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Fractional Choquard-Kirchhoff problems with critical nonlinearity and Hardy potential

Wenjing Chen[1] · Vicenţiu D. Rădulescu[2,3] · Binlin Zhang[4]

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Abstract

In this paper, we investigate the following fractional p-Kirchhoff type problem

$$\begin{cases} \left(a + b[u]_{s,p}^{p(\theta-1)}\right)(-\Delta)_p^s u = \left(\mathcal{I}_\mu * |u|^q\right)|u|^{q-2}u + \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha}, \ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy$, Ω is a bounded smooth domain of $\mathbb{R}^N$ containing 0 with Lipschitz boundary, $(-\Delta)_p^s$ denotes the fractional p-Laplacian, $0 \le \alpha < ps < N$ with $s \in (0,1)$, $p > 1$, $a \ge 0$, $b > 0$, $1 < \theta \le p_\alpha^*/p$, $p_\alpha^* = \frac{(N-\alpha)p}{N-ps}$ is the fractional critical Hardy-Sobolev exponent, $\mathcal{I}_\mu(x) = |x|^{-\mu}$ is the Riesz potential of order $\mu \in (0, \min\{N, 2ps\})$, $q \in (1, Np/(N-ps))$ satisfies some restrictions. By the concentration-compactness principle and mountain pass theorem, we obtain the existence of a positive weak solution for the above problem as q satisfies suitable ranges.

✉ Binlin Zhang
zhangbinlin2012@163.com

Wenjing Chen
wjchen@swu.edu.cn

Vicenţiu D. Rădulescu
radulescu@inf.ucv.ro

1 School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China

2 Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland

3 Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania

4 College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao 266590, People's Republic of China

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1 Introduction and main results

This paper is devoted to the qualitative analysis of solutions for a class of p–fractional Choquard–Kirchhoff problems. The features of this paper are the following:

(1) we are concerned with the existence of solutions if the reaction has a *critical* growth;
(2) the problem has a *singular* behavior due to the presence of a Hardy potential;
(3) the *lack of compactness* is overcome by using the concentration-compactness principle.

In the first part of this section, we recall some significant historical comments related to the development of Choquard-type problems. The main result and some related comments are described in the second part of the present section.

1.1 Historical comments

The Choquard equation

$$-\Delta u+u=\left(\frac{1}{|x|}*|u|^2\right)u,\ \text{ in } \mathbb{R}^3. \tag{1.1}$$

was first introduced in the pioneering work of Fröhlich [1] and Pekar [2] for the modeling of quantum polaron. This model corresponds to the study of how free electrons in an ionic lattice interact with phonons associated to deformations of the lattice or with the polarisation that it creates on the medium (interaction of an electron with its own hole). In the approximation to Hartree-Fock theory of one component plasma, Choquard used Eq. (1.1) to describe an electron trapped in its own hole,

The Choquard equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. The equation can also be derived from the Einstein-Klein-Gordon and Einstein-Dirac system. Such a model was proposed for boson stars and for the collapse of galaxy fluctuations of scalar field dark matter. We refer for details to Elgart and Schlein [3], Giulini and Großardt [4], Jones [5], and Schunck and Mielke [6]. Penrose [7,8] proposed Eq. (1.1) as a model of self-gravitating matter in which quantum state reduction was understood as a gravitational phenomenon.

As pointed out by Lieb [9], Choquard used Eq. (1.1) to study steady states of the one component plasma approximation in the Hartree-Fock theory. Classification of solutions of (1.1) was first studied by Ma and Zhao [10]. For a broad survey of Choquard equations we refer to Moroz and van Schaftingen [11] and references therein. We also refer to D'Avenia and Squassina [42], Cassani and Zhang [12], Mingqi, Rădulescu and

Zhang [13] and Seok [14] as recent relevant contributions to the study of Choquard-type problems.

1.2 Main result and related remarks

In this article, we consider the following fractional p-Kirchhoff type problem

$$\begin{cases} \left(a+b[u]_{s,p}^{p(\theta-1)}\right)(-\Delta)_p^s u = \left(\mathcal{I}_\mu * |u|^q\right)|u|^{q-2}u + \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha}, \quad u>0, & \text{in } \Omega, \\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ containing 0 with Lipschitz boundary, $0 \leq \alpha < ps < N$ with $s \in (0,1)$, $p > 1$, $a \geq 0$, $b > 0$, $\theta > 1$, and $\mathcal{I}_\mu(x) = |x|^{-\mu}$ is the Riesz potential of order $\mu \in (0, \min\{N, 2ps\})$, the fractional p–Laplacian operator $(-\Delta)_p^s$ is the differential of the convex functional

$$u \mapsto \frac{1}{p}[u]_{s,p}^p := \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy$$

defined on the Banach space (with respect to the norm $[u]_{s,p}$ defined above)

$$W_0^{s,p}(\Omega) := \left\{u \in L^p_{\text{loc}}(\mathbb{R}^N) : u \equiv 0 \text{ in} \mathbb{R}^N \setminus \Omega \text{ and } [u]_{s,p} < +\infty\right\}.$$

This definition is consistent, up to a normalization constant, with the linear fractional Laplacian $(-\Delta)^s$ for the case $p = 2$. Moreover, $p_\alpha^* = \frac{(N-\alpha)p}{N-ps}$ is the fractional critical Hardy–Sobolev exponent, which arises from the general fractional Hardy–Sobolev inequality

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{p_\alpha^*}}{|x|^\alpha}\,dx\right)^{1/p_\alpha^*} \leq C(N,p,\alpha)[u]_{s,p}. \tag{1.3}$$

The latter is a scale invariant inequality and as such is critical for the embedding

$$W_0^{s,p}(\Omega) \hookrightarrow L^q\left(\Omega, \frac{dx}{|x|^\alpha}\right)$$

in the sense that the latter is continuous for any $q \in [1, p_\alpha^*]$ but (as long as $0 \in \Omega$, as we are assuming) is compact if and only if $q < p_\alpha^*$. As a result, the energy functional does not satisfy the Palais-Smale condition globally for the critical case, but it is true for the energy functional in a suitable range related to the best fractional critical Hardy-Sobolev constant in the embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q\left(\Omega, \frac{dx}{|x|^\alpha}\right)$. To do this, let us define the best fractional critical Hardy-Sobolev constant S_α as

$$S_\alpha = \inf\left\{\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy :\in W_0^{s,p}(\Omega) \text{ with } \int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx = 1\right\}. \tag{1.4}$$

For the critical fractional p-Laplacian case, the main difficulty is the lack of an explicit formula for a minimizer of S_α. We can overcome this difficulty by the asymptotic estimates for minimizers obtained by Marano and Mosconi in [15], we will recall them in Sect. 3.

This paper is motivated by some works which have been focused on the study of Kirchhoff type problems and the Choquard equation. On the one hand, Fiscella and Valdinoci [16] first proposed a stationary fractional Kirchhoff variational model as follows

$$\begin{cases} M\left([u]_{s,2}^2\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2^*-2}u, & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $2^* = 2N/(N-2s)$, $N > 2s$ with $s \in (0,1)$, M and f are two continuous functions under some suitable assumptions. In [16], the authors first provided a detailed discussion about the physical meaning underlying the fractional Kirchhoff problems and their applications. They supposed that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing and continuous function, and there exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for all $t \in \mathbb{R}^+$. Based on the truncated skill and the mountain pass theorem, they obtained the existence of a non-negative solution to problem (1.5) for any $\lambda > \lambda^* > 0$, where λ^* is an appropriate threshold. Subsequently, there were many extension and complement results, see for example [17] for the existence and the asymptotic behavior of non–negative solutions to problem (1.5) under different assumptions on M. In particular, the Kirchhoff function M may be zero at zero; that is, the Kirchhoff-type problem is degenerate. We also refer to [18–24] and references therein, for some recent results on the existence, uniqueness and multiplicity of solutions for Kirchhoff-type fractional p-Laplacian problems. For the case involving the Hardy term, Fiscella and Pucci in [25] considered the following Kirchhoff–Hardy problem:

$$\begin{cases} M\left([u]_{s,p}^p\right)(-\Delta)_p^s u = \lambda\omega(x)|u|^{q-2}u + \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha} & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.6}$$

where $p\theta < q < p_\alpha^*$, λ is a real parameter, $\omega(x) \in L^{p^*/(p^*-q)}(\mathbb{R}^N)$ with $p^* := p_0^*$. As a result, the existence of nontrivial mountain pass solution for (1.6) is obtained as λ is large enough, see [25, Theorem 1.2] for more details. We refer the interested reader to [26,27] for more related results. The existence of infinitely many solutions for p-fractional Kirchhoff equations with critical Hardy-Sobolev nonlinearities can be seen in [25–29].

On the other hand, there are some interesting results about the Choquard equation. We refer to [11] for a good survey of the Choquard equation. In the setting of the fractional Choquard equations, the following problem has been investigated recently

$$(-\Delta)^s u + V(x)u = (\mathcal{I}_\mu * F(u))f(u), \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

The existence, regularity and asymptotic behavior of solutions to problem (1.7) with f satisfying some mild assumptions have been obtained, we refer to [30–32] and references therein. In the Kirchhoff setting, Mingqi *et al.* in [33] firstly considered the following Choquard-type fractional p-Laplacian problem

$$\begin{aligned} &\left(a + b\|u\|_s^{p(\theta-1)}\right)\left[(-\Delta)_p^s u + V(x)|u|^{p-2}u\right] = \lambda f(x,u) \\ &+ \left(\mathcal{K}_\mu * |u|^{p^*_{\mu,s}}\right)|u|^{p^*_{\mu,s}-2}u, \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1.8}$$

where $\|u\|_s = \left([u]_{s,p}^p + \int_{\mathbb{R}^N} V(x)|u|^p dx\right)^{1/p}$, $\mathcal{K}_\mu(x) = |x|^{-\mu}$, $a, b \geq 0$ with $a + b > 0$, $p^*_{\mu,s} = p(N - \mu/2)/(N - ps)$ is the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality. Under some appropriate assumptions, existence of mountain pass solutions for problem (1.8) is obtained when f satisfies the sublinear or superlinear growth condition in the non-degenerate or degenerate case. Subsequently, some existence results for problems like (1.8) with the extra magnetic field are presented in [13].

It is worth mentioning that Chen *et al.* in [34] studied the following fractional p-Laplacian problem with critical Hardy-Sobolev nonlinearity:

$$\begin{cases} (-\Delta)_p^s u = \lambda|u|^{r-2}u + \frac{|u|^{p^*_\alpha-2}u}{|x|^\alpha}, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.9}$$

where $p \leq r < Np/(N - ps)$. By finding the minimizer of the corresponding energy functional on positive Nehari and sign-changing Nehari sets, the existence of positive and sign-changing least energy solutions for problem (1.9) was established. Very recently, Chen in [35] was interested in the existence of positive solutions for the following problem:

$$\begin{cases} M([u]_{s,p}^p)(-\Delta)_p^s u = \lambda\left(\mathcal{I}_\mu * F(u)\right)f(u) + \frac{|u|^{p^*_\alpha-2}u}{|x|^\alpha}, & u > 0, \text{ in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.10}$$

where $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a Kirchhoff function, $f \in C^1(\mathbb{R}, \mathbb{R})$ fulfills the Ambrosetti-Rabinowitz type condition, $F(u) = \int_0^u f(t)dt$. Note that the authors in [35] and [31] assumed the parameter $\lambda > 0$ is large enough, in order to make the level of the energy functional below a suitable compactness threshold.

Inspired by the above works, especially by [13,25,35], we are devoted to studying the existence of positive solutions for problem (1.2) in the possibly degenerate case.

Here we suppose that the parameter is a fixed number, and then consider the existence of positive solutions by choosing q in some suitable range. It is worth pointing out that the related non-Choquard cases involving the critical Kirchhoff exponents, i.e. $\theta p = p_s^*$, have been investigated in [24,36] as $\alpha = 0$.

We first give the definition of weak solutions for problem (1.2).

Definition 1.1 We say that $u \in W_0^{s,p}(\Omega)$ is a (weak) solution of problem (1.2), if

$$\left(a + b[u]_{s,p}^{p(\theta-1)}\right) \langle u, \varphi\rangle_{s,p} = \int_\Omega \int_\Omega \frac{|u(y)|^q}{|x-y|^\mu} |u(x)|^{q-2} u(x)\varphi(x)\, dxdy + \int_\Omega \frac{|u(x)|^{p_\alpha^*-2}u(x)\varphi(x)}{|x|^\alpha} dx,$$

$$\langle u, \varphi\rangle_{s,p} := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\, dx\, dy,$$

for all $\varphi \in W_0^{s,p}(\Omega)$, where space $W_0^{s,p}(\Omega)$ will be introduced in Sect. 2.

Now we can state our main result as follows.

Theorem 1.1 *Assume that* $0 \le \alpha < ps < N$ *with* $s \in (0, 1)$*,* $p > 1$*,* $0 < \mu < \min\{N, 2ps\}$*, and one of the following cases holds*

(1) $1 < \theta < \frac{N-\alpha}{N-ps}$*,* $a = 0$ *and* $b > 0$*;*

(2) $\theta = \frac{N-\alpha}{N-ps}$*,* $a > 0$ *and* $0 < b < S_\alpha^{-p_\alpha^*/p}$*, where* S_α *is the best Sobolev embedding constant of* $W_0^{s,p}(\Omega) \hookrightarrow L^{p_\alpha^*}(\Omega, dx/|x|^\alpha)$*, which is defined in* (1.4)*;*

(3) $\theta = \frac{N-(ps+\alpha)/2}{N-ps}$*,* $a > 0$ *and* $b > 0$*.*

Moreover, assume that q *satisfies*

$$\max\left\{\frac{(2N-\mu)p}{2N}, \frac{p_\alpha^*}{2}, p_{\mu,s}^* - \frac{p}{2(p-1)}\right\} < q < p_{\mu,s}^*, \tag{1.11}$$

or

$$\max\left\{\frac{(2N-\mu)p}{2N}, \frac{p\theta}{2}, p_{\mu,s}^* - \frac{p}{2(p-1)}\right\} < q \le \frac{p_\alpha^*}{2}. \tag{1.12}$$

Then problem (1.2) *admits a positive solution in* $W_0^{s,p}(\Omega)$*.*

Remark 1.1 We would like to point out that there is no similar result exists for the local counterpart of problem (1.2), that is, the case $s = 1$. As far as we know, Ghoussoub and Yuan [37] investigated the existence and multiplicity of solutions for the quasi-linear problem

$$-\Delta_p u = \lambda |u|^{r-2}u + \mu \frac{|u|^{q-2}u}{|x|^\alpha} \quad \text{in } \Omega,$$

where $\lambda, \mu > 0$, and $1 < p < N$, $p \le q \le (N-\alpha)p/(N-p)$ and $p \le r \le Np/(N-p)$.

Remark 1.2 Comparing our main result with the result of [35], advantages of our result are threefold:

(a) Our result is more delicate than that of [35]. In fact, our result may cover the degenerate case of Kirchhoff–type problems.
(b) Our approach is quite different from that of [35]. Note that the existence of positive solutions in our result heavily depends on the exponent q in some certain range instead of the adjusted parameter λ in (1.10).
(c) Our result covers larger range of q in [35]. In fact, f in [35] satisfies the Ambrosetti–Rabinowitz type condition and hence $q \in \left(\frac{(2N-\mu)p}{2N}, \frac{(2N-\mu)p}{2(N-ps)}\right)$, see (1.11) and (1.12).

The main tool in this paper is variational methods. More precisely, we will use the mountain pass theorem, which is proposed by Ambrosetti and Rabinowitz in the celebrated paper [38]. The key point is to overcome the compactness for the associated Lagrange-Euler functional, namely, the Palais-Smale ((PS) for short) condition. Since the nonlinearity term in problem (1.2) contains the critical Hardy-Sobolev term, the functional does not satisfy the Palais-Smale condition in all range, we will use a fractional version of the concentration compactness principle to show that the energy functional satisfies the local $(PS)_c$ condition for c less than some critical level when q is in some suitable range.

The paper is organized as follows: In Sect. 2, we give some definitions and preliminaries. In Sect. 3, we recall the decay properties for the Aubin-Talenti functions optimizing the Hardy–Sobolev inequality, and give some estimates for suitable truncations of the latter. Section 4 is devoted to proving a compactness result. Finally, we complete the proof of Theorem 1.1 in Sect. 5.

Throughout this paper, the positive constant C may vary from line to line.

2 Abstract setting and preliminary properties

In this section, we introduce some useful notations. The fractional Sobolev space $W_0^{s,p}(\Omega)$ is defined by

$$W_0^{s,p}(\Omega) = \Big\{u \in L^p_{\mathrm{loc}}(\mathbb{R}^N) : [u]_{s,p}^p < +\infty,\ u \equiv 0 \text{ a.e. on } \mathbb{R}^N \backslash \Omega\Big\}$$

and the homogeneous fractional Sobolev space

$$D^{s,p}(\mathbb{R}^N) = \Big\{u \in L^{p^*}(\mathbb{R}^N) : [u]_{s,p} < +\infty\Big\} \supset W_0^{s,p}(\Omega).$$

For $p > 1$, $W_0^{s,p}(\Omega)$ and $D^{s,p}(\mathbb{R}^N)$ are separable reflexive Banach spaces with respect to the norm $[\,\cdot\,]_{s,p}$ and both can also be seen as the completion with respect to the norm $[\,\cdot\,]_{s,p}$ of $C_c^\infty(\mathbb{R}^N)$ (see e.g. [39, Theorem 2.1]). The topological dual of $W_0^{s,p}(\Omega)$ will be denoted by $W^{-s,p'}(\Omega)$, with corresponding duality pairing $\langle\,\cdot\,,\,\cdot\,\rangle : W^{-s,p'}(\Omega) \times W_0^{s,p}(\Omega) \to \mathbb{R}$. The weak and weak* convergence in $W^{-s,p'}(\Omega)$ coincide because of reflexivity.

We first recall the following fractional Hardy-Sobolev inequality and Hardy-Littlewood-Sobolev inequality.

Lemma 2.1 ([34, Lemma 2.1]) (Hardy-Sobolev inequality) *Assume that* $0 \leq \alpha \leq ps < N$. *Then there exists a positive constant* C *such that*

$$\Big(\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx\Big)^{1/p_\alpha^*} \leq C\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}, \qquad \textit{for every } u \in W_0^{s,p}(\Omega).$$

As a consequence of Lemma 2.1, it is easy to see that the optimization problem (1.4) has a solution in $D^{s,p}(\mathbb{R}^N)$.

Lemma 2.2 ([40, Theorem 4.3]) (Hardy-Littlewood-Sobolev inequality) *Assume that* $1 < r, t < \infty$, $0 < \mu < N$ *and*

$$\frac{1}{r} + \frac{1}{t} + \frac{\mu}{N} = 2.$$

Then there exists $C(N, \mu, r, t) > 0$ *such that*

$$\iint_{\mathbb{R}^{2N}} \frac{|g(x)||h(y)|}{|x-y|^\mu}\,dx\,dy \leq C(N, \mu, r, t)\|g\|_r\|h\|_t$$

for all $g \in L^r(\mathbb{R}^N)$ *and* $h \in L^t(\mathbb{R}^N)$.

As a consequence, the integral

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}\,dx\,dy$$

is well defined if

$$\tilde{p}_{\mu,s} := \frac{(2N-\mu)p}{2N} < q < \frac{(2N-\mu)p}{2(N-ps)} := p^*_{\mu,s}.$$

Hence, $\tilde{p}_{\mu,s}$ is called the lower critical exponent and $p^*_{\mu,s}$ is said to be the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

The energy functional $\mathcal{J} : W_0^{s,p}(\Omega) \to \mathbb{R}$ associated with problem (1.2) is

$$\begin{aligned}\mathcal{J}(u) &= \frac{a}{p}\|u\|^p + \frac{b}{p\theta}\|u\|^{p\theta} - \frac{1}{2q}\int_\Omega\int_\Omega \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}\,dx\,dy - \frac{1}{p_\alpha^*}\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx \\ &:= \Phi(u) - \Psi_q(u) - H_\alpha(u),\end{aligned}$$

with

$$\Phi(u) = \frac{a}{p}\|u\|^p + \frac{b}{p\theta}\|u\|^{p\theta},$$

$$\Psi_q(u) = \frac{1}{2q}\int_\Omega\int_\Omega \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}\,dx\,dy,$$
$$H_\alpha(u) = \frac{1}{p_\alpha^*}\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx,$$

where and in what follows, $\|\cdot\| = [\,\cdot\,]_{s,p}$ denotes the norm of the space $W_0^{s,p}(\Omega)$. We have the following result.

Lemma 2.3 *Let* $0 \le \alpha \le ps < N$ *with* $s \in (0,1)$, $0 < \mu < \min\{N, 2ps\}$ *and* $\frac{(2N-\mu)p}{2N} < q < \frac{(2N-\mu)p}{2(N-ps)}$. *Then the functional* $\mathcal{J}$ *is of class* $C^1(W_0^{s,p}(\Omega), \mathbb{R})$ *and*

$$\mathcal{J}'(u)\varphi = \left(a + b\|u\|^{p(\theta-1)}\right)\langle u,\varphi\rangle_{s,p} - \int_\Omega\int_\Omega \frac{|u(y)|^q}{|x-y|^\mu}|u(x)|^{q-2}u(x)\varphi(x)\,dxdy$$
$$- \int_\Omega \frac{|u(x)|^{p_\alpha^*-2}u(x)\varphi(x)}{|x|^\alpha}dx,$$

for all $\varphi \in W_0^{s,p}(\Omega)$. *Moreover,* $\mathcal{J}$ *is sequentially weakly lower semi–continuous in* $W_0^{s,p}(\Omega)$*, and the operator* $\mathcal{J}' : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ *is sequentially weak to weak continuous.*

Proof It easily follows from [19, Lemma 2], [31, Lemma 2.2] and [34, Lemma 2.3], so we omit the details. □

Clearly, the critical points of $\mathcal{J}$ are exactly the weak solutions of problem (1.2).

3 Hardy optimizers and some estimates

In this section, we recall the known decay properties for the Aubin–Talenti functions optimizing the Hardy–Sobolev inequality, and give some estimates for suitable truncations of the latter.

In [39] the existence and properties of solutions for the minimization problem (1.4) when $\alpha = 0$ was investigated. For $0 \le \alpha < ps$, one can get the following results, see [15, Theorem 1.1].

Proposition 3.1 (Existence and properties) *Let* $0 \le \alpha < ps < N$. *Then the following facts hold.*

(1) *Problem* (1.4) *admits constant sign solutions, and any solution is bounded;*
(2) *For every nonnegative* $U_\alpha \in D^{s,p}(\mathbb{R}^N)$ *solving problem* (1.4)*, there exist* $x_0 \in \mathbb{R}^N$ *and a non-increasing* $u : \mathbb{R}^+ \to \mathbb{R}$ *such that* $U_\alpha(x) = u(|x-x_0|)$*;*
(3) *Every nonnegative minimizer* $U_\alpha \in D^{s,p}(\mathbb{R}^N)$ *of problem* (1.4) *weakly solves*

$$(-\Delta)_p^s U_\alpha = S_\alpha \frac{U_\alpha^{p_\alpha^*-1}}{|x|^\alpha} \quad \text{in } \mathbb{R}^N.$$

i.e.,

$$\langle (-\Delta_p)^s U_\alpha, \varphi \rangle = S_\alpha \int_{\mathbb{R}^N} \frac{U_\alpha^{p_\alpha^*-1}}{|x|^\alpha} \varphi \, dx, \quad \textit{for all } \varphi \in D^{s,p}(\mathbb{R}^N)$$

and the last integrand is absolutely integrable.

Next we *fix* N, p, s, α and a positive radially symmetric decreasing minimizer $U_\alpha = U_\alpha(r)$ for S_α as in (1.4). Multiplying U_α by a positive constant, we may assume that

$$(-\Delta)_p^s U_\alpha = \frac{U_\alpha^{p_\alpha^*-1}}{|x|^\alpha} \quad \text{weakly in } \mathbb{R}^N. \tag{3.1}$$

Testing this equation by U_α and using (1.4), we get

$$\|U_\alpha\|^p = \int_{\mathbb{R}^N} \frac{U_\alpha^{p_\alpha^*}}{|x|^\alpha} dx = S_\alpha^{\frac{N-\alpha}{ps-\alpha}}. \tag{3.2}$$

In [39] the asymptotic behavior for U_α was obtained when $\alpha = 0$, while in [15] the asymptotic for U_α with $0 < \alpha < ps$ was derived by similar arguments.

Lemma 3.1 (Optimal decay) *There exist* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\frac{c_1}{r^{\frac{N-ps}{p-1}}} \le U_\alpha(r) \le \frac{c_2}{r^{\frac{N-ps}{p-1}}}, \quad \textit{for all } r \ge 1.$$

Furthermore, there exists $\kappa > 1$ *such that*

$$U_\alpha(\kappa r) \le \frac{1}{2} U_\alpha(r) \quad \textit{for all } r \ge 1. \tag{3.3}$$

For any $\varepsilon > 0$, the function

$$U_{\alpha,\varepsilon}(x) := \varepsilon^{-\frac{N-ps}{p}} U_\alpha\left(\frac{x}{\varepsilon}\right) \tag{3.4}$$

is also a minimizer for S_α satisfying (3.1). We note that c_1, c_2, κ are universal since we *fixed* $N, p, s, \alpha, U_\alpha$. In general, they depend upon these entries.

In what follows, $0 \le \alpha < ps < N$, U_α is a fixed minimizer for (1.4), κ is the constant in Lemma 3.1 depending only on N, p, s, α and U_α. For every $\delta \ge \varepsilon > 0$, let us set

$$m_{\varepsilon,\delta} := \frac{U_{\alpha,\varepsilon}(\delta)}{U_{\alpha,\varepsilon}(\delta) - U_{\alpha,\varepsilon}(\kappa\delta)}.$$

Due to (3.3) and the definition (3.4), it readily follows that $m_{\varepsilon,\delta} \leq 2$. Furthermore, let us set

$$g_{\varepsilon,\delta}(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq U_{\alpha,\varepsilon}(\kappa\delta), \\ m^p_{\varepsilon,\delta}(t - U_{\alpha,\varepsilon}(\kappa\delta)), & \text{if } U_{\alpha,\varepsilon}(\kappa\delta) \leq t \leq U_{\alpha,\varepsilon}(\delta), \\ t + U_{\alpha,\varepsilon}(\delta)(m^{p-1}_{\varepsilon,\delta} - 1), & \text{if } t \geq U_{\alpha,\varepsilon}(\delta), \end{cases}$$

as well as

$$G_{\varepsilon,\delta}(t) = \int_0^t g'_{\varepsilon,\delta}(\tau)^{\frac{1}{p}}\, d\tau = \begin{cases} 0, & \text{if } 0 \leq t \leq U_{\alpha,\varepsilon}(\kappa\delta), \\ m_{\varepsilon,\delta}(t - U_{\alpha,\varepsilon}(\kappa\delta)), & \text{if } U_{\alpha,\varepsilon}(\kappa\delta) \leq t \leq U_{\alpha,\varepsilon}(\delta), \\ t, & \text{if } t \geq U_{\alpha,\varepsilon}(\delta). \end{cases}$$

The functions $g_{\varepsilon,\delta}$ and $G_{\varepsilon,\delta}$ are nondecreasing and absolutely continuous. Consider now the radially symmetric nonincreasing function

$$u_{\alpha,\varepsilon,\delta}(r) := G_{\varepsilon,\delta}(U_{\alpha,\varepsilon}(r)), \tag{3.5}$$

which satisfies

$$u_{\alpha,\varepsilon,\delta}(r) = \begin{cases} U_{\alpha,\varepsilon}(r), & \text{if } r \leq \delta, \\ 0, & \text{if } r \geq \kappa\delta. \end{cases}$$

Then $u_{\alpha,\varepsilon,\delta} \in W_0^{s,p}(\Omega)$, for any $\delta < \kappa^{-1}\delta_\Omega := \kappa^{-1}\text{dist}(0, \partial\Omega)$. We have the following estimates.

Lemma 3.2 ([34, Lemma 2.10]) *There exists $C > 0$ such that, for any $0 < 2\varepsilon \leq \delta < \kappa^{-1}\delta_\Omega$, there holds*

$$\|u_{\alpha,\varepsilon,\delta}\|^p \leq S_\alpha^{\frac{N-\alpha}{ps-\alpha}} + C\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-ps}{p-1}}, \tag{3.6}$$

and

$$\int_{\mathbb{R}^N} \frac{u_{\alpha,\varepsilon,\delta}(x)^{p^*_\alpha}}{|x|^\alpha}\, dx \geq S_\alpha^{\frac{N-\alpha}{ps-\alpha}} - C\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-\alpha}{p-1}}. \tag{3.7}$$

Using Lemma 3.1, we can get the following estimate.

Lemma 3.3 *We have*

$$\int_\Omega \int_\Omega \frac{|u_{\alpha,\varepsilon,\delta}(x)|^q |u_{\alpha,\varepsilon,\delta}(y)|^q}{|x-y|^\mu} dxdy \geq C\varepsilon^{2N-\mu-2\frac{N-ps}{p}q}, \tag{3.8}$$

for some positive constant C.

Proof The desired conclusion follows immediately from the following observation:

$$
\begin{aligned}
\int_\Omega\int_\Omega \frac{|u_{\alpha,\varepsilon,\delta}(x)|^q|u_{\alpha,\varepsilon,\delta}(y)|^q}{|x-y|^\mu}dxdy &\geq \int_{B_\delta}\int_{B_\delta} \frac{|u_{\alpha,\varepsilon,\delta}(x)|^q|u_{\alpha,\varepsilon,\delta}(y)|^q}{|x-y|^\mu}dxdy \\
&= \int_{B_\delta}\int_{B_\delta} \frac{|U_{\alpha,\varepsilon}(x)|^q|U_{\alpha,\varepsilon}(y)|^q}{|x-y|^\mu}dxdy \\
&= \varepsilon^{2N-\mu-2\frac{N-ps}{p}q}\int_{B_{\frac{\delta}{\varepsilon}}}\int_{B_{\frac{\delta}{\varepsilon}}} \frac{|U_\alpha(x)|^q|U_\alpha(y)|^q}{|x-y|^\mu}dxdy \\
&\geq C\varepsilon^{2N-\mu-2\frac{N-ps}{p}q}\int_{B_{\frac{\delta}{\varepsilon}}\setminus B_1} \\
&\int_{B_{\frac{\delta}{\varepsilon}}\setminus B_1} \frac{1}{(1+|x|)^{\frac{(N-ps)q}{p-1}}}\frac{1}{(1+|y|)^{\frac{(N-ps)q}{p-1}}}\frac{1}{|x-y|^\mu}dxdy \\
&= O\Big(\varepsilon^{2N-\mu-2\frac{N-ps}{p}q}\Big),
\end{aligned}
$$

where B_δ denotes the ball with center at 0 and radius $\delta > 0$. □

4 Compactness result

This section is devoted to proving a compactness result. We start with the following definition.

Definition 4.1 ([38]) For $c \in \mathbb{R}$, we say that $\mathcal{J}$ satisfies the $(PS)_c$ condition if for any sequence $\{u_n\} \subset W_0^{s,p}(\Omega)$ with

$$\mathcal{J}(u_n) \to c, \quad \mathcal{J}'(u_n) \to 0 \text{ in } W^{-s,p'}(\Omega)$$

has a convergent subsequence.

Lemma 4.1 (Palais-Smale condition) *The functional $\mathcal{J}$ satisfies $(PS)_c$ condition for any $c < c_*$, where c_* is defined as*

$$
c_* := \begin{cases}
\left(\frac{1}{p}-\frac{1}{p_\alpha^*}\right)\frac{(aS_\alpha)^{\frac{N-\alpha}{ps-\alpha}}}{\left(1-bS_\alpha^{\frac{N-\alpha}{N-ps}}\right)^{\frac{N-ps}{ps-\alpha}}}, & \text{if } \theta=\frac{N-\alpha}{N-ps},\ a>0,\ 0<b<S_\alpha^{-\frac{p_\alpha^*}{p}}, \\
\left(\frac{1}{p\theta}-\frac{1}{p_\alpha^*}\right)\left(bS_\alpha^\theta\right)^{\frac{p_\alpha^*}{p_\alpha^*-p\theta}}, & \text{if } 1<\theta<\frac{N-\alpha}{N-ps},\ a=0,\ b>0, \\
\left[a\left(\frac{1}{p}-\frac{1}{p\theta}\right)\Lambda^{\frac{2(N-ps)}{ps-\alpha}}+\left(\frac{1}{p\theta}-\frac{1}{p_\alpha^*}\right)\Lambda^{\frac{2(N-\alpha)}{ps-\alpha}}\right]S_\alpha^{\frac{N-\alpha}{ps-\alpha}}, & \text{if } \theta=\frac{N-(ps+\alpha)/2}{N-ps},\ a>0,\ b>0,
\end{cases}
\tag{4.1}
$$

with

$$\Lambda := \Lambda(a,b,S_\alpha,p,s,\alpha,N) = \frac{bS_\alpha^{\frac{N-\alpha}{2(N-ps)}}+\sqrt{b^2S_\alpha^{\frac{N-\alpha}{N-ps}}+4a}}{2}. \tag{4.2}$$

Proof Assume that $\{u_n\} \subset W_0^{s,p}(\Omega)$ is the $(PS)_c$ sequence of $\mathcal{J}$, that is

$$\mathcal{J}(u_n) \to c \quad \text{and} \quad \mathcal{J}'(u_n) \to 0 \;\text{ in } W^{-s,p'}(\Omega).$$

If q satisfies (1.11), we have $2q > p_\alpha^*$, then there exists $C > 0$ such that

$$\begin{aligned} C + \|u_n\| &\geq \mathcal{J}(u_n) - \frac{1}{p_\alpha^*}\mathcal{J}'(u_n)u_n \\ &= a\left(\frac{1}{p} - \frac{1}{p_\alpha^*}\right)\|u_n\|^p + b\left(\frac{1}{p\theta} - \frac{1}{p_\alpha^*}\right)\|u_n\|^{p\theta} \\ &\quad - \left(\frac{1}{2q} - \frac{1}{p_\alpha^*}\right)\int_\Omega\int_\Omega \frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\mu}\,dx\,dy \\ &\geq \begin{cases} b\left(\frac{1}{p\theta} - \frac{1}{p_\alpha^*}\right)\|u_n\|^{p\theta}, & \text{if } 1 < \theta < \frac{N-\alpha}{N-ps}, a \geq 0, b > 0, \\ a\left(\frac{1}{p} - \frac{1}{p_\alpha^*}\right)\|u_n\|^p, & \text{if } \theta = \frac{N-\alpha}{N-ps}, a > 0, b > 0, \end{cases} \end{aligned} \tag{4.3}$$

for n large enough.

If q satisfies (1.12), we have $p\theta < 2q \leq p_\alpha^*$, then there exists $C > 0$ such that

$$\begin{aligned} C + \|u_n\| &\geq \mathcal{J}(u_n) - \frac{1}{p\theta}\mathcal{J}'(u_n)u_n \\ &= a\left(\frac{1}{p} - \frac{1}{p\theta}\right)\|u_n\|^p - \left(\frac{1}{p_\alpha^*} - \frac{1}{p\theta}\right)\int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}dx \\ &\quad - \left(\frac{1}{2q} - \frac{1}{p\theta}\right)\int_\Omega\int_\Omega \frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\mu}\,dx\,dy \\ &\geq a\left(\frac{1}{p} - \frac{1}{p\theta}\right)\|u_n\|^p. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4), we obtain that $\{u_n\}$ is bounded in $W_0^{s,p}(\Omega)$. By the concentration-compactness principle (see [41, Theorem 2.5]), there exist $u \in W_0^{s,p}(\Omega)$, two Borel regular measures σ and ν, Λ denumerable, at most countable set $\{x_j\}_{j\in\Lambda} \subseteq \bar{\Omega}$, and non-negative numbers $\{\sigma_j\}_{j\in\Lambda}, \{\nu_j\}_{j\in\Lambda} \subset [0,\infty)$ such that, up to subsequence

$$\begin{aligned} &u_n \rightharpoonup u \;\text{ in } W_0^{s,p}(\Omega), \\ &u_n \to u \;\text{ in } L^r(\Omega), \quad \text{for } p \leq r < p_0^*, \\ &u_n \to u \;\text{ a. e. in } \Omega, \end{aligned} \tag{4.5}$$

as $n \to \infty$. Moreover,

$$\|u_n\|^p \rightharpoonup^* \sigma, \qquad \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha} \rightharpoonup^* \nu, \tag{4.6}$$

$$d\sigma \geq \|u\|^p + \sum_{j\in\Lambda}\sigma_j\delta_{x_j}, \quad \sigma_j := \sigma(\{x_j\}), \tag{4.7}$$

$$d\nu = \frac{|u|^{p_\alpha^*}}{|x|^\alpha} + \sum_{j\in\Lambda} \nu_j \delta_{x_j}, \quad \nu_j := \nu(\{x_j\}), \tag{4.8}$$

$$\sigma_j \geq S_\alpha \nu_j^{p/p_\alpha^*}, \qquad \text{for all } j \in \Lambda. \tag{4.9}$$

Fix $i_0 \in \Lambda$, we prove that either $\nu_{i_0} = 0$ or

$$\nu_{i_0} \geq \begin{cases} \left(\frac{aS_\alpha}{1-bS_\alpha^\theta}\right)^{\frac{\theta}{\theta-1}}, & \text{if } \theta = \frac{N-\alpha}{N-ps},\ a>0,\ 0<b<S_\alpha^{-\theta}, \\ \left(bS_\alpha^\theta\right)^{\frac{p_\alpha^*}{p_\alpha^*-p\theta}}, & \text{if } 1<\theta<\frac{N-\alpha}{N-ps},\ a=0,\ b>0, \\ \left(\frac{bS_\alpha^\theta+\sqrt{b^2S_\alpha^{2\theta}+4aS_\alpha}}{2}\right)^{\frac{2\theta-1}{\theta-1}}, & \text{if } \theta = \frac{N-(ps+\alpha)/2}{N-ps},\ a>0,\ b>0. \end{cases} \tag{4.10}$$

In fact, let $\varphi_\epsilon \in C_0^\infty(B_{2\epsilon}(x_{i_0}))$ satisfy $0 \leq \varphi_\epsilon \leq 1$, $\varphi_\epsilon|_{B_\epsilon(x_{i_0})} = 1$, and $\|\nabla\varphi_\epsilon\|_\infty \leq C/\epsilon$. Clearly $\{\varphi_\epsilon u_n\}$ is bounded in $W_0^{s,p}(\Omega)$. It follows from $\langle \mathcal{J}'(u_n), \varphi_\epsilon u_n\rangle \to 0$ as $n \to \infty$ that

$$\begin{aligned} &\left(a + b\|u_n\|^{p(\theta-1)}\right) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_\epsilon(x)u_n(x)-\varphi_\epsilon(y)u_n(y))}{|x-y|^{N+ps}} dxdy \\ &= \int_\Omega\int_\Omega \frac{|u_n(y)|^q}{|x-y|^\mu}|u_n(x)|^{q-2}u_n(x)\varphi_\epsilon(x)dxdy + \int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}\varphi_\epsilon(x)dx + o(1). \end{aligned} \tag{4.11}$$

On the one hand,

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))\big[\varphi_\epsilon(x)u_n(x)-\varphi_\epsilon(y)u_n(y)\big]}{|x-y|^{N+ps}}\, dx\, dy \\ &= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))\big[u_n(x)-u_n(y)\big]\varphi_\epsilon(x)}{|x-y|^{N+ps}}\, dx\, dy \\ &+ \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))u_n(y)\big[\varphi_\epsilon(x)-\varphi_\epsilon(y)\big]}{|x-y|^{N+ps}}\, dx\, dy. \end{aligned} \tag{4.12}$$

Since

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p\varphi_\epsilon(x)}{|x-y|^{N+ps}}\, dx\, dy \to \int_{\mathbb{R}^N} \varphi_\epsilon(x)d\sigma,$$

as $n \to \infty$. Taking $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p\varphi_\epsilon(x)}{|x-y|^{N+ps}}\, dx\, dy = \sigma_{i_0}.$$

This gives that

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\lim_{n\to\infty}\left(a+b\|u_n\|^{p(\theta-1)}\right)\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\varphi_\epsilon(x)dxdy\\
&\geq \lim_{\epsilon\to 0}\lim_{n\to\infty}\left[a\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\varphi_\epsilon(x)dxdy\right.\\
&\left.+b\left(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\varphi_\epsilon(x)dxdy\right)^\theta\right]\\
&= a\sigma_{i_0}+b\sigma_{i_0}^\theta. \qquad (4.13)
\end{aligned}
$$

By using Hölder's inequality and Lemma 2.3 in [24], we find

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\lim_{n\to\infty}\left|\left(a+b\|u_n\|^{p(\theta-1)}\right)\right.\\
&\left.\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_\epsilon(x)-\varphi_\epsilon(y))u_n(y)}{|x-y|^{N+ps}}dxdy\right|\\
&\leq C\lim_{\epsilon\to 0}\lim_{n\to\infty}\left(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{p-1}{p}}\\
&\left(\iint_{\mathbb{R}^{2N}}\frac{|(\varphi_\epsilon(x)-\varphi_\epsilon(y))u_n(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}}\\
&\leq C\lim_{\epsilon\to 0}\lim_{n\to\infty}\left(\iint_{\mathbb{R}^{2N}}\frac{|(\varphi_\epsilon(x)-\varphi_\epsilon(y))u_n(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}}=0. \qquad (4.14)
\end{aligned}
$$

Moreover, by Lemma 2.2 in [31], we get

$$
\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega\int_\Omega\frac{|u_n(y)|^q}{|x-y|^\mu}|u_n(x)|^{q-2}u_n(x)\varphi_\epsilon(x)dxdy=0. \qquad (4.15)
$$

Furthermore, by (4.8), one has

$$
\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega\frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}\varphi_\epsilon dx=\lim_{\epsilon\to 0}\int_\Omega\varphi_\epsilon d\nu=\nu_{i_0}. \qquad (4.16)
$$

Therefore, taking the limit for $n\to\infty$ and $\epsilon\to 0$ in (4.11), from (4.13), (4.15) and (4.16), we obtain

$$
\nu_{i_0}\geq a\sigma_{i_0}+b\sigma_{i_0}^\theta.
$$

This together with (4.9) implies that $\nu_{i_0}=0$ or (4.10) holds.

Next we claim that (4.10) cannot occur.

Indeed, by contradiction, we assume that there exists $i_0 \in \Lambda$ such that (4.10) holds. From (4.7) and (4.8), we have

$$
\begin{aligned}
c &= \lim_{n\to\infty}\left[\mathcal{J}(u_n) - \frac{1}{p\theta}\langle \mathcal{J}'(u_n), u_n\rangle\right]\\
&= \lim_{n\to\infty}\left[a\Big(\frac{1}{p} - \frac{1}{p\theta}\Big)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N-ps}}dxdy\right.\\
&\quad\left. +\Big(\frac{1}{p\theta} - \frac{1}{p_\alpha^*}\Big)\int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}dx - \Big(\frac{1}{2q} - \frac{1}{p\theta}\Big)\int_\Omega\int_\Omega \frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\mu}dxdy\right]\\
&\geq \lim_{\epsilon\to 0}\lim_{n\to\infty}\left[a\Big(\frac{1}{p} - \frac{1}{p\theta}\Big)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N-ps}}\varphi_\epsilon(x)dxdy\right.\\
&\quad +\Big(\frac{1}{p\theta} - \frac{1}{p_\alpha^*}\Big)\int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}\varphi_\epsilon(x)dx - \Big(\frac{1}{2q}\\
&\quad\left. -\frac{1}{p\theta}\Big)\int_\Omega\int_\Omega \frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\mu}\varphi_\epsilon(x)dxdy\right]\\
&= a\Big(\frac{1}{p} - \frac{1}{p\theta}\Big)\sigma_{i_0} + \Big(\frac{1}{p\theta} - \frac{1}{p_\alpha^*}\Big)\nu_{i_0}
\end{aligned}
$$

Using this fact and (4.9), (4.10), we deduce that $c \geq c_*$.

This is a contradiction. Thus the claim holds. Hence $\nu_j \equiv 0$ for all $j \in \Lambda$, then

$$
\int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}dx \to \int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx \quad \text{as } n\to\infty. \tag{4.17}
$$

Finally, we show that $u_n \to u$ strongly in $W_0^{s,p}(\Omega)$.

We first assume that $d := \inf_{n\geq 1}\|u_n\| > 0$. For simplicity, let $\psi \in W_0^{s,p}(\Omega)$ be fixed and B_ψ be the linear functional on $W_0^{s,p}(\Omega)$ defined by

$$
B_\psi(v) = \iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^{p-2}(\psi(x)-\psi(y))}{|x-y|^{N+ps}}(v(x)-v(y))\,dx\,dy
$$

for all $v \in W_0^{s,p}(\Omega)$. By the Hölder inequality, we have

$$
\left|B_\psi(v)\right| \leq [\psi]_{s,p}^{p-1}[v]_{s,p},
$$

for all $v \in W_0^{s,p}(\Omega)$. Since $\mathcal{J}'(u_n) \to 0$ in $W^{-s,p'}(\Omega)$ and $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$, we have

$$
\langle \mathcal{J}'(u_n) - \mathcal{J}'(u), u_n - u\rangle \to 0 \quad \text{as } n\to\infty.
$$

That is,

$$
o(1) = \langle \mathcal{J}'(u_n) - \mathcal{J}'(u), u_n - u\rangle
$$

$$
\begin{aligned}
&= \left(a+b\|u_n\|^{p(\theta-1)}\right) B_{u_n}(u_n-u) - \left(a+b\|u\|^{p(\theta-1)}\right) B_u(u_n-u) \\
&\quad - \int_\Omega \Big[(\mathcal{I}_\mu * |u_n|^q)|u_n|^{q-2}u_n - (\mathcal{I}_\mu * |u|^q)|u|^{q-2}u\Big](u_n-u)dx \\
&\quad - \int_\Omega \Big[\frac{|u_n|^{p_\alpha^*-2}u_n}{|x|^\alpha} - \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha}\Big](u_n-u)dx \\
&= \left(a+b\|u_n\|^{p(\theta-1)}\right)\Big[B_{u_n}(u_n-u) - B_u(u_n-u)\Big] \\
&\quad + \Big[\left(a+b\|u_n\|^{p(\theta-1)}\right) - \left(a+b\|u\|^{p(\theta-1)}\right)\Big] B_u(u_n-u) \\
&\quad - \int_\Omega \Big[(\mathcal{I}_\mu * |u_n|^q)|u_n|^{q-2}u_n - (\mathcal{I}_\mu * |u|^q)|u|^{q-2}u\Big](u_n-u)dx \\
&\quad - \int_\Omega \Big[\frac{|u_n|^{p_\alpha^*-2}u_n}{|x|^\alpha} - \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha}\Big](u_n-u)dx. \qquad (4.18)
\end{aligned}
$$

The boundedness of $\{u_n\}_n$ and (4.5) give that

$$
\begin{aligned}
&\lim_{n\to\infty} \left(a+b\|u_n\|^{p(\theta-1)}\right) B_u(u_n-u) = 0, \\
&\lim_{n\to\infty} \left(a+b\|u\|^{p(\theta-1)}\right) B_u(u_n-u) = 0. \qquad (4.19)
\end{aligned}
$$

From Lemma 2.2 in [31], we have

$$
\int_\Omega \Big[(\mathcal{I}_\mu * |u_n|^q)|u_n|^{q-2}u_n - (\mathcal{I}_\mu * |u|^q)|u|^{q-2}u\Big](u_n-u)dx \to 0, \quad \text{as } n\to\infty. \qquad (4.20)
$$

Moreover, from (4.17) and the well-known Brézis-Lieb Lemma, we get

$$
\int_\Omega \frac{|u_n-u|^{p_\alpha^*}}{|x|^\alpha}dx = \int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}dx - \int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx + o(1) \to 0
$$

as $n\to\infty$. This together with the Hölder inequality implies

$$
\int_\Omega \Big[\frac{|u_n|^{p_\alpha^*-2}u_n}{|x|^\alpha} - \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha}\Big](u_n-u)dx \to 0, \quad \text{as } n\to\infty. \qquad (4.21)
$$

From (4.18)–(4.21), we obtain

$$
\lim_{n\to\infty} \left(a+b\|u_n\|^{p(\theta-1)}\right)\Big[B_{u_n}(u_n-u) - B_u(u_n-u)\Big] = 0.
$$

By the assumptions $\inf_{n\geq 1}\|u_n\| > 0$, $a \geq 0$ and $b > 0$, we have

$$
\lim_{n\to\infty} \Big[B_{u_n}(u_n-u) - B_u(u_n-u)\Big] = 0. \qquad (4.22)
$$

Let us now recall the well-known Simon inequalities. There exist positive numbers c_p and C_p, depending only on p, such that

$$|\xi-\eta|^p \leq \begin{cases} c_p(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta) & \text{for } p\geq 2,\\ C_p\big[(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)\big]^{p/2}(|\xi|^p+|\eta|^p)^{(2-p)/2} & \text{for } 1<p<2, \end{cases} \tag{4.23}$$

for all $\xi,\eta\in\mathbb{R}^N$. According to the Simon inequality, we divide the discussion into two cases.

Case (I) $p\geq 2$: From (4.22) and (4.23), we have

$$\begin{aligned}\|u_n-u\|^p &= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u(x)-u_n(y)+u(y)|^p}{|x-y|^{N+ps}}dxdy\\ &\leq c_p\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\\ &\times\Big(u_n(x)-u(x)-u_n(y)+u(y)\Big)dxdy\\ &= c_p\Big[B_{u_n}(u_n-u)-B_u(u_n-u)\Big]=o(1),\end{aligned}$$

as $n\to\infty$.

Case (2) $1<p<2$: taking $\xi=u_n(x)-u_n(y)$ and $\eta=u(x)-u(y)$ in (4.23), as $n\to\infty$, we have

$$\begin{aligned}\|u_n-u\|^p &\leq C_p\big[B_{u_n}(u_n-u)-B_u(u_n-u)\big]^{p/2}(\|u_n\|^p+\|u\|^p)^{(2-p)/2}\\ &\leq C\big[B_{u_n}(u_n-u)-B_u(u_n-u)\big]^{p/2}(\|u_n\|^{p(2-p)/2}+\|u\|^{p(2-p)/2})=o(1).\end{aligned}$$

Here we used (4.22) and the fact that $\|u_n\|$ and $\|u\|$ are bounded, and the elementary inequality $(a+b)^{(2-p)/2}\leq a^{(2-p)/2}+b^{(2-p)/2}$ for all $a,b\geq 0$ and $1<p<2$. Thus $u_n\to u$ strongly in $W_0^{s,p}(\Omega)$ as $n\to\infty$.

We now consider the case $\inf_{n\geq 1}\|u_n\|=0$, then either 0 is an accumulation point of the sequence $\{u_n\}$ and so there exists a subsequence of $\{u_n\}$ strongly converging to $u=0$, or 0 is an isolated point of the sequence $\{u_n\}$ and so there exists a subsequence, still denoted by $\{u_n\}$ such that $\inf_{n\geq 1}\|u_n\|>0$. We are done the first case, while in the second case we can process as before. The proof is thus complete. □

5 Proof of Theorem 1.1

The existence of positive solutions for problem (1.2) follows the standard mountain pass approach. The next result shows that the functional $\mathcal{J}$ has the geometric structure of the mountain pass theorem.

Lemma 5.1 *Assume the conditions in Theorem 1.1 hold. Then*

(1) *There exists $\vartheta,\rho>0$ such that $\mathcal{J}(u)\geq\vartheta$ for all $u\in W_0^{s,p}(\Omega)$ with $\|u\|=\rho$.*

(2) *There exists* $e \in W_0^{s,p}(\Omega)$ *such that* $\|e\| > \rho$ *and* $\mathcal{J}(e) < 0$.

Proof (1) By using the Hardy-Littlewood-Sobolev inequality, fractional Sobolev embedding and the definition of S_α, we have

$$\begin{aligned}\mathcal{J}(u) &= \frac{a}{p}\|u\|^p + \frac{b}{p\theta}\|u\|^{p\theta} - \frac{1}{2q}\int_\Omega\int_\Omega \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}\,dx\,dy - \frac{1}{p_\alpha^*}\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx \\ &\geq \frac{a}{p}\|u\|^p + \frac{b}{p\theta}\|u\|^{p\theta} - C\|u\|^{2q} - \frac{1}{p_\alpha^*}S_\alpha^{-\frac{p_\alpha^*}{p}}\|u\|^{p_\alpha^*}.\end{aligned}$$

If $1 < \theta < \frac{N-\alpha}{N-ps}$ i.e. $p\theta < p_\alpha^*$, in this case, we assume that $a \geq 0$ and $b > 0$, then we have

$$\mathcal{J}(u) \geq \frac{b}{p\theta}\|u\|^{p\theta} - \frac{1}{p_\alpha^*}S_\alpha^{-\frac{p_\alpha^*}{p}}\|u\|^{p_\alpha^*} - C\|u\|^{2q}.$$

Notice that if q satisfies (1.11), we have $p\theta < p_\alpha^* < 2q$. If q satisfies (1.12), we have $p\theta < 2q \leq p_\alpha^*$. Thus the claim follows if we choose $\rho > 0$ small enough.

If $\theta = \frac{N-\alpha}{N-ps}$ i.e. $p\theta = p_\alpha^*$, in this case, we assume that $a > 0$ and $0 < b < S_\alpha^{-\frac{p_\alpha^*}{p}}$, then we have

$$\mathcal{J}(u) \geq \frac{a}{p}\|u\|^p + \frac{b - S_\alpha^{-\frac{p_\alpha^*}{p}}}{p\theta}\|u\|^{p\theta} - C\|u\|^{2q} \geq \frac{a}{p}\|u\|^p - C\|u\|^{2q}.$$

Since $p < 2q$, the claim follows if we choose $\rho > 0$ small enough.

(2) Taking $u_{\alpha,\varepsilon,\delta} \in W_0^{s,p}(\Omega)$ given in formula (3.5), without loss of generality, we can consider $\delta = 1$. From Lemma 3.2, we have

$$\begin{aligned}\mathcal{J}(tu_{\alpha,\varepsilon,1}) &= \frac{a}{p}t^p\|u_{\alpha,\varepsilon,1}\|^p + \frac{b}{p\theta}t^{p\theta}\|u_{\alpha,\varepsilon,1}\|^{p\theta} \\ &\quad - \frac{t^{2q}}{2q}\int_\Omega\int_\Omega \frac{|u_{\alpha,\varepsilon,1}(x)|^q|u_{\alpha,\varepsilon,1}(y)|^q}{|x-y|^\mu}\,dx\,dy - \frac{t^{p_\alpha^*}}{p_\alpha^*}\int_\Omega \frac{|u_{\alpha,\varepsilon,1}|^{p_\alpha^*}}{|x|^\alpha}dx \\ &\leq \frac{a}{p}t^p\|u_{\alpha,\varepsilon,1}\|^p + \frac{b}{p\theta}t^{p\theta}\|u_{\alpha,\varepsilon,1}\|^{p\theta} - \frac{t^{p_\alpha^*}}{p_\alpha^*}\int_\Omega \frac{|u_{\alpha,\varepsilon,1}|^{p_\alpha^*}}{|x|^\alpha}dx \\ &\leq \frac{a}{p}t^p\left(S_\alpha^{\frac{N-\alpha}{ps-\alpha}} + C\varepsilon^{\frac{N-ps}{p-1}}\right) + \frac{b}{p\theta}t^{p\theta}\left(S_\alpha^{\frac{N-\alpha}{ps-\alpha}} + C\varepsilon^{\frac{N-ps}{p-1}}\right)^\theta \\ &\quad - \frac{t^{p_\alpha^*}}{p_\alpha^*}\left(S_\alpha^{\frac{N-\alpha}{ps-\alpha}} - C\varepsilon^{\frac{N-\alpha}{p-1}}\right) \\ &\leq \left(1 + C\varepsilon^{\frac{N-ps}{p-1}}\right)\left[\frac{a}{p}t^p S_\alpha^{\frac{N-\alpha}{ps-\alpha}} + \frac{b}{p\theta}t^{p\theta}S_\alpha^{\theta\frac{N-\alpha}{ps-\alpha}} - \frac{t^{p_\alpha^*}}{p_\alpha^*}S_\alpha^{\frac{N-\alpha}{ps-\alpha}}\right].\end{aligned}$$

If $1 < \theta < \frac{N-\alpha}{N-ps}$ i.e. $p\theta < p_\alpha^*$, we obtain that there exists $t > 0$ large enough such that $\|tu_{\alpha,\varepsilon,1}\| > \rho$ and $\mathcal{J}(tu_{\alpha,\varepsilon,1}) < 0$.

If $\theta = \frac{N-\alpha}{N-ps}$ i.e. $p\theta = p_\alpha^*$, we have

$$\begin{aligned}\mathcal{J}(tu_{\alpha,\varepsilon,1}) &\le \left(1 + C\varepsilon^{\frac{N-ps}{p-1}}\right)\left[\frac{a}{p}t^p S_\alpha^{\frac{N-\alpha}{ps-\alpha}} + \frac{b}{p\theta}t^{p\theta} S_\alpha^{\theta\frac{N-\alpha}{ps-\alpha}} - \frac{t^{p_\alpha^*}}{p_\alpha^*} S_\alpha^{\frac{N-\alpha}{ps-\alpha}}\right]\\ &= \left(1 + C\varepsilon^{\frac{N-ps}{p-1}}\right) S_\alpha^{\frac{N-\alpha}{ps-\alpha}}\left\{\frac{a}{p}t^p - \frac{t^{p_\alpha^*}}{p_\alpha^*}\left[1 - bS_\alpha^{\frac{p_\alpha^*}{p}}\right]\right\}.\end{aligned}$$

It follows from $a > 0$, $0 < b < S_\alpha^{-p_\alpha^*/p}$ that there exists $t > 0$ large enough such that $\|tu_{\alpha,\varepsilon,1}\| > \rho$ and $\mathcal{J}(tu_{\alpha,\varepsilon,1}) < 0$. □

Lemma 5.2 *Under the assumptions in Theorem 1.1. There exists $u_0 \in W_0^{s,p}(\Omega)\backslash\{0\}$ such that*

$$\sup_{t\ge 0} \mathcal{J}(tu_0) < c_*, \tag{5.1}$$

where c_* is given in (4.1).

Proof Write $\mathcal{J}(u) = I(u) - \Psi_q(u)$, where the functions $I : W_0^{s,p}(\Omega) \to \mathbb{R}$ and $\Psi_q : W_0^{s,p}(\Omega) \to \mathbb{R}$ are defined by

$$\begin{aligned} I(u) &= \frac{a}{p}\|u\|^p + \frac{b}{p\theta}\|u\|^{p\theta} - \frac{1}{p_\alpha^*}\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx,\\ \Psi_q(u) &= \frac{1}{2q}\int_\Omega\int_\Omega \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}\,dx\,dy.\end{aligned}$$

Without loss of generality, we can consider $\delta = 1$ in the definition of $u_{\alpha,\varepsilon,\delta} \in W_0^{s,p}(\Omega)$ given in formula (3.5), for any sufficiently small $0 < \varepsilon < 1$, set $u_0 = u_{\alpha,\varepsilon,1}$. The map $h(t) := I(tu_0)$ satisfies $h(t) > 0$ for $t > 0$ small, and $h(t) < 0$ for $t > 0$ large. Note that

$$\frac{d}{dt}h(t) = t^{p-1}\left[a\|u_0\|^p + bt^{p\theta-p}\|u_0\|^{p\theta} - t^{p_\alpha^*-p}\int_\Omega \frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx\right].$$

There exists $\tilde{t} > 0$ such that $h(t)$ attains maximum value at the point $\tilde{t}$, with $\tilde{t}$ satisfying

$$a\|u_0\|^p + b\tilde{t}^{p\theta-p}\|u_0\|^{p\theta} - \tilde{t}^{p_\alpha^*-p}\int_\Omega \frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx = 0.$$

• *In the case* $\theta = \frac{N-\alpha}{N-ps}$, $a > 0$, $0 < b < S_\alpha^{-\frac{p_\alpha^*}{p}}$. Direct calculations give that

$$\tilde{t} = \left(\frac{a\|u_0\|^p}{\int_\Omega \frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx - b\|u_0\|^{p\theta}} \right)^{\frac{1}{p(\theta-1)}},$$

and

$$\sup_{t\geq 0} I(tu_0) = I(\tilde{t}u_0) = \left(\frac{1}{p} - \frac{1}{p\theta}\right) \frac{(a\|u_0\|^p)^{\frac{\theta}{\theta-1}}}{\left(\int_\Omega \frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx - b\|u_0\|^{p\theta}\right)^{\frac{1}{\theta-1}}}.$$

By the estimations in Lemma 3.2, we get

$$\sup_{t\geq 0} I(tu_0) \leq \left(\frac{1}{p} - \frac{1}{p_\alpha^*}\right) \frac{(aS_\alpha)^{\frac{N-\alpha}{ps-\alpha}}}{\left(1 - bS_\alpha^{\frac{N-\alpha}{N-ps}}\right)^{\frac{N-ps}{ps-\alpha}}} + O\left(\varepsilon^{\frac{N-ps}{p-1}}\right). \tag{5.2}$$

• *In the case* $1 < \theta < \frac{N-\alpha}{N-ps}$, $a = 0$, $b > 0$, we have

$$\tilde{t} = \left(\frac{b\|u_0\|^{p\theta}}{\int_\Omega \frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx} \right)^{\frac{1}{p_\alpha^* - p\theta}},$$

and

$$\sup_{t\geq 0} I(tu_0) = I(\tilde{t}u_0) = \left(\frac{1}{p\theta} - \frac{1}{p_\alpha^*}\right) \frac{\left(b\|u_0\|^{p\theta}\right)^{\frac{p_\alpha^*}{p_\alpha^*-p\theta}}}{\left(\int_\Omega \frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx\right)^{\frac{p\theta}{p_\alpha^*-p\theta}}}.$$

By Lemma 3.2, we have

$$\sup_{t\geq 0} I(tu_0) \leq \left(\frac{1}{p\theta} - \frac{1}{p_\alpha^*}\right) \left(bS_\alpha^\theta\right)^{\frac{p_\alpha^*}{p_\alpha^*-p\theta}} + O\left(\varepsilon^{\frac{N-ps}{p-1}}\right). \tag{5.3}$$

• *In the case* $\theta = \frac{N-(ps+\alpha)/2}{N-ps}$, $a > 0$, $b > 0$, we have $p_\alpha^* - p = 2(p\theta - p)$ and $\tilde{t}$ satisfies the following equality

$$\left(\int_\Omega \frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx\right) \tilde{t}^{2(p\theta-p)} - b\|u_0\|^{p\theta}\tilde{t}^{p\theta-p} - a\|u_0\|^p = 0. \tag{5.4}$$

By solving the above equation, we get

$$\tilde{t}=\left(\frac{b\|u_0\|^{p\theta}+\sqrt{(b\|u_0\|^{p\theta})^2+4a\|u_0\|^p\int_\Omega\frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx}}{2\int_\Omega\frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx}\right)^{\frac{1}{p(\theta-1)}}$$

$$=\left(\frac{\mathcal{B}+\sqrt{\mathcal{B}^2+4a\|u_0\|^p}}{2\left(\int_\Omega\frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx\right)^{1/2}}\right)^{\frac{1}{p(\theta-1)}},\qquad \text{with } \mathcal{B}=\frac{b\|u_0\|^{p\theta}}{\left(\int_\Omega\frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx\right)^{1/2}}.$$

Therefore,

$$\begin{aligned}\sup_{t\geq 0} I(tu_0)=I(\tilde{t}u_0)&=\left(\frac{1}{p}-\frac{1}{p\theta}\right)a\|u_0\|^p\tilde{t}^p+\left(\frac{1}{p\theta}-\frac{1}{p_\alpha^*}\right)\tilde{t}^{(2\theta-1)p}\int_\Omega\frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx\\ &=\left(\frac{1}{p}-\frac{1}{p\theta}\right)a\|u_0\|^p\left(\frac{\mathcal{B}+\sqrt{\mathcal{B}^2+4a\|u_0\|^p}}{2\left(\int_\Omega\frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx\right)^{1/2}}\right)^{\frac{1}{\theta-1}}\\ &\quad+\left(\frac{1}{p\theta}-\frac{1}{p_\alpha^*}\right)\left(\frac{\mathcal{B}+\sqrt{\mathcal{B}^2+4a\|u_0\|^p}}{2\left(\int_\Omega\frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx\right)^{1/2}}\right)^{\frac{2\theta-1}{\theta-1}}\int_\Omega\frac{|u_0|^{p_\alpha^*}}{|x|^\alpha}dx,\end{aligned}$$

From Lemma 3.2, we have

$$\sup_{t\geq 0} I(tu_0)\leq\left[a\left(\frac{1}{p}-\frac{1}{p\theta}\right)\Lambda^{\frac{2(N-ps)}{ps-\alpha}}+\left(\frac{1}{p\theta}-\frac{1}{p_\alpha^*}\right)\Lambda^{\frac{2(N-\alpha)}{ps-\alpha}}\right]S_\alpha^{\frac{N-\alpha}{ps-\alpha}}+O\left(\varepsilon^{\frac{N-ps}{p-1}}\right). \tag{5.5}$$

We note that

$$\mathcal{J}(tu_0)\leq a\frac{t^p}{p}\|u_0\|^p+b\frac{t^{p\theta}}{p\theta}\|u_0\|^{p\theta}\quad\text{for } t\geq 0.$$

Thus there exists $t_0>0$ such that

$$\sup_{0\leq t\leq t_0}\mathcal{J}(tu_0)<c_*.$$

From (5.2), (5.3) and (5.5), we obtain

$$\sup_{t\geq t_0}\mathcal{J}(tu_0)\leq c_*+O\left(\varepsilon^{\frac{N-ps}{p-1}}\right)-\frac{t_0^{2q}}{2q}\int_\Omega\int_\Omega\frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}dx\,dy.$$

By Lemma 3.3, we find

$$\sup_{t\geq t_0} \mathcal{J}(tu_0) \leq c_* + O\left(\varepsilon^{\frac{N-ps}{p-1}}\right) - O\left(\varepsilon^{2N-\mu-2\frac{N-ps}{p}q}\right).$$

Either condition (1.11) or (1.12) implies that

$$\frac{N-ps}{p-1} > 2N - \mu - 2\frac{N-ps}{p}q.$$

Therefore, we get that $\sup_{t\geq 0} \mathcal{J}(tu_0) < c_*$. □

Proof of Theorem 1.1 By means of Lemmas 4.1, 5.1 and 5.2 , the existence of a positive solution for problem (1.2) follows from the well-known mountain pass theorem (see [38]). □

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