L^p-norm constraint:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \mu |u|^{q-2}u + g(u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p\, dx = a^p, \end{cases} \tag{1.7}$$

where $N > 1$, $a > 0$, $1 < p < q \le \tilde{p}$, $\mu \in \mathbb{R}$, $g \in C(\mathbb{R}, \mathbb{R})$. Assume that g is odd and L^p-supercritical. When $q < \tilde{p}$ and $\mu > 0$, using Schwarz rearrangement and Ekeland variational principle, they proved the existence of positive radial ground states for suitable μ. When $q = \tilde{p}$ and $\mu > 0$ or $q \le \tilde{p}$ and $\mu \le 0$, with an additional condition of g, they proved a positive radial ground state if μ lies in a suitable range by the Schwarz rearrangement and minimax theorems. Via a fountain theorem type argument, with suitable $\mu \in \mathbb{R}$, they showed the existence of infinitely many radial solutions for any $N \geq 2$ and the existence of infinitely many nonradial sign-changing solutions for $N = 4$ or $N \geq 6$. In addition, Baldelli and Yang in [7] were concerned with the existence of normalized solutions to the following $(2, q)$-Laplacian equation in all possible cases according to the value of p with respect to the critical exponent $2\left(1 + \frac{2}{N}\right)$

$$\begin{cases} -\Delta u - \Delta_q u = \lambda u + |u|^{p-2}u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2\, dx = c^2. \end{cases} \tag{1.8}$$

In the L^2-subcritical case, they studied a global minimization problem and obtained a ground state solution. While in the L^2-critical case, they proved several nonexistence results, extended also in the L^q-critical case. For the L^2-supercritical case, they derived a ground state and infinitely many radial solutions.

Inspired by the above literature, we want to study the existence of normalized solutions to the (p, q)-Laplacian equation (1.1) with L^p-constraint. Under mild conditions on $f \in C(\mathbb{R}, \mathbb{R})$, we can introduce the C^1 functional

$$I(u) := \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\, dx - \int_{\mathbb{R}^N} F(u)\, dx$$

on E, where $F(t) := \int_0^t f(\tau)\, d\tau$ for $t \in \mathbb{R}$. For any $c > 0$, we let

$$S_c := \left\{ u \in E : \int_{\mathbb{R}^N} |u|^p\, dx = c^p \right\}.$$

Obviously, solutions to (1.1) correspond to critical points of the functional I constrained to the sphere S_c. if u is a solution to equation (1.1), then the following Nehari identity holds

$$\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \int_{\mathbb{R}^N} |\nabla u|^q\, dx + \lambda \int_{\mathbb{R}^N} |u|^p\, dx = \int_{\mathbb{R}^N} f(u)u\, dx. \tag{1.9}$$

Here by a ground state it is intended a solution u to (1.1) that minimizes the functional I among all the solutions to (1.1):

$$dI|_{S_c}(u) = 0 \quad \text{and} \quad I(u) = \inf\left\{ I(v) : dI|_{S_c}(v) = 0 \right\}.$$

From [7], we also know that if u is a solution to equation (1.1), then u satisfies the following Pohozaev identity

$$\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla u|^q\, dx + \frac{\lambda N}{p}\int_{\mathbb{R}^N} |u|^p\, dx = N\int_{\mathbb{R}^N} F(u)\, dx. \tag{1.10}$$

Combining with (1.9), (1.10), we obtain that solution u satisfies

$$P(u)=0,$$

where

$$P(u):=\int_{\mathbb{R}^N}|\nabla u|^p\,dx+\left(\frac{N}{p}-\frac{N}{q}+1\right)\int_{\mathbb{R}^N}|\nabla u|^q\,dx-\frac{N}{p}\int_{\mathbb{R}^N}\bar{F}(u)\,dx$$

and $\bar{F}(u)=f(u)u-pF(u)$.

To find the normalized solutions of equation (1.1), for given $c>0$, we identify the suspected ground state energy

$$E_c:=\inf_{u\in\mathcal{P}_c}I(u), \tag{1.11}$$

where $\mathcal{P}_c$ is the Pohozaev manifold defined by

$$\mathcal{P}_c:=\{u\in S_c:P(u)=0\}.$$

Throughout this paper, we introduce some relevant results about the Sobolev spaces. For $p\in(1,\infty)$ and $N>p$, we define $D^{1,p}(\mathbb{R}^N)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to $\|\nabla u\|_p:=\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{\frac{1}{p}}$. Let $W^{1,p}(\mathbb{R}^N)$ be the usual Sobolev space endowed with the standard norm $\|u\|_{W^{1,p}(\mathbb{R}^N)}:=\left(\int_{\mathbb{R}^N}|\nabla u|^p+|u|^p\,dx\right)^{\frac{1}{p}}$. For equation (1.1), we introduce the working space E endowed with the norm

$$\|u\|_{p,q}:=\|u\|_{W^{1,p}(\mathbb{R}^N)}+\|u\|_{W^{1,q}(\mathbb{R}^N)}.$$

Next, we need to give the well-known Sobolev embedding theorem and Gagliardo–Nirenberg inequality.

Lemma 1.1. *[1] Let $N>p$. There exists a constant $S>0$ such that, for any $u\in D^{1,p}(\mathbb{R}^N)$,*

$$\|u\|_{p^*}^p\le S^{-1}\|\nabla u\|_p^p.$$

Moreover, $W^{1,p}(\mathbb{R}^N)$ is embedded continuously into $L^m(\mathbb{R}^N)$ for any $m\in[p,p^*]$ and compactly into $L^m_{loc}(\mathbb{R}^N)$ for any $m\in[1,p^*)$, where $p^*:=\frac{Np}{N-p}$.

Lemma 1.2. *[3] The space E is embedded continuously into $L^m(\mathbb{R}^N)$ for $m\in[p,q^*]$ and compactly into $L^m_{loc}(\mathbb{R}^N)$ for $m\in[1,q^*)$.*

Lemma 1.3. *[2,37] The following results hold:*

(i) *Let $m\in(p,p^*)$. There exists a sharp constant $C_{N,m}>0$ such that*

$$\|u\|_m\le C_{N,m}\|\nabla u\|_p^{\delta_m}\|u\|_p^{1-\delta_m},\quad \forall u\in W^{1,p}(\mathbb{R}^N), \tag{1.12}$$

where $\delta_m:=\frac{N}{p}-\frac{N}{m}$.

(ii) *Let* $1 < q < N$ *and* $1 \leq p < m < q^*$. *Then there exists a sharp constant* $K_{N,m} > 0$ *such that*

$$\|u\|_m \leq K_{N,m} \|\nabla u\|_q^{\gamma_m} \|u\|_p^{1-\gamma_m}, \quad \forall u \in E, \tag{1.13}$$

where $\gamma_m := \frac{Nq(m-p)}{m[Nq-p(N-q)]}$.

From Lemma 1.3, combining with the definition of energy functional $I(u)$, we know that $\bar{p} := \frac{pq}{N} + q$ is mass critical exponent to equation (1.1).

Before stating the main results of this paper, we present our conditions on f.

- (f_1) $\lim_{t\to 0} \frac{f(t)}{|t|^{\bar{p}-1}} = 0$ and $\lim_{t\to\infty} \frac{f(t)}{|t|^{p^*-1}} = 0$, where $\bar{p} < p^*$.
- (f_2) $\lim_{t\to\infty} \frac{F(t)}{|t|^{\bar{p}}} = +\infty$.
- (f_3) $t \mapsto \frac{\bar{F}(t)}{|t|^{\bar{p}}}$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$.
- (f_4) $f(t)t < p^* F(t)$ for all $t \in \mathbb{R}\backslash\{0\}$.
- (f_5) $\lim_{t\to 0} \frac{f(t)t}{|t|^{p^*}} = +\infty$.

Conditions (f_1) and (f_2) show that (1.1) is Sobolev subcritical but mass supercritical. Hypotheses (f_3)-(f_5) play a crucial role in ensuring the Lagrange multipliers are positive and guaranteeing that certain bounded Palais-Smale sequences are strongly convergent up to a subsequence and up to translations if necessary.

As an example of the nonlinearity that fulfills (f_1)-(f_5), setting $\alpha_{N,p} = \frac{p^2}{N(N-p)}$, we get the odd continuous function

$$f(t) := \left[\bar{p} \ln(1 + |t|^{\alpha_{N,p}}) + \frac{\alpha_{N,p} |t|^{\alpha_{N,p}}}{1 + |t|^{\alpha_{N,p}}} \right] |t|^{\bar{p}-2} t$$

with the primitive function $F(t) := |t|^{\bar{p}} \ln(1 + |t|^{\alpha_{N,p}})$.

Inspired by [8], we introduce some analytical techniques. To be more precise, for any $u \neq 0$ and $s \in \mathbb{R}$, let $(s * u)(x) := e^{\frac{Ns}{p}} u(e^s x)$ for almost everywhere $x \in \mathbb{R}^N$ and define the free functional

$$\Psi_u(s) := I(s * u) = \frac{1}{p} e^{ps} \|\nabla u\|_p^p + \frac{1}{q} e^{q(\delta_q+1)s} \|\nabla u\|_q^q - e^{-Ns} \int_{\mathbb{R}^N} F\left(e^{\frac{Ns}{p}} u\right) dx$$

on $E\backslash\{0\}$. We shall see that critical points of Ψ_u allow to project a function on the Pohozaev manifold $\mathcal{P}_c$. Thus, the properties of Ψ_u strongly affect the structure of $\mathcal{P}_c$, which will be reflected in subsequent proofs.

The main results read as follows.

Theorem 1.4. *Assume that* $1 < p < q < N$ *and* $f \in C(\mathbb{R}, \mathbb{R})$ *satisfies* (f_1)-(f_4). *Then equation* (1.1) *admits a ground state for any* $c > 0$ *with the associated Lagrange multiplier* $\lambda > 0$. *Moreover, when* f *is odd, equation* (1.1) *admits a positive ground state for any* $c > 0$ *with the associated Lagrange multiplier* $\lambda > 0$.

Theorem 1.5. *Assume that* $1 < p < q < N$ *and* $f \in C(\mathbb{R}, \mathbb{R})$ *satisfies* (f_1)-(f_3). *Then the function* $c \mapsto E_c$ *is positive, continuous, nonincreasing and* $\lim_{c\to 0^+} E_c = +\infty$. *Moreover, when* f *also satisfies* (f_4) *and* (f_5), *the following results hold:*

(i) E_c *is strictly decreasing in* $c > 0$.
(ii) $\lim_{c\to\infty} E_c = 0$.

Remark 1.6. Let us explain the strategy for the proof of Theorems 1.4 and 1.5. First, we show that the Pohozaev manifold $\mathcal{P}_c$ is nonempty and the ground state energy $E_c > 0$. Since $\mathcal{P}_c$ contains all the possible critical points of I restricted to S_c, Our main task is to show that E_c is a critical level of $I|_{S_c}$. In this process, we try to construct a bounded Palais-Smale sequence of $I|_{S_c}$ at the level E_c and deal with the lack of compactness. In particular, the compactness would be proved with aid of the monotonicity of the function $c \mapsto E_c$. So the study of the behavior of the function $c \mapsto E_c$ arises as a fundamental problem. We will develop robust arguments which can be used to treat other constrained problems in general mass supercritical settings.

Remark 1.7. To our best knowledge, it seems to be the first work on the existence of normalized solutions for the (p, q)-Laplacian equation with general nonlinearities. Compared with [25,48], the appearance of two differential operators with different growth will affect the geometry of the problem. We need to introduce the new Pohozaev manifold and use new analysis tools to judge the change of the energy of the (p, q)-Laplacian equation with respect to L^p-constraint constant c. In addition, compared with [7], we consider the general nonlinearity f, which satisfies weak mass supercritical conditions. This leads us to estimate the Pohozaev equality and describe the relationship between the general nonlinearity term and the local term in more detail.

Remark 1.8. Because of the unbounded domain, the main difficulty we encounter in proving the existence of normalized solutions is the lack of compactness. Since the embedding $E \hookrightarrow L^\nu(\mathbb{R}^N)$ is not compact with $\nu \in (p, q^*)$. Then we need to rely on the monotonicity of energy E_c to equation (1.1) and some compactness lemmas to over this difficulty.

Compared with the conditions (f_1)-(f_5), inspired by [14], although we need $\bar{F}$ to be of C^1-class, the more general growth conditions are given as follows. We want to study the existence of normalized ground state solutions from a different point of view than Theorems 1.4 and 1.5.

Theorem 1.9. *Assume that* $1 < p < q < N$ *and* $f \in C(\mathbb{R}, \mathbb{R})$ *satisfies*

(g_1) $\bar{F}'(u)$ *are continuous and there exists* $C > 0$ *such that*

$$|\bar{F}'(t)| \leq C(|t|^{p-1} + |t|^{p^*-1}) \quad \textit{for } t \in \mathbb{R}.$$

(g_2) $\limsup_{|t|\to 0} \frac{F(t)}{|t|^{\bar{p}}} < +\infty$.
(g_3) $\lim_{|t|\to\infty} \frac{F(t)}{|t|^{\bar{p}}} = \infty$.
(g_4) $\lim_{|t|\to\infty} \frac{F(t)}{|t|^{p^*}} = 0$, *where* $\bar{p} < p^*$.
(g_5) $\bar{p}\bar{F}(t) < \bar{F}'(t)t$ *for* $t \in \mathbb{R}$.
(g_6) $(\bar{p} - p)F(t) \leq \bar{F}(t) \leq (p^* - p)F(t)$ *for* $t \in \mathbb{R}$.
(g_7) $\bar{F}(\xi_0) > 0$ *for some* $\xi_0 \neq 0$.

Then there exists $u \in \mathcal{M}_c$ such that $\bar{E}_c := \inf_{\mathcal{M}_c} I > 0$ and $\inf_{\mathcal{P}_c} I = \inf_{\mathcal{M}_c} I$. Moreover, if f is odd, then $u \in \mathcal{P}_c$ is a positive, radially symmetric normalized ground state solution to (1.1), where

$$\mathcal{M}_c := \{u \in A_c : P(u) = 0\}, \quad A_c := \left\{u \in E : \|u\|_p^p \le c^p\right\}.$$

Remark 1.10. Note that (g_1) implies that $I(u)$ and $P(u)$ are of class C^1. Furthermore, assuming in addition (g_3) and (g_6) hold, $F(u) > 0$, $\bar{F}(u) > 0$ for $u \neq 0$ and (g_7) hold. Observe that (g_2) admits L^p-critical growth of $F(u)$ close to 0, but (g_3) excludes the pure L^p-critical case. In addition, (g_4) excludes the Sobolev critical case.

In order to illustrate Theorem 1.9, we provide the following examples and properties with regard to our assumptions (g_1)-(g_7). Suppose that f satisfies (g_1)-(g_7) and f is odd, e.g. $F(u) = \frac{1}{m}|u|^m$ with $\bar{p} < m < p^*$. Then f is of class $\mathcal{C}^1$ on $(-\infty, 0) \cup (0, \infty)$ and note that $f'(\zeta) > 0$ for some $\zeta > 0$. On the one hand, we assume for simplicity $\zeta = 1$. Then we define $\check{f} : \mathbb{R} \mapsto \mathbb{R}$ such that $\check{f}(0) = 0$ and

$$\check{f}'(t) := \begin{cases} f'(1)|t|^{p^*-2} & \text{if } |t| \le 1 \\ f'(t) & \text{if } |t| > 1. \end{cases}$$

Hence $\check{F}(u) = \int_0^u \check{f}(s)\,ds$ and $\bar{\check{F}}(u) := \check{f}(u)u - p\check{F}(u)$ satisfy (g_1)-(g_7). On the other hand, we observe that $\check{F}(u) = \xi|u|^{\bar{p}} + F(u)$, $\xi \ge 0$ and $\bar{\check{F}}(u) := \check{f}(u)u - p\check{F}(u)$ satisfy (g_1)-(g_7). In particular, we can deal with the case of $f(u) = \xi|u|^{\bar{p}-2}u + |u|^{m-2}u$, $\bar{p} < m < p^*$.

Remark 1.11. Compared with Theorems 1.4 and 1.5, we consider the minimization problem on the closed L^p-ball in E of radius c^p (instead of the sphere S_c) intersected with $\mathcal{P}$ in Theorem 1.9, where $\mathcal{P}$ is introduced in (5.1). More precisely, we briefly sketch our strategy to prove Theorem 1.9. First of all, we show that $I(u)$ is bounded away from 0 on $\mathcal{M}_c$ and coercive on $\mathcal{M}_c$. Next, if $\{u_n\} \subset \mathcal{M}_c$ is a minimizing sequence, then by means of the profile decomposition theorem, we will find a sequence of translations $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + y_n)$ weakly and a.e. converges to a minimizer u of I on $\mathcal{M}_c$. Consequently, by the standard compactness lemma and the Schwartz symmetrization, we may find a nonnegative and radially symmetric minimizer. In the last, we show that for any $u \in (A_c \backslash S_c) \cap \mathcal{P}$, the crucial inequality holds $\inf_{\mathcal{P}_c} I < I(u)$. Thus the minimizer u of I on $\mathcal{M}_c$ is achieved in $\mathcal{P}_c$. Moreover, by analyzing Lagrange multipliers λ and μ for constraints S_c and $\mathcal{P}$ respectively, we obtain that $\mu = 0$ and u is a normalized ground state solution to (1.1).

Remark 1.12. From Theorem 1.9, we appropriately weaken the conditions of Theorem 1.1 in [14]. In particular, we do not need the inequality in (g_6) to be strict due to the effect of two differential operators. We only need to make full use of the Pohozaev equality and Nehari-type equality to calculate the energy of equation (1.1).

The remaining part of this paper is organized as follows. In Section 2, we show some preliminary results and then study in Section 3 some properties of the function $c \mapsto E_c$. In Section 4, we prove Theorems 1.4 and 1.5. Section 5 is dedicated to characterizing the relationship between E_c and $\bar{E}_c$ and proving Theorem 1.9.

2. Preliminary results

In this section, we prepare several technical results for the proof of our main results.

Lemma 2.1. *Assume that $1 < p < q < N$ and f satisfies (f_1). Then the following results hold.*

(i) *For any $c > 0$, there exists $\delta = \delta(N, c) > 0$ small enough such that*

$$\frac{1}{2p}\|\nabla u\|_p^p + \frac{1}{2q}\|\nabla u\|_q^q \le I(u) \le \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q$$

for all $u \in A_c$ satisfying $\|\nabla u\|_p^p \le \delta$, where A_c is defined in Theorem 1.9.

(ii) *Suppose that $\{u_n\}$ is a bounded sequence in E. If $\lim_{n\to\infty} \|u_n\|_{\bar{p}} = 0$, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)\, dx = 0 = \lim_{n\to\infty} \int_{\mathbb{R}^N} \bar{F}(u_n)\, dx.$$

(iii) *Suppose that $\{u_n\}$, $\{v_n\}$ are two bounded sequences in E. If $\lim_{n\to\infty} \|v_n\|_{\bar{p}} = 0$, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)v_n\, dx = 0.$$

Proof. (i) On the one hand, we observe that

$$I(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_{\mathbb{R}^N} F(u)\, dx \le \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q. \tag{2.1}$$

On the other hand, using (f_1), there exists $C_\epsilon > 0$ such that $|F(t)| \le \epsilon|t|^{\bar{p}} + C_\epsilon|t|^{p^*}$ for all $t \in \mathbb{R}$, where $\epsilon > 0$ is arbitrary. For any $u \in A_c$, by Lemma 1.1, (1.13), one has

$$\begin{aligned}\int_{\mathbb{R}^N} |F(u)|\, dx &\le \epsilon\|u\|_{\bar{p}}^{\bar{p}} + C_\epsilon\|u\|_{p^*}^{p^*} \\ &\le \epsilon K_{N,\bar{p}}\|\nabla u\|_q^q c^{\bar{p}(1-\gamma_{\bar{p}})} + C_\epsilon S^{-\frac{N}{N-p}}\|\nabla u\|_p^{\frac{Np}{N-p}}.\end{aligned}$$

Then

$$\begin{aligned}I(u) &\ge \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \epsilon K_{N,\bar{p}}\|\nabla u\|_q^q c^{\bar{p}(1-\gamma_{\bar{p}})} - C_\epsilon S^{-\frac{N}{N-p}}\|\nabla u\|_p^{\frac{Np}{N-p}} \\ &= \left(\frac{1}{p} - C_\epsilon S^{-\frac{N}{N-p}}\|\nabla u\|_p^{\frac{p^2}{N-p}}\right)\|\nabla u\|_p^p + \left(\frac{1}{q} - \epsilon K_{N,\bar{p}} c^{\bar{p}(1-\gamma_{\bar{p}})}\right)\|\nabla u\|_q^q.\end{aligned}$$

So we take ϵ, δ small enough, so that

$$I(u) \geq \frac{1}{2p}\|\nabla u\|_p^p + \frac{1}{2q}\|\nabla u\|_q^q. \tag{2.2}$$

Combining with (2.1), (2.2), we know that (i) holds.

(ii) Using (f_1), there exists $D_\epsilon > 0$ such that $|F(u)| + |\bar{F}(t)| \leq \epsilon |t|^{\bar{p}} + D_\epsilon |t|^{p^*}$, which implies that

$$\int_{\mathbb{R}^N} |F(u)| + |\bar{F}(u)|\, dx \leq \epsilon \|u\|_{p^*}^{p^*} + D_\epsilon \|u\|_{\bar{p}}^{\bar{p}}.$$

By the boundedness of $\{u_n\}$, $\lim_{n\to\infty} \|u_n\|_{\bar{p}} = 0$ and the arbitrariness of ϵ, we infer that (ii) holds.

(iii) Using (f_1), there exists $\bar{D}_\epsilon > 0$ such that $|f(u)| \leq \epsilon |u|^{p^*-1} + \bar{D}_\epsilon |u|^{\bar{p}-1}$. Thus

$$\int_{\mathbb{R}^N} |f(u_n)||v_n|\, dx \leq \epsilon \left(\int_{\mathbb{R}^N} |u_n|^{p^*}\, dx\right)^{\frac{p^*-1}{p^*}} \cdot \|v_n\|_{p^*} + \bar{D}_\epsilon \left(\int_{\mathbb{R}^N} |u_n|^{\bar{p}}\, dx\right)^{\frac{\bar{p}-1}{\bar{p}}} \|v_n\|_{\bar{p}}.$$

It follows from the boundedness of $\{u_n\}$ and $\{v_n\}$, $\lim_{n\to\infty} \|v_n\|_{\bar{p}} = 0$ and the arbitrariness of ϵ that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n) v_n\, dx = 0.$$

Hence (iii) holds. □

Lemma 2.2. *Assume that $1 < p < q < N$ and f satisfies (f_1) and (f_2). For any $u \in E\backslash\{0\}$, one has*

(i) $\Psi_u(s) \to 0^+$ *as* $s \to -\infty$.
(ii) $\Psi_u(s) \to -\infty$ *as* $s \to +\infty$.

Proof. (i) Since $s * u \in S_c \subset A_c$ and

$$\|\nabla(s*u)\|_p^p = e^{ps}\|\nabla u\|_p^p, \quad \|\nabla(s*u)\|_q^q = e^{(\delta_q+1)s}\|\nabla u\|_q^q,$$

by Lemma 2.1(i), it follows that

$$\frac{1}{2p}e^{ps}\|\nabla u\|_p^p + \frac{1}{2q}e^{q(\delta_q+1)s}\|\nabla u\|_q^q \leq I(s*u) \leq \frac{1}{p}e^{ps}\|\nabla u\|_p^p + \frac{1}{q}e^{q(\delta_q+1)s}\|\nabla u\|_q^q$$

as $s \to -\infty$. Therefore, $\lim_{s\to-\infty} \Psi_u(s) = 0^+$.

(ii) For any $\lambda \geq 0$, we define a function $g_\mu : \mathbb{R} \to \mathbb{R}$ as follows:

$$g_\mu(t) := \begin{cases} \frac{F(t)}{|t|^{\bar{p}}} + \mu, & \text{for } t \neq 0, \\ \mu, & \text{for } t = 0. \end{cases}$$

Obviously, $F(t) = g_\mu(t)|t|^{\bar{p}} - \mu|t|^{\bar{p}}$ for all $t \in \mathbb{R}$. Moreover, it follows from (f_1) and (f_2) that g_μ is continuous and

$$g_\mu(t) \to +\infty \quad \text{as } t \to \infty.$$

Hence we take $\mu > 0$ large enough such that $g_\mu(t) \geq 0$ for any $t \in \mathbb{R}$. Then

$$\lim_{s\to+\infty} \int_{\mathbb{R}^N} g_\mu \left(e^{\frac{Ns}{p}} u\right) |u|^{\bar{p}}\, dx = +\infty.$$

Since when $s \to +\infty$,

$$\begin{aligned}\Psi_u(s) &= \frac{1}{p}\|\nabla(s*u)\|_p^p + \frac{1}{q}\|\nabla(s*u)\|_q^q + \mu\|s*u\|_{\bar{p}}^{\bar{p}} - \int_{\mathbb{R}^N} g_\lambda(s*u)|s*u|^{\bar{p}}\, dx \\ &= \frac{e^{ps}}{p}\|\nabla u\|_p^p + e^{q(\delta_q+1)s}\left[\frac{1}{q}\|\nabla u\|_q^q + \mu\|u\|_{\bar{p}}^{\bar{p}} - \int_{\mathbb{R}^N} g_\lambda\left(e^{\frac{Ns}{p}}u\right)|u|^{\bar{p}}\, dx\right],\end{aligned}$$

we deduce that $\Psi_u(s) \to -\infty$ as $s \to +\infty$. □

Lemma 2.3. *Assume that $1 < p < q < N$. If f satisfies (f_1)-(f_3), then*

$$f(t)t > \bar{p}F(t) \quad \text{for all } t \neq 0.$$

Proof. We divide the proof of Lemma 2.3 into five steps.

Step 1. $F(t) > 0$ for any $t \neq 0$. If there exists $t_0 \neq 0$ such that $F(t_0) \leq 0$, then by (f_1) and (f_2), the function $\frac{F(t)}{|t|^{\bar{p}}}$ reaches the global minimum at some $t_1 \neq 0$ satisfying $F(t_1) \leq 0$ and

$$\left[\frac{F(t)}{|t|^{\bar{p}}}\right]'_{t=t_1} = \frac{f(t_1)t_1 - \bar{p}F(t_1)}{|t_1|^{\bar{p}+1}\operatorname{sign}(t_1)} = 0.$$

In addition, it follows from (f_1) and (f_3) that $f(t)t > pF(t)$ for any $t \neq 0$. So

$$0 < f(t_1)t_1 - pF(t_1) = (\bar{p} - p)F(t_1) \leq 0.$$

This is impossible. Hence $F(t) > 0$ for any $t \neq 0$.

Step 2. There exists a positive sequence $\{t_n^+\}$ and a negative sequence $\{t_n^-\}$ such that $|t_n^\pm| \to 0$ and $f(t_n^\pm)t_n^\pm > \bar{p}F(t_n^\pm)$ for each $n \geq 1$. We mainly focus on the positive case since the negative case is similar. If we suppose that there exists $t_2 > 0$ small enough such that $f(t)t \leq \bar{p}F(t)$ for any $t \in (0, t_2]$. Based on Step 1, we deduce that

$$\frac{F(t)}{t^{\bar{p}}} \geq \frac{F(t_2)}{t_2^{\bar{p}}} > 0 \quad \text{for all } t \in (0, t_2].$$

Observe that $\lim_{t\to 0}\frac{F(t)}{t^{\bar{p}}}=0$ by (f_1). This is a contradiction. Hence we complete the proof of Step 2.

Step 3. There exists a positive sequence $\{\tau_n^+\}$ and a negative sequence $\{\tau_n^-\}$ such that $|\tau_n^{\pm}|\to +\infty$ and $f(\tau_n^{\pm})\tau_n^{\pm}>\bar{p}F(\tau_n^{\pm})$ for each $n\geq 1$. Since the two cases are similar, we only need to show the existence of $\{\tau_n^+\}$. Assume by contradiction that there exists $t_3>0$ such that $f(t)t\leq \bar{p}F(t)$ for any $t\geq t_3$. Then

$$\frac{F(t)}{|t|^{\bar{p}}}\leq \frac{F(t_3)}{|t_3|^{\bar{p}}}<+\infty \quad \text{for all } t>t_3,$$

which contradicts with (f_2). So the sequence $\{\tau_n^+\}$ exists and the proof of Step 3 is completed.

Step 4. $f(t)t\geq \bar{p}F(t)$ for any $t\neq 0$. We can assume by contradiction that there exists a $t_4\neq 0$ such that $f(t_4)t_4<\bar{p}F(t_4)$. Without loss of generality, we can further assume that $t_4>0$. Based on Step 2 and Step 3, there exist $\tau_1,\ \tau_2\in\mathbb{R}$ such that $0<\tau_1<t_4<\tau_2$,

$$f(t)t<\bar{p}F(t) \quad \text{for all } t\in(\tau_1,\tau_2) \tag{2.3}$$

and

$$f(t)t=\bar{p}F(t) \quad \text{when } t=\tau_1,\ \tau_2. \tag{2.4}$$

On the one hand, by (2.3), we get

$$\frac{F(\tau_1)}{|\tau_1|^{\bar{p}}}>\frac{F(\tau_2)}{|\tau_2|^{\bar{p}}}. \tag{2.5}$$

On the other hand, it follows from (2.4) and (f_3) that

$$\frac{F(\tau_1)}{|\tau_1|^{\bar{p}}}=(\bar{p}-p)\frac{\bar{F}(\tau_1)}{|\tau_1|^{\bar{p}}}>(\bar{p}-p)\frac{\bar{F}(\tau_2)}{|\tau_2|^{\bar{p}}}=\frac{F(\tau_2)}{|\tau_2|^{\bar{p}}},$$

which contradicts with (2.5) and hence the proof of Step 4 is completed.

Step 5. $f(t)t>\bar{p}F(t)$ for any $t\neq 0$. Based on Step 4, the function $\frac{F(t)}{|t|^{\bar{p}}}$ is nonincreasing on $(-\infty,0)$ and nondecreasing on $(0,\infty)$. Then in view of (f_4), the function $\frac{f(t)}{|t|^{\bar{p}-1}}$ is strictly increasing on $(-\infty,0)$ and $(0,\infty)$. For any $t\neq 0$, we infer that

$$\bar{p}F(t)=\bar{p}\int_0^t f(s)\,ds<\bar{p}\frac{f(t)}{|t|^{\bar{p}-1}}\int_0^t |s|^{\bar{p}-1}\,ds=f(t)t$$

and this proves Step 5.

From Steps 1-5, we complete the proof of Lemma 2.3. □

Lemma 2.4. *Assume that* $1<p<q<N$ *and* f *satisfies* (f_1)-(f_3). *For any* $u\in E\backslash\{0\}$, *the following results hold:*

(i) *There exists a unique* $s_u\in\mathbb{R}$ *such that* $P(s_u*u)=0$.

(ii) $\Psi_u(s_u) > \Psi_u(s)$ *for any* $s \neq s_u$*. In particular,* $\Psi_u(s_u) > 0$.
(iii) *The mapping* $u \mapsto s_u$ *is continuous in* $u \in E\backslash\{0\}$.
(iv) $s_{u(\cdot+y)} = s_u$ *for any* $y \in \mathbb{R}^N$*. If* f *is odd, then one has* $s_{-u} = s_u$.

Proof. (i) Since

$$\Psi_u(s) = \frac{e^{ps}}{p}\|\nabla u\|_p^p + \frac{e^{q(\delta_q+1)s}}{q}\|\nabla u\|_q^q - e^{-Ns}\int\limits_{\mathbb{R}^N} F\left(e^{\frac{Ns}{p}}u\right)dx,$$

we easily find that $I(s * u)$ is of class C^1 and by direct calculation,

$$\frac{d}{ds}\Psi_u(s) = e^{ps}\|\nabla u\|_p^p + (\delta_q+1)e^{q(\delta_q+1)s}\|\nabla u\|_q^q - \frac{N}{p}e^{-Ns}\int\limits_{\mathbb{R}^N} \bar{F}\left(e^{\frac{Ns}{p}}u\right)dx = P(s*u).$$

From Lemma 2.2, we know that

$$\lim_{s\to-\infty}\Psi_u(s) = 0^+ \quad \text{and} \quad \lim_{s\to+\infty}\Psi_u(s) = -\infty.$$

Hence $\Psi_u(s)$ reaches the global maximum at some $s_u \in \mathbb{R}$ and then

$$P(s_u * u) = \frac{d}{ds}\Big|_{s_u}\Psi_u(s) = 0.$$

To prove the uniqueness of s_u, we define a continuous function $h : \mathbb{R} \to \mathbb{R}$ as follows:

$$h(t) := \begin{cases} \frac{\bar{F}(t)}{|t|^{\bar{p}}}, & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Moreover, it is not difficult for us to see that h is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$. Then $\bar{F}(t) = g(t)|t|^{\bar{p}}$ for all $t \in \mathbb{R}$. It follows that

$$P(s*u) = e^{ps}\|\nabla u\|_p^p + e^{q(\delta_q+1)s}\left[(\delta_q+1)\|\nabla u\|_q^q - \frac{N}{p}\int\limits_{\mathbb{R}^N} h\left(e^{\frac{Ns}{p}}u\right)|u|^{\bar{p}}\,dx\right].$$

Obviously, for fixed $t \in \mathbb{R}\backslash\{0\}$, the function $t \mapsto h\left(e^{\frac{Ns}{p}}t\right)$ is strictly increasing by (f_3). So we conclude that s_u is unique.

(ii) Based on (i), by Lemma 2.2, we know that $\Psi_u(s)$ reaches the global maximum at s_u, s_u is unique and $\Psi_u(s_u) > 0$. Therefore $\Psi_u(s_u) > \Psi_u(s)$ for any $s \neq s_u$.

(iii) Based on (i), we find that the mapping $u \mapsto s_u$ is well-defined. Let $u \in E\backslash\{0\}$ and $\{u_n\} \subset E\backslash\{0\}$ be any sequence such that $u_n \to u$ in E. Denoting $s_n := s_{u_n}$ for any $n \geq 1$, we only need to prove that up to a subsequence $s_n \to s_u$ as $n \to \infty$. First of all, we claim that $\{s_n\}$ is bounded. Recall the continuous coercive function g_μ defined by Lemma 2.2. It follows from Lemma 2.3

that $g_0(t) \geq 0$ for any $t \in \mathbb{R}$. Using the Fatou's lemma and the fact that $u_n \to u \neq 0$ almost everywhere in $\mathbb{R}^N$, we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g_0\left(e^{\frac{Ns_n}{p}} u_n\right) |u_n|^{\bar{p}}\, dx = +\infty.$$

Then from (ii), we obtain that

$$\begin{aligned} 0 &\leq e^{-q(\delta_q+1)s_n} \Psi_{u_n}(s_n) \\ &= \frac{e^{ps_n - q(\delta_q+1)s_n}}{p} \|\nabla u_n\|_p^p + \frac{1}{q}\|\nabla u_n\|_q^q - \int_{\mathbb{R}^N} g_0\left(e^{\frac{Ns_n}{p}} u\right) |u_n|^{\bar{p}}\, dx \to -\infty, \end{aligned} \tag{2.6}$$

which is a contradiction and the sequence $\{s_n\}$ is bounded. In addition, by (ii), one has

$$\Psi_{u_n}(s_n) \geq \Psi_{u_n}(s_u) \quad \text{for any } n \geq 1.$$

Since $s_u * u_n \to s_u * u$ in E, it follows that

$$\Psi_{u_n}(s_n) = \Psi_u(s_u) + o_n(1).$$

Hence

$$\liminf_{n\to\infty} \Psi_{u_n}(s_n) \geq \Psi_u(s_u) > 0. \tag{2.7}$$

Using the fact that $\{s_n * u_n\} \subset A_c$ for $c > 0$ large enough,

$$\|\nabla(s_n * u_n)\|_p = e^{s_n} \|\nabla u_n\|_p$$

and (2.7), in view of Lemma 2.1(i), we obtain that $\{s_n\}$ is bounded from below. Then there exists a $s_1 \in \mathbb{R}$ such that $s_n \to s_1$. Since $u_n \to u$ in E, we get $s_n * u_n \to s_1 * u$ in E. On the other hand, $P(s_n * u_n) = 0$ for any $n \geq 1$, it follows that $P(s_1 * u) = 0$. Based on (i), we find that $s_1 = s_u$ and (iii) is completed.

(iv) For any $y \in \mathbb{R}^N$, Note the fact that

$$P(s_u * u(\cdot + y)) = P(s_u * u) = 0.$$

So it follows from (i) that $s_{u(\cdot+y)} = s_u$. In particular, if f is odd, then

$$P(s_u * (-u)) = P(-(s_u * u)) = P(s_u * u) = 0,$$

which yields that $s_{-u} = s_u$. □

Lemma 2.5. *Assume that $1 < p < q < N$ and f satisfies (f_1)-(f_3). Then*

(i) $\mathcal{P}_c \neq \emptyset$.

(ii) $\inf_{u\in\mathcal{P}_c} \|\nabla u\|_p > 0$.
(iii) $\inf_{u\in\mathcal{P}_c} I(u) > 0$.
(iv) *I is coercive on $\mathcal{P}_c$, namely, $I(u_n) \to +\infty$ for any $\{u_n\} \subset \mathcal{P}_c$ with $\|u_n\|_{p,q} \to \infty$.*

Proof. (i) It follows from the definition of $\mathcal{P}_c$ and Lemma 2.4(i) that $\mathcal{P}_c \neq \emptyset$.

(ii) If there exists $\{u_n\} \subset \mathcal{P}_c$ such that $\|\nabla u_n\|_p \to 0$, then $P(u_n) = 0$ as $n \to \infty$. On the other hand, similar to the proof of Lemma 2.1(i), we obtain that for n large enough,

$$P(u_n) \geq \frac{1}{2p}\|\nabla u_n\|_p^p > 0.$$

Combining with the above two aspects, we know that this is a contradiction and $\inf_{u\in\mathcal{P}_c} \|\nabla u\|_p > 0$.

(iii) For any $u \in \mathcal{P}_c$, it follows from Lemma 2.4(i)(ii) that

$$I(u) = \Psi_u(0) \geq \Psi_u(s) \quad \text{for all } s \in \mathbb{R}.$$

Let $\delta > 0$ introduced by Lemma 2.1(i) and $\bar{s} := \ln\left(\frac{\delta}{\|\nabla u\|_p}\right)$. Then using $\|\nabla(\bar{s} * u)\|_p = \delta$ and Lemma 2.1(i), we infer that

$$I(u) \geq \Psi_u(\bar{s}) \geq \frac{1}{2p}\|\nabla(\bar{s} * u)\|_p^p = \frac{1}{2p}\delta^2.$$

Therefore, the proof of (iii) is completed.

(iv) We assume by contradiction that there exists $\{u_n\} \subset \mathcal{P}_c$ such that $\|u_n\|_{p,q} \to \infty$, $\sup_{n\geq 1} I(u_n) \leq d$ for some $d \in (0, +\infty)$. Without loss of generality, we further assume that

$$\|\nabla u_n\|_q^{\frac{p}{\delta_q+1}} \gg \|\nabla u_n\|_p^p \to +\infty.$$

For any $n \geq 1$, we set

$$\tilde{s}_n := \frac{1}{(\delta_q + 1)}\ln(\|\nabla u_n\|_q) \quad \text{and} \quad \omega_n := (-\tilde{s}_n) * u_n.$$

Obviously, $\tilde{s}_n \to +\infty$, $\{\omega_n\} \subset S_c$, $\|\nabla \omega_n\|_p \leq 1$ and $\|\nabla \omega_n\|_q = 1$ for any $n \geq 1$. Let

$$\rho := \limsup_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B(y,1)} |\omega_n|^p\, dx\right).$$

In the following argument, we distinguish the two cases: $\rho > 0$ and $\rho = 0$.

Case 1. $\rho > 0$. Up to a subsequence, there exists $\{y_n\} \subset \mathbb{R}^N$ and $\bar{\omega} \in E\backslash\{0\}$ such that

$$\bar{\omega}_n := \omega_n(\cdot + y_n) \rightharpoonup \bar{\omega} \quad \text{in } E$$

and

$$\bar{\omega}_n \to \bar{\omega} \quad \text{a.e. in } \mathbb{R}^N.$$

Since $\tilde{s}_n \to \infty$, it follows from the continuous coercive function g_μ, Lemma 2.3 and the Fatou's lemma that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g_0\left(e^{\frac{Ns_n}{p}} \bar{\omega}_n\right) |\bar{\omega}_n|^{\bar{p}}\, dx = +\infty.$$

Hence in view of (iii), we deduce that

$$\begin{aligned} 0 \le e^{-q(\delta_q+1)\tilde{s}_n} I(u_n) &= e^{-q(\delta_q+1)\tilde{s}_n} \Psi_{\omega_n}(\tilde{s}_n) \\ &\le \frac{1}{p} e^{(p-q(\delta_q+1))\tilde{s}_n} + \frac{1}{q} - \int_{\mathbb{R}^N} g_0\left(e^{\frac{N\tilde{s}_n}{p}} \omega_n\right) |\omega_n|^{\bar{p}}\, dx \\ &= \frac{1}{p} e^{(p-q(\delta_q+1))\tilde{s}_n} + \frac{1}{q} - \int_{\mathbb{R}^N} g_0\left(e^{\frac{N\tilde{s}_n}{p}} \bar{\omega}_n\right) |\bar{\omega}_n|^{\bar{p}}\, dx \\ &\to -\infty. \end{aligned}$$

This is impossible.

Case 2. $\rho = 0$. In this case, using Lemma I.1 in [29], we see that $\omega_n \to 0$ in $L^{\bar{p}}(\mathbb{R}^N)$. Then it follows from Lemma 2.1(ii) that

$$\lim_{n\to\infty} e^{-Ns} \int_{\mathbb{R}^N} F(e^{\frac{Ns}{p}} \omega_n)\, dx = 0 \quad \text{for any } s \in \mathbb{R}.$$

Moreover, it follows that $P(\tilde{s}_n * \omega_n) = P(u_n) = 0$ and Lemma 2.4(i)(ii) that for any $s \in \mathbb{R}$,

$$\begin{aligned} d \ge I(u_n) = \Psi_{\omega_n}(\tilde{s}_n) &\ge \Psi_{\omega_n}(s) \\ &\ge \frac{1}{q} e^{q(\delta_q+1)s} \|\nabla \omega_n\|_q^q - e^{-Ns} \int_{\mathbb{R}^N} F(e^{\frac{Ns}{p}} \omega_n)\, dx = \frac{1}{q} e^{q(\delta_q+1)s} + o_n(1). \end{aligned}$$

Clearly, this leads a contradiction for $s > \frac{\ln(qd)}{q(\delta_q+1)}$. Thus I is coercive on $\mathcal{P}_c$. $\square$

Remark 2.6. From Lemma 2.5(iv), suppose that $1 < p < q < N$ and f satisfies (f_1)-(f_3). For any sequence $\{u_n\} \subset E\backslash\{0\}$ such that

$$P(u_n) = 0, \quad \sup_{n\ge1} \|u_n\|_p < +\infty, \quad \text{and} \quad \sup_{n\ge1} I(u_n) < +\infty.$$

Then we know that $\{u_n\}$ is bounded in E.

3. The behavior of the function $c \mapsto E_c$

When $1 < p < q < N$ and f satisfies (f$_1$)-(f$_3$), for given $c > 0$, it follows from Lemma 2.5 that the infimum E_c is well defined and strictly positive. Next we will prove that E_c is continuous and nonincreasing in $c > 0$.

Lemma 3.1. *Assume that $1 < p < q < N$ and f satisfies (f_1)-(f_3). Then the function $c \mapsto E_c$ is continuous at each $c > 0$.*

Proof. The result of Lemma 3.1 is equivalent to prove that for a given $c > 0$ and any positive sequence $\{c_n\}$ such that $c_n \to c$ as $n \to \infty$, one has $\lim_{n\to\infty} E_{c_n} = E_c$. Then we first claim that

$$\limsup_{n\to\infty} E_{c_n} \le E_c. \tag{3.1}$$

For any $u \in \mathcal{P}_c$, we define

$$u_n := \frac{c_n}{c} \cdot u \in S_{c_n}, \quad n \in \mathbb{N}^+.$$

Then $u_n \to u$ in E. Combining Lemma 2.4(iii), we deduce that $\lim_{n\to\infty} s_{u_n} = s_u = 0$ and

$$s_{u_n} * u_n \to s_u * u = u \quad \text{in } E \text{ as } n \to \infty,$$

which implies that

$$\limsup_{n\to\infty} E_{c_n} \le \limsup_{n\to\infty} \Psi_{u_n}(s_{u_n}) = I(u).$$

Observe that $u \in \mathcal{P}_c$ is arbitrary, hence (3.1) holds.

Next we show that

$$\liminf_{n\to\infty} E_{c_n} \ge E_c. \tag{3.2}$$

For each $n \in \mathbb{N}^+$, there exists $v_n \in \mathcal{P}_{c_n}$ such that

$$I(v_n) \le E_{c_n} + \frac{1}{n}. \tag{3.3}$$

Denoting

$$t_n := \left(\frac{c^p}{c_n^p}\right)^{\frac{1}{N}} \quad \text{and} \quad \bar{v}_n := v_n\left(\frac{\cdot}{t_n}\right) \in S_c,$$

it follows from Lemma 2.4 and (3.3) that

$$
\begin{aligned}
E_c \le \Psi_{\bar{v}_n}(s_{\bar{v}_n}) &\le \Psi_{v_n}(s_{\bar{v}_n}) + \left|\Psi_{\bar{v}_n}(s_{\bar{v}_n}) - \Psi_{v_n}(s_{\bar{v}_n})\right| \\
&\le I(v_n) + \left|\Psi_{\bar{v}_n}(s_{\bar{v}_n}) - \Psi_{v_n}(s_{\bar{v}_n})\right| \\
&\le E_{c_n} + \frac{1}{n} + \left|\Psi_{\bar{v}_n}(s_{\bar{v}_n}) - \Psi_{v_n}(s_{\bar{v}_n})\right| \\
&= E_{c_n} + \frac{1}{n} + \bar{C}_n,
\end{aligned}
$$

where $\bar{C}_n := \left|\Psi_{\bar{v}_n}(s_{\bar{v}_n}) - \Psi_{v_n}(s_{\bar{v}_n})\right|$. Obviously, if

$$
\lim_{n\to\infty} \bar{C}_n = 0, \tag{3.4}
$$

then (3.2) holds. Noting that $s * \left(u\left(\frac{\cdot}{t}\right)\right) = (s*u)\left(\frac{\cdot}{t}\right)$, we deduce that

$$
\begin{aligned}
\bar{C}_n =& \left| \frac{1}{p}\left(t_n^{N-p}-1\right)\|\nabla(s_{\bar{v}_n} * v_n)\|_p^p + \frac{1}{q}\left(t_n^{N-q}-1\right)\|\nabla(s_{\bar{v}_n} * v_n)\|_q^q \right. \\
&\left. - \left(t_n^N - 1\right)\int_{\mathbb{R}^N} F(s_{\bar{v}_n} * v_n)\,dx \right| \\
\le& \frac{1}{p}\left|t_n^{N-p}-1\right| \cdot \|\nabla(s_{\bar{v}_n} * v_n)\|_p^p + \frac{1}{q}\left|t_n^{N-q}-1\right| \|\nabla(s_{\bar{v}_n} * v_n)\|_q^q \\
&+ \left|t_n^N - 1\right| \cdot \int_{\mathbb{R}^N} |F(s_{\bar{v}_n} * v_n)|\,dx \\
=& \frac{1}{p}\left|t_n^{N-p}-1\right| \cdot \mathcal{A}_n + \frac{1}{q}\left|t_n^{N-q}-1\right| \cdot \mathcal{B}_n + \left|t_n^N - 1\right| \cdot \mathcal{C}_n,
\end{aligned}
$$

where

$$
\mathcal{A}_n := \|\nabla(s_{\bar{v}_n} * v_n)\|_p^p, \quad \mathcal{B}_n := \|\nabla(s_{\bar{v}_n} * v_n)\|_q^q, \quad \text{and} \quad \mathcal{C}_n := \int_{\mathbb{R}^N} |F(s_{\bar{v}_n} * v_n)|\,dx.
$$

From $t_n \to 1$ as $n \to \infty$, we find that (3.4) is reduced to show that

$$
\limsup_{n\to\infty} \mathcal{A}_n < +\infty, \quad \limsup_{n\to\infty} \mathcal{B}_n < +\infty \quad \text{and} \quad \limsup_{n\to\infty} \mathcal{C}_n < +\infty. \tag{3.5}
$$

To verify (3.5), we prove below three steps in turn.

Step 1. The sequence $\{v_n\}$ is bounded in E. It follows from (3.1) and (3.3) that $\limsup_{n\to\infty} I(v_n) \le E_c$. Since $v_n \in \mathcal{P}_{c_n}$ and $c_n \to c$, based on Remark 2.6, we see that Step 1 is completed.

Step 2. The sequence $\{\bar{v}_n\}$ is bounded in E, and there exists $\{y_n\} \subset \mathbb{R}^N$ and $v \in E$ such that up to a subsequence $\bar{v}_n(\cdot + y_n) \to v \ne 0$ almost everywhere in $\mathbb{R}^N$. Indeed, by the fact that $t_n \to 1$ and Step 1, we infer that $\{\bar{v}_n\}$ is bounded in E. Set

$$\bar{\rho} := \limsup_{n\to\infty} \left(\sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |\bar{v}_n|^p \, dx \right).$$

Then we only need to exclude the case of $\bar{\rho} = 0$. If $\bar{p} = 0$, then by Lemma I.1 in [29], we get $\bar{v}_n \to 0$ in $L^{\bar{p}}(\mathbb{R}^N)$. Consequently,

$$\|v_n\|_{\bar{p}}^{\bar{p}} = \|\bar{v}_n(t_n\cdot)\|_{\bar{p}}^{\bar{p}} = t_n^{-N}\|\bar{v}_n\|_{\bar{p}}^{\bar{p}} \to 0.$$

Combining Lemma 2.1(ii) with that $P(v_n) = 0$, we deduce that

$$\|\nabla v_n\|_p^p + (\delta_q + 1)\|\nabla v_n\|_q^q = \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(v_n)\, dx \to 0. \tag{3.6}$$

On the other hand, similar to the proof of Lemma 2.1(i), we obtain

$$P(v_n) \geq \frac{1}{2p}\|\nabla v_n\|_p^p > 0$$

for n large enough, which along with (3.6) yields that $0 = P(v_n) \geq \frac{1}{2p}\|\nabla v_n\|_p^p > 0$ for n large enough. This is impossible. So we complete the proof of Step 2.

Step 3. $\limsup_{n\to\infty} s_{\bar{v}_n} < +\infty$. We assume by contradiction that

$$s_{\bar{v}_n} \to +\infty \tag{3.7}$$

as $n \to +\infty$. On the one hand, using Step 2, we see that up to a subsequence,

$$\bar{v}_n(\cdot + y_k) \to v \neq 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{3.8}$$

On the other hand, it follows from Lemma 2.4(iv) and (3.7) that

$$s_{\bar{v}_n(\cdot+y_n)} = s_{\bar{v}_n} \to +\infty, \tag{3.9}$$

which along with Lemma 2.4(ii) implies that

$$I(s_{\bar{v}_n(\cdot+y_k)} * \bar{v}_k(\cdot + y_k)) \geq 0. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), similar to (2.6), we can obtain a contradiction and Step 3 is completed.

Now, from Steps 1-3, we find that

$$\limsup_{n\to\infty} \|s_{\bar{v}_n} * v_n\|_E < +\infty,$$

which along with the conditions (f_1), (f_2) yields that (3.5) holds. So the proof of Lemma 3.1 is completed. □

Lemma 3.2. *Assume that $1 < p < q < N$ and f satisfies (f_1)-(f_3). Then the function $c \mapsto E_c$ is nonincreasing on $(0, \infty)$.*

Proof. The result of Lemma 3.2 is equivalent to that for any $c_1 > c_2 > 0$ and any arbitrary $\epsilon > 0$ one has

$$E_{c_1} \leq E_{c_2} + \epsilon. \tag{3.11}$$

By the definition of E_{c_2}, there exists $u \in \mathcal{P}_{c_2}$ such that

$$I(u) \leq E_{c_2} + \frac{\epsilon}{2}. \tag{3.12}$$

Suppose that $\gamma \in C_0^\infty(\mathbb{R}^N)$ is radial and satisfies

$$\gamma(x) = \begin{cases} 1, & |x| \leq 1, \\ \in [0,1], & |x| \in (1,2), \\ 0, & |x| \geq 2. \end{cases}$$

Then for any small $\delta > 0$, we define $u_\delta(x) = u(x) \cdot \gamma(\delta x) \in E \backslash \{0\}$. Noting that $u_\delta \to u$ in E as $\delta \to 0^+$, by Lemma 2.4(iii), one has $\lim_{\delta \to 0^+} s_{u_\delta} = s_u = 0$. It follows that

$$s_{u_\delta} * u_\delta \to s_u * u = u \quad \text{in } E \text{ as } \delta \to 0^+.$$

Then we can take a $\delta > 0$ small enough such that

$$I(s_{u_\delta} * u_\delta) \leq I(u) + \frac{\epsilon}{4}. \tag{3.13}$$

In addition, we take $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $\mathrm{supp}(\chi) \subset B\left(0, 1 + \frac{4}{\delta}\right) \backslash B\left(0, \frac{4}{\delta}\right)$ and set

$$\bar{\chi} := \left(\frac{c_1^p - \|u_\delta\|_p^p}{\|\chi\|_p^p} \right)^{\frac{1}{p}} \chi.$$

For any $\nu \leq 0$, we define $w_\nu := u_\delta + \lambda * \bar{v}$. Noting that

$$\mathrm{supp}(u_\delta) \cap \mathrm{supp}(\nu * \bar{\chi}) = \emptyset,$$

we have $w_\nu \in S_{c_1}$. Now we claim that s_{w_ν} is bounded from above as $\nu \to -\infty$. Indeed, we assume by contradiction that $|s_{w_\nu}| \to +\infty$. Observe that $I(s_{w_\nu} * w_\nu) \geq 0$ by Lemma 2.4(ii) and that $w_\nu \to u_\delta \neq 0$ almost everywhere in $\mathbb{R}^N$ as $\nu \to -\infty$. Similar to (2.6), we can obtain a contradiction. That is, the claim is true. Next, since

$$s_{w_\nu} + \nu \to -\infty \quad \text{as } \nu \to -\infty,$$

we deduce that

$$\|\nabla[(s_{w_\nu}+\nu)*\bar{\chi}]\|_p \to 0, \quad \|\nabla[(s_{w_\nu}+\nu)*\bar{\chi}]\|_q \to 0 \quad \text{and} \quad (s_{w_\nu}+\nu)*\bar{v} \to 0 \quad \text{in } L^{\bar{p}}(\mathbb{R}^N).$$

Then it follows from Lemma 2.1(ii) that

$$I((s_{w_\nu}+\nu)*\bar{\chi}) \le \frac{\epsilon}{4} \tag{3.14}$$

for $\nu < 0$ small enough. In the last, in view of Lemma 2.4(ii), (3.11), (3.12) and (3.13), we get

$$\begin{aligned} E_{c_1} \le I(s_{w_\nu} * w_\nu) &= I(s_{w_\nu} * u_\delta) + I(s_{w_\nu} * (\nu * \bar{\chi})) \\ &\le I(s_{u_\delta} * u_\delta) + I((s_{w_\nu}+\nu)*\bar{\chi}) \\ &\le I(u) + \frac{\epsilon}{2} \le E_{c_2} + \epsilon. \end{aligned}$$

Hence (3.10) holds and the proof of Lemma 3.2 is completed. □

Lemma 3.3. *Assume that $1 < p < q < N$ and f satisfies (f_1)-(f_3). If there exist $u \in S_c$ and $\lambda \in \mathbb{R}$ such that*

$$-\Delta_p u - \Delta_q u + \lambda |u|^{p-2} u = f(u)$$

and $I(u) = E_c$, then $E_c > E_{\bar{c}}$ for any $\bar{c} > c$ close enough to c if $\lambda > 0$ and for each $\bar{c} < c$ near enough to c if $\lambda < 0$.

Proof. For any $t > 0$ and $s \in \mathbb{R}$, we set $u_{t,s} := s * (tu) \in S_{ct}$. Define

$$a(t,s) := I(u_{t,s}) = \frac{1}{p} t^p e^{ps} \|\nabla u\|_p^p + \frac{1}{q} t^q e^{q(\delta_q+1)s} \|\nabla u\|_q^q - e^{-Ns} \int_{\mathbb{R}^N} F\left(t e^{\frac{Ns}{p}} x\right) dx.$$

By direct calculation, it is clear that

$$\begin{aligned} \frac{\partial}{\partial t} a(t,s) &= t^{p-1} e^{ps} \|\nabla u\|_p^p + t^{q-1} e^{q(\delta_q+1)s} \|\nabla u\|_q^q - e^{-Ns} \int_{\mathbb{R}^N} f\left(t^{\frac{Ns}{p}} u\right) e^{\frac{Ns}{p}} u \, dx \\ &= t^{-1} I'(u_{t,s}) u_{t,s}. \end{aligned}$$

When $\lambda > 0$, combining the fact that $u_{t,s} \to u$ in E as $(t,s) \to (1,0)$ and that

$$I'(u)u = -\lambda \|u\|_p^p = -\lambda c^p < 0,$$

we can fix a $\tilde{\delta} > 0$ small enough such that

$$\frac{\partial}{\partial t} a(t,s) < 0 \quad \text{for any } (t,s) \in (1, 1+\tilde{\delta}] \times [-\tilde{\delta}, \tilde{\delta}].$$

It follows from the mean value theorem that

$$a(t,s) = a(1,s) + (t-1)\cdot \frac{\partial}{\partial t} a(\zeta, s) < a(1,s), \tag{3.15}$$

where $1 < \zeta < t \le 1+\tilde{\delta}$ and $|s| \le \tilde{\delta}$. From Lemma 2.4(iii), one has $s_{tu} \to s_u = 0$ as $t \to 1^+$. In particular, for any $\bar{c} > c$ close enough to c, we choose

$$t := \frac{\bar{c}}{c} \in (1, 1+\tilde{\delta}] \quad \text{and} \quad s := s_{tu} \in [-\tilde{\delta}, \tilde{\delta}].$$

In view of (3.15) and Lemma 2.4(ii), one has

$$E_{\bar{c}} \le a(t, s_{tu}) < a(1, s_{tu}) = I(s_{tu} * u) \le I(u) = E_c.$$

The case of $\lambda < 0$ can be discussed similarly. So the proof Lemma 3.3 is completed. □

Lemma 3.4. *Assume that* $1 < p < q < N$ *and* f *satisfies* (f_1)-(f_3). *Then* $E_c \to +\infty$ *as* $c \to 0^+$.

Proof. From the result of Lemma 3.4, it is sufficient to prove that for any sequence $\{u_n\} \subset E \backslash \{0\}$ such that

$$P(u_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_p = 0,$$

one infers that $I(u_n) \to +\infty$ as $n \to \infty$. Denote

$$\hat{s}_n := \ln(\|\nabla u_n\|_p) \quad \text{and} \quad \hat{v}_n := (-\hat{s}_n) * u_n.$$

Obviously, $\|\nabla \hat{v}_n\|_p = 1$ and $\|\hat{v}_n\|_p = \|u_n\|_p \to 0$. Noting that $\hat{v}_n \to 0$ in $L^{\bar{p}}(\mathbb{R}^N)$, using Lemma 2.1(ii), we obtain that

$$\lim_{n\to\infty} e^{-Ns} \int_{\mathbb{R}^N} F\left(e^{\frac{Ns}{p}} \hat{v}_n\right) dx = 0 \quad \text{for any } s \in \mathbb{R}.$$

Combining the fact that $P(\hat{s}_n * \hat{v}_n) = P(u_n) = 0$, Lemma 2.4(i),(ii), we derive

$$\begin{aligned} I(u_n) &= I(\hat{s}_n * \hat{v}_n) \ge I(s * \hat{v}_n) \\ &\ge \frac{1}{p} e^{ps} - e^{-Ns} \int_{\mathbb{R}^N} F\left(e^{\frac{Ns}{p}} \hat{v}_n\right) dx = \frac{1}{p} e^{ps} + o_n(1), \end{aligned}$$

which yields that $I(u_n) \to +\infty$ due to the arbitrariness of $s \in \mathbb{R}$. We complete the proof of Lemma 3.4. □

Lemma 3.5. *Assume that* $1 < p < q < N$ *and* f *satisfies* (f_1)-(f_3) *and* (f_5). *Then* $E_c \to 0$ *as* $c \to \infty$.

Proof. Fix $u \in S_1 \cap L^\infty(\mathbb{R}^N)$ and set $u_c := c \cdot u \in S_c$ for any $c > 1$. It follows from Lemma 2.4(i) that there exists a unique $s_c \in \mathbb{R}$ such that $s_c * u_c \in \mathcal{P}_c$. In addition, F is nonnegative by Lemma 2.3. Then

$$0 < E_c \leq I(s_c * u_c) \leq \frac{1}{p} c^p e^{ps_c} \|\nabla u\|_p^p + \frac{1}{q} c^q e^{q(\delta_q+1)s_c} \|\nabla u\|_q^q.$$

Now we only need to show that

$$\lim_{c\to\infty} c^p e^{ps_c} = 0. \tag{3.16}$$

Since $P(s_c * u_c) = 0$, it follows that

$$e^{ps_c}\|\nabla u_c\|_p^p + (\delta_q + 1)e^{q(\delta_q+1)s_c}\|\nabla u_c\|_q^q = \frac{N}{p} e^{-Ns_c} \int_{\mathbb{R}^N} \bar{F}\left(e^{\frac{Ns_c}{p}} u_c\right) dx,$$

which along with the definition of h yields that

$$c^p e^{ps_c}\|\nabla u\|_p^p + (\delta_q + 1)c^q e^{q(\delta_q+1)s_c}\|\nabla u\|_q^q = \frac{N}{p} c^{\bar{p}} e^{q(\delta_q+1)s_c} \int_{\mathbb{R}^N} h\left(c \cdot e^{\frac{Ns_c}{p}} u\right) |u|^{\bar{p}}\, dx$$

This means that

$$(\delta_q + 1)\|\nabla u\|_q^q \leq \frac{N}{p} c^{\bar{p}-q} \int_{\mathbb{R}^N} h\left(c \cdot e^{\frac{Ns_c}{p}} u\right) |u|^{\bar{p}}\, dx$$

and thus

$$\lim_{c\to\infty} c \cdot e^{\frac{Ns_c}{p}} = 0 \tag{3.17}$$

Finally, in view of Lemma 2.3 and (f_5), there exists $\bar{\delta} > 0$ small enough such that $\bar{F}(t) \geq (\bar{p} - p)F(t) \geq \epsilon^{-1}|t|^{p^*}$ for any $|t| \leq \bar{\delta}$. Based on $P(s_c * u_c) = 0$ and (3.17), we derive

$$\begin{aligned} c^p e^{ps_c}\|\nabla u\|_p^p + (\delta_q + 1)c^q e^{q(\delta_q+1)s_c}\|\nabla u\|_q^q &= \frac{N}{p} \cdot e^{-Ns_c} \int_{\mathbb{R}^N} \bar{F}\left(c \cdot e^{\frac{Ns_c}{p}} u\right) dx \\ &\geq \frac{N}{p} \epsilon^{-1} e^{-Ns_c} \cdot \left|c \cdot e^{\frac{Ns_c}{p}}\right|^{p^*} \|u\|_{p^*}^{p^*}, \end{aligned}$$

which implies that (3.16) holds. Thus the proof of Lemma 3.5 is completed. □

4. Proof of Theorems 1.4 and 1.5

In this section, in order to establish the existence of ground states to (1.1), we need to construct a Palais-Smale sequence for the constrained functional $I|_{S_c}$ at the level E_c. Inspired by [20], we give the following technical result.

Definition 4.1. Let B be a closed subset of a metric space X. We say that a class $\mathcal{G}$ of compact subsets of X is a homotopy stable family with closed boundary B provided

(i) every set in $\mathcal{G}$ contains B.
(ii) for any set $A \in \mathcal{G}$ and any homotopy $\eta \in C([0,1] \times X, X)$ that satisfies $\eta(t,u) = u$ for all $(t,u) \in (\{0\} \times X) \cup ([0,1] \times B)$, one has $\eta(\{1\} \times A) \in \mathcal{G}$.

We remark that the case $B = \emptyset$ is admissible.

Inspired by [8], we introduce the free functional $\bar{\Psi} : E\backslash\{0\} \to \mathbb{R}$,

$$\bar{\Psi}(u) := I(s_u * u) = \frac{e^{ps_u}}{p}\|\nabla u\|_p^p + \frac{e^{q(\delta_q+1)s_u}}{q}\|\nabla u\|_q^q - e^{-Ns_u}\int_{\mathbb{R}^N} F\left(e^{\frac{Ns_u}{p}} u\right) dx,$$

where $s_u \in \mathbb{R}$ is the unique number guaranteed by Lemma 2.4. Moreover, inspired by [43,44], we find $\bar{\Psi}$ is of class C^1 and

$$\begin{aligned} d\bar{\Psi}(u)[\varphi] &= e^{ps_u}\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi\, dx + e^{q(\delta_q+1)s_u}\int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u\nabla\varphi\, dx \\ &\quad - e^{-Ns_u}\int_{\mathbb{R}^N} f(e^{\frac{Ns_u}{2}}u)e^{\frac{Ns_u}{2}}\varphi\, dx \\ &= dI(s_u * u)[s_u * \varphi] \end{aligned}$$

for any $u \in E\backslash\{0\}$ and $\varphi \in E$. In addition, for given $c > 0$, we define the constrained functional

$$J := \bar{\Psi}|_{S_c} : S_c \to \mathbb{R}.$$

Clearly, the functional $J : S_c \to \mathbb{R}$ is of class C^1 and

$$dJ(u)[\varphi] = d\bar{\Psi}(u)[\varphi] = dI(s_u * u)[s_u * \varphi]$$

for any $u \in S_c$ and $\varphi \in T_u S_c$. Then we have the following result.

Lemma 4.1. *Assume that $\mathcal{G}$ is a homotopy stable family of compact subsets of S_c(with $B = \emptyset$) and set*

$$E_{c,\mathcal{G}} := \inf_{A\in\mathcal{G}} \max_{u\in A} J(u).$$

If $E_{c,\mathcal{G}} > 0$, then there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{P}_c$ for the constrained functional $I|_{S_c}$ at the level $E_{c,\mathcal{G}}$. Moreover, when f is odd and $\mathcal{G}$ is the class of all singletons included in S_c, we have $\|u_n^-\|_p \to 0$, where u^- stands for the negative part of u.

Proof. Suppose that $\{A_n\} \subset \mathcal{G}$ is an arbitrary minimizing sequence of $E_{c,\mathcal{G}}$. By Lemma 2.4(iii), we define the continuous mapping

$$\eta : [0,1] \times S_c \to S_c, \quad \eta(t,u) = (ts_u) * u$$

satisfying $\eta(t,u) = u$ for all $(t,u) \in \{0\} \times S_c$. It follows from the definition of $\mathcal{G}$ that

$$D_n := \eta_n(1, A_n) = \{s_u * u | u \in A_n\} \in \mathcal{G}.$$

Obviously, $D_n \subset \mathcal{P}_c$ for every $n \in \mathbb{N}^+$. Moreover,

$$\max_{D_n} J = \max_{A_n} J \to E_{c,\mathcal{G}}$$

and thus $\{D_n\} \subset \mathcal{G}$ is another minimizing sequence of $E_{c,\mathcal{G}}$. Applying Theorem 3.2 in [20], we obtain a Palais-Smale sequence $\{v_n\} \subset S_c$ for J at the level $E_{c,\mathcal{G}}$ such that $\mathrm{dist}_E(v_n, D_n) \to 0$ as $n \to \infty$. Denote $u_n := s_{v_n} * v_n$. Now we claim that there exists $\tilde{C} > 0$ such that $e^{-ps_{v_n}} \le \tilde{C}$ for every n. Observe that $e^{-ps_{v_n}} = \frac{\|\nabla v_n\|_p^p}{\|\nabla u_n\|_p^p}$. From $\{u_n\} \subset \mathcal{P}_c$ and Lemma 2.5(ii), we see that $\{\|\nabla u_n\|_p\}$ is bounded from below by a positive constant. In addition, since $D_n \subset \mathcal{P}_c$ for every n, we deduce that

$$\max_{D_n} I = \max_{D_n} J \to E_{c,\mathcal{G}}.$$

In view of Lemma 2.5(iv), we find that $\{D_n\}$ is uniformly bounded in E. On the other hand, it follows from $\mathrm{dist}_E(v_n, D_n) \to 0$ that $\sup_n \|\nabla v_n\|_p < \infty$. Combining the above argument, we know that the claim is true. Note that

$$I(u_n) = J(u_n) = J(v_n) \to E_{c,\mathcal{G}}.$$

Then it is sufficient for us to prove that $\{u_n\}$ is a Palais-Smale sequence for I on S_c. For any $\psi \in T_{u_n} S_c$, we can easily obtain that $(-s_{v_n}) * \psi \in T_{v_n} S_c$. By the boundedness of $e^{-ps_{v_n}}$, there exists $\hat{C} > 0$ such that

$$\|(-s_{v_n}) * \psi\|_E \le \hat{C} \|\psi\|_E.$$

Then denoting by $\|\cdot\|_{u,*}$ the dual norm of $(T_u S_c)^*$, we have

$$\begin{aligned}\|dI(u_n)\|_{u_n,*} &= \sup_{\psi\in T_{u_n}S_c,\|\psi\|_E\le 1} |dI(u_n)[\psi]| \\ &= \sup_{\psi\in T_{u_n}S_c,\|\psi\|_E\le 1} |dI(s_{v_n}*v_n)[s_n*((-s_n)*\psi)]| \\ &= \sup_{\psi\in T_{u_n}S_c,\|\psi\|_E\le 1} |dJ(v_n)[(-s_{v_n})*\psi]| \\ &\le \|dJ(v_n)\|_{v_n,*}\cdot \sup_{\psi\in T_{u_n}S_c,\|\psi\|_E\le 1} \|(-s_n)*\psi\|_{p,q} \\ &\le \hat{C}\|dJ(v_n)\|_{v_n,*},\end{aligned}$$

which along with the fact that $\{v_n\}\subset S_c$ is a Palais-Smale sequence of J yields that $\|dI(u_n)\|_{u_n,*}\to 0$.

In the last, we observe that the class of all singletons included in S_c is a homotopy stable family of compact subsets of S_c(with $B=\emptyset$). if f is odd, then by Lemma 2.4(iv), $J(u)$ is even. Based on the above argument, we take a minimizing sequence $\{A_n\}\subset\mathcal{G}$, which includes non-negative functions and the sequence $\{D_n\}$ also has this property. Combining $\operatorname{dist}_E(v_n,D_n)\to 0$, we can find a Palais-Smale sequence $\{u_n\}\subset\mathcal{P}_c$ for $I|_{S_c}$ at the level $E_{c,\mathcal{G}}$ satisfying the property

$$\|u_n^-\|_p^p=\|s_{v_n}*v_n^-\|_p^p=\|v_n^-\|_p^p\to 0.$$

Hence the proof of Lemma 4.1 is completed. □

Lemma 4.2. *There exists a Palais-Smale sequence $\{u_n\}\subset\mathcal{P}_c$ for the constrained functional $I|_{S_c}$ at the level E_c. When f is odd, we have $\|u_n^-\|_p\to 0$ as $n\to\infty$.*

Proof. In view of Lemma 4.1, by $E_c>0$, we only need to show that $E_{c,\mathcal{G}}=E_c$. Indeed, note that

$$E_{c,\mathcal{G}}=\inf_{A\in\mathcal{G}}\max_{u\in A}J(u)=\inf_{u\in S_c}I(s_u*u).$$

On the one hand, for any $u\in S_c$, we get $I(s_u*u)\ge E_c$ by $s_u*u\in\mathcal{P}_c$, which implies that $E_{c,\mathcal{G}}\ge E_c$. On the other hand, for any $u\in\mathcal{P}_c$, we have $s_u=0$ and thus $I(u)=I(0*u)\ge E_{c,\mathcal{G}}$, which yields that $E_c\ge E_{c,\mathcal{G}}$. Combining the above aspects, we know that the proof of Lemma 4.2 is completed. □

Lemma 4.3. *Assume that $\{u_n\}\subset S_c$ is any bounded Palais-Smale sequence for the constrained functional $I|_{S_c}$ at the level $E_c>0$, satisfying $P(u_n)\to 0$. If (f_4) holds, then there exists $u\in S_c$ and $\lambda>0$ such that, up to the extraction of a subsequence and up to translations in $\mathbb{R}^N$, $u_n\to u$ strongly in E and*

$$-\Delta_p u-\Delta_q u+\lambda|u|^{p-2}u=f(u).$$

Proof. Since $\{u_n\}\subset S_c$ is bounded in E, we can obtain the existence of limits to $\|\nabla u_n\|_p^p$, $\|\nabla u_n\|_q^q$, $\int_{\mathbb{R}^N}F(u_n)\,dx$ and $\int_{\mathbb{R}^N}f(u_n)u_n\,dx$. Applying Lemma 3 in [13] and the condition that $\|dI(u_n)\|_{u_n,*}$, we deduce that

$$-\Delta_p u_n - \Delta_q u_n + \lambda_n |u_n|^{p-2} u_n - f(u_n) \to 0 \quad \text{in } E^*,$$

where

$$\lambda_n := \frac{1}{c^p}\left(\int_{\mathbb{R}^N} f(u_n)u_n\,dx - \|\nabla u_n\|_p^p - \|\nabla u_n\|_q^q\right).$$

Since $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$, we get

$$-\Delta_p u_n(\cdot+y_n) - \Delta_q u_n(\cdot+y_n) + \lambda|u_n(\cdot+y_n)|^{p-2}u_n(\cdot+y_n) - f(u_n(\cdot+y_n)) \to 0 \quad \text{in } E^* \tag{4.1}$$

for any $\{y_n\} \subset \mathbb{R}^N$. Now we claim that $\{u_n\}$ is non-vanishing. Indeed, we assume by contradiction that $\{u_n\}$ is vanishing. Then applying Lemma I.1 in [29], we derive $u_n \to 0$ in $L^{\bar{p}}(\mathbb{R}^N)$. Combining with Lemma 2.1(ii) and $P(u_n) \to 0$, we get $\int_{\mathbb{R}^N} F(u_n)\,dx \to 0$ and

$$\|\nabla u_n\|_p^p + (\delta_q + 1)\|\nabla u_n\|_q^q = P(u_n) + \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(u_n)\,dx \to 0.$$

Hence $E_c = \lim_{n\to\infty} I(u_n) = 0$, which contradicts with $E_c > 0$. So the claim is true. That is, $\{u_n\}$ is non-vanishing, up to a subsequence, there exists $\{y_n^1\} \subset \mathbb{R}^N$ and $u^1 \in A_c\backslash\{0\}$ such that $u_n(\cdot + y_n^1) \rightharpoonup u^1$ in E, $u_n(\cdot + y_n^1) \to u^1$ in $L^{\nu}_{\text{loc}}(\mathbb{R}^N)$ for any $\nu \in [1, q^*)$ and $u_n(\cdot + y_n^1) \to u^1$ almost everywhere in $\mathbb{R}^N$. Then with the aid of Lemma A.I in [12] and compactness Lemma 2 in [41], one infers that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\left|[f(u_n(\cdot+y_n^1)) - f(u^1)]\varphi\right|dx \\
&\le \|\varphi\|_{L^\infty(\mathbb{R}^N)}\lim_{n\to\infty}\int_{\operatorname{supp}(\varphi)}\left|f(u_n(\cdot+y_n^1)) - f(u^1)\right|dx = 0
\end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. So by (4.1), we know that u^1 satisfies

$$-\Delta_p u^1 - \Delta_q u^1 + \lambda|u^1|^{p-2}u^1 = f(u^1). \tag{4.2}$$

Similarly, $P(u^1) = 0$. Let $v_n^1 := u_n - u^1(\cdot - y_n^1)$ for every $n \in \mathbb{N}^+$. It results that $v_n^1(\cdot + y_n^1) \rightharpoonup 0$ in E and

$$c^p = \lim_{n\to\infty}\|v_n^1(\cdot+y_n^1) + u^1\|_p^p = \lim_{n\to\infty}\|v_n^1\|_p^p + \|u^1\|_p^p. \tag{4.3}$$

Similarly,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(u_n(\cdot+y_n^1))\,dx = \int_{\mathbb{R}^N} F(u^1)\,dx + \lim_{n\to\infty}\int_{\mathbb{R}^N} F(v_n^1(\cdot+y_n^1))\,dx.$$

In addition, by [4], we obtain that

$$\lim_{n\to\infty}\left\|\nabla\left(v_n^1(\cdot+y_n^1)+u^1\right)\right\|_i^i=\left\|\nabla(u^1)\right\|_i^i+\lim_{n\to\infty}\left\|\nabla(v_n^1)\right\|_i^i,$$

where $i=p,q$. It follows that

$$\begin{aligned}E_c=\lim_{n\to\infty}I(u_n)&=\lim_{n\to\infty}I(u_n(\cdot+y_n^1))\\&=I(u^1)+\lim_{n\to\infty}I(v_n^1(\cdot+y_n^1))=I(u^1)+\lim_{n\to\infty}I(v_n^1).\end{aligned}\tag{4.4}$$

Next we claim that $\lim_{n\to\infty}I(v_n^1)\geq 0$. Otherwise, $\lim_{n\to\infty}I(v_n^1)<0$. Then $\{v_n^1\}$ is not-vanishing. So up to a subsequence, there exists a sequence $\{y_n^2\}\subset\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{B(y_n^2,1)}|v_n^1|^p\,dx>0.$$

Based on the fact that $v_n^1(\cdot+y_n^1)\to 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$, one has $|y_n^2-y_n^1|\to\infty$. Therefore, up to a subsequence, there exists $u^2\in A_c\backslash\{0\}$ such that $v_n^1(\cdot+y_n^2)\rightharpoonup u^2$ in E. Then

$$u_n(\cdot+y_n^2)=v_n^1(\cdot+y_n^2)+u^1(\cdot-y_n^1+y_n^2)\rightharpoonup u^2\quad\text{in } E.$$

Similarly, we also obtain that $P(u^2)=0$ and $I(u^2)>0$. Set

$$v_n^2:=v_n^1-u^2(\cdot-y_n^2)=u_n-\sum_{j=1}^{2}u^j(\cdot-y_n^j).$$

Consequently,

$$0>\lim_{n\to\infty}I(v_n^1)=I(u^2)+\lim_{n\to\infty}I(v_n^2)>\lim_{n\to\infty}I(v_n^2).$$

We can continue this way to obtain an infinite sequence $\{u^k\}\subset A_c\backslash\{0\}$ such that $P(u^k)=0$ and

$$\sum_{j=1}^{k}\|\nabla u^j\|_p^p\leq\lim_{n\to\infty}\|\nabla u_n\|_p^p<+\infty$$

for any $k\in\mathbb{N}^+$. This is impossible since similar to the proof of Lemma 2.1, we see the fact that there exists a $\delta>0$ such that $\|\nabla u\|_p\geq\delta$ for any $u\in A_c\backslash\{0\}$ satisfying $P(u)=0$. Thus $\lim_{n\to\infty}I(v_n^1)\geq 0$. In the following, we set $\tilde{c}:=\|u^1\|_p\in(0,c]$. It follows from $\lim_{n\to\infty}I(v_n^1)\geq 0$, $u^1\in\mathcal{P}_{\tilde{c}}$ and (4.4) that

$$E_c=I(u^1)+\lim_{n\to\infty}I(v_n^1)\geq I(u^1)\geq E_{\tilde{c}}.$$

Then using Lemma 3.2, one infers that

$$I(u^1) = E_{\tilde{c}} = E_c \qquad \lim_{n\to\infty} I(v_n^1) = 0. \tag{4.5}$$

Combining (4.2), (4.5) and Lemmas 3.2, 3.3, we deduce that $\lambda \geq 0$. Based on (f_4), it follows from $P(u^1) = 0$ and (4.2) that

$$\lambda = \frac{1}{c^p}\left(\int_{\mathbb{R}^N} NF(u^1) - \frac{N-p}{p} f(u^1)u^1\,dx + \delta_q \|\nabla u^1\|_q^q\right) > 0.$$

If $\tilde{c} < c$, using (4.2) and Lemmas 3.2, 3.3 again, we have $I(u^1) = E_{\tilde{c}} > E_c$, which contradicts (4.5). So $\tilde{c} := \|u^1\|_p = c$ and $\|v_n^1\|_p \to 0$ by (4.3). From Lemma 2.1(ii), we see that $\lim_{n\to\infty}\int_{\mathbb{R}^N} F(v_n^1)\,dx = 0$, which along with (4.5) yields that $\|\nabla v_n^1\|_p \to 0$ and $\|\nabla v_n^1\|_q \to 0$. Then $u_n(\cdot + y_n^1) \to u^1$ strongly in E. The proof of Lemma 4.3 is completed. □

Proof of Theorem 1.4. Applying Lemmas 2.5(iv) and 4.1, we obtain a bounded Palais-Smale sequence $\{u_n\} \subset \mathcal{P}_c$ for the constrained functional $I|_{S_c}$ at the level $E_c > 0$. By Lemma 4.3, we get the existence of a ground state $u \in S_c$ at the level E_c. Moreover, when f is odd, using Lemma 4.2, we deduce that $\|u_n^-\|_p \to 0$. Then in view of Lemma 4.3, we obtain a nonnegative ground state $u \in S_c$ at the level E_c. using the regularity in [22] and Harnack's inequality in [45], we can conclude that $u > 0$. □

Proof of Theorem 1.5. Based on Theorem 1.4, E_c is achieved by a ground state of (1.1) with the associated Lagrange multiplier being positive. Applying Lemmas 3.1, 3.2 and 3.4, we know that the function $c \mapsto E_c$ is positive, continuous, nonincreasing and $\lim_{c\to 0^+} E_c = +\infty$. Moreover, applying Lemmas 3.3, 3.5 and 4.3, we derive that E_c is strictly decreasing in $c > 0$ and $\lim_{c\to\infty} E_c = 0$. □

5. Proof of Theorem 1.9

Compared with Section 4, we want to study the existence of solutions for (1.1) with the level $\bar{E}_c := \inf_{\mathcal{M}_c} I$ and discuss the relationship between $\bar{E}_c$ and E_c, where $\mathcal{M}_c$ is defined in Theorem 1.9. In addition, using (g_7) and argument in [12], for any $R > 0$, one can find a radial function $u \in W_0^{1,p}(B(0,R)) \cap W_0^{1,q}(B(0,R)) \cap L^\infty(B(0,R))$ such that $\int_{\mathbb{R}^N} \bar{F}(u)\,dx > 0$. Then let

$$G(t) := t^p\|\nabla u\|_p^p + (\delta_q + 1)t^q\|\nabla u\|_q^q - \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(u)\,dx.$$

Since $1 < p < q < N$, we can easily know that there exists $t(u) \in \mathbb{R}$ such that $G(t(u)) = 0$. That is

$$u(t(u)\cdot) \in \mathcal{P}, \tag{5.1}$$

where

$$\mathcal{P} := \{u \in E\backslash\{0\} : P(u) = 0\}.$$

Hence $\mathcal{P}$ is nonempty.

Lemma 5.1. *Assume that* (g_1), (g_2), (g_4), (g_6) *and* (g_7) *hold. There holds* $\inf_{u\in\mathcal{M}_c}\|\nabla u\|_p>0$.

Proof. Using (g_2), (g_4) and (g_6), for any $\epsilon>0$, there exists $C_\epsilon>0$ such that

$$\bar{F}(u)\le(p^*-p)F(u)\le(p^*-p)\left(\epsilon|u|^{p^*}+(\epsilon+\kappa)|u|^{\bar{p}}+C_\epsilon|u|^m\right)$$

for any $u\in E$, where $m\in(\bar{p},p^*)$ and $\kappa:=\limsup_{|u|\to0}\frac{F(u)}{|u|^{\bar{p}}}$. Then since $u\in\mathcal{M}_c$, by (1.12), we get

$$\begin{aligned}
&\|\nabla u\|_p^p+(\delta_q+1)\|\nabla u\|_q^q\\
&=\frac{N}{p}\int_{\mathbb{R}^N}\bar{F}(u)\,dx\\
&\le\frac{N}{p}(p^*-p)\left[\epsilon\left(\|u\|_{p^*}^{p^*}+\|u\|_{\bar{p}}^{\bar{p}}\right)+\kappa\|u\|_{\bar{p}}^{\bar{p}}+C_\epsilon C_{N,m}^m c^{m(1-\delta_m)}\|\nabla u\|_p^{m\delta_m}\right]\\
&=p^*\left[\epsilon\left(\|u\|_{p^*}^{p^*}+\|u\|_{\bar{p}}^{\bar{p}}\right)+\kappa\|u\|_{\bar{p}}^{\bar{p}}+C_\epsilon C_{N,m}^m c^{m(1-\delta_m)}\|\nabla u\|_p^{m\delta_m}\right]\\
&\le\epsilon p^*S^{-\frac{p^*}{p}}\|\nabla u\|_p^{p^*}+\epsilon p^*C_{N,\bar{p}}^{\bar{p}}c^{\bar{p}(1-\delta_{\bar{p}})}\|\nabla u\|_p^{\bar{p}\delta_{\bar{p}}}+\kappa p^*C_{N,\bar{p}}^{\bar{p}}c^{\bar{p}(1-\delta_{\bar{p}})}\|\nabla u\|_p^{\bar{p}\delta_{\bar{p}}}\\
&\quad+p^*C_\epsilon C_{N,m}^m c^{m(1-\delta_m)}\|\nabla u\|_p^{\bar{p}\delta_{\bar{p}}},
\end{aligned}$$

which implies that $\|\nabla u\|_p$ is bounded away from 0 on $\mathcal{M}_c$. So the proof of Lemma 5.1 is completed. □

Now we define

$$H(r):=I\left(r^{\frac{N}{p}}u(r\cdot)\right),\quad r\in(0,\infty),\ u\in E\backslash\{0\}.$$

Lemma 5.2. *Assume that* (g_2), (g_4)-(g_6) *hold. Then there exists a unique* $r_0>0$ *such that* $r_0^{\frac{N}{p}}u(r_0\cdot)\in\mathcal{P}$.

Proof. Fix $u\in E\backslash\{0\}$. From (g_2),

$$H(r)=\frac{r^p}{p}\|\nabla u\|_p^p+\frac{r^{q(\delta_q+1)}}{q}\|\nabla u\|_q^q-r^{-N}\int_{\mathbb{R}^N}F(r^{\frac{N}{p}}u)\,dx\to0$$

as $r\to0^+$. In addition, set $R:=\|u\|_p=\|r^{\frac{N}{p}}u(r\cdot)\|_p>0$. Then it follows from (g_2), (g_4), (g_6) and (1.13) that for every $\epsilon>0$, there exists $C_\epsilon>0$ such that

$$\int_{\mathbb{R}^N}F(u)\,dx\le(\epsilon+\kappa)\|u\|_{\bar{p}}^{\bar{p}}+C_\epsilon\|u\|_{p^*}^{p^*}\le(\epsilon+\kappa)K_{N,\bar{p}}^{\bar{p}}\|\nabla u\|_q^{\bar{p}\gamma_{\bar{p}}}R^{\bar{p}(1-\gamma_{\bar{p}})}+C_\epsilon S^{-\frac{p^*}{p}}\|\nabla u\|_p^{p^*}.$$

Therefore,

$$\begin{aligned}\frac{H(r)}{r^{q(\delta_q+1)}} &= \frac{r^{p-q(\delta_q+1)}}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - r^{-q(\delta_q+1)}\int_{\mathbb{R}^N} F\left(r^{\frac{N}{p}}u(rx)\right)dx \\ &\geq \frac{1}{q}\|\nabla u\|_q^q - (\epsilon+\kappa)r^{\bar{p}\gamma_{\bar{p}}(\delta_q+1)-q(\delta_q+1)}K_{N,\bar{p}}^{\bar{p}}\|\nabla u\|_q^{\bar{p}\delta_{\bar{p}}}R^{\bar{p}(1-\gamma_{\bar{p}})} \\ &\quad - C_\epsilon r^{p^*-q(\delta_q+1)}S^{-\frac{p^*}{p}}\|\nabla u\|_p^{p^*}\end{aligned}$$

as $r \to 0^+$, which yields that $H(r) > 0$ for sufficiently small $r > 0$. On the other hand, it follows from (g_3) that

$$\frac{H(r)}{r^{q(\delta_q+1)}} = \frac{r^{p-q(\delta_q+1)}}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_{\mathbb{R}^N} \frac{F\left(r^{\frac{N}{p}}u\right)}{\left(r^{\frac{N}{p}}\right)^{\bar{p}}}dx \to -\infty$$

as $r \to \infty$. So H has a maximum at some $r_0 > 0$ and $H'(r_0) = 0$. That is,

$$0 = H'(r_0) = r_0^{p-1}\|\nabla u\|_p^p + (\delta_q+1)r_0^{q(\delta_q+1)-1}\|\nabla u\|_q^q - \frac{N}{p}r_0^{-N-1}\int_{\mathbb{R}^N}\bar{F}\left(r_0^{\frac{N}{p}}u\right)dx.$$

This means that $r_0^{\frac{N}{p}}u(r_0\cdot) \in \mathcal{P}$. Moreover,

$$\begin{aligned}H'(r) &= r^{p-1}\|\nabla u\|_p^p + (\delta_q+1)r^{q(\delta_q+1)-1}\|\nabla u\|_q^q - \frac{N}{p}r^{q(\delta_q+1)-1}\int_{\mathbb{R}^N}\frac{\bar{F}\left(r^{\frac{N}{p}}u\right)}{\left(r^{\frac{N}{p}}\right)^{\bar{p}}}dx \\ &= r^{q(\delta_q+1)-1}\left[r^{p-q(\delta_q+1)}\|\nabla u\|_p^p + (\delta_q+1)\|\nabla u\|_q^q - \frac{N}{p}\int_{\mathbb{R}^N}\frac{\bar{F}\left(r^{\frac{N}{p}}u\right)}{\left(r^{\frac{N}{p}}\right)^{\bar{p}}}dx\right]\end{aligned}$$

From (g_5), we see that $r \mapsto \int_{\mathbb{R}^N}\frac{\bar{F}\left(r^{\frac{N}{p}}u\right)}{\left(r^{\frac{N}{p}}\right)^{\bar{p}}}dx$ is strictly increasing. Thus we know that r_0 is unique. So the proof of Lemma 5.2 is completed. □

Lemma 5.3. *Assume that (g_1)-(g_6) hold. Then I is coercive on $\mathcal{M}_c$.*

Proof. First of all, for $u \in \mathcal{M}_c$, from (g_6), we have

$$I(u) = I(u) - \frac{1}{q(\delta_q+1)}P(u)$$

$$= \left(\frac{1}{p} - \frac{1}{q(\delta_q+1)}\right)\|\nabla u\|_p^p + \frac{N}{pq(\delta_q+1)}\int_{\mathbb{R}^N} \bar{F}(u)\,dx - \int_{\mathbb{R}^N} F(u)\,dx \geq 0,$$

which implies that $I(u)$ is bounded from below on $\mathcal{M}_c$. Next similar to the arguments in Lemma 2.5, we suppose that $\{u_n\} \subset \mathcal{M}_c$ is a sequence such that $\|u_n\|_E \to \infty$ and $I(u_n)$ is bounded from above. Without loss of generality, we assume that

$$\|\nabla u_n\|_q^{\frac{p}{\delta_q+1}} \gg \|\nabla u_n\|_p^p \to +\infty.$$

Then we set $r_n := \|\nabla u_n\|_q^{-\frac{1}{\delta_q+1}} > 0$ and define $v_n := r_n^{\frac{N}{p}} u_n(r_n\cdot)$. Observe that $r_n \to 0^+$ as $n \to \infty$. Then

$$\|v_n\|_p^p = \|u_n\|_p^p \leq c^p.$$

Furthermore,

$$\|\nabla v_n\|_p^p = r_n^p\|\nabla u_n\|_p^p = \frac{\|\nabla u_n\|_p^p}{\|\nabla u_n\|_q^{\frac{p}{\delta_q+1}}} \leq 1$$

and

$$\|\nabla v_n\|_q^q = r_n^{q(\delta_q+1)}\|\nabla u_n\|_q^q = 1.$$

Then $\{v_n\}$ is bounded in E. If we suppose that

$$\limsup_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B(y,1)} |v_n|^p\,dx\right) > 0,$$

then up to a subsequence, we can find translations $\{y_n\} \subset \mathbb{R}^N$ such that

$$v_n(\cdot + y_n) \rightharpoonup v \neq 0 \quad \text{in } E$$

and $v_n(x + y_n) \to v(x)$ for a.e. $x \in \mathbb{R}^N$. Then using (g_3), we deduce that

$$\begin{aligned}
0 \leq \frac{I(u_n)}{\|\nabla u_n\|_q^q} &= \frac{1}{p}\frac{\|\nabla u_n\|_p^p}{\|\nabla u_n\|_q^q} + \frac{1}{q} - \int_{\mathbb{R}^N} \frac{F(u_n)}{\|\nabla u_n\|_q^q}\,dx \\
&= \frac{r_n^{q(\delta_q+1)-p}}{p}\frac{\|\nabla v_n\|_p^p}{\|\nabla v_n\|_q^q} + \frac{1}{q} - r_n^{q(\delta_q+1)} \cdot r_n^N \int_{\mathbb{R}^N} F(u_n(r_n x))\,dx \\
&\leq \frac{r_n^{q(\delta_q+1)-p}}{p} + \frac{1}{q} - r_n^{N+q(\delta_q+1)}\int_{\mathbb{R}^N} F\left(r_n^{-\frac{N}{p}} v_n\right)dx
\end{aligned}$$

$$= \frac{r_n^{q(\delta_q+1)-p}}{p} + \frac{1}{q} - r_n^{N+q(\delta_q+1)} \int_{\mathbb{R}^N} \frac{F\left(r_n^{-\frac{N}{p}} v_n\right)}{\left|r_n^{-\frac{N}{p}} v_n\right|^{\bar{p}}} \cdot \left|r_n^{-\frac{N}{p}} v_n\right|^{\bar{p}} dx$$

$$= \frac{r_n^{q(\delta_q+1)-p}}{p} + \frac{1}{q} - \int_{\mathbb{R}^N} \frac{F\left(r_n^{-\frac{N}{p}} v_n\right)}{\left|r_n^{-\frac{N}{p}} v_n\right|^{\bar{p}}} \cdot |v_n|^{\bar{p}} dx$$

$$= \frac{r_n^{q(\delta_q+1)-p}}{p} + \frac{1}{q} - \int_{\mathbb{R}^N} \frac{F\left(r_n^{-\frac{N}{p}} v_n(x+y_n)\right)}{\left|r_n^{-\frac{N}{p}} v_n(x+y_n)\right|^{\bar{p}}} \cdot |v_n(x+y_n)|^{\bar{p}} dx \to -\infty$$

as $n \to \infty$. This is a contradiction. Hence we may assume that

$$\sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |v_n|^p dx \to 0.$$

Using Lemma I.1 in [29], we see that $v_n \to 0$ in $L^{\bar{p}}(\mathbb{R}^N)$. Note that $u_n = r_n^{-\frac{N}{p}} v_n\left(\frac{\cdot}{r_n}\right) \in \mathcal{M}_c$. Then it follows from Lemma 5.2 that

$$\begin{aligned} I(u_n) = I\left(r_n^{-\frac{N}{p}} v_n\left(\frac{\cdot}{r_n}\right)\right) \geq I\left(r^{\frac{N}{p}} v_n(r\cdot)\right) \\ &= \frac{r^p}{p}\|\nabla v_n\|_p^p + \frac{r^{q(\delta_q+1)}}{q}\|\nabla v_n\|_q^q - r^{-N}\int_{\mathbb{R}^N} F\left(r^{\frac{N}{p}} v_n\right) dx \\ &\geq \frac{r^{q(\delta_q+1)}}{q} - r^{-N}\int_{\mathbb{R}^N} F\left(r^{\frac{N}{p}} v_n\right) dx. \end{aligned}$$

Combining the property that $r^{-N}\int_{\mathbb{R}^N} F\left(r^{\frac{N}{p}} v_n\right) dx \to 0$ as $n \to \infty$, by taking sufficiently large $r > 0$, we can obtain a contradiction. So I is coercive on $\mathcal{M}_c$. □

Lemma 5.4. *Assume that* (g_1), (g_2), (g_4)-(g_7) *hold. Then*

$$\bar{E}_c := \inf_{\mathcal{M}_c} I > 0.$$

Proof. For any $u \in \mathcal{M}_c$, it follows from (1.12) that

$$\int_{\mathbb{R}^N} F(u)\, dx \leq (\epsilon + \kappa)\|u\|_{\bar{p}}^{\bar{p}} + C_\epsilon \|u\|_{p^*}^{p^*}$$

$$
\begin{aligned}
&\leq (\epsilon+\kappa) C_{N,\bar{p}}^{\bar{p}} c^{(1-\delta_{\bar{p}})\bar{p}} \|\nabla u\|_p^{\bar{p}\delta_{\bar{p}}} + C_\epsilon \cdot S^{-\frac{p^*}{p}} \|\nabla u\|_p^{p^*} \\
&= \left((\epsilon+\kappa) C_{N,\bar{p}}^{\bar{p}} c^{(1-\delta_{\bar{p}})\bar{p}} \|\nabla u\|_p^{\bar{p}\delta_{\bar{p}}-p} + C_\epsilon \cdot S^{-\frac{p^*}{p}} \|\nabla u\|_p^{p^*-p} \right) \|\nabla u\|_p^p.
\end{aligned}
$$

Hence

$$
\begin{aligned}
I(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_{\mathbb{R}^N} F(u)\,dx \\
&\geq \left(\frac{1}{p} - (\epsilon+\kappa) C_{N,\bar{p}}^{\bar{p}} c^{(1-\delta_{\bar{p}})\bar{p}} \|\nabla u\|_p^{\bar{p}\delta_{\bar{p}}-p} - C_\epsilon \cdot S^{-\frac{p^*}{p}} \|\nabla u\|_p^{p^*-p} \right) \|\nabla u\|_p^p.
\end{aligned}
$$

Then there exists $\hat{\delta} > 0$ such that $\|\nabla u\|_p \leq \hat{\delta}$ and $I(u) \geq \frac{1}{2p}\|\nabla u\|_p^p$. Fix $u \in \mathcal{M}_c$, it follows from Lemma 5.2 that for every $r > 0$, $I(u) \geq I\left(r^{\frac{N}{p}} u(r\cdot)\right)$. In particular, we choose $\tilde{r} := \frac{\hat{\delta}}{\|\nabla u\|_p} > 0$ and let $v = \tilde{r}^{\frac{N}{p}} u(\tilde{r}\cdot)$. Obviously, $\|v\|_p = \|u\|_p$ and $v \in S_c$. In addition, $\|\nabla v\|_p = \hat{\delta}$. Thus

$$
I(u) \geq I(v) \geq \frac{1}{2p}\|\nabla v\|_p^p = \frac{1}{2p}\hat{\delta}^p > 0.
$$

This means that $\bar{E}_c > 0$. The proof of Lemma 5.4 is completed. □

Lemma 5.5. *Suppose that $\{u_n\} \subset E$ is bounded. Then there exist sequences $\{\hat{u}_i\}_{i=0}^\infty \subset E$, $\{y_n^i\}_{i=0}^\infty \subset \mathbb{R}^N$ for any $n \geq 1$, such that $y_n^0 = 0$, $|y_n^i - y_n^j| \to \infty$ as $n \to \infty$ for $i \neq j$, and passing to a subsequence, the following results hold for any $i \geq 0$:*

$$
\begin{aligned}
&u_n(\cdot + y_n^i) \rightharpoonup \hat{u}_i \quad \text{in } E \text{ as } n \to \infty, \\
&\lim_{n\to\infty} \|\nabla u_n\|_p^p = \sum_{j=0}^{i} \|\nabla \hat{u}_j\|_p^p + \lim_{n\to\infty} \|\nabla v_n^i\|_p^p, \\
&\lim_{n\to\infty} \|\nabla u_n\|_q^q = \sum_{j=0}^{i} \|\nabla \hat{u}_j\|_q^q + \lim_{n\to\infty} \|\nabla v_n^i\|_q^q,
\end{aligned}
\tag{5.2}
$$

where $v_n^i := u_n - \sum_{j=0}^{i} \hat{u}_j(\cdot - y_n^j)$ and if

$$
\lim_{t\to 0} \frac{\bar{F}(t)}{|t|^p} = \lim_{|t|\to\infty} \frac{\bar{F}(t)}{|t|^{p^*}} = 0, \tag{5.3}
$$

then

$$
\limsup_{n\to\infty} \int_{\mathbb{R}^N} \bar{F}(u_n)\,dx = \sum_{j=0}^{\infty} \int_{\mathbb{R}^N} \bar{F}(\hat{u}_j)\,dx. \tag{5.4}
$$

Proof. Since the proof of Lemma 5.5 is similar to Theorem 1.4 in [34], for the convenience of the reader, we show the details of the proof. We claim that, passing to a subsequence, there exist sequences $k \in \mathbb{N} \cup \{\infty\}$, $\{\hat{u}_i\}_{i=0}^k \subset E$, for $0 \le i < k+1$ (if $k = \infty$, then $k+1 = \infty$ as well), $\{v_n^i\} \subset E$, $\{y_n^i\} \subset \mathbb{R}^N$ and positive numbers $\{c_i\}_{i=0}^k$, $\{r_i\}_{i=0}^k$ such that $y_n^0 = 0$, $r_0 = 0$ and for any $0 \le i < k+1$ one has

(1) $u_n(\cdot + y_n^i) \rightharpoonup \hat{u}_i$ in E and $u_n(\cdot + y_n^i)\chi_{B(0,n)} \to \hat{u}_i$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$.
(2) $\hat{u}_i \neq 0$ if $i \ge 1$.
(3) $|y_n^i - y_n^j| \ge n - r_i - r_j$ for $0 \le j \ne i < k+1$ and sufficiently large n.
(4) $v_n^{-1} := u_n$ and $v_n^i := v_n^{i-1} - \hat{u}_i(\cdot - y_n^i)$ for $n \ge 1$
(5) $\int_{B(y_n^i, r_i)} |v_n^{i-1}|^p\,dx \ge c_i \ge \frac{1}{2}\sup_{y\in\mathbb{R}^N}\int_{B(y,r_i)} |v_n^{i-1}|^p\,dx$ for sufficiently large n, $r_i \ge \max\{i, r_{i-1}\}$, if $i \ge 1$, and

$$c_i = \frac{3}{4}\lim_{r\to\infty}\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,r)} |v_n^{i-1}|^p\,dx > 0$$

(6)

$$\lim_{n\to\infty}\|\nabla u_n\|_p^p = \sum_{j=0}^{i}\|\nabla \hat{u}_j\|_p^p + \lim_{n\to\infty}\|\nabla v_n^i\|_p^p,$$

$$\lim_{n\to\infty}\|\nabla u_n\|_q^q = \sum_{j=0}^{i}\|\nabla \hat{u}_j\|_q^q + \lim_{n\to\infty}\|\nabla v_n^i\|_q^q,$$

Let $\{u_n\} \subset E$ be a bounded sequence. Passing to a subsequence, we may assume that $\lim_{n\to\infty}\|\nabla u_n\|_p^p$, $\lim_{n\to\infty}\|\nabla u_n\|_q^q$ exists and

$$u_n \rightharpoonup \hat{u}_0 \quad \text{in } E,$$
$$u_n\chi_{B(0,n)} \to \hat{u}_0 \quad \text{in } L^p(\mathbb{R}^N),$$

where $\chi_{B(0,n)}$ is the characteristic function of $B(0,n)$. Take $v_n^0 := u_n - \hat{u}_0$ and if

$$\lim_{n\to\infty}\sup_{y\mathbb{R}^N}\int_{B(y,r)} |v_n^0|^p\,dx = 0$$

for every $r \ge 1$, then we can finish the proof of our claim with $k = 0$. Otherwise we get

$$\infty > \sup_{n\ge 1}\int_{\mathbb{R}^N} |v_n^0|^p\,dx \ge c_1 := \frac{3}{4}\lim_{r\to\infty}\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,r)} |v_n^0|^p\,dx > 0$$

and there exists $r_1 \ge 1$ and, passing to subsequence, we find $\{y_n^1\} \subset \mathbb{R}^N$ such that

$$\int_{B(y_n^1, r_1)} |v_n^0|^p\,dx \ge c_1 \ge \frac{1}{2}\sup_{y\in\mathbb{R}^N}\int_{B(y,r_1)} |v_n^0|^p\,dx. \tag{5.5}$$

Observe that $\{y_n^1\}$ is unbounded and we may suppose that $|y_n^1| \geq n - r_1$. Since $\{u_n(\cdot + y_n^1)\}$ is bounded in E, up to a subsequence, we find that there exists $\hat{u}_1 \in E$ such that

$$u_n(\cdot + y_n^1) \rightharpoonup \hat{u}_1 \quad \text{in } E.$$

By (5.5), we deduce that $\hat{u}_1 \neq 0$, and we may suppose that $u_n(\cdot + y_n^1)\chi_{B(0,n)} \to \hat{u}_1$ in $L^p(\mathbb{R}^N)$. Since

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N} |\nabla(u_n - \hat{u}_0)(\cdot + y_n^1)|^p\, dx - \int_{\mathbb{R}^N} |\nabla v_n^1(\cdot + y_n^1)|^p\, dx\right) = \int_{\mathbb{R}^N} |\nabla \hat{u}_1|^p\, dx$$

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N} |\nabla(u_n - \hat{u}_0)(\cdot + y_n^1)|^q\, dx - \int_{\mathbb{R}^N} |\nabla v_n^1(\cdot + y_n^1)|^q\, dx\right) = \int_{\mathbb{R}^N} |\nabla \hat{u}_1|^q\, dx,$$

where $v_n^1 := v_n^0 - \hat{u}_1(\cdot - y_n^1) = u_n - \hat{u}_0 - \hat{u}_1(\cdot - y_n^1)$, one has

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^p\, dx = \int_{\mathbb{R}^N} |\nabla \hat{u}_0|^p\, dx + \int_{\mathbb{R}^N} |\nabla \hat{u}_1|^p\, dx + \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n^1|^p\, dx,$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^q\, dx = \int_{\mathbb{R}^N} |\nabla \hat{u}_0|^q\, dx + \int_{\mathbb{R}^N} |\nabla \hat{u}_1|^q\, dx + \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n^1|^q\, dx.$$

If

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,r)} |v_n^1|^p\, dx = 0$$

for every $r \geq \max\{2, r_1\}$, then we can complete the proof of our claim with $k = 1$. Otherwise,

$$c_2 := \frac{3}{4}\lim_{r\to\infty}\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,r)} |v_n^1|^p\, dx > 0.$$

Then there exists $r_2 \geq \max\{2, r_1\}$ and, passing to a subsequence, we find $\{y_n^2\} \subset \mathbb{R}^N$ such that

$$\int_{B(y_n^2,r_2)} |v_n^1|^p\, dx \geq c_2 \geq \frac{1}{2}\sup_{y\in\mathbb{R}^N}\int_{B(y,r_2)} |v_n^1|^p\, dx \tag{5.6}$$

and $|y_n^2| \geq n - r_2$. Furthermore, $|y_n^2 - y_n^1| \geq n - r_2 - r_1$. Otherwise, $B(y_n^2, r_2) \subset B(y_n^1, n)$ and the convergence $u_n(\cdot + y_n^1)\chi_{B(0,n)} \to \hat{u}_1$ in $L^p(\mathbb{R}^N)$, which contradicts with (5.6). Then, passing to a subsequence, we find $\hat{u}_2 \neq 0$ such that

$$v_n^1(\cdot + y_n^2),\ u_n(\cdot + y_n^2) \rightharpoonup \hat{u}_2 \quad \text{in } E$$

and

$$u_n(\cdot + y_n^2)\chi_{B(0,n)} \to \hat{u}_2 \quad \text{in } L^p(\mathbb{R}^N).$$

Similarly, if

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |v_n^2|^p \, dx = 0$$

for every $r \geq \max\{3, r_2\}$, where $v_n^2 := v_n^1 - \hat{u}_2(\cdot - y_n^2)$, then we complete the proof with $k = 2$. Continuing the above procedure, for each $i \geq 1$, we find a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, satisfies (1)-(6). Similarly as above, if there exists $i \geq 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |v_n^i|^p \, dx = 0 \tag{5.7}$$

for every $r \geq \max\{n, r_{i-1}\}$, then $k = i$ and we complete the proof of claim. Otherwise, $k = \infty$. Using the standard diagonal method and passing to a subsequence, we show that (1)-(6) are satisfied for every $i \geq 0$.

Next, we show that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \bar{F}(u_n)\, dx = \sum_{j=0}^{i} \int_{\mathbb{R}^N} \bar{F}(\hat{u}_j)\, dx + \limsup_{n\to\infty} \int_{\mathbb{R}^N} \bar{F}(v_n^i)\, dx. \tag{5.8}$$

It follows from Vitali's convergence theorem that

$$\begin{aligned}
\int_{\mathbb{R}^N} \bar{F}(u_n) - \bar{F}(v_n^0)\, dx &= \int_{\mathbb{R}^N} \int_0^1 -\frac{d}{ds}\bar{F}(u_n - s\hat{u}_0)\, ds dx \\
&= \int_{\mathbb{R}^N} \int_0^1 \bar{F}'(u_n - s\hat{u}_0)\hat{u}_0\, ds dx \\
&= \int_0^1 \int_{\mathbb{R}^N} \bar{F}'(\hat{u}_0 - s\hat{u}_0)\hat{u}_0\, ds dx + o_n(1) \\
&= \int_{\mathbb{R}^N} \int_0^1 -\frac{d}{ds}\bar{F}(\hat{u}_0 - s\hat{u}_0)\, ds dx + o_n(1) \\
&= \int_{\mathbb{R}^N} \bar{F}(\hat{u}_0)\, dx + o_n(1),
\end{aligned}$$

which yields that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\bar{F}(u_n)-\bar{F}(v_n^0)\,dx=\int_{\mathbb{R}^N}\bar{F}(\hat{u}_0)\,dx.$$

As a consequence,

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\bar{F}(u_n)\,dx=\int_{\mathbb{R}^N}\bar{F}(\hat{u}_0)\,dx+\limsup_{n\to\infty}\int_{\mathbb{R}^N}\bar{F}(v_n^0)\,dx. \tag{5.9}$$

So (5.8) holds for $i=0$. Similarly, we show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\bar{F}((u_n-\hat{u}_0)(\cdot+y_n^1))-\bar{F}(v_n^1(\cdot+y_n^1))\,dx=\int_{\mathbb{R}^N}\bar{F}(\hat{u}_1)\,dx.$$

Based on (5.9), we derive that

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\bar{F}(u_n)\,dx&=\int_{\mathbb{R}^N}\bar{F}(\hat{u}_0)\,dx+\limsup_{n\to\infty}\int_{\mathbb{R}^N}\bar{F}(u_n-\hat{u}_0)\,dx\\&=\int_{\mathbb{R}^N}\bar{F}(\hat{u}_0)\,dx+\int_{\mathbb{R}^N}\bar{F}(\hat{u}_1)\,dx+\limsup_{n\to\infty}\int_{\mathbb{R}^N}\bar{F}(v_n^1)\,dx.\end{aligned}$$

Continuing the above procedure we obtain that (5.8) holds for every $i\geq 0$. In the last, we show that

$$\lim_{i\to\infty}\left(\limsup_{n\to\infty}\int_{\mathbb{R}^N}\bar{F}(v_n^i)\,dx\right)=0. \tag{5.10}$$

Observe that if there exists $i\geq 0$ such that (5.7) holds for every $r\geq\max\{i,r_i\}$, then $k=i$. If (5.3) holds, then we can easily obtain that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\bar{F}(v_n^i)\,dx=0.$$

Hence we complete the proof by setting $\hat{u}_j=0$ for $j>i$. Otherwise $k=\infty$. From (5), we infer that

$$c_{l+1}\leq\int_{B(y_n^{l+1},r_{l+1})}|v_n^l|^p\,dx$$

$$\leq p \int_{B(y_n^{l+1},r_{l+1})} |v_n^i|^p\,dx + p \int_{B(y_n^{l+1},r_{l+1})} \left|\sum_{j=i+1}^{l} \hat{u}_j(\cdot - y_n^j)\right|^p dx$$

$$\leq p \sup_{y\in\mathbb{R}^N} \int_{B(y,r_{l+1})} |v_n^i|^p\,dx + p(k-i)\sum_{j=i+1}^{k} \int_{B(y_n^{l+1}-y_n^j,r_{l+1})} |\hat{u}_j|^p\,dx$$

for any $0 \leq i < l$, which along with (3) yields that letting $n \to \infty$, we get $c_{l+1} \leq \frac{4p}{3} c_{i+1}$. Choose $l \geq 1$ and sufficiently large $n > 4r_l$ such that (3) and (5) are satisfied. Then

$$\begin{aligned}
&\frac{3}{32} \sup_{y\in\mathbb{R}^N} \int_{B(y,r_{l+1})} |v_n^l|^p\,dx \\
&\leq \frac{3}{16} c_{l+1} \leq \frac{1}{2l}\sum_{i=0}^{l-1} c_{i+1} \leq \frac{1}{2l}\sum_{i=0}^{l-1} \int_{B(y_n^{i+1},r_{i+1})} |v_n^i|^p\,dx \\
&\leq \frac{p}{2l}\sum_{i=0}^{l-1} \int_{B(y_n^{i+1},r_{i+1})} \left(|u_n|^p + \left|\sum_{j=0}^{i} \hat{u}_j(\cdot - y_n^j)\right|^p\right) dx \\
&= \frac{p}{2l} \int_{\cup_{i=0}^{l-1} B(y_n^{i+1},r_{i+1})} |u_n|^p\,dx + \frac{p}{2l}\int_{\mathbb{R}^N} \left|\sum_{i=0}^{l-1}\sum_{j=0}^{i} \hat{u}_j(\cdot - y_n^j)\chi_{B(y_n^{i+1},r_{i+1})}\right|^p dx \\
&\leq \frac{p}{2l}\|u_n\|_p^p + \frac{p}{2l}\left\|\sum_{i=0}^{l-1}\sum_{j=0}^{i} \hat{u}_j(\cdot - y_n^j)\chi_{B(y_n^{i+1},r_{i+1})}\right\|_p^p .
\end{aligned}$$

In addition, using (3) and since $n > 4r_l$, we have

$$B(y_n^{i+1} - y_n^i, r_{i+1}) \subset \mathbb{R}^N \setminus B(0, n-3r_l) \quad \text{for } 0 \leq j < i < l$$

and

$$\begin{aligned}
\left\|\sum_{i=0}^{l-1}\sum_{j=0}^{i} \hat{u}_j(\cdot - y_n^j)\chi_{B(y_n^{i+1},r_{i+1})}\right\|_p &\leq \sum_{i=0}^{l-1}\sum_{j=0}^{i} \|\hat{u}_j \chi_{B(y_n^{i+1}-y_n^i,r_{i+1})}\|_p \\
&\leq \sum_{i=0}^{l-1}\sum_{j=0}^{i} \|\hat{u}_j \chi_{\mathbb{R}^N\setminus B(0,n-3r_l)}\|_p \\
&\leq l \sum_{j=0}^{l-1} \|\hat{u}_j \chi_{B(y_n^{i+1}-y_n^i,r_{i+1})}\|_p \to 0
\end{aligned}$$

as $n \to \infty$. So

$$\limsup_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int\limits_{B(y,r_{l+1})}|v_n^l|^p\,dx\right)\le\frac{16p}{3l}\limsup_{n\to\infty}\|u_n\|_p^p. \tag{5.11}$$

Assume by contradiction that (5.10) does not hold, namely, there exists a $\delta > 0$ such that

$$\limsup_{i\to\infty}\left(\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N}\bar{F}(v_n^i)\,dx\right)>\delta. \tag{5.12}$$

Then we see that increasing sequences $\{i_l\}$, $\{n_l\}\subset\mathbb{N}$ such that

$$\int\limits_{\mathbb{R}^N}\bar{F}(v_{n_l}^{i_l})\,dx>\delta$$

and

$$\sup_{y\in\mathbb{R}^N}\int\limits_{B(y,r_{l+1})}|v_{n_l}^{i_l}|^p\le\limsup_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int\limits_{B(y,r_{l+1})}|v_n^{i_l}|^p\,dx\right)+\frac{1}{i_l}.$$

Based on (5.11), we obtain that

$$\lim_{l\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int\limits_{B(y,r_{l+1})}|v_{n_l}^{i_l}|^p\,dx\right)=0.$$

Then

$$\lim_{l\to\infty}\int\limits_{\mathbb{R}^N}\bar{F}(v_{n_l}^{i_l})\,dx=0.$$

This is impossible. Thus (5.10) holds. It results to (5.4). The proof of Lemma 5.5 is completed. □

Lemma 5.6. *Assume that* (g_1)*-*(g_6) *hold. Then* $\bar{E}_c=\inf_{\mathcal{M}_c} I$ *is achieved. Moreover, if* f *is odd, then* $\bar{E}_c$ *is achieved by a nonnegative and radially symmetric function in* $\mathcal{M}_c$*.*

Proof. First of all, we choose any $\{u_n\}\subset\mathcal{M}_c$ such that $I(u_n)\to\bar{E}_c$. By Lemma 5.3, we know that $\{u_n\}$ is bounded in E. Then it follows from (g_2), (g_4), (g_6) and Lemma 5.5 that there exist a profile decomposition of $\{u_n\}$ satisfying (5.2) and (5.4). Now we claim that

$$0 < \|\nabla \hat{u}_i\|_p^p + (\delta_q + 1)\|\nabla \hat{u}_i\|_q^q \le \frac{N}{p} \int_{\mathbb{R}^N} \bar{F}(\hat{u}_i)\, dx$$

for some $i \ge 0$. Let $\mathcal{I} := \left\{ i \ge 0 : \hat{u}_i \ne 0 \right\}$. Based on Lemma 5.1 and (5.4), $\mathcal{I} \ne \emptyset$. Assume by contradiction that

$$\|\nabla \hat{u}_i\|_p^p + (\delta_q + 1)\|\nabla \hat{u}_i\|_q^q > \frac{N}{p} \int_{\mathbb{R}^N} \bar{F}(\hat{u}_i)\, dx$$

for all $i \in \mathcal{I}$. Then by (5.2) and (5.4), we derive that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{N}{p} \int_{\mathbb{R}} \bar{F}(u_n)\, dx &= \limsup_{n\to\infty} \|\nabla u_n\|_p^p + \limsup_{n\to\infty} (\delta_q + 1)\|\nabla u_n\|_q^q \\
&\ge \sum_{j=0}^{\infty} \|\nabla \hat{u}_j\|_p^p + (\delta_q + 1) \sum_{j=0}^{\infty} \|\nabla \hat{u}_j\|_q^q \\
&= \sum_{j\in\mathcal{I}} \|\nabla \hat{u}_j\|_p^p + (\delta_q + 1) \sum_{j\in\mathcal{I}} \|\nabla \hat{u}_j\|_q^q \\
&\ge \sum_{j=0}^{\infty} \frac{N}{p} \int_{\mathbb{R}^N} \bar{F}(\hat{u}_j)\, dx = \limsup_{n\to\infty} \frac{N}{p} \int_{\mathbb{R}^N} \bar{F}(u_n)\, dx.
\end{aligned}$$

This is a contradiction. Thus there is $i \in \mathcal{I}$ such that $t(\hat{u}_i) \ge 1$ and $\hat{u}_i(t(\hat{u}_i)\cdot) \in \mathcal{P}$, where $t(u)$ is defined in (5.1). In addition,

$$\|\hat{u}_i(t(\hat{u}_i)\cdot)\|_p^p = t(\hat{u}_i)^{-N} \|\hat{u}_i\|_p^p \le t(\hat{u}_i)^{-N} c^p \le c^p.$$

Then $\hat{u}_i(t(\hat{u}_i)\cdot) \in \mathcal{M}_c$. In particular, if $t(\hat{u}_i) > 1$, then passing to a subsequence $u_n(x + y_n^i) \to \hat{u}_i(x)$ for a.e. $x \in \mathbb{R}^N$. Based on the Fatou's lemma,

$$\begin{aligned}
0 < \inf_{\mathcal{M}_c} I &\le I(\hat{u}_i(t(\hat{u}_i)\cdot)) \\
&= t^{p-N}(\hat{u}_i) \left(\frac{1}{p} - \frac{1}{q(\delta_q + 1)} \right) \|\nabla \hat{u}_i\|_p^p \\
&\quad + t^{-N}(\hat{u}_i) \left[\frac{N}{pq(\delta_q + 1)} \int_{\mathbb{R}^N} \bar{F}(\hat{u}_i)\, dx - \int_{\mathbb{R}^N} F(\hat{u}_i)\, dx \right] \\
&< \left(\frac{1}{p} - \frac{1}{q(\delta_q + 1)} \right) \|\nabla \hat{u}_i\|_p^p + \left(\frac{N}{pq(\delta_q + 1)} \int_{\mathbb{R}^N} \bar{F}(\hat{u}_i)\, dx - \int_{\mathbb{R}^N} F(\hat{u}_i)\, dx \right) \\
&\le \liminf_{n\to\infty} \left[\left(\frac{1}{p} - \frac{1}{q(\delta_q + 1)} \right) \|\nabla u_n(\cdot + y_n^i)\|_p^p \right.
\end{aligned}$$

$$+\left(\frac{N}{pq(\delta_q+1)}\int_{\mathbb{R}^N}\bar{F}(u_n(\cdot+y_n^i))\,dx-\int_{\mathbb{R}^N}F(u_n(\cdot+y_n^i))\,dx\right)\Bigg]$$
$$=\liminf_{n\to\infty} I(u_n)=\bar{E}_c=\inf_{\mathcal{M}_c} I.$$

This is impossible. Therefore $t(\hat{u}_i)=1$, $\hat{u}_i\in\mathcal{M}_c$ and $I(\hat{u}_i)=c$.

Suppose that f is odd. Then $F(u)$ and $\bar{F}(u)$ are even, namely, $F(|u|)=F(u)$ and $\bar{F}(|u|)=\bar{F}(u)$ for all $u\in E$. Set $\hat{v}_i:=|\hat{u}_i|^*$ as the Schwarz symmetrization of $|\hat{u}_i|$. Then $\|\hat{v}_i\|_p=\|\hat{u}_i\|_p$ and $\hat{v}_i\in S_c$. Furthermore, by

$$\|\nabla(\hat{v}_i)\|_p^p+(\delta_q+1)\|\nabla(\hat{v}_i)\|_q^q\le\|\nabla(\hat{u}_i)\|_p^p+(\delta_q+1)\|\nabla(\hat{u}_i)\|_q^q=\frac{N}{p}\int_{\mathbb{R}^N}\bar{F}(\hat{u}_i)\,dx$$
$$=\frac{N}{p}\int_{\mathbb{R}^N}\bar{F}(\hat{v}_i)\,dx$$

Similarly, we get $t(\hat{v}_i)=1$ and $\hat{v}_i\in\mathcal{M}_c$, where $t(\hat{v}_i)$ is defined in (5.1). Moreover, $I(\hat{v}_i)=\inf_{\mathcal{M}_c} I$, $\hat{v}_i\ge 0$ and $\hat{v}_i$ is radially symmetric. □

Lemma 5.7. *Assume that* (g_1)-(g_6) *hold. Then for any* $u\in(A_c\backslash S_c)\cap\mathcal{P}$, *there holds*

$$\inf_{\mathcal{P}_c} I<I(u).$$

Proof. Assume by contradiction that there exists $\check{u}\in\mathcal{P}$ such that $\|\check{u}\|_p<c$ and $\bar{E}_c=I(\check{u})\le\inf_{\mathcal{P}_c} I$. Hence $\check{u}$ is a local minimizer for I on $\mathcal{M}_c$. On the other hand, $(A_c\backslash S_c)\cap\mathcal{P}$ is an open set in $\mathcal{P}$, we find that $\check{u}$ is a local minimizer of I on $\mathcal{P}$. Hence there is a Lagrange multiplier $\check{\lambda}\in\mathbb{R}$ such that

$$I'(\check{u})v+\check{\lambda}\cdot\left(p\int_{\mathbb{R}^N}|\nabla\check{u}|^{p-2}\nabla\check{u}\nabla v\,dx+q(\delta_q+1)\int_{\mathbb{R}^N}|\nabla\check{u}|^{q-2}\nabla\check{u}\nabla v\,dx-\frac{N}{p}\int_{\mathbb{R}^N}\bar{F}'(\check{u})v\,dx\right)=0$$

for any $v\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$. Hence $\check{u}$ is a weak solution to

$$-(1+\check{\lambda}p)\Delta_p\check{u}-(1+\check{\lambda}q(\delta_q+1))\Delta_q\check{u}=f(\check{u})+\frac{N\check{\lambda}}{p}\bar{F}'(\check{u}).$$

In particular, $\check{u}$ satisfies the following Nehari-type identity

$$(1+\check{\lambda}p)\|\nabla\check{u}\|_p^p+(1+\check{\lambda}q(\delta_q+1))\|\nabla\check{u}\|_q^q=\int_{\mathbb{R}^N}f(\check{u})\check{u}\,dx+\frac{N\check{\lambda}}{p}\int_{\mathbb{R}^N}\bar{F}'(\check{u})\check{u}\,dx$$

If we take $\check{\lambda} = -\frac{1}{q(\delta_q+1)}$, then

$$\left(1 - \frac{p}{q(\delta_q+1)}\right)\|\nabla \check{u}\|_p^p = \int_{\mathbb{R}^N} f(\check{u})\check{u}\,dx - \frac{N}{pq(\delta_q+1)}\int_{\mathbb{R}^N} \bar{F}'(\check{u})\check{u}\,dx. \tag{5.13}$$

In addition, it follows from (g_5) and (g_6) that

$$\begin{aligned}\int_{\mathbb{R}^N} f(\check{u})\check{u}\,dx - \frac{N}{pq(\delta_q+1)}\int_{\mathbb{R}^N} \bar{F}'(\check{u})\check{u}\,dx &\le \int_{\mathbb{R}^N} f(\check{u})\check{u}\,dx - \frac{N}{pq(\delta_q+1)}\int_{\mathbb{R}^N} \bar{p}\bar{F}(\check{u})\,dx \\ &= p\int_{\mathbb{R}^N} F(\check{u})\,dx - \frac{p}{\bar{p}-p}\int_{\mathbb{R}^N} \bar{F}(\check{u})\,dx \le 0,\end{aligned}$$

which contradicts with (5.13). So $\check{\lambda} \neq -\frac{1}{q(\delta_q+1)}$. Moreover, on the one hand, since $\check{u} \in \mathcal{P}$, we get

$$\|\nabla \check{u}\|_p^p + (\delta_q+1)\|\nabla \check{u}\|_q^q = \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(\check{u})\,dx. \tag{5.14}$$

On the other hand, $\check{u}$ satisfies Nehari-type and Pohozaev identities. That is, $\check{u}$ satisfies

$$\begin{aligned}&(1+\check{\lambda}p)\|\nabla \check{u}\|_p^p + (\delta_q+1)(1+\check{\lambda}q(\delta_q+1))\|\nabla \check{u}\|_q^q \\ &\quad = \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(\check{u})\,dx + \check{\lambda}\int_{\mathbb{R}^N} \frac{N^2}{p^2}\bar{F}'(\check{u})\check{u} - \frac{N^2}{p}\bar{F}(\check{u})\,dx.\end{aligned} \tag{5.15}$$

Combining (5.14) and (5.15), we deduce that

$$\check{\lambda}(p - q(\delta_q+1))\|\nabla \check{u}\|_p^p = \check{\lambda}\frac{N^2}{p^2}\int_{\mathbb{R}^N} \bar{F}'(\check{u})\check{u} - \bar{p}\bar{F}(\check{u})\,dx.$$

In view of (g_5), we find that $\check{\lambda} = 0$ and $\check{u}$ is a weak solution to

$$-\Delta_p \check{u} - \Delta_q \check{u} = f(\check{u}).$$

Similarly, we also obtain that $\check{u}$ satisfies

$$\|\nabla \check{u}\|_p^p + \|\nabla \check{u}\|_q^q = \int_{\mathbb{R}^N} f(\check{u})\check{u}\,dx \tag{5.16}$$

and

$$\|\nabla \check{u}\|_p^p + (\delta_q+1)\|\nabla \check{u}\|_q^q = \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(\check{u})\,dx \tag{5.17}$$

Combining (5.16) and (5.17), we deduce that

$$\delta_q \|\nabla \check{u}\|_q^q = \frac{N-p}{p}\left[\int_{\mathbb{R}^N} \bar{F}(\check{u})\,dx - (p^* - p)\int_{\mathbb{R}^N} F(\check{u})\,dx\right].$$

By (g_6), we obtain a contradiction. So for any $u \in (A_c \backslash S_c) \cap \mathcal{P}$, there holds

$$\inf_{\mathcal{P}_c} I < I(u).$$

The proof of Lemma 5.7 is completed. □

Proof of Theorem 1.9. Using Lemmas 5.6 and 5.7, we derive that $E_c = \inf_{\mathcal{P}_c} I$ is attained. Moreover, if f is odd, then by the regularity in [22] and Harnack's inequality in [45], we know that $E_c = \inf_{\mathcal{P}_c} I$ is achieved by $\tilde{u} > 0$, which is a radially symmetric function. Now there exist Lagrange multipliers $\lambda,\ \mu \in \mathbb{R}$ such that $\tilde{u} \in \mathcal{P}_c$ solves

$$\begin{aligned} &-\Delta_p \tilde{u} - \Delta_q \tilde{u} - f(\tilde{u}) + \lambda |\tilde{u}|^{p-2}\tilde{u} + \mu \\ &\quad \cdot \left(-p\Delta_p \tilde{u} - q(\delta_q + 1)\Delta_q \tilde{u} - \frac{N}{p}\bar{F}'(\tilde{u})\right) = 0, \end{aligned}$$

namely,

$$-(1+\mu p)\Delta_p \tilde{u} - \big(1 + \mu \cdot q(\delta_q + 1)\big)\,\Delta_q \tilde{u} + \lambda |\tilde{u}|^{p-2}\tilde{u} = f(\tilde{u}) + \frac{N}{p}\mu \bar{F}'(\tilde{u}).$$

Similar to Lemma 5.7, we can find that $\mu = 0$. Finally, by (g_5) and (g_6), one has

$$\begin{aligned} \lambda \|\tilde{u}\|_p^p &= -\left(1 - \frac{p}{q(\delta_q+1)}\right)\|\nabla \tilde{u}\|_p^p + \int_{\mathbb{R}^N} f(\tilde{u})\tilde{u}\,dx - \frac{N}{pq(\delta_q+1)}\int_{\mathbb{R}^N} \bar{F}'(\tilde{u})\tilde{u}\,dx \\ &\le -\left(1 - \frac{p}{q(\delta_q+1)}\right)\|\nabla \tilde{u}\|_p^p + p\int_{\mathbb{R}^N} F(\tilde{u})\,dx - \frac{p}{\bar{p}-p}\int_{\mathbb{R}^N} \bar{F}(\tilde{u})\,dx < 0, \end{aligned}$$

which implies that $\lambda < 0$. The proof of Theorem 1.9 is completed. □

Data availability

No data was used for the research described in the article.

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