Calc. Var. 10, 355–387 (2000)

Multiple solutions of hemivariational inequalities with area-type term

Marco Degiovanni[1⋆], **Marco Marzocchi**[1], **Vicenţiu D. Rădulescu**[2⋆⋆]

[1] Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, Via Trieste 17, I 25121 Brescia, Italy
(e-mail: m.degiovanni@dmf.bs.unicatt.it, m.marzocchi@dmf.bs.unicatt.it)

[2] Department of Mathematics, University of Craiova, RO-1100 Craiova, Romania
(e-mail: varadulescu@hotmail.com)

Received: December 24, 1998 / Accepted: October 1, 1999

Abstract. Hemivariational inequalities containing both an area-type and a non-locally Lipschitz term are considered. Multiplicity results are obtained by means of techniques of nonsmooth critical point theory.

Mathematics Subject Classification (1991): 35J85

1 Introduction

The theory of variational inequalities appeared in the middle 60's in connection with the notion of subdifferential in the sense of Convex analysis (see e.g. [6, 22, 33] for the main aspects of this theory). All the inequality problems treated to the beginning 80's were related to convex energy functionals and therefore strictly connected to monotonicity: for instance, only monotone (possibly multivalued) boundary conditions and stress-strain laws could be studied.

Nonconvex inequality problems first appeared in [35] in the setting of Global analysis and were related to the subdifferential introduced in [17] (see A. Marino [34] for a survey of the developments in this direction).

In the setting of Continuum mechanics, P. D. Panagiotopoulos started the study of nonconvex and nonsmooth potentials by using Clarke's subdifferential for locally Lipschitz functionals. Due to the lack of convexity,

⋆ The research of the first two authors was partially supported by Ministero dell'Università e della Ricerca Scientifica e Tecnologica (40%–1997).

⋆⋆ The research of the third author was partially supported by G.N.A.F.A. - Consiglio Nazionale delle Ricerche, Italy.

new types of inequality problems, called hemivariational inequalities, have been generated. Roughly speaking, mechanical problems involving nonmonotone stress-strain laws or boundary conditions derived by nonconvex superpotentials lead to hemivariational inequalities. We refer the reader to [41, 42] for the main aspects of this theory.

A typical feature of nonconvex problems is that, while in the convex case the stationary variational inequalities give rise to minimisation problems for the potential or for the energy, in the nonconvex case the problem of the stationarity of the potential emerges and therefore it becomes reasonable to expect results also in the line of critical point theory.

For hemivariational inequalities, several contributions have been recently obtained by techniques of nonsmooth critical point theory (see [5, 23, 25–28, 38–40, 43] and references therein). The associated functional f is typically of the form $f = f_0 + f_1$, where f_0 is the principal part satisfying some standard coerciveness condition and f_1 is locally Lipschitz. In such a setting, the main abstract tool is constituted by the nonsmooth critical point theory developed in [12] for locally Lipschitz functionals.

The aim of our paper is to obtain existence and multiplicity results for hemivariational inequalities associated with functionals which come from the relaxation of, say,

$$f(u) = \int_\Omega \sqrt{1 + |Du|^2}\, dx + \int_\Omega G(x, u)\, dx\,,$$
$$u \in W_0^{1,1}(\Omega; \mathbf{R}^N),\ \Omega \text{ open in } \mathbf{R}^n,\ n \geq 2\,.$$

The first feature is that the functional f does not satisfy the Palais-Smale condition in $BV(\Omega; \mathbf{R}^N)$, the natural domain of f, as it is already known in the case of equations (see e.g. [36]). Therefore we extend f to $L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N)$ with value $+\infty$ outside $BV(\Omega; \mathbf{R}^N)$. This larger space is better behaved for the compactness properties, but the nonsmoothness of the functional increases. The second feature is that the assumptions we impose on G imply the second term of f to be continuous on $L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N)$, but not locally Lipschitz. More precisely, the function $\{s \mapsto G(x, s)\}$ is supposed to be locally Lipschitz for a.e. $x \in \Omega$, but the growth conditions we impose do not ensure the corresponding property for the integral on $L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N)$. Because of these facts, we will take advantage of the nonsmooth techniques developed in [7, 16, 19], which have been already applied in the setting of equations (see [8–10, 15, 18, 20, 21, 23, 36, 37] and references therein) and turn out to be suitable also for our setting.

In Sect. 2 we recall the main tools we will need, while in Sect. 3 we prove some general results for a class of lower semicontinuous functionals $f : L^p(\Omega; \mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$. In Sect. 4 we show that the area-type integrals fall into the class considered in Sect. 3. By the way, we also prove a relation between the convergence in the so-called intermediate topologies of $BV(\Omega;\mathbf{R}^N)$ and the convergence in $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$ (see Theorem 4.10), which seems to be new. Finally, in sections 5 and 6 we apply the general setting of Sect. 3 to obtain multiplicity results of Clark and Ambrosetti-Rabinowitz type. Of course, we believe that our approach could be equally applied to other situations with different geometries.

2 Recalls of nonsmooth analysis

Let X be a metric space endowed with the metric d and let $f : X \to \overline{\mathbf{R}}$ be a function. We denote by $\mathrm{B}_r(u)$ the open ball of centre u and radius r and we set

$$\mathrm{epi}\,(f) = \{(u,\lambda) \in X \times \mathbf{R} :\ f(u) \leq \lambda\}\ .$$

In the following, $X \times \mathbf{R}$ will be endowed with the metric

$$d\left((u,\lambda),(v,\mu)\right) = \left(d(u,v)^2 + (\lambda-\mu)^2\right)^{\frac{1}{2}}$$

and $\mathrm{epi}\,(f)$ with the induced metric.

Definition 2.1 *For every $u \in X$ with $f(u) \in \mathbf{R}$, we denote by $|df|\,(u)$ the supremum of the σ's in $[0,+\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : (\mathrm{B}_\delta\,(u, f(u)) \cap \mathrm{epi}\,(f)) \times [0,\delta] \to X$$

satisfying

$$d(\mathcal{H}((w,\mu),t),w) \leq t\,, \qquad f(\mathcal{H}((w,\mu),t)) \leq \mu - \sigma t\,,$$

whenever $(w,\mu) \in \mathrm{B}_\delta\,(u, f(u)) \cap \mathrm{epi}\,(f)$ and $t \in [0,\delta]$.

The extended real number $|df|\,(u)$ is called the weak slope *of f at u.*

The above notion has been introduced in [19], following an equivalent approach. When f is continuous, it has been independently introduced also in [32], while a variant has been considered in [30,31]. The version we have recalled here is taken from [7].

Now, according to [17], we define a function $\mathcal{G}_f : \mathrm{epi}\,(f) \to \mathbf{R}$ by $\mathcal{G}_f(u,\lambda) = \lambda$. Of course, $\mathcal{G}_f$ is Lipschitz continuous of constant 1.

Proposition 2.2 *For every $u \in X$ with $f(u) \in \mathbf{R}$, we have $f(u) = \mathcal{G}_f(u, f(u))$ and*

$$|df|\,(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|\,(u,f(u))}{\sqrt{1-|d\mathcal{G}_f|\,(u,f(u))^2}} & \text{if } |d\mathcal{G}_f|\,(u,f(u)) < 1\,, \\ +\infty & \text{if } |d\mathcal{G}_f|\,(u,f(u)) = 1\,. \end{cases}$$

Proof. See [7, Proposition 2.3]. □

The previous proposition allows us to reduce, at some extent, the study of the general function f to that of the continuous function $\mathcal{G}_f$.

Definition 2.1 can be simplified, when f is continuous.

Proposition 2.3 *Let $f : X \to \mathbf{R}$ be continuous. Then $|df|\,(u)$ is the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : \mathrm{B}_\delta\,(u) \times [0, \delta] \to X$$

satisfying

$$d(\mathcal{H}(w,t), w) \le t\,, \qquad f(\mathcal{H}(w,t)) \le f(w) - \sigma t\,, \tag{2.4}$$

whenever $w \in \mathrm{B}_\delta\,(u)$ and $t \in [0, \delta]$.

Proof. See [7, Proposition 2.2]. □

We need also, in a particular case, the notion of equivariant weak slope (see e.g. [10] for the general definition).

Definition 2.5 *Let X be a normed space and $f : X \to \overline{\mathbf{R}}$ an even function with $f(0) < +\infty$. For every $(0, \lambda) \in \mathrm{epi}\,(f)$ we denote by $|d_{\mathbf{Z}_2}\mathcal{G}_f|\,(0, \lambda)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) : (\mathrm{B}_\delta\,(0, \lambda) \cap \mathrm{epi}\,(f)) \times [0, \delta] \to \mathrm{epi}\,(f)$$

satisfying

$$d\,(\mathcal{H}((w,\mu),t), (w,\mu)) \le t\,, \qquad \mathcal{H}_2((w,\mu),t) \le \mu - \sigma t\,,$$

$$\mathcal{H}_1((-w,\mu),t) = -\mathcal{H}_1((w,\mu),t)\,,$$

whenever $(w, \mu) \in \mathrm{B}_\delta\,(0, \lambda) \cap \mathrm{epi}\,(f)$ and $t \in [0, \delta]$.

Remark 2.6 In Proposition 2.3, if there exist $\varrho > 0$ and a continuous map $\mathcal{H}$ satisfying

$$d(\mathcal{H}(w,t), w) \le \varrho t\,, \qquad f(\mathcal{H}(w,t)) \le f(w) - \sigma t\,,$$

instead of (2.4), we can deduce that $|df|\,(u) \ge \sigma/\varrho$.

A similar remark applies to Definition 2.5.

By means of the weak slope, we can now introduce the two main notions of critical point theory.

Definition 2.7 *We say that* $u \in X$ *is* a (lower) critical point *of* f, *if* $f(u) \in \mathbf{R}$ *and* $|df|(u) = 0$. *We say that* $c \in \mathbf{R}$ *is* a (lower) critical value *of* f, *if there exists a (lower) critical point* $u \in X$ *of* f *with* $f(u) = c$.

Definition 2.8 *Let* $c \in \mathbf{R}$. *A sequence* (u_h) *in* X *is said to be* a Palais-Smale sequence at level c *(*$(PS)_c-$sequence, *for short) for* f, *if* $f(u_h) \to c$ *and* $|df|(u_h) \to 0$.

We say that f *satisfies* the Palais-Smale condition at level c *(*$(PS)_c$, *for short), if every* $(PS)_c-$*sequence* (u_h) *for* f *admits a convergent subsequence* (u_{h_k}) *in* X.

The main feature of the weak slope is that it allows to prove natural extensions of the classical critical point theory for general continuous functions defined on complete metric spaces. Moreover, one can try to reduce the study of a lower semicontinuous function f to that of the continuous function $\mathcal{G}_f$. Actually, Proposition 2.2 suggests to exploit the bijective correspondence between the set where f is finite and the graph of f. This approach can be successful, if we can ensure that the remaining part of $\mathrm{epi}\,(f)$ does not carry much information. The next notion turns out to be useful for this purpose.

Definition 2.9 *Let* $c \in \mathbf{R}$. *We say that* f *satisfies condition* $(epi)_c$, *if there exists* $\varepsilon > 0$ *such that*

$$\inf\left\{|d\mathcal{G}_f|(u,\lambda) :\ f(u) < \lambda,\ |\lambda - c| < \varepsilon\right\} > 0\,.$$

The next two results may help in dealing with condition $(epi)_c$.

Proposition 2.10 *Let* $(u,\lambda) \in \mathrm{epi}\,(f)$. *Assume that there exist* $\varrho, \sigma, \delta, \varepsilon > 0$ *and a continuous map*

$$\mathcal{H} : \{w \in \mathrm{B}_\delta\,(u) : f(w) < \lambda + \delta\} \times [0,\delta] \to X$$

satisfying

$$d(\mathcal{H}(w,t), w) \leq \varrho t\,, \qquad f(\mathcal{H}(w,t)) \leq \max\{f(w) - \sigma t, \lambda - \varepsilon\}$$

whenever $w \in \mathrm{B}_\delta\,(u)$, $f(w) < \lambda + \delta$ *and* $t \in [0,\delta]$.

Then we have

$$|d\mathcal{G}_f|(u,\lambda) \geq \frac{\sigma}{\sqrt{\varrho^2 + \sigma^2}}\,.$$

If moreover X *is a normed space,* f *is even,* $u = 0$ *and* $\mathcal{H}(-w,t) = -\mathcal{H}(w,t)$, *then we have*

$$|d_{\mathbf{Z}_2}\mathcal{G}_f|(0,\lambda) \geq \frac{\sigma}{\sqrt{\varrho^2 + \sigma^2}}\,.$$

Proof. Let $\delta' \in]0,\delta]$ be such that $\delta' + \sigma\delta' \leq \varepsilon$ and let

$$\mathcal{K} : \big(\mathrm{B}_{\delta'}(u,\lambda) \cap \mathrm{epi}\,(f)\big) \times [0,\delta'] \to \mathrm{epi}\,(f)$$

be defined by $\mathcal{K}\big((w,\mu),t\big) = \big(\mathcal{H}(w,t),\mu - \sigma t\big)$. If $(w,\mu) \in \mathrm{B}_{\delta'}(u,\lambda) \cap \mathrm{epi}\,(f)$ and $t \in [0,\delta']$, we have

$$\lambda - \varepsilon \leq \lambda - \delta' - \sigma\delta' < \mu - \sigma t\,, \qquad f(w) - \sigma t \leq \mu - \sigma t\,,$$

hence

$$f(\mathcal{H}(w,t)) \leq \max\{f(w) - \sigma t, \lambda - \varepsilon\} \leq \mu - \sigma t\,.$$

Therefore $\mathcal{K}$ actually takes its values in $\mathrm{epi}\,(f)$. Furthermore, it is

$$d\big(\mathcal{K}\big((w,\mu),t\big),(w,\mu)\big) \leq \sqrt{\varrho^2 + \sigma^2}\,t\,,$$

$$\mathcal{G}_f\big(\mathcal{K}\big((w,\mu),t\big)\big) = \mu - \sigma t = \mathcal{G}_f(w,\mu) - \sigma t\,.$$

Taking into account Proposition 2.3 and Remark 2.6, the first assertion follows.

In the symmetric case, $\mathcal{K}$ automatically satisfies the further condition required in Definition 2.5. □

Corollary 2.11 *Let $(u,\lambda) \in \mathrm{epi}\,(f)$ with $f(u) < \lambda$. Assume that for every $\varrho > 0$ there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : \{w \in \mathrm{B}_\delta(u) : f(w) < \lambda + \delta\} \times [0,\delta] \to X$$

satisfying

$$d(\mathcal{H}(w,t),w) \leq \varrho t\,, \qquad f(\mathcal{H}(w,t)) \leq f(w) + t\big(f(u) - f(w) + \varrho\big)$$

whenever $w \in \mathrm{B}_\delta(u)$, $f(w) < \lambda + \delta$ and $t \in [0,\delta]$.

Then we have $|d\mathcal{G}_f|\,(u,\lambda) = 1$. If moreover X is a normed space, f is even, $u = 0$ and $\mathcal{H}(-w,t) = -\mathcal{H}(w,t)$, then we have $|d_{\mathbf{Z}_2}\mathcal{G}_f|\,(0,\lambda) = 1$.

Proof. Let $\varepsilon > 0$ with $\lambda - 2\varepsilon > f(u)$, let $0 < \varrho < \lambda - f(u) - 2\varepsilon$ and let δ and $\mathcal{H}$ be as in the hypothesis. By reducing δ, we may also assume that

$$\delta \leq 1\,, \qquad \delta\big(|\lambda - 2\varepsilon| + |f(u) + \varrho|\big) \leq \varepsilon\,.$$

Now consider $w \in \mathrm{B}_\delta(u)$ with $f(w) < \lambda + \delta$ and $t \in [0,\delta]$. If $f(w) \leq \lambda - 2\varepsilon$, we have

$$\begin{aligned} f(w) + t(f(u) - f(w) + \varrho) &= (1-t)f(w) + t(f(u) + \varrho) \leq \\ &\leq (1-t)(\lambda - 2\varepsilon) + t(f(u) + \varrho) \leq \\ &\leq \lambda - 2\varepsilon + t|\lambda - 2\varepsilon| + t|f(u) + \varrho| \leq \lambda - \varepsilon\,, \end{aligned}$$

while, if $f(w) > \lambda - 2\varepsilon$, we have

$$f(w) + t(f(u) - f(w) + \varrho) \leq f(w) - (\lambda - f(u) - 2\varepsilon - \varrho)t\,.$$

In any case it follows

$$f(\mathcal{H}(w,t)) \leq \max\{f(w) - (\lambda - f(u) - 2\varepsilon - \varrho)t, \lambda - \varepsilon\}\,.$$

From Proposition 2.10 we get

$$|d\mathcal{G}_f|\,(u,\lambda) \geq \frac{\lambda - f(u) - 2\varepsilon - \varrho}{\sqrt{\varrho^2 + (\lambda - f(u) - 2\varepsilon - \varrho)^2}}$$

and the first assertion follows by the arbitrariness of ϱ.

The same proof works also in the symmetric case. □

Now we recall two critical point theorems we will apply later. The first one is an adaptation of a result of D. C. Clark (see [13] and [44, Theorem 9.1]) to our setting.

Theorem 2.12 *Let X be a Banach space and $f : X \to \mathbf{R} \cup \{+\infty\}$ an even lower semicontinuous function. Assume that*

(a) *f is bounded from below;*
(b) *for every $c < f(0)$, the function f satisfies $(PS)_c$ and $(epi)_c$;*
(c) *there exist $k \geq 1$ and an odd continuous map $\psi : S^{k-1} \to X$ such that*

$$\sup\left\{f(\psi(x)) :\, x \in S^{k-1}\right\} < f(0)\,,$$

where S^{k-1} denotes the unit sphere in $\mathbf{R}^k$.

Then f admits at least k pairs $(u_1, -u_1), \ldots, (u_k, -u_k)$ of critical points with $f(u_j) < f(0)$.

Proof. See [20, Theorem 2.5]. □

The next result is an adaptation of the classical Theorem of Ambrosetti-Rabinowitz [1,44,48].

Theorem 2.13 *Let X be a Banach space and $f : X \to \mathbf{R} \cup \{+\infty\}$ an even lower semicontinuous function. Assume that there exists a strictly increasing sequence (V_h) of finite-dimensional subspaces of X with the following properties:*

(a) *there exist a closed subspace Z of X, $\varrho > 0$ and $\alpha > f(0)$ such that $X = V_0 \oplus Z$ and*

$$\forall u \in Z :\, \|u\| = \varrho \implies f(u) \geq \alpha\,;$$

(b) *there exists a sequence* (R_h) *in* $]\varrho, +\infty[$ *such that*

$$\forall u \in V_h : \|u\| \geq R_h \implies f(u) \leq f(0)\,;$$

(c) *for every* $c \geq \alpha$*, the function* f *satisfies* $(PS)_c$ *and* $(epi)_c$*;*
(d) *we have* $|d_{\mathbf{Z}_2}\mathcal{G}_f|\,(0, \lambda) \neq 0$ *whenever* $\lambda \geq \alpha$*.*

Then there exists a sequence (u_h) *of critical points of* f *with* $f(u_h) \to +\infty$*.*

Proof. Because of assumption (c), the function $\mathcal{G}_f$ satisfies $(PS)_c$ for any $c \geq \alpha$. Then the assertion follows from [36, Theorem (2.7)]. □

Now assume that X is a normed space over $\mathbf{R}$ and $f : X \to \overline{\mathbf{R}}$ a function.

Definition 2.14 *For every* $u \in X$ *with* $f(u) \in \mathbf{R}$*,* $v \in X$ *and* $\varepsilon > 0$*, let* $f^\circ_\varepsilon(u; v)$ *be the infimum of* r*'s in* $\overline{\mathbf{R}}$ *such that there exist* $\delta > 0$ *and a continuous map*

$$\mathcal{V} : (\mathrm{B}_\delta\,(u, f(u)) \cap \mathrm{epi}\,(f)) \times]0, \delta] \to \mathrm{B}_\varepsilon\,(v)$$

satisfying

$$f(z + t\,\mathcal{V}((z, \mu), t)) \leq \mu + rt$$

whenever $(z, \mu) \in \mathrm{B}_\delta\,(u, f(u)) \cap \mathrm{epi}\,(f)$ *and* $t \in]0, \delta]$*. Then let*

$$f^\circ\,(u; v) = \sup_{\varepsilon > 0} f^\circ_\varepsilon\,(u; v)\ .$$

Let us recall that the function $f^\circ\,(u; \cdot)$ is convex, lower semicontinuous and positively homogeneous of degree 1 (see [7, Corollary 4.6]).

Definition 2.15 *For every* $u \in X$ *with* $f(u) \in \mathbf{R}$*, we set*

$$\partial f(u) = \{u^* \in X^* : \langle u^*, v\rangle \leq f^\circ\,(u; v) \quad \forall v \in X\}\,.$$

It turns out that $f^\circ\,(u; v)$ is greater than or equal to the generalized directional derivative in the sense of Rockafellar (see [14, 47]). Consequently, $\partial f(u)$ contains the subdifferential of f at u in the sense of Clarke. These modified notions of $f^\circ\,(u; v)$ and $\partial f(u)$ have been introduced in [7, 18], because they are better related with the notion of weak slope and hence more suitable for critical point theory, as the next result shows.

Theorem 2.16 *If* $u \in X$ *and* $f(u) \in \mathbf{R}$*, the following facts hold:*

(a) $|df|\,(u) < +\infty \iff \partial f(u) \neq \emptyset$*;*
(b) $|df|\,(u) < +\infty \implies |df|\,(u) \geq \min\,\{\|u^*\| : u^* \in \partial f(u)\}$.

Proof. See [7, Theorem 4.13]. $\square$

However, if $f : X \to \mathbf{R}$ is locally Lipschitz, these notions agree with those of Clarke (see [7, Corollary 4.10]). Thus, in such a case, $f^\circ(u;\cdot)$ is also Lipschitz continuous and we have that

$$\forall u, v \in X : \; f^\circ(u;v) = \limsup_{\substack{z\to u,\, w\to v\\ t\to 0^+}} \frac{f(z+tw)-f(z)}{t}\,, \tag{2.17}$$

$$\{(u,v) \mapsto f^\circ(u;v)\} \text{ is upper semicontinuous on } X \times X\,. \tag{2.18}$$

3 The general framework

Let $n \geq 1$, $N \geq 1$, Ω be an open subset of $\mathbf{R}^n$ and $1 < p < \infty$. In the following, we will denote by $\|\cdot\|_q$ the usual norm in L^q $(1 \leq q \leq \infty)$. We now define the functional setting we are interested in.

Let $\mathcal{E} : L^p(\Omega;\mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ be a functional such that:

$(\mathcal{E}_1)$ $\mathcal{E}$ is convex, lower semicontinuous and $0 \in \mathcal{D}(\mathcal{E})$, where

$$\mathcal{D}(\mathcal{E}) = \left\{u \in L^p(\Omega;\mathbf{R}^N) : \; \mathcal{E}(u) < +\infty\right\}\,;$$

$(\mathcal{E}_2)$ there exists $\vartheta \in C_c(\mathbf{R}^N)$ with $0 \leq \vartheta \leq 1$ and $\vartheta(0) = 1$ such that

$$(\mathcal{E}_2.1) \qquad \forall u \in \mathcal{D}(\mathcal{E}),\ \forall v \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega;\mathbf{R}^N),\ \forall c > 0 :$$
$$\lim_{h\to\infty} \left[\sup_{\substack{\|z-u\|_p \leq c\\ \mathcal{E}(z)\leq c}} \mathcal{E}\left(\vartheta\left(\frac{z}{h}\right) v\right)\right] = \mathcal{E}(v)\,;$$

$$(\mathcal{E}_2.2) \qquad \forall u \in \mathcal{D}(\mathcal{E}) : \; \lim_{h\to\infty} \mathcal{E}\left(\vartheta\left(\frac{u}{h}\right) u\right) = \mathcal{E}(u)\,.$$

Moreover, let $G : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a function such that

(G_1) $G(\cdot, s)$ is measurable for every $s \in \mathbf{R}^N$;
(G_2) for every $t > 0$ there exists $\alpha_t \in L^1(\Omega)$ such that

$$|G(x,s_1) - G(x,s_2)| \leq \alpha_t(x)|s_1 - s_2|$$

for a.e. $x \in \Omega$ and every $s_1, s_2 \in \mathbf{R}^N$ with $|s_j| \leq t$; for a.e. $x \in \Omega$ we set

$$G^\circ(x,s;\hat{s}) = \gamma^\circ(s;\hat{s})\,, \qquad \partial_s G(x,s) = \partial\gamma(s)\,,$$

where $\gamma(s) = G(x,s)$;

(G_3) there exist $a_0 \in L^1(\Omega)$ and $b_0 \in \mathbf{R}$ such that

$$G(x,s) \geq -a_0(x) - b_0|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,;$$

(G_4) there exist $a_1 \in L^1(\Omega)$ and $b_1 \in \mathbf{R}$ such that

$$G^\circ(x,s;-s) \leq a_1(x) + b_1|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,.$$

Because of $(\mathcal{E}_1)$ and (G_3), we can define a lower semicontinuous functional $f : L^p(\Omega;\mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ by

$$f(u) = \mathcal{E}(u) + \int_\Omega G(x,u(x))\,dx\,.$$

Remark 3.1 According to $(\mathcal{E}_1)$, the functional $\mathcal{E}$ is lower semicontinuous. Condition $(\mathcal{E}_2)$ ensures that $\mathcal{E}$ is continuous at least on some particular restrictions.

Remark 3.2 If $\{s \mapsto G(x,s)\}$ is of class C^1 for a.e $x \in \Omega$, the estimates in (G_2) and in (G_4) are respectively equivalent to

$$|s| \leq t \implies |D_sG(x,s)| \leq \alpha_t(x)\,,$$

$$D_sG(x,s) \cdot s \geq -a_1(x) - b_1|s|^p\,.$$

Because of (G_2), for a.e. $x \in \Omega$ and any $t > 0$ and $s \in \mathbf{R}^N$ with $|s| < t$ we have

$$\forall \hat{s} \in \mathbf{R}^N :\ |G^\circ(x,s;\hat{s})| \leq \alpha_t(x)|\hat{s}|\,; \tag{3.3}$$

$$\forall s^* \in \partial_s G(x,s) :\ |s^*| \leq \alpha_t(x)\,. \tag{3.4}$$

In the following, we set $\vartheta_h(s) = \vartheta(s/h)$, where ϑ is a function as in $(\mathcal{E}_2)$, and we fix $M > 0$ such that $\vartheta = 0$ outside $B_M(0)$. Therefore

$$\forall s \in \mathbf{R}^N :\ |s| \geq hM \implies \vartheta_h(s) = 0\,. \tag{3.5}$$

Our first result concerns the connection between the notions of generalized directional derivative and subdifferential in the functional space $L^p(\Omega;\mathbf{R}^N)$ and the more concrete setting of hemivariational inequalities, which also involves the notion of generalized directional derivative, but in $\mathbf{R}^N$.

If $u, v \in L^p(\Omega; \mathbf{R}^N)$, we can define $\int_\Omega G^\circ(x,u;v)\,dx$ if we agree, as in [46], that

$$\int_\Omega G^\circ(x,u;v)\,dx = +\infty \quad \text{whenever}$$
$$\int_\Omega [G^\circ(x,u;v)]^+\,dx = \int_\Omega [G^\circ(x,u;v)]^-\,dx = +\infty\,.$$

With this convention, $\left\{v \mapsto \int_\Omega G^\circ(x,u;v)\,dx\right\}$ is a convex functional from $L^p(\Omega; \mathbf{R}^N)$ into $\overline{\mathbf{R}}$.

Theorem 3.6 *Let $u \in \mathcal{D}(f)$. Then the following facts hold:*

(a) *for every $v \in \mathcal{D}(\mathcal{E})$ there exists a sequence (v_h) in $\mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega; \mathbf{R}^N)$ satisfying $[G^\circ(x,u;v_h - u)]^+ \in L^1(\Omega)$, $\|v_h - v\|_p \to 0$ and $\mathcal{E}(v_h) \to \mathcal{E}(v)$;*

(b) *for every $v \in \mathcal{D}(\mathcal{E})$ we have*

$$f^\circ(u; v-u) \leq \mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u;v-u)\,dx\,; \tag{3.7}$$

(c) *if $\partial f(u) \neq \emptyset$, we have $G^\circ(x,u;-u) \in L^1(\Omega)$ and*

$$\mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u;v-u)\,dx \geq \int_\Omega u^* \cdot (v-u)\,dx \tag{3.8}$$

for every $u^* \in \partial f(u)$ and $v \in \mathcal{D}(\mathcal{E})$ (the dual space of $L^p(\Omega; \mathbf{R}^N)$ is identified with $L^{p'}(\Omega; \mathbf{R}^N)$ in the usual way);

(d) *if $N = 1$, we have $[G^\circ(x,u;v-u)]^+ \in L^1(\Omega)$ for every $v \in L^\infty(\Omega; \mathbf{R}^N)$.*

Proof. (a) Let us set $\mathcal{G}(v) = \int_\Omega G/x, v)dx$. Given $\varepsilon > 0$, by $(\mathcal{E}_2.2)$ we have $\|\vartheta_h(v)v - v\|_p < \varepsilon$ and $|\mathcal{E}(\vartheta_h(v)v) - \mathcal{E}(v)| < \varepsilon$ for h large enough. Then, by $(\mathcal{E}_2.1)$ we get $\|\vartheta_k(u)\vartheta_h(v)v - v\|_p < \varepsilon$ and $|\mathcal{E}(\vartheta_k(u)\vartheta_h(v)v) - \mathcal{E}(v)| < \varepsilon$ for k large enough. Of course $\vartheta_k(u)\vartheta_h(v)v \in L^\infty(\Omega; \mathbf{R}^N)$ and by (3.3) we have

$$\begin{aligned} G^\circ(x,u;\vartheta_k(u)\vartheta_h(v)v - u) &\leq \vartheta_k(u)\vartheta_h(v)G^\circ(x,u;v-u) + \\ &\quad +(1 - \vartheta_k(u)\vartheta_h(v))G^\circ(x,u;-u) \leq \\ &\leq (h+k)M\alpha_{kM}(x) + [G^\circ(x,u;-u)]^+\,. \end{aligned}$$

From (G_4) we infer that $[G^\circ(x,u;-u)]^+ \in L^1(\Omega)$ and assertion (a) follows.

(b) Without loss of generality, we may assume that $[G^\circ(x,u;v-u)]^+ \in L^1(\Omega)$. Suppose first that $v \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega; \mathbf{R}^N)$ and take $\varepsilon > 0$.

We claim that for every $z \in L^p(\Omega; \mathbf{R}^N)$, $t \in]0, 1/2]$ and $h \geq 1$ with $hM > \|v\|_\infty$, we have

$$\frac{G(x, z + t(\vartheta_h(z)v - z)) - G(x, z)}{t} \leq$$
$$\leq 2\left(\|v\|_\infty \alpha_{hM} + a_1 + b_1(|z| + |v|)^p\right) . \tag{3.9}$$

In fact, for a.e. $x \in \Omega$, by Lebourg's Theorem (see e.g. [14]) there exist $\bar{t} \in]0, t[$ and $u^* \in \partial_s G(x, z + \bar{t}(\vartheta_h(z)v - z))$ such that

$$\begin{aligned} &\frac{G(x, z + t(\vartheta_h(z)v - z)) - G(x, z)}{t} = \\ &\quad = u^* \cdot (\vartheta_h(z)v - z) = \\ &\quad = \frac{1}{1 - \bar{t}}\left[\vartheta_h(z)u^* \cdot v - u^* \cdot (z + \bar{t}(\vartheta_h(z)v - z))\right] . \end{aligned}$$

By (3.4) and (3.5), it easily follows that

$$\frac{|\vartheta_h(z)u^* \cdot v|}{1 - \bar{t}} \leq 2\,\|v\|_\infty\, \alpha_{hM} .$$

On the other hand, from (G_4) we deduce that for a.e. $x \in \Omega$

$$\begin{aligned} \frac{u^* \cdot (z + \bar{t}(\vartheta_h(z)v - z))}{1 - \bar{t}} &\geq -\frac{1}{1 - \bar{t}} G^\circ(x, z + \bar{t}(\vartheta_h(z)v - z); \\ -(z + \bar{t}(\vartheta_h(z)v - z)) &\geq -\frac{1}{1 - \bar{t}}(a_1 + b_1|z + \bar{t}(\vartheta_h(z)v - z)|^p) \\ &\geq -2\left(a_1 + b_1(|z| + |v|)^p\right) . \end{aligned}$$

Then (3.9) easily follows.

For a.e. $x \in \Omega$ we have

$$\begin{aligned} G^\circ(x, u; \vartheta_h(u)v - u) \leq \vartheta_h(u) &G^\circ(x, u; v - u) + \\ &+(1 - \vartheta_h(u))G^\circ(x, u; -u) \leq \\ &\leq [G^\circ(x, u; v - u)]^+ + [G^\circ(x, u; -u)]^+ . \end{aligned}$$

Furthermore, for a.e. $x \in \Omega$ and every $s \in \mathbf{R}^N$, (G_2) implies $G^\circ(x, s; \cdot)$ to be Lipschitz continuous, so in particular

$$\lim_h G^\circ(x, u; \vartheta_h(u)v - u) = G^\circ(x, u; v - u) \qquad \text{a.e. in } \Omega .$$

Then, given

$$\lambda > \int_\Omega G^\circ(x, u; v - u)\, dx ,$$

by Fatou's Lemma there exists $\overline{h} \geq 1$ such that

(3.10)
$$\forall h \geq \overline{h}: \int_\Omega G^\circ(x,u;\vartheta_h(u)v-u)\,dx < \lambda \quad \text{and} \quad \|\vartheta_h(u)v-v\|_p < \varepsilon\,.$$

By the lower semicontinuity of $\mathcal{G}$, there exists $\overline{\delta} \in]0,1/2]$ such that for every $z \in \mathrm{B}_{\overline{\delta}}(u)$ it is $\mathcal{G}(z) \geq \mathcal{G}(u) - \frac{1}{2}$. Then for every $(z,\mu) \in \mathrm{B}_{\overline{\delta}}(u,f(u)) \cap \mathrm{epi}\,(f)$ it follows

$$\mathcal{E}(z) \leq \mu - \mathcal{G}(z) \leq \mu + \frac{1}{2} - \mathcal{G}(u) \leq f(u) + \overline{\delta} - \mathcal{G}(u) + \frac{1}{2} \leq \mathcal{E}(u) + 1\,.$$

Let now $\sigma > 0$. By assumptions $(\mathcal{E}_1)$ and $(\mathcal{E}_2.1)$ there exist $h \geq \overline{h}$ and $\delta \leq \overline{\delta}$ such that

$$\|v\|_\infty < hM,$$

$$\mathcal{E}(z) > \mathcal{E}(u)-\sigma,\ \ \mathcal{E}(\vartheta_h(z)v) < \mathcal{E}(v)+\sigma,\ \ \|(\vartheta_h(z)v-z)-(v-u)\|_p < \varepsilon,$$

for any $z \in \mathrm{B}_\delta(u)$ with $\mathcal{E}(z) \leq \mathcal{E}(u)+1$.

Taking into account (2.17), (3.9) and (3.10), we deduce by Fatou's Lemma that, possibly reducing δ, for any $t \in]0,\delta]$ and for any $z \in \mathrm{B}_\delta(u)$ we have

$$\int_\Omega \frac{G(x,z+t(\vartheta_h(z)v-z)) - G(x,z)}{t}\,dx < \lambda\,.$$

Now let $\mathcal{V} : (\mathrm{B}_\delta(u,f(u)) \cap \mathrm{epi}\,(f)) \times]0,\delta] \to \mathrm{B}_\varepsilon(v-u)$ be defined setting

$$\mathcal{V}((z,\mu),t) = \vartheta_h(z)v - z\,.$$

Since $\mathcal{V}$ is evidently continuous and

$$\begin{aligned} f(z+t\mathcal{V}((z,\mu),t)) &= f(z+t(\vartheta_h(z)v-z)) \leq \\ &\leq \mathcal{E}(z) + t\,(\mathcal{E}(\vartheta_h(z)v) - \mathcal{E}(z)) + \\ &\quad +\mathcal{G}(z+t(\vartheta_h(z)v-z)) \leq \\ &\leq \mathcal{E}(z) + (\mathcal{E}(v) - \mathcal{E}(u) + 2\sigma)t + \mathcal{G}(z) + \lambda t = \\ &= f(z) + (\mathcal{E}(v) - \mathcal{E}(u) + \lambda + 2\sigma)t\,, \end{aligned}$$

we have

$$f^\circ_\varepsilon(u;v-u) \leq \mathcal{E}(v) - \mathcal{E}(u) + \lambda + 2\sigma\,.$$

By the arbitrariness of $\sigma > 0$ and $\lambda > \int_\Omega G^\circ(x,u;v-u)\,dx$, it follows

$$f^\circ_\varepsilon(u;v-u) \leq \mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u;v-u)\,dx\,.$$

Passing to the limit as $\varepsilon \to 0^+$, we get (3.7) when $v \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega; \mathbf{R}^N)$.

Let us now treat the general case. If we set $v_h = \vartheta_h(v)v$, we have $v_h \in L^\infty(\Omega; \mathbf{R}^N)$. Arguing as before, it is easy to see that

$$G^\circ(x, u; v_h - u) \leq [G^\circ(x, u; v - u)]^+ + [G^\circ(x, u; -u)]^+ ,$$

so that

$$\limsup_h \int_\Omega G^\circ(x, u; v_h - u)\, dx \leq \int_\Omega G^\circ(x, u; v - u)\, dx .$$

On the other hand, by the previous step it holds

$$f^\circ(u; v_h - u) \leq \mathcal{E}(v_h) - \mathcal{E}(u) + \int_\Omega G^\circ(x, u; v_h - u)\, dx .$$

Passing to the lower limit as $h \to \infty$ and taking into account the lower semicontinuity of $f^\circ(u, \cdot)$ and $(\mathcal{E}_2.2)$, we get (3.7).
(c) We already know that $[G^\circ(x, u; -u)]^+ \in L^1(\Omega)$. If we choose $v = 0$ in (3.7), we obtain

$$f^\circ(u; -u) \leq \mathcal{E}(0) - \mathcal{E}(u) + \int_\Omega G^\circ(x, u; -u)\, dx .$$

Since $\partial f(u) \neq \emptyset$, it is $f^\circ(u; -u) > -\infty$, hence

$$\int_\Omega [G^\circ(x, u; -u)]^-\, dx < +\infty .$$

Finally, if $u^* \in \partial f(u)$ we have by definition that

$$f^\circ(u; v - u) \geq \int_\Omega u^* \cdot (v - u)\, dx$$

and (3.8) follows from (3.7).
(d) From (3.3) it readily follows that $G^\circ(x, u; v - u)$ is summable where $|u(x)| \leq \|v\|_\infty$. On the other hand, where $|u(x)| > \|v\|_\infty$ we have

$$G^\circ(x, u; v - u) = \left(1 - \frac{v}{u}\right) G^\circ(x, u; -u)$$

and the assertion follows from (G_4). $\square$

Since f is only lower semicontinuous, we are interested in the verification of the condition $(epi)_c$. For this purpose, we consider an assumption (G_3') on G stronger than (G_3).

Theorem 3.11 *Assume that*

(G_3') *there exist* $a \in L^1(\Omega)$ *and* $b \in \mathbf{R}$ *such that*

$$|G(x, s)| \leq a(x) + b|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N .$$

Then for every $(u,\lambda) \in \mathrm{epi}\,(f)$ *with* $\lambda > f(u)$ *it is* $|d\mathcal{G}_f|\,(u,\lambda) = 1$. *Moreover, if* $\mathcal{E}$ *and* $G(x,\cdot)$ *are even, for every* $\lambda > f(0)$ *we have* $|d_{\mathbf{Z}_2}\mathcal{G}_f|\,(0,\lambda) = 1$.

Proof. Let $\varrho > 0$. Since

$$\forall \tau \in [0,1] : \, G^\circ(x,u;\tau u - u) = (1-\tau)G^\circ(x,u;-u) \leq [G^\circ(x,u;-u)]^+,$$

by $(\mathcal{E}_2.2)$ and (G_4) there exists $\overline{h} \geq 1$ such that

$$\|\vartheta_{\overline{h}}(u)u - u\|_p < \varrho\,, \qquad \mathcal{E}(\vartheta_{\overline{h}}(u)u) < \mathcal{E}(u) + \varrho\,,$$

$$\forall h \geq \overline{h} : \int_\Omega G^\circ(x,u;\vartheta_h(u)\vartheta_{\overline{h}}(u)u - u)\,dx < \varrho\,.$$

Set $v = \vartheta_{\overline{h}}(u)u$.

By $(\mathcal{E}_2.1)$ there exist $h \geq \overline{h}$ and $\delta \in]0,1]$ such that

$$\|\vartheta_h(z)v - z\|_p < \varrho\,, \qquad \mathcal{E}(\vartheta_h(z)v) < \mathcal{E}(u) + \varrho\,,$$

whenever $\|z-u\|_p < \delta$ and $\mathcal{E}(z) \leq \lambda + 1 - \mathcal{G}(u) + \varrho$.

By decreasing δ, from (G_3'), (3.9) and (2.17) we deduce that

$$|\mathcal{G}(z) - \mathcal{G}(u)| < \varrho\,, \qquad \int_\Omega \frac{G(x, z + t(\vartheta_h(z)v - z)) - G(x,z)}{t}\,dx < \varrho$$

whenever $\|z-u\|_p < \delta$ and $0 < t \leq \delta$.

Define a continuous map

$$\mathcal{H} : \{z \in \mathrm{B}_\delta\,(u) : f(z) < \lambda + \delta\} \times [0,\delta] \to X$$

by $\mathcal{H}(z,t) = z + t(\vartheta_h(z)v - z)$. It is readily seen that $\|\mathcal{H}(z,t) - z\|_p \leq \varrho t$.

If $z \in \mathrm{B}_\delta\,(u)$, $f(z) < \lambda + \delta$ and $0 \leq t \leq \delta$, we have

$$\mathcal{E}(z) = f(z) - \mathcal{G}(z) < \lambda + \delta - \mathcal{G}(u) + \varrho \leq \lambda + 1 - \mathcal{G}(u) + \varrho\,,$$

hence, taking into account the convexity of $\mathcal{E}$,

$$\begin{aligned} \mathcal{E}(z + t(\vartheta_h(z)v - z)) &\leq \mathcal{E}(z) + t(\mathcal{E}(\vartheta_h(z)v) - \mathcal{E}(z)) \\ &\leq \mathcal{E}(z) + t(\mathcal{E}(u) - \mathcal{E}(z) + \varrho)\,. \end{aligned}$$

Moreover, we also have

$$\mathcal{G}(z + t(\vartheta_h(z)v - z)) \leq \mathcal{G}(z) + t\varrho \leq \mathcal{G}(z) + t(\mathcal{G}(u) - \mathcal{G}(z) + 2\varrho)\,.$$

Therefore

$$f(z + t(\vartheta_h(z)v - z)) \leq f(z) + t(f(u) - f(z) + 3\varrho)$$

and the first assertion follows by Corollary 2.11.

Now assume that $\mathcal{E}$ and $G(x,\cdot)$ are even and that $u = 0$. Then, in the previous argument, we have $v = 0$, so that $\mathcal{H}(-z,t) = -\mathcal{H}(z,t)$ and the second assertion also follows. □

Now we want to provide a criterion which helps in the verification of the Palais-Smale condition. For this purpose, we consider further assumptions on $\mathcal{E}$, which ensure a suitable coerciveness, and a new condition (G_4') on G, stronger than (G_4), which is a kind of one-sided subcritical growth condition.

Theorem 3.12 *Let $c \in \mathbf{R}$. Assume that*

$(\mathcal{E}_3)$ *for every (u_h) bounded in $L^p(\Omega;\mathbf{R}^N)$ with $(\mathcal{E}(u_h))$ bounded, there exists a subsequence (u_{h_k}) and a function $u \in L^p(\Omega;\mathbf{R}^N)$ such that*

$$\lim_{k\to\infty} u_{h_k}(x) = u(x) \qquad \text{for a.e. } x \in \Omega\,;$$

$(\mathcal{E}_4)$ *if (u_h) is a sequence in $L^p(\Omega;\mathbf{R}^N)$ weakly convergent to $u \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(u_h)$ converges to $\mathcal{E}(u)$, then (u_h) converges to u strongly in $L^p(\Omega;\mathbf{R}^N)$;*

(G_4') *for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ such that*

$$G^\circ(x,s;-s) \le a_\varepsilon(x) + \varepsilon|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,.$$

Then any $(PS)_c$-sequence (u_h) for f bounded in $L^p(\Omega;\mathbf{R}^N)$ admits a subsequence strongly convergent in $L^p(\Omega;\mathbf{R}^N)$.

Proof. From (G_3) we deduce that $(\mathcal{G}(u_h))$ is bounded from below. Taking into account $(\mathcal{E}_1)$, it follows that $(\mathcal{E}(u_h))$ is bounded. By $(\mathcal{E}_3)$ there exists a subsequence, still denoted by (u_h), converging weakly in $L^p(\Omega;\mathbf{R}^N)$ and a.e. to some $u \in \mathcal{D}(\mathcal{E})$.

Given $\varepsilon > 0$, by $(\mathcal{E}_2.2)$ and (G_4) we may find $k_0 \ge 1$ such that

$$\mathcal{E}(\vartheta_{k_0}(u)u) < \mathcal{E}(u) + \varepsilon\,,$$

$$\int_\Omega (1-\vartheta_{k_0}(u))G^\circ(x,u;-u)\,dx < \varepsilon\,.$$

Since $\vartheta_{k_0}(u)u \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega;\mathbf{R}^N)$, by $(\mathcal{E}_2.1)$ there exists $k_1 \ge k_0$ such that

$$\forall h \in \mathbf{N}: \quad \mathcal{E}(\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u) < \mathcal{E}(u) + \varepsilon\,, \tag{3.13}$$

$$\int_\Omega (1-\vartheta_{k_1}(u)\vartheta_{k_0}(u))G^\circ(x,u;-u)\,dx < \varepsilon\,.$$

It follows that $\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u \in \mathcal{D}(\mathcal{E})$. Moreover, from (3.3) and (G_4') we get

$$\begin{aligned} G^\circ(x,u_h;\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u-u_h) &\le \\ &\le \vartheta_{k_1}(u_h)G^\circ(x,u_h;\vartheta_{k_0}(u)u-u_h)+ \\ &\quad +(1-\vartheta_{k_1}(u_h))G^\circ(x,u_h;-u_h)\le \\ &\le \alpha_{k_1M}(x)(k_0M+k_1M)+a_\varepsilon(x)+\varepsilon|u_h|^p\,. \end{aligned}$$

From (2.18) and Fatou's Lemma we deduce that

$$\begin{aligned} \limsup_{h\to\infty}\int_\Omega \left[G^\circ(x,u_h;\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u-u_h)-\varepsilon|u_h|^p\right]dx &\le \\ \le \int_\Omega \left[G^\circ(x,u;\vartheta_{k_1}(u)\vartheta_{k_0}(u)u-u)-\varepsilon|u|^p\right]dx &\le \\ \le \int_\Omega (1-\vartheta_{k_1}(u)\vartheta_{k_0}(u))G^\circ(x,u;-u)\,dx &< \varepsilon\,, \end{aligned}$$

hence

(3.14)
$$\limsup_{h\to\infty}\int_\Omega G^\circ(x,u_h;\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u-u_h)\,dx<\varepsilon\sup_h\|u_h\|_p^p+\varepsilon\,.$$

Since (u_h) is a $(PS)_c$-sequence, by Theorem 2.16 there exists $u_h^*\in\partial f(u_h)$ with $\|u_h^*\|_{p'}\le|df|\,(u_h)$, so that $\lim\limits_{h\to\infty}\|u_h^*\|_{p'}=0$. Applying (c) of Theorem 3.6, we get

$$\begin{aligned} \mathcal{E}(\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u)\ge \mathcal{E}(u_h)-\int_\Omega G^\circ(x,u_h;\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u-u_h)\,dx+ \\ +\int_\Omega u_h^*\cdot(\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u-u_h)\,dx\,. \end{aligned}$$

Taking into account (3.13), (3.14) and passing to the upper limit, we obtain

$$\limsup_{h\to\infty}\mathcal{E}(u_h)\le\mathcal{E}(u)+2\varepsilon+\varepsilon\sup_h\|u_h\|_p^p\,.$$

By the arbitrariness of $\varepsilon>0$, we finally have

$$\limsup_{h\to\infty}\mathcal{E}(u_h)\le\mathcal{E}(u)$$

and the strong convergence of (u_h) to u follows from $(\mathcal{E}_4)$. □

4 Area type functionals

Let $n \geq 2$, $N \geq 1$, Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary and let

$$\Psi : \mathbf{R}^{nN} \to \mathbf{R}$$

be a convex function satisfying

$$(\Psi) \qquad \begin{cases} \Psi(0) = 0, \Psi(\xi) > 0 \text{ for any } \xi \neq 0 \text{ and} \\ \text{there exists } c > 0 \text{ such that } \Psi(\xi) \leq c|\xi| \text{ for any } \xi \in \mathbf{R}^{nN}. \end{cases}$$

We want to study the functional $\mathcal{E} : L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\mathcal{E}(u) = \begin{cases} \displaystyle\int_\Omega \Psi(Du^a)\, dx + \int_\Omega \Psi^\infty\left(\frac{Du^s}{|Du^s|}\right) d|Du^s|(x) + \\ \displaystyle + \int_{\partial\Omega} \Psi^\infty(u \otimes \nu)\, d\mathcal{H}^{n-1}(x) & \text{if } u \in BV(\Omega; \mathbf{R}^N), \\ +\infty & \text{if } u \in L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N) \setminus BV(\Omega; \mathbf{R}^N), \end{cases}$$

where $Du = Du^a\, dx + Du^s$ is the Lebesgue decomposition of Du, $|Du^s|$ is the total variation of Du^s, $Du^s/|Du^s|$ is the Radon-Nikodym derivative of Du^s with respect to $|Du^s|$, Ψ^∞ is the recession functional associated with Ψ, ν is the outer normal to Ω and the trace of u on $\partial\Omega$ is still denoted by u (see e.g. [4,29]).

Theorem 4.1 *The functional $\mathcal{E}$ satisfies conditions $(\mathcal{E}_1)$, $(\mathcal{E}_2)$, $(\mathcal{E}_3)$ and $(\mathcal{E}_4)$.*

The section will be devoted to the proof of this result. We begin establishing some technical lemmas. For notions concerning the space BV, such as those of $\tilde{u}$, S_u, u^+ and u^-, we refer the reader to [2,3].

In $BV(\Omega; \mathbf{R}^N)$ we will consider the norm

$$\|u\|_{BV} = \int_\Omega |Du^a|\, dx + |Du^s|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1}(x)\,,$$

which is equivalent to the standard norm of $BV(\Omega; \mathbf{R}^N)$.

Lemma 4.2 *For every $u \in BV(\Omega; \mathbf{R}^N)$ and every $\varepsilon > 0$ there exists $v \in C_c^\infty(\Omega; \mathbf{R}^N)$ such that*

$$\|v - u\|_{\frac{n}{n-1}} < \varepsilon\,, \quad \left| \int_\Omega |Dv|\, dx - \|u\|_{BV} \right| < \varepsilon\,,$$

$$|\mathcal{E}(v) - \mathcal{E}(u)| < \varepsilon\,, \quad \|v\|_\infty \leq \operatorname*{ess\,sup}_\Omega |u|\,.$$

Proof. Let $\delta > 0$, let $R > 0$ with $\overline{\Omega} \subseteq \mathrm{B}_R(0)$ and let

$$\vartheta_h(x) = 1 - \min\left\{\max\left\{\frac{h+1}{h}[1 - h\,d(x, \mathbf{R}^n \setminus \Omega)], 0\right\}, 1\right\}.$$

Define $\hat{u} \in BV(\mathrm{B}_R(0); \mathbf{R}^N)$ by

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega\,, \\ 0 & \text{if } x \in \mathrm{B}_R(0) \setminus \Omega\,. \end{cases}$$

According to [11, Lemma 7.4 and formula (7.2)], if h is sufficiently large, we have that $\vartheta_h u \in BV(\Omega; \mathbf{R}^N)$, $\|\vartheta_h u - u\|_{\frac{n}{n-1}} < \delta$ and

$$\begin{aligned} &\int_\Omega \sqrt{1 + |D(\vartheta_h u)^a|^2}\, d\mathcal{L}^n + |D(\vartheta_h u)^s|(\Omega) < \\ &\qquad < \int_\Omega \sqrt{1 + |Du^a|^2}\, d\mathcal{L}^n + |Du^s|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} + \delta = \\ &\qquad = \int_{\mathrm{B}_R(0)} \sqrt{1 + |D\hat{u}^a|^2}\, d\mathcal{L}^n + |D\hat{u}^s|(\mathrm{B}_R(0)) + \delta\,. \end{aligned}$$

Moreover, $\vartheta_h u$ has compact support in Ω and $\operatorname*{ess\,sup}_\Omega |\vartheta_h u| \leq \operatorname*{ess\,sup}_\Omega |u|$.

If we regularize $\vartheta_h u$ by convolution, we easily get $v \in C_c^\infty(\Omega; \mathbf{R}^N)$ with

$$\|v\|_\infty \leq \operatorname*{ess\,sup}_\Omega |u|\,, \qquad \|v - u\|_{\frac{n}{n-1}} < \delta$$

and

$$\int_\Omega \sqrt{1 + |Dv|^2}\, d\mathcal{L}^n < \int_{\mathrm{B}_R(0)} \sqrt{1 + |D\hat{u}^a|^2}\, d\mathcal{L}^n + |D\hat{u}^s|(\mathrm{B}_R(0)) + \delta\,.$$

Since

$$\|u\|_{BV} = \int_{\mathrm{B}_R(0)} |D\hat{u}^a|\, dx + |D\hat{u}^s|(\mathrm{B}_R(0))\,,$$

$$\mathcal{E}(u) = \int_{\mathrm{B}_R(0)} \Psi(D\hat{u}^a)\, dx + \int_{\mathrm{B}_R(0)} \Psi^\infty\left(\frac{D\hat{u}^s}{|D\hat{u}^s|}\right) d|D\hat{u}^s|\,,$$

by the results of [45] the assertion follows (see also [4, Fact 3.1]). □

Lemma 4.3 *The following facts hold:*

(a) $\Psi : \mathbf{R}^{nN} \to \mathbf{R}$ *is Lipschitz continuous of some constant* $\mathrm{Lip}(\Psi) > 0$*;*
(b) *for any* $\xi \in \mathbf{R}^{nN}$ *and* $s \in [0,1]$ *we have* $\Psi(s\xi) \leq s\Psi(\xi)$*;*

(c) *for every* $\sigma > 0$ *there exists* $d_\sigma > 0$ *such that*

$$\forall \xi \in \mathbf{R}^{nN} : \quad \Psi(\xi) \geq d_\sigma(|\xi| - \sigma)\,;$$

(d) $\mathcal{E} : BV(\Omega; \mathbf{R}^N) \to \mathbf{R}$ *is Lipschitz continuous of constant* $\mathrm{Lip}(\Psi)$*;*
(e) *if* σ *and* d_σ *are as in* (c)*, we have*

$$\forall u \in BV(\Omega; \mathbf{R}^N) : \quad \mathcal{E}(u) \geq d_\sigma \Big(\|u\|_{BV} - \sigma \mathcal{L}^n(\Omega) \Big)\,.$$

Proof. Properties (a) and (b) easily follow from the convexity of Ψ and assumption (Ψ).

To prove (c), assume by contradiction that $\sigma > 0$ and (ξ_h) is a sequence with $\Psi(\xi_h) < \frac{1}{h}(|\xi_h| - \sigma)$. If $|\xi_h| \to +\infty$, we have eventually

$$\Psi\left(\frac{\xi_h}{|\xi_h|}\right) \leq \frac{\Psi(\xi_h)}{|\xi_h|} < \frac{1}{h}\left(1 - \frac{\sigma}{|\xi_h|}\right)\,.$$

Up to a subsequence, $(\xi_h/|\xi_h|)$ is convergent to some $\eta \neq 0$ with $\Psi(\eta) \leq 0$, which is impossible. Since $|\xi_h|$ is bounded, up to a subsequence we have $\xi_h \to \xi$ with $|\xi| \geq \sigma$ and $\Psi(\xi) \leq 0$, which is again impossible.

Finally, (d) easily follows from (a) and the definition of $\|\cdot\|_{BV}$, while (e) follows from (c) (see e.g. [37, Lemma 4.1]). $\square$

Let now $\vartheta \in C^1_c(\mathbf{R}^N)$ with $0 \leq \vartheta \leq 1$, $\|\nabla\vartheta\|_\infty \leq 2$, $\vartheta(s) = 1$ for $|s| \leq 1$ and $\vartheta(s) = 0$ for $|s| \geq 2$. Define $\vartheta_h : \mathbf{R}^N \to \mathbf{R}$ and $T_h, R_h : \mathbf{R}^N \to \mathbf{R}^N$ by

$$\vartheta_h(s) = \vartheta\left(\frac{s}{h}\right)\,, \quad T_h(s) = \vartheta_h(s)s\,, \quad R_h(s) = (1 - \vartheta_h(s))s\,.$$

Lemma 4.4 *There exists a constant* $c_\Psi > 0$ *such that*

$$\mathcal{E}\left(\vartheta\left(\frac{u}{h}\right)v\right) \leq \mathcal{E}(v) + \frac{c_\Psi}{h}\|v\|_\infty \|u\|_{BV}\,,$$

$$\begin{aligned}
\mathcal{E}(T_h \circ u) \leq \mathcal{E}(u) + c_\Psi \Bigg[& |Du|(\{x \in \Omega \setminus S_u : |\tilde{u}(x)| > h\}) + \\
& + \int_{\{x \in S_u : |u^+(x)| > h \text{ or } |u^-(x)| > h\}} |u^+ - u^-| \, d\mathcal{H}^{n-1}(x) + \\
& + \int_{\{x \in \partial\Omega : |u(x)| > h\}} |u| \, d\mathcal{H}^{n-1}(x) \Bigg]\,,
\end{aligned}$$

$$\mathcal{E}(T_h \circ w) + \mathcal{E}(R_h \circ w) \leq \mathcal{E}(w) + c_\Psi \int_{\{x \in \Omega : h < |w(x)| < 2h\}} |Dw| \, dx$$

whenever $h \geq 1$, $u \in BV(\Omega; \mathbf{R}^N)$, $v \in BV(\Omega; \mathbf{R}^N) \cap L^\infty(\Omega; \mathbf{R}^N)$ *and* $w \in C^\infty_c(\Omega; \mathbf{R}^N)$.

Proof. Suppose first that $u, v \in C_c^\infty(\Omega; \mathbf{R}^N)$. Then, since

$$D\left[\vartheta\left(\frac{u}{h}\right)v\right] = \vartheta\left(\frac{u}{h}\right)Dv + \frac{1}{h}\,v \otimes \left[D\vartheta\left(\frac{u}{h}\right)Du\right]\,,$$

by (Ψ) and Lemma 4.3 it follows that

$$\mathcal{E}\left(\vartheta\left(\frac{u}{h}\right)v\right) \leq \mathcal{E}(v) + \mathrm{Lip}(\Psi)\frac{\|D\vartheta\|_\infty}{h}\|v\|_\infty \int_\Omega |Du|\,dx\,. \tag{4.5}$$

In the general case, let us consider two sequences (u_k), (v_k) in $C_c^\infty(\Omega; \mathbf{R}^N)$ converging to u, v in $L^1(\Omega; \mathbf{R}^N)$ with $\int_\Omega |Du_k|\,dx \to \|u\|_{BV}$, $\mathcal{E}(v_k) \to \mathcal{E}(v)$ and $\|v_k\|_\infty \leq \|v\|_\infty$. Passing to the lower limit in (4.5), we obtain the first inequality in the assertion.

To prove the second inequality, we first observe that by Lemma 4.3 we have

$$\mathcal{E}(T_h \circ u) \leq \mathcal{E}(u) + \mathrm{Lip}(\Psi)\|R_h \circ u\|_{BV}\,. \tag{4.6}$$

In order to estimate the last term in (4.6), we apply the chain rule of [2,49]. Since $R_h(s) = 0$ if $|s| \leq h$ and $\|DR_h\|_\infty \leq k_\vartheta$ for some $k_\vartheta > 0$, we have

$$\begin{aligned}\int_\Omega |D(R_h(u))^a|\,dx &\leq \int_{\Omega\setminus S_u} |DR_h(\tilde{u})||Du^a|\,dx \leq \\ &\leq k_\vartheta \int_{\{x\in\Omega\setminus S_u:|\tilde{u}(x)|>h\}} |Du^a|\,dx\,,\end{aligned}$$

$$\begin{aligned}\Big|D(R_h(u))^s\Big|(\Omega) \leq& \int_{\Omega\setminus S_u} |DR_h(\tilde{u})|\,d|Du^s|(x) + \\ &+ \int_{S_u} |R_h(u^+) - R_h(u^-)|\,d\mathcal{H}^{n-1}(x) \leq \\ &\leq k_\vartheta\Bigg(|Du^s|\left(\{x \in \Omega \setminus S_u : |\tilde{u}(x)| > h\}\right) + \\ &+ \int_{\{x\in S_u:|u^+(x)|>h \text{ or } |u^-(x)|>h\}} |u^+ - u^-|\,d\mathcal{H}^{m-1}(x)\Bigg)\end{aligned}$$

and

$$\int_{\partial\Omega} |R_h(u)|\,d\mathcal{H}^{n-1}(x) \leq k_\vartheta \int_{\{x\in\partial\Omega:|u(x)|>h\}} |u|\,d\mathcal{H}^{n-1}(x)\,.$$

Combining these three estimates, we get

$$
\begin{aligned}
\|R_h \circ u\|_{BV} \leq k_\vartheta \Bigg(& \int_{\{x \in \Omega \setminus S_u : |\tilde{u}(x)| > h\}} |Du^a| \, dx + \\
& + |Du^s|(\{x \in \Omega \setminus S_u : |\tilde{u}(x)| > h\}) + \\
& + \int_{\{x \in S_u : |u^+(x)| > h \text{ or } |u^-(x)| > h\}} |u^+ - u^-| \, d\mathcal{H}^{n-1}(x) + \\
& + \int_{\{x \in \partial\Omega : |u(x)| > h\}} |u| \, d\mathcal{H}^{n-1}(x) \Bigg) .
\end{aligned} \tag{4.7}
$$

Then the second inequality follows from (4.6) and (4.7).

Again, since Ψ is Lipschitz continuous, we have

$$
\begin{aligned}
\left| \int_\Omega \Psi(D(T_h \circ w)) \, dx - \int_\Omega \Psi(\vartheta_h(w) Dw) \, dx \right| & \leq \\
\leq \frac{\operatorname{Lip}(\Psi)}{h} \int_\Omega \left| D\vartheta \left(\frac{w}{h} \right) Dw \right| |w| \, dx & \leq \\
\leq 2 \operatorname{Lip}(\Psi) \|\nabla\vartheta\|_\infty \int_{\{h < |w| < 2h\}} |Dw| \, dx \, . &
\end{aligned}
$$

In a similar way, it is also

$$
\begin{aligned}
\left| \int_\Omega \Psi(D(R_h \circ w)) \, dx - \int_\Omega \Psi((1 - \vartheta_h(w)) Dw) \, dx \right| & \leq \\
\leq 2 \operatorname{Lip}(\Psi) \|\nabla\vartheta\|_\infty \int_{\{h < |w| < 2h\}} |Dw| \, dx \, . &
\end{aligned}
$$

Hence, combining the last two estimates and taking into account (b) of Lemma 4.3, we get

$$
\begin{aligned}
\int_\Omega \Psi\big(D(T_h \circ w)\big) \, dx + \int_\Omega \Psi\big(D(R_h \circ w)\big) \, dx & \leq \\
\leq \int_\Omega \Psi(Dw) \, dx + 4 \operatorname{Lip}(\Psi) \|\nabla\vartheta\|_\infty \int_{\{h < |w| < 2h\}} |Dw| \, dx &
\end{aligned}
$$

and the proof is complete. □

Lemma 4.8 *Let (u_h) be a sequence in $C_c^\infty(\Omega; \mathbf{R}^N)$ and assume that (u_h) is bounded in $BV(\Omega; \mathbf{R}^N)$.*

Then for every $\varepsilon > 0$ and every $\overline{k} \in \mathbf{N}$ there exists $k \geq \overline{k}$ such that

$$
\liminf_{h \to \infty} \int_{\{k < |u_h| < 2k\}} |Du_h| \, dx < \varepsilon \, .
$$

Proof. Let $m \geq 1$ be such that

$$\sup_h \int_\Omega |Du_h|\,dx \leq \frac{m\varepsilon}{2}$$

and let $i_0 \in \mathbf{N}$ with $2^{i_0} \geq \overline{k}$. Then, since

$$\sum_{i=i_0}^{i_0+m-1} \int_{\{2^i<|u_h|<2^{i+1}\}} |Du_h|\,dx \leq \int_\Omega |Du_h|\,dx \leq \frac{m\varepsilon}{2}\,,$$

there exists i_h between i_0 and i_0+m-1 such that

$$\int_{\{2^{i_h}<|u_h|<2^{i_h+1}\}} |Du_h|\,dx \leq \frac{\varepsilon}{2}\,.$$

Passing to a subsequence (i_{h_j}), we can suppose $i_{h_j} \equiv i \geq i_0$, and setting $k = 2^i$ we get

$$\forall j \in \mathbf{N}: \quad \int\limits_{\{k<|u_{h_j}|<2k\}} |Du_{h_j}|\,dx \leq \frac{\varepsilon}{2}\,.$$

Then the assertion follows. $\square$

Lemma 4.9 *Let (u_h) be a sequence in $C_c^\infty(\Omega;\mathbf{R}^N)$ and let $u \in BV(\Omega;\mathbf{R}^N)$ with $\|u_h - u\|_1 \to 0$ and $\mathcal{E}(u_h) \to \mathcal{E}(u)$.*

Then for every $\varepsilon > 0$ and every $\overline{k} \in \mathbf{N}$ there exists $k \geq \overline{k}$ such that

$$\liminf_{h\to\infty} \|R_k \circ u_h\|_{BV} < \varepsilon\,.$$

Proof. Given $\varepsilon > 0$, let $d > 0$ be such that

$$\forall \xi \in \mathbf{R}^{nN}: \quad \Psi(\xi) \geq d\left(|\xi| - \frac{\varepsilon}{3\mathcal{L}^n(\Omega)}\right)\,,$$

according to Lemma 4.3. Let also $c_\Psi > 0$ be as in Lemma 4.4. By (4.7) and Lemma 4.8, there exists $k \geq \overline{k}$ such that

$$\|R_k \circ u\|_{BV} < \frac{d\varepsilon}{3\mathrm{Lip}(\Psi)}\,,$$

$$\liminf_{h\to\infty} \int\limits_{\{k<|u_h|<2k\}} |Du_h|\,dx < \frac{d\varepsilon}{3c_\Psi}\,.$$

From Lemma 4.4 we deduce that

$$
\begin{aligned}
&\mathcal{E}(T_k \circ u) + \liminf_{h\to\infty} \mathcal{E}(R_k \circ u_h) \le \\
&\quad \le \liminf_{h\to\infty} \mathcal{E}(T_k \circ u_h) + \liminf_{h\to\infty} \mathcal{E}(R_k \circ u_h) \le \\
&\quad \le \liminf_{h\to\infty} \Big(\mathcal{E}(T_k \circ u_h) + \mathcal{E}(R_k \circ u_h) \Big) \le \\
&\quad \le \mathcal{E}(u) + c_\Psi \liminf_{h\to\infty} \int\limits_{\{k<|u_h|<2k\}} |Du_h|\, dx < \\
&\quad < \mathcal{E}(u) + \frac{d\varepsilon}{3} \le \mathcal{E}(T_k \circ u) + \mathrm{Lip}(\Psi)\|R_k \circ u\|_{BV} + \frac{d\varepsilon}{3} < \\
&\quad < \mathcal{E}(T_k \circ u) + \frac{2}{3} d\varepsilon \,,
\end{aligned}
$$

whence

$$
\liminf_{h\to\infty} \mathcal{E}(R_k \circ u_h) < \frac{2}{3} d\varepsilon \,.
$$

On the other hand, by Lemma 4.3 we have

$$
\mathcal{E}(R_k \circ u_h) \ge d \left(\|R_k \circ u_h\|_{BV} - \frac{\varepsilon}{3} \right)
$$

and the assertion follows. □

Now we can prove the main auxiliary result we need for the proof of Theorem 4.1. It is a property of the space BV which could be interesting also in itself.

Theorem 4.10 *Let (u_h) be a sequence in $BV(\Omega; \mathbf{R}^N)$ and let $u \in BV(\Omega; \mathbf{R}^N)$ with $\|u_h - u\|_1 \to 0$ and $\mathcal{E}(u_h) \to \mathcal{E}(u)$.*

Then (u_h) is strongly convergent to u in $L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N)$.

Proof. By Lemma 4.2 we may find $v_h \in C_c^\infty(\Omega; \mathbf{R}^N)$ with

$$
\|v_h - u_h\|_1 < \frac{1}{h} \,, \quad \|v_h - u_h\|_{\frac{n}{n-1}} < \frac{1}{h} \,, \quad |\mathcal{E}(v_h) - \mathcal{E}(u_h)| < \frac{1}{h} \,.
$$

Therefore it is sufficent to treat the case in which $u_h \in C_c^\infty(\Omega; \mathbf{R}^N)$.

By contradiction, up to a subsequence we may assume that there exists $\varepsilon > 0$ such that $\|u_h - u\|_{\frac{n}{n-1}} \ge \varepsilon$. Let $\tilde{c}$ be a constant such that $\|w\|_{\frac{n}{n-1}} \le \tilde{c}\|w\|_{BV}$ for any $w \in BV(\Omega; \mathbf{R}^N)$ (see [24, Theorem 1.28]). According to Lemma 4.9, let $k \in \mathbf{N}$ be such that

$$
\|R_k \circ u\|_{\frac{n}{n-1}} < \frac{\varepsilon}{2}, \quad \liminf_{h\to\infty} \|R_k \circ u_h\|_{\frac{n}{n-1}} \le \tilde{c} \liminf_{h\to\infty} \|R_k \circ u_h\|_{BV} < \frac{\varepsilon}{2} \,.
$$

Then we have

$$
\begin{aligned}
(4.11) \qquad \|u_h - u\|_{\frac{n}{n-1}} \leq \|R_k \circ u_h\|_{\frac{n}{n-1}} + \|T_k \circ u_h - T_k \circ u\|_{\frac{n}{n-1}} + \\
+\|R_k \circ u\|_{\frac{n}{n-1}} .
\end{aligned}
$$

Since $T_k \circ u_h \to T_k \circ u$ in $L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N)$ as $h \to \infty$, passing to the lower limit in (4.11) we get

$$
\liminf_{h\to\infty} \|u_h - u\|_{\frac{n}{n-1}} < \varepsilon ,
$$

whence a contradiction. □

Proof of Theorem 4.1. It is well known that $\mathcal{E}$ satisfies condition $(\mathcal{E}_1)$. Conditions $(\mathcal{E}_2)$ are an immediate consequence of Lemma 4.4. From (e) of Lemma 4.3 and Rellich's Theorem (see [24, Theorem 1.19]) it follows that $\mathcal{E}$ satisfies condition $(\mathcal{E}_3)$. To prove $(\mathcal{E}_4)$, let (u_h) be a sequence in $L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N)$ weakly convergent to $u \in BV(\Omega; \mathbf{R}^N)$ such that $\mathcal{E}(u_h)$ converges to $\mathcal{E}(u)$. Again by (e) of Lemma 4.3 and Rellich's Theorem we deduce that (u_h) is strongly convergent to u in $L^1(\Omega; \mathbf{R}^N)$. Then the assertion follows from Theorem 4.10. □

5 A result of Clark type

Let $n \geq 2$ and Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary, let $\Psi : \mathbf{R}^{nN} \to \mathbf{R}$ be an even convex function satisfying (Ψ) and let $G : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a function satisfying (G_1), (G_2), (G'_3), (G'_4) with $p = \frac{n}{n-1}$ and the following conditions:

(5.1)
$$
\begin{cases}
\text{there exist } \tilde{a} \in L^1(\Omega) \text{ and } \tilde{b} \in L^n(\Omega) \text{ such that} \\
G(x,s) \geq -\tilde{a}(x) - \tilde{b}(x)|s| \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N ;
\end{cases}
$$

$$
(5.2) \qquad \lim_{|s|\to\infty} \frac{G(x,s)}{|s|} = +\infty \qquad \text{for a.e. } x \in \Omega ;
$$

$$
(5.3) \qquad \{s \longmapsto G(x,s)\} \text{ is even for a.e. } x \in \Omega .
$$

Finally, define $\mathcal{E}$ as in Section 4. The main result of this section is:

Theorem 5.4 *For every $k \in \mathbf{N}$ there exists Λ_k such that for any $\lambda \geq \Lambda_k$ the problem*

$$\begin{cases} u \in BV(\Omega;\mathbf{R}^N) \\ \mathcal{E}(v) - \mathcal{E}(u) + \displaystyle\int_\Omega G^\circ(x,u;v-u)\,dx \geq \\ \qquad \geq \lambda \displaystyle\int_\Omega \frac{u}{\sqrt{1+|u|^2}} \cdot (v-u)\,dx \quad \forall v \in BV(\Omega;\mathbf{R}^N) \end{cases}$$

admits at least k pairs $(u,-u)$ of distinct solutions.

For the proof we need the following

Lemma 5.5 *Let (u_h) be a bounded sequence in $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$, which is convergent a.e. to u, and let (ϱ_h) be a positively divergent sequence of real numbers.*

Then we have

$$\lim_h \int_\Omega \frac{G(x,\varrho_h u_h)}{\varrho_h}\,dx = +\infty \text{ if } u \neq 0\,,$$
$$\liminf_h \int_\Omega \frac{G(x,\varrho_h u_h)}{\varrho_h}\,dx \geq 0 \text{ if } u = 0\,.$$

Proof. If $u=0$, the assertion follows directly from (5.1). If $u \neq 0$, we have

$$\int_\Omega \frac{G(x,\varrho_h u_h)}{\varrho_h}\,dx \geq \int_{\{u\neq 0\}} \frac{G(x,\varrho_h u_h)}{\varrho_h}\,dx - \\ -\frac{1}{\varrho_h}\int_{\{u=0\}} \tilde{a}\,dx - \int_{\{u=0\}} \tilde{b}|u_h|\,dx\,.$$

From (5.1), (5.2) and Fatou's Lemma, we deduce that

$$\lim_h \int_{\{u\neq 0\}} \frac{G(x,\varrho_h u_h)}{\varrho_h}\,dx = +\infty\,,$$

whence the assertion. □

Proof of Theorem 5.4. First of all, set

$$\widetilde{G}(x,s) = G(x,s) - \lambda\left(\sqrt{1+|s|^2} - 1\right).$$

It is easy to see that also $\widetilde{G}$ satisfies (G_1), (G_2), (G_3'), (G_4'), (5.1), (5.2), (5.3) and that

$$\widetilde{G}^\circ(x,s;\hat{s}) = G^\circ(x,s;\hat{s}) - \lambda \frac{s}{\sqrt{1+|s|^2}} \cdot \hat{s}\,.$$

Now define a lower semicontinuous functional $f : L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ by

$$f(u) = \mathcal{E}(u) + \int_\Omega \widetilde{G}(x,u)\,dx\,.$$

Then f is even by (5.3) and satisfies condition $(epi)_c$ by Theorem 3.11. We claim that

$$\lim_{\|u\|_{\frac{n}{n-1}}\to\infty} f(u) = +\infty\,. \tag{5.6}$$

To prove it, let (u_h) be a sequence in $BV(\Omega;\mathbf{R}^N)$ with $\|u_h\|_{\frac{n}{n-1}} = 1$ and let $\varrho_h \to +\infty$. By (e) of Lemma 4.3 there exist $\tilde{c} > 0$ and $\tilde{d} > 0$ such that

$$\forall u \in BV(\Omega;\mathbf{R}^N): \quad \mathcal{E}(u) \geq \tilde{d}\Big(\|u\|_{BV} - \tilde{c}\mathcal{L}^n(\Omega)\Big)\,.$$

If $\|u_h\|_{BV} \to +\infty$, it readily follows from (5.1) that $f(\varrho_h u_h) \to +\infty$. Otherwise, up to a subsequence, u_h is convergent a.e. and the assertion follows from the previous Lemma and the inequality

$$f(\varrho_h u_h) \geq \varrho_h \left[\tilde{d}\left(\|u_h\|_{BV} - \frac{\tilde{c}}{\varrho_h}\mathcal{L}^n(\Omega)\right) + \int_\Omega \frac{\widetilde{G}(x,\varrho_h u_h)}{\varrho_h}\,dx\right]\,.$$

Since f is bounded below on bounded subsets of $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$, it follows from (5.6) that f is bounded below on all $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$; furthermore, it also turns out from (5.6) that any $(PS)_c$ sequence is bounded, hence f satisfies $(PS)_c$ by Theorem 3.12.

Finally, let $k \geq 1$, let $w_1, \ldots, w_k$ be linearly independent elements of $BV(\Omega;\mathbf{R}^N)$ and let $\psi : S^{k-1} \to L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$ be the odd continuous map defined by

$$\psi(\xi) = \sum_{j=1}^k \xi_j w_j\,.$$

Because of (G_3'), it is easily seen that

$$\sup\left\{\mathcal{E}(u) + \int_\Omega G(x,u)\,dx : u \in \psi(S^{k-1})\right\} < +\infty$$

and

$$\inf\left\{\int_\Omega \big(\sqrt{1+|u|^2} - 1\big)\,dx : u \in \psi(S^{k-1})\right\} > 0\,.$$

Therefore there exists $\Lambda_k > 0$ such that $\sup\limits_{\xi \in S^{k-1}} f(\psi(\xi)) < 0$ whenever $\lambda \geq \Lambda_k$.

Applying Theorem 2.12, it follows that f admits at least k pairs $(u_k, -u_k)$ of critical points. Therefore, by Theorem 2.16, for any u_k it is possible to apply Theorem 3.6 (with $\widetilde{G}$ instead of G), whence the assertion. □

6 A superlinear potential

Let $n \geq 2$ and Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary, let $\Psi : \mathbf{R}^{nN} \to \mathbf{R}$ be an even convex function satisfying (Ψ) and let $G : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a function satisfying (G_1), (G_2), (G_3'), (G_4'), (5.3) with $p = \frac{n}{n-1}$ and the following condition:

$$
(6.1) \qquad \begin{cases} \text{there exist } q > 1 \text{ and } R > 0 \text{ such that} \\ G^\circ(x, s; s) \leq qG(x, s) < 0 \\ \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N \text{ with } |s| \geq R\,. \end{cases}
$$

Define $\mathcal{E}$ as in Sect. 4 and an even lower semicontinuous functional $f : L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ by

$$
f(u) = \mathcal{E}(u) + \int_\Omega G(x, u)\, dx\,.
$$

Theorem 6.2 *There exists a sequence (u_h) of solutions of the problem*

$$
\begin{cases} u \in BV(\Omega; \mathbf{R}^N) \\ \mathcal{E}(v) - \mathcal{E}(u) + \displaystyle\int_\Omega G^\circ(x, u; v - u)\, dx \geq 0 \quad \forall v \in BV(\Omega; \mathbf{R}^N) \end{cases}
$$

with $f(u_h) \to +\infty$.

Proof. According to (3.3), we have

$$
|s| < R \quad \Longrightarrow \quad |G^\circ(x, s; s)| \leq \alpha_R(x)|s|\,.
$$

Combining this fact with (6.1) and (G_3'), we deduce that there exists $a_0 \in L^1(\Omega)$ such that

$$
(6.3) \qquad \begin{aligned} & G^\circ(x, s; s) \leq qG(x, s) + a_0(x) \\ & \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,. \end{aligned}
$$

Moreover, from (6.1) and Lebourg's Theorem [14] it follows that for every $s \in \mathbf{R}^N$ with $|s| = 1$ the function $\{t \to t^{-q}G(x, ts)\}$ is nonincreasing on $[R,+\infty[$. Taking into account (G_3') and possibly substituting a_0 with another function in $L^1(\Omega)$, we deduce that

$$G(x,s) \leq a_0(x) - b_0(x)|s|^q \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,, \tag{6.4}$$

where

$$b_0(x) = \inf_{|s|=1} (-R^{-q} G(x, Rs)) > 0 \qquad \text{for a.e. } x \in \Omega\,.$$

Finally, since $\{\hat{s} \to G^\circ(x,s;\hat{s})\}$ is a convex function vanishing at the origin, we have $G^\circ(x,s;s) \geq -G^\circ(x,s;-s)$. Combining (6.3) with (G_4'), we deduce that for every $\varepsilon > 0$ there exists $\tilde{a}_\varepsilon \in L^1(\Omega)$ such that

$$G(x,s) \geq -\tilde{a}_\varepsilon(x) - \varepsilon|s|^{\frac{n}{n-1}} \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,. \tag{6.5}$$

By Theorem 3.11 we have that f satisfies $(epi)_c$ for any $c \in \mathbf{R}$ and that $|d_{\mathbf{Z}_2}\mathcal{G}_f|(0,\lambda) = 1$ for any $\lambda > f(0)$.

We also recall that, since Ψ is Lipschitz continuous, there exists $M \in \mathbf{R}$ such that

$$(q+1)\Psi(\xi) - \Psi(2\xi) \geq \frac{q-1}{2}\Psi(\xi) - M\,, \tag{6.6}$$

$$(q+1)\Psi^\infty(\xi) - \Psi^\infty(2\xi) \geq \frac{q-1}{2}\Psi^\infty(\xi) \tag{6.7}$$

(see also [36]).

We claim that f satifies the condition $(PS)_c$ for every $c \in \mathbf{R}$. Let (u_h) be a $(PS)_c$-sequence for f. By Theorem 2.16 there exists a sequence (u_h^*) in $L^n(\Omega;\mathbf{R}^N)$ with $u_h^* \in \partial f(u_h)$ and $\|u_h^*\|_n \to 0$. According to Theorem 3.6 and (6.3), we have

$$\begin{aligned}\mathcal{E}(2u_h) &\geq \mathcal{E}(u_h) - \int_\Omega G^\circ(x,u_h;u_h)\,dx + \int_\Omega u_h^* \cdot u_h\,dx \geq \\ &\geq \mathcal{E}(u_h) - q\int_\Omega G(x,u_h)\,dx + \int_\Omega u_h^* \cdot u_h\,dx - \int_\Omega a_0(x)\,dx\,.\end{aligned}$$

By the definition of f, it follows

$$qf(u_h) + \|u_h^*\|_n \|u_h\|_{\frac{n}{n-1}} + \int_\Omega a_0(x)\,dx \geq (q+1)\mathcal{E}(u_h) - \mathcal{E}(2u_h)\,.$$

Finally, applying (6.6) and (6.7) we get

$$qf(u_h) + \|u_h^*\|_n \|u_h\|_{\frac{n}{n-1}} + \int_\Omega a_0(x)\,dx \geq \frac{q-1}{2}\mathcal{E}(u_h) - M\mathcal{L}^n(\Omega)\,.$$

By (e) of Lemma 4.3 we deduce that (u_h) is bounded in $BV(\Omega;\mathbf{R}^N)$, hence in $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$. Applying Theorem 3.12 we get that (u_h) admits a strongly convergent subsequence and $(PS)_c$ follows.

By [36, Lemma 3.8], there exist a strictly increasing sequence (W_h) of finite-dimensional subspaces of $BV(\Omega;\mathbf{R}^N)\cap L^\infty(\Omega;\mathbf{R}^N)$ and a strictly decreasing sequence (Z_h) of closed subspaces of $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$ such that $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N) = W_h \oplus Z_h$ and $\bigcap\limits_{h=0}^{\infty} Z_h = \{0\}$. By (e) of Lemma 4.3 there exists $\varrho > 0$ such that

$$\forall u \in L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N): \qquad \|u\|_{\frac{n}{n-1}} = \varrho \quad \Longrightarrow \quad \mathcal{E}(u) \geq 1\,.$$

We claim that

$$\lim_h \Big(\inf\{f(u) : u \in Z_h, \|u\|_{\frac{n}{n-1}} = \varrho\}\Big) > f(0)\,.$$

Actually, assume by contradiction that (u_h) is a sequence with $u_h \in Z_h$, $\|u_h\|_{\frac{n}{n-1}} = \varrho$ and

$$\limsup_h f(u_h) \leq f(0)\,.$$

Taking into account (G_3') and Lemma 4.3, we deduce that $(\mathcal{E}(u_h))$ is bounded, so that (u_h) is bounded in $BV(\Omega;\mathbf{R}^N)$. Therefore, up to a subsequence, (u_h) is convergent a.e. to 0. From (6.5) it follows that

$$\liminf_h \int_\Omega \Big(G(x,u_h) + \varepsilon |u_h|^{\frac{n}{n-1}}\Big)\,dx \geq \int_\Omega G(x,0)\,dx\,,$$

hence

$$\liminf_h \int_\Omega G(x,u_h)\,dx \geq \int_\Omega G(x,0)\,dx$$

by the boundedness of (u_h) in $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$ and the arbitrariness of ε. Therefore

$$\limsup_h \mathcal{E}(u_h) \leq \mathcal{E}(0) = 0$$

which contradicts the choice of ϱ.

Now, fix $\overline{h}$ with

$$\inf\{f(u) : u \in Z_{\overline{h}}, \|u\|_{\frac{n}{n-1}} = \varrho\}\Big) > f(0)$$

and set $Z = Z_{\overline{h}}$ and $V_h = W_{\overline{h}+h}$. Then Z satisfies assumption (a) of Theorem 2.13 for some $\alpha > f(0)$.

Finally, since V_h is finite-dimensional,

$$\|u\|_G := \left(\int_\Omega b_0|u|^q dx\right)^{\frac{1}{q}}$$

is a norm on V_h equivalent to the norm of $BV(\Omega;\mathbf{R}^N)$. Then, combining (6.4) with (d) of Lemma 4.3, we see that also assumption (b) of Theorem 2.13 is satisfied.

Therefore there exists a sequence (u_h) of critical points for f with $f(u_h) \to +\infty$ and, by Theorems 2.16 and 3.6, the result follows. $\square$

References

1. A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. **14** (1973), 349–381.
2. L. Ambrosio, G. Dal Maso, A general chain rule for distributional derivatives, Proc. Amer. Math. Soc. **108** (1990), 691–702.
3. L. Ambrosio, S. Mortola, V.M. Tortorelli, Functionals with linear growth defined on vector valued BV functions, J. Math. Pures Appl. **70** (1991), 269-323.
4. G. Anzellotti, The Euler equation for functionals with linear growth, Trans. Amer. Math. Soc. **290** (1985), 483–501.
5. M.F. Bocea, Multiple solutions for a class of eigenvalue problems involving a nonlinear monotone operator in hemivariational inequalities, Appl. Anal. **65** (1997), 395–407.
6. H. Brezis, “Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert”, North-Holland Mathematics Studies, **5**, Notas de Matemàtica (50), North-Holland, Amsterdam-London, 1973.
7. I. Campa, M. Degiovanni, Subdifferential calculus and nonsmooth critical point theory, SIAM J. Optim., in press.
8. A. Canino, Multiplicity of solutions for quasilinear elliptic equations, Topol. Methods Nonlinear Anal. **6** (1995), 357–370.
9. A. Canino, On a jumping problem for quasilinear elliptic equations, Math.Z. **226** (1997), 193–210.
10. A. Canino, M. Degiovanni, Nonsmooth critical point theory and quasilinear elliptic equations, Topological Methods in Differential Equations and Inclusions (Montreal, PQ, 1994), 1–50, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **472**, Kluwer Acad. Publ., Dordrecht, 1995.
11. M. Carriero, G. Dal Maso, A. Leaci, E. Pascali, Relaxation of the non-parametric Plateau problem with an obstacle, J. Math. Pures Appl. **67** (1988), 359–396.
12. K.C. Chang, Variational methods for non-differentiable functionals and their applications to partial differential equations, J. Math. Anal. Appl. **80** (1981), 102–129.
13. D.C. Clark, A variant of the Ljusternik-Schnirelmann theory, Indiana Univ. Math.J. **22** (1972), 65–74.
14. F.H. Clarke, “Optimization and Nonsmooth Analysis”, Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1983.
15. J.-N. Corvellec, M. Degiovanni, Nontrivial solutions of quasilinear equations via nonsmooth Morse theory, J. Differential Equations **136** (1997), 268–293.

16. J.-N. Corvellec, M. Degiovanni, M. Marzocchi, Deformation properties for continuous functionals and critical point theory, Topol. Methods Nonlinear Anal. **1** (1993), 151-171.
17. E. De Giorgi, A. Marino, M. Tosques, Problemi di evoluzione in spazi metrici e curve di massima pendenza, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **68** (1980), 180–187.
18. M. Degiovanni, Nonsmooth critical point theory and applications, Second World Congress of Nonlinear Analysts (Athens, 1996), Nonlinear Anal. **30** (1997), 89–99.
19. M. Degiovanni, M. Marzocchi, A critical point theory for nonsmooth functionals, Ann. Mat. Pura Appl. (4) **167** (1994), 73–100.
20. M. Degiovanni, F. Schuricht, Buckling of nonlinearly elastic rods in the presence of obstacles treated by nonsmooth critical point theory, Math. Ann. **311** (1998), 675–728.
21. M. Degiovanni, S. Zani, Multiple solutions of semilinear elliptic equations with one-sided growth conditions, in “Advanced Topics in Nonlinear Operator Theory”, Math. Comput. Modelling, in press.
22. I. Ekeland, R. Temam, “Convex analysis and variational problems”, Studies in Mathematics and its Applications, **1**, North-Holland Publishing Co., Amsterdam-Oxford, 1976.
23. F. Gazzola, V.D. Rădulescu, A nonsmooth critical point theory approach to some nonlinear elliptic equations in $\mathbf{R}^n$, Differential Integral Equations, in press.
24. E. Giusti, “Minimal Surfaces and Functions of Bounded Variation”, Birkhäuser, Basel, 1984.
25. D. Goeleven, D. Motreanu, P.D. Panagiotopoulos, Multiple solutions for a class of eigenvalue problems in hemivariational inequalities, Nonlinear Anal. **29** (1997), 9–26.
26. D. Goeleven, D. Motreanu, P.D. Panagiotopoulos, Multiple solutions for a class of hemivariational inequalities involving periodic energy functionals, Math. Methods Appl. Sci. **20** (1997), 547–568.
27. D. Goeleven, D. Motreanu, P.D. Panagiotopoulos, Semicoercive variational-hemivariational inequalities, Appl. Anal. **65** (1997), 119–134.
28. D. Goeleven, D. Motreanu, P.D. Panagiotopoulos, Eigenvalue problems for variational-hemivariational inequalities at resonance, Nonlinear Anal. **33** (1998), 161–180.
29. G. Goffman, J. Serrin, Sublinear functions of measures and variational integrals, Duke Math.J. **31** (1964), 159-178.
30. A. Ioffe, E. Schwartzman, Metric critical point theory I. Morse regularity and homotopic stability of a minimum, J. Math. Pures Appl. (9) **75** (1996), 125–153.
31. A. Ioffe, E. Schwartzman, Metric critical point theory II. Deformation techniques. New Results in Operator Theory and its Applications, 131–144, Oper. Theory Adv. Appl., **98**, Birkhäuser, Basel, 1997.
32. G. Katriel, Mountain pass theorems and global homeomorphism theorems, Ann. Inst.H. Poincaré Anal. Non Linéaire **11** (1994), 189–209.
33. D. Kinderlehrer, G. Stampacchia, “An introduction to variational inequalities and their applications”, Pure and Applied Mathematics, **88**, Academic Press, New York-London-Toronto, Ont., 1980.
34. A. Marino, The calculus of variations and some semilinear variational inequalities of elliptic and parabolic type, Partial Differential Equations and the Calculus of Variations. Volume II. Essays in Honor of Ennio De Giorgi, Birkäuser, Boston, 1989.
35. A. Marino, D. Scolozzi, Geodetiche con ostacolo, Boll. Un. Mat. Ital. B (6) **2** (1983), 1–31.
36. M. Marzocchi, Multiple solutions of quasilinear equations involving an area-type term, J. Math. Anal. Appl. **196** (1995), 1093–1104.

37. M. Marzocchi, Nontrivial solutions of quasilinear equations in BV, Well-posed Problems and Stability in Optimization (Marseille, 1995), Serdica Math.J. **22** (1996), 451-470.
38. D. Motreanu, Z. Naniewicz, Discontinuous semilinear problems in vector-valued function spaces, Differential Integral Equations **9** (1996), 581–598.
39. D. Motreanu, P.D. Panagiotopoulos, A minimax approach to the eigenvalue problem of hemivariational inequalities and applications, Appl. Anal. **58** (1995), 53–76.
40. D. Motreanu, P.D. Panagiotopoulos, Double eigenvalue problems for hemivariational inequalities, Arch. Rational Mech. Anal. **140** (1997), 225–251.
41. Z. Naniewicz, P.D. Panagiotopoulos, "Mathematical Theory of Hemivariational Inequalities and Applications", Monographs and Textbooks in Pure and Applied Mathematics, **188**, Marcel Dekker, Inc., New York, 1995.
42. P.D. Panagiotopoulos, "Hemivariational Inequalities. Applications in Mechanics and Engineering", Springer-Verlag, Berlin, 1993.
43. P.D. Panagiotopoulos, V.D. Rădulescu, Perturbations of hemivariational inequalities with constraints and applications, J. Global Optim. **12** (1998), 285–297.
44. P.H. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations", CBMS Reg. Conf. Series Math. **65**, Amer. Math. Soc., Providence, RI, 1986.
45. Yu.G. Reshetnyak, Weak convergence of completely additive vector functions on a set, Siberian Math.J. **9** (1968), 1039-1045 (translated from Sibirsk. Mat. Zh. **9** (1968), 1386-1394).
46. R.T. Rockafellar, Integral functionals, normal integrands and measurable selections, Nonlinear Operators and the Calculus of Variations (Bruxelles, 1975), 157–207, Lecture Notes in Math., **543**, Springer, Berlin-New York, 1976.
47. R.T. Rockafellar, Generalized directional derivatives and subgradients of nonconvex functions, Canad.J. Math. **32** (1980), 257–280.
48. M. Struwe, "Variational Methods", Springer-Verlag, Berlin, 1990.
49. A.I. Vol'pert, S.I. Hudjaev, "Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics", Martinus Nijhoff Publisher, Dordrecht, 1985.