Available online at www.sciencedirect.com
ScienceDirect
J. Math. Pures Appl. 87 (2007) 563–581
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

Lane–Emden–Fowler equations with convection and singular potential

Louis Dupaigne [a], Marius Ghergu [b], Vicenţiu Rădulescu [b,c,*]

[a] *LAMFA, Faculté de Mathématiques et d'Informatique, Université de Picardie Jules Verne, 33 rue Saint-Leu, 80039 Amiens, France*
[b] *Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, RO-014700, Bucharest, Romania*
[c] *Department of Mathematics, University of Craiova, 200585 Craiova, Romania*

Received 20 February 2007

Available online 12 March 2007

Abstract

We are concerned with singular elliptic problems of the form $-\Delta u \pm p(d(x))g(u) = \lambda f(x,u) + \mu|\nabla u|^a$ in Ω, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $d(x) = \operatorname{dist}(x, \partial\Omega)$, $\lambda > 0$, $\mu \in \mathbb{R}$, $0 < a \leqslant 2$, and f is a nondecreasing function. We assume that $p(d(x))$ is a positive weight with possible singular behavior on the boundary of Ω and that the nonlinearity g is unbounded around the origin. Taking into account the competition between the anisotropic potential $p(d(x))$, the convection term $|\nabla u|^a$, and the singular nonlinearity g, we establish various existence and nonexistence results.

Résumé

On considère des problèmes singuliers du type $-\Delta u \pm p(d(x))g(u) = \lambda f(x,u) + \mu|\nabla u|^a$ dans un domaine borné régulier Ω de $\mathbb{R}^N$, où $d(x) = \operatorname{dist}(x, \partial\Omega)$, $\lambda > 0$, $\mu \in \mathbb{R}$, $0 < a \leqslant 2$, et f est une fonction croissante. Nous supposons que $p(d(x))$ est un potentiel positif singulier sur $\partial\Omega$ et que la non-linéarité g est non bornée autour de l'origine. Compte tenu de la compétition entre le potentiel anisotrope $p(d(x))$, le terme de convection $|\nabla u|^a$ et la non-linéarité singulière g, nous établissons plusieurs résultats d'existence et de non-existence.

MSC: 35B50; 35J65; 58J55

Keywords: Lane–Emden–Fowler equations; Convection term; Singular potential

* Corresponding author.
E-mail addresses: louis.dupaigne@u-picardie.fr (L. Dupaigne), marius.ghergu@imar.ro (M. Ghergu), vicentiu.radulescu@math.cnrs.fr (V. Rădulescu).

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2007.03.002

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded domain with smooth boundary. We are concerned in this paper with singular elliptic problems of the following type:

$$\begin{cases} -\Delta u \pm p(d(x))g(u) = \lambda f(x,u) + \mu|\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P)^{\pm}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$, $\lambda > 0$, $\mu \in \mathbb{R}$, and $0 < a \leqslant 2$.

We assume that $g \in C^1(0,\infty)$ is a positive decreasing function, and

$(g1)$ $\lim_{t\to 0^+} g(t) = +\infty$.

As remarked by Serrin [27], Choquet-Bruhat and Leray [8], and Kazdan and Warner [24], the requirement that the nonlinearity $|\nabla u|^a$ grows at most quadratically is natural in order to apply the maximum principle.

Throughout this paper we suppose that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is nondecreasing with respect to the second variable and such that f is positive on $\overline{\Omega} \times (0,\infty)$. The analysis we develop in this paper concerns the cases where f is either linear or f is sublinear with respect to the second variable. This last case means that f fulfills the hypotheses:

$(f1)$ the mapping $(0,\infty) \ni t \mapsto \frac{f(x,t)}{t}$ is nonincreasing for all $x \in \overline{\Omega}$;
$(f2)$ $\lim_{t\to 0^+} \frac{f(x,t)}{t} = +\infty$ and $\lim_{t\to+\infty} \frac{f(x,t)}{t} = 0$, uniformly for $x \in \overline{\Omega}$.

Such singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids or boundary layer phenomena for viscous fluids (we refer for more details to [5–7,11,13,14] and the more recent papers [9,15,21–23,25,28,29,31]). We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases.

To the best of our knowledge, there does not exist a qualitative theory for the study of singular boundary value problems with nonlinearities in the Kato class $K_N^{\mathrm{loc}}(\mathbb{R}^N)$. This theory was introduced by Aizenman and Simon in [2] to describe wide classes of functions arising in Potential Theory. We refer to the recent paper [26] for existence and bifurcation results on Dirichlet boundary value problems with indefinite nonlinearities.

The main features of this paper are the presence of the convection term $|\nabla u|^a$ combined with the singular weight $p : (0,\infty) \to (0,\infty)$ which is supposed to be nonincreasing and Hölder continuous.

The results in this paper complete the study developed in [17,16,18] since here we deal with singular weights. One of our purposes is to give a necessary and sufficient condition on the weight p in order to obtain a classical solution of problems $(P)^{\pm}$. By classical solution we understand a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that fulfills $(P)^{\pm}$.

Dealing with problem $(P)^+$ we show that when $\mu \leqslant 0$, then a necessary condition in order to have classical solution is:

$$\int_0^1 p(t)g(t)\,\mathrm{d}t < +\infty. \qquad (1)$$

In the case where f is sublinear, that is, f fulfills the hypotheses $(f1)$ and $(f2)$, condition (1) is also sufficient for existence of a classical solutions of $(P)^+$ provided λ and μ belong to a certain range (see Theorem 2.2). Obviously, (1) implies the following Keller–Osserman type condition around the origin (see the proof of Theorem 2.2):

$$\text{(KO)} \qquad \int_0^1 \left(\int_0^t \Phi(s)\,\mathrm{d}s \right)^{-1/2} \mathrm{d}t < +\infty, \quad \text{where } \Phi(s) = p(s)g(s), \text{ for all } s > 0.$$

As proved by Bénilan, Brezis and Crandall [4], condition (KO) is equivalent to the *property of compact support*, that is, for every $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and

$$-\Delta u + \Phi(u) = h \quad \text{a.e. in } \mathbb{R}^N.$$

The results are completely different for problem $(P)^-$. Our results in this case generalize those established in [32], in the sense that in the present paper we do not prescribe the behavior of the singular nonlinearity g around the origin. Also, we proved in [17] that if $p \equiv 1$, then the existence of a classical solution to $(P)^-$ does not depend on the asymptotic behavior of g near the origin, whereas the exponent a of the convection term $|\nabla u|^a$ plays a crucial role. In our case, the potential $p(d(x))$ also affects the existence of classical solutions to $(P)^-$.

Many papers have been devoted to the case $p \equiv 1$ and $\mu = 0$ (see [10,12,15,28] and the references therein). One of the first works in the literature dealing with singular weights in connection with singular nonlinearities is due to Taliaferro [30]. In [30] the following problem has been considered:

$$\begin{cases} -y'' = \varphi(x) y^{-\beta} & \text{in } (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{2}$$

where $\beta > 0$ and $\varphi(x)$ is positive and continuous on $(0,1)$. It was proved that problem (2) has solutions if and only if $\int_0^1 t(1-t)\varphi(t)\,\mathrm{d}t < +\infty$. Later, Agarwal and O'Regan [1, Section 2] studied the more general problem:

$$\begin{cases} H''(t) = -p(t) g(H(t)) & \text{in } (0,1), \\ H > 0 & \text{in } (0,1), \\ H(0) = H(1) = 0, \end{cases} \tag{3}$$

where g satisfies $(g1)$ and p is positive and continuous on $(0,1)$. It is shown in [1] that if

$$\int_0^1 t(1-t)p(t)\,\mathrm{d}t < +\infty, \tag{4}$$

then (3) has at least one classical solution. In our framework, p is continuous at $t = 1$ so condition (4) shows that

$$\int_0^1 tp(t)\,\mathrm{d}t < +\infty. \tag{5}$$

In this paper we prove that the assumption (5) is also necessary in order that problem $(P)^-$ with $\mu \geqslant 0$ has classical solutions. Furthermore, we argue in Section 3 that the existence of a classical solution of $(P)^-$ when f is sublinear depends on the asymptotic behavior of the gradient term $|\nabla u|^a$. In this sense, we prove that if $0 < a < 1$, then $(P)^-$ has at least one classical solution for all $\mu \in \mathbb{R}$. In turn, if $1 < a \leqslant 2$, then $(P)^-$ has no solutions for large values of μ.

A special attention is payed to the case where $a = 1$. This case was left as an open question in [17]. We prove in Theorem 3.3 that if Ω is a ball centered at the origin, then $(P)^-$ has at least one solution for all $\mu \in \mathbb{R}$ provided $a = 1$.

The existence of a solution to $(P)^{\pm}$ is achieved by the sub and super-solution method. In particular, the super-solution of $(P)^-$ is expressed in terms of H. In the case of pure power nonlinearities, a careful analysis of (3) allows us to give boundary estimates of the solution.

The outline of the paper is as follows. In Section 2 we obtain existence and nonexistence results for problem $(P)^+$. Section 3 concerns the problem $(P)^-$ in which we discuss separately the case where f is linear or sublinear. At the end of this section we present the case $p(t) = t^{-\alpha}$ and $g(t) = t^{-\beta}$, and we give some estimates for the solution at the boundary. To make the results clearer, we assume that $\lambda = 1$ and f is sublinear. Thus, problem $(P)^-$ becomes:

$$\begin{cases} -\Delta u = d(x)^{-\alpha} u^{-\beta} + f(x,u) + \mu |\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

2. The problem $(P)^+$

We first establish the following nonexistence result related to problem $(P)^+$.

Theorem 2.1. *Assume that* $\int_0^1 p(t)g(t)\,\mathrm{d}t = +\infty$. *Let* $\Phi : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ *be a continuous function such that* $\Phi \not\equiv 0$. *Then the inequality boundary value problem*:

$$\begin{cases} -\Delta u + p(d(x))g(u) \leqslant \Phi(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

has no classical solutions.

Proof. Assume that (7) has a classical solution u and let $C = \max_{\overline{\Omega}} \Phi(x,u) > 0$. Let also $v \in C^2(\overline{\Omega})$ be the unique solution of

$$\begin{cases} -\Delta v = C & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Moreover, there exist $c_1, c_2 > 0$ such that

$$c_1 d(x) \leqslant v \leqslant c_2 d(x), \quad \text{for all } x \in \Omega. \tag{9}$$

By the maximum principle, it follows that $u \leqslant v$ in Ω. Next we consider the perturbed problem:

$$\begin{cases} -\Delta u + p(d(x)+\varepsilon)g(u+\varepsilon) = C & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Then, u and v are, respectively, sub- and super-solution of (10). By standard arguments and elliptic regularity (see [19]), there exists $u_\varepsilon \in C^2(\overline{\Omega})$ a solution of (10) such that $u \leqslant u_\varepsilon \leqslant v$ in Ω. Integrating in (10) we obtain:

$$-\int_\Omega \Delta u_\varepsilon \,\mathrm{d}x + \int_\Omega p\big(d(x)+\varepsilon\big)g(u_\varepsilon+\varepsilon)\,\mathrm{d}x = C|\Omega|.$$

Hence

$$-\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}\,\mathrm{d}s + \int_\Omega p\big(d(x)+\varepsilon\big)g(u_\varepsilon+\varepsilon)\,\mathrm{d}x \leqslant M, \tag{11}$$

where M is a positive constant. Taking into account that $\frac{\partial u_\varepsilon}{\partial n} \leqslant 0$ on $\partial\Omega$, relation (11) yields:

$$\int_\Omega p\big(d(x)+\varepsilon\big)g(u_\varepsilon+\varepsilon)\,\mathrm{d}x \leqslant M.$$

Since g is decreasing and $u_\varepsilon \leqslant v$ in $\overline{\Omega}$, the last inequality implies $\int_\Omega p(d(x)+\varepsilon)g(v+\varepsilon)\,\mathrm{d}x \leqslant M$. Thus, for any compact subset $\omega \Subset \Omega$ we have:

$$\int_\omega p\big(d(x)+\varepsilon\big)g(v+\varepsilon)\,\mathrm{d}x \leqslant M.$$

Passing to the limit with $\varepsilon \to 0^+$ we obtain $\int_\omega p(d(x))g(v)\,\mathrm{d}x \leqslant M$, for all $\omega \Subset \Omega$. Therefore

$$\int_\Omega p\big(d(x)\big)g(v)\,\mathrm{d}x \leqslant M. \tag{12}$$

On the other hand, using (9) and the hypothesis $\int_0^1 p(t)g(t)\,\mathrm{d}s = +\infty$, it follows that

$$M \geqslant \int_\Omega p\big(d(x)\big)g(v)\,\mathrm{d}x \geqslant \int_\Omega p\big(d(x)\big)g\big(c_2 d(x)\big)\,\mathrm{d}x = +\infty,$$

which is a contradiction. Hence, problem (7) has no classical solutions and the proof of Theorem 2.1 is complete. □

Corollary 2.1. *Assume that $\int_0^1 p(t)g(t)\,\mathrm{d}t = +\infty$. Then, for all $\mu \leqslant 0$, the problem $(P)^+$ has no classical solutions.*

Several times in this paper we apply the following auxiliary result (we refer to [16, Lemma 2.1] for a complete proof).

Lemma 2.1. *Let $\Psi : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ be a Hölder continuous function such that the mapping $(0,\infty) \ni s \mapsto \frac{\Psi(x,s)}{s}$ is strictly decreasing for each $x \in \Omega$. Assume that there exist v, $w \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

(a) $\Delta w + \Psi(x,w) \leqslant 0 \leqslant \Delta v + \Psi(x,v)$ *in* Ω;
(b) $v, w > 0$ *in* Ω *and* $v \leqslant w$ *on* $\partial\Omega$;
(c) $\Delta v \in L^1(\Omega)$ *or* $\Delta w \in L^1(\Omega)$.

Then $v \leqslant w$ in Ω.

Next, we prove that condition (1) is sufficient for the existence of a classical solution to $(P)^+$ provided $\mu \leqslant 0$ and $\lambda > 0$ is large enough. We have the:

Theorem 2.2. *Assume that $\int_0^1 p(t)g(t)\,\mathrm{d}t < +\infty$ and f fulfills $(f1)$–$(f2)$.*

(i) *If $\mu = -1$, then there exists $\lambda^* > 0$ such that $(P)^+$ has at least one classical solution if $\lambda > \lambda^*$ and no solution exists if $0 < \lambda < \lambda^*$.*
(ii) *If $\mu = +1$ and $0 < a < 1$, then there exists $\lambda^* > 0$ such that $(P)^+$ has at least one classical solution for all $\lambda > \lambda^*$ and no solution exists if $0 < \lambda < \lambda^*$.*

Proof. (i) We split the proof into several steps.

Step 1: *Existence of a solution for λ large*. By virtue of [28, Lemma 2.4] (see also [29, Theorem 2.2]), the problem:

$$\begin{cases} -\Delta U = \lambda f(x,U) & \text{in } \Omega, \\ U > 0 & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

has at least one classical solution U_λ, for all $\lambda > 0$. Using the regularity of f it follows that $U_\lambda \in C^2(\overline{\Omega})$ and there exist $c_1, c_2 > 0$ depending on λ such that

$$c_1 d(x) \leqslant U_\lambda(x) \leqslant c_2 d(x) \quad \text{in } \Omega. \tag{14}$$

Fix $\lambda > 0$ and notice that U_λ is a super-solution of $(P)^+$. The main point is to find a sub-solution $\underline{u}_\lambda$ of $(P)^+$ such that $\underline{u}_\lambda \leqslant U_\lambda$ in Ω. For this purpose, let $\Phi(t) = p(t)g(t)$, $t > 0$, and define

$$\Psi : [0,\infty) \to [0,\infty), \quad \Psi(t) = \int_0^t \frac{1}{\sqrt{2\int_0^s \Phi(\tau)\,\mathrm{d}\tau}}\,\mathrm{d}s.$$

Remark first that Ψ is well defined, since $\Phi \in L^1(0,1)$. Indeed, if $M := \Phi(1) > 0$, then $\Phi(s) \geqslant M$, for all $0 < s < 1$. This yields $(\int_0^s \Phi(\tau)\,\mathrm{d}\tau)^{-1/2} \leqslant (\sqrt{Ms}\,)^{-1}$, for all $0 < s < 1$ which implies the Keller–Osserman condition (KO) around the origin:

$$\int_0^1 \left(\int_0^t \Phi(s)\,\mathrm{d}s \right)^{-1/2} \mathrm{d}t < +\infty.$$

We claim that Ψ is a bijective map. Indeed, Ψ is increasing, and

$$\int_0^s \Phi(\tau)\,\mathrm{d}\tau \leqslant \int_0^1 \Phi(\tau)\,\mathrm{d}\tau + M(s-1), \quad \forall s \geqslant 1.$$

Thus, there exists $c > 0$ such that

$$\int_0^s \Phi(\tau)\,\mathrm{d}\tau \leqslant Ms + c, \quad \forall s \geqslant 1.$$

It follows that

$$\Psi(t) \geqslant \int_1^t \frac{1}{\sqrt{2(Ms+c)}}\,\mathrm{d}s \geqslant \frac{1}{M}\bigl(\sqrt{2(Mt+c)} - c_1\bigr), \quad \forall t \geqslant 1.$$

This gives $\lim_{t\to+\infty} \Psi(t) = +\infty$ and the claim follows.

Let $h : [0,\infty) \to [0,\infty)$ be the inverse of Ψ. Then h satisfies:

$$\begin{cases} h > 0 & \text{in } (0,\infty), \\ h'(t) = \sqrt{2\int_0^{h(t)} \Phi(s)\,\mathrm{d}s} & \text{in } (0,\infty), \\ h''(t) = \Phi(h(t)) & \text{in } (0,\infty), \\ h(0) = h'(0) = 0. \end{cases} \tag{15}$$

Hence $h \in C^2(0,\infty) \cap C^1[0,\infty)$. Let $\varphi_1 > 0$ be the first eigenfunction of $(-\Delta)$ in $H_0^1(\Omega)$. It is well known that there exists $C > 0$ such that

$$Cd(x) \leqslant \varphi_1 \leqslant \frac{1}{C} d(x) \quad \text{for all } x \in \Omega. \tag{16}$$

The key result for this part of the proof is the following:

Lemma 2.2. *There exist two positive constants $c > 0$ and $M > 0$ such that $\underline{u}_\lambda := Mh(c\varphi_1)$ is a sub-solution of $(P)^+$ provided $\lambda > 0$ is large enough.*

Proof. Since $h \in C^1[0,\infty)$ and $h(0) = 0$, we can take $c > 0$ small enough such that

$$h(c\varphi_1) \leqslant d(x) \quad \text{in } \Omega. \tag{17}$$

By Hopf's maximum principle, there exist $\delta > 0$ and $\omega \Subset \Omega$ such that $|\nabla\varphi_1| \geqslant \delta$ in $\Omega \setminus \omega$. Let

$$M = \max\bigl\{1, 2(c\delta)^{-2}\bigr\}. \tag{18}$$

Since

$$\lim_{d(x)\to 0^+} \bigl\{-p\bigl(d(x)\bigr)g\bigl(h(c\varphi_1)\bigr) + Mc\lambda_1\varphi_1 h'(c\varphi_1) + \bigl(Mch'(c\varphi_1)|\nabla\varphi_1|\bigr)^a\bigr\} = -\infty,$$

we can assume that

$$-p\bigl(d(x)\bigr)g\bigl(h(c\varphi_1)\bigr) + Mc\lambda_1\varphi_1 h'(c\varphi_1) + \bigl(Mch'(c\varphi_1)|\nabla\varphi_1|\bigr)^a < 0 \quad \text{in } \Omega \setminus \omega. \tag{19}$$

We are now able to show that $\underline{u}_\lambda := Mh(c\varphi_1)$ is a sub-solution of $(P)^+$ provided $\lambda > 0$ is sufficiently large. Indeed, using the monotonicity of g and (17) we have:

$$\begin{aligned} &-\Delta\underline{u}_\lambda + p\bigl(d(x)\bigr)g(\underline{u}_\lambda) + |\nabla\underline{u}_\lambda|^a \\ &\quad = -Mc^2 p\bigl(h(c\varphi_1)\bigr)g\bigl(h(c\varphi_1)\bigr)|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 h'(c\varphi_1) + p\bigl(d(x)\bigr)g\bigl(Mh(c\varphi_1)\bigr) + \bigl(Mch'(c\varphi_1)|\nabla\varphi_1|\bigr)^a \\ &\quad \leqslant p\bigl(d(x)\bigr)g\bigl(h(c\varphi_1)\bigr)\bigl(1 - Mc^2|\nabla\varphi_1|^2\bigr) + Mc\lambda_1\varphi_1 h'(c\varphi_1) + \bigl(Mch'(c\varphi_1)|\nabla\varphi_1|\bigr)^a. \end{aligned} \tag{20}$$

Taking into account the definition of M and (19), we find:

$$\begin{aligned}
&-\Delta \underline{u}_\lambda + p\big(d(x)\big)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \\
&\quad \leqslant -p\big(d(x)\big)g\big(h(c\varphi_1)\big) + Mc\lambda_1\varphi_1 h'(c\varphi_1) + \big(Mch'(c\varphi_1)|\nabla\varphi_1|\big)^a \\
&\quad < 0 \quad \text{in } \Omega\setminus\omega.
\end{aligned} \tag{21}$$

On the other hand, from (20) and for all $x \in \omega$, we have:

$$-\Delta \underline{u}_\lambda + p\big(d(x)\big)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \leqslant p\big(d(x)\big)g\big(h(c\varphi_1)\big) + Mc\lambda_1\varphi_1 h'(c\varphi_1) + \big(Mch'(c\varphi_1)|\nabla\varphi_1|\big)^a. \tag{22}$$

Since $\varphi_1 > 0$ in $\overline{\omega}$ and f is positive on $\overline{\omega}\times(0,+\infty)$, we may choose $\lambda > 0$ such that for all $x \in \overline{\omega}$, we have:

$$\lambda f\big(x, Mh(c\varphi_1)\big) \geqslant p\big(d(x)\big)g\big(h(c\varphi_1)\big) + Mc\lambda_1\varphi_1 h'(c\varphi_1) + \big(Mch'(c\varphi_1)|\nabla\varphi_1|\big)^a. \tag{23}$$

From (22) and (23) we deduce:

$$-\Delta \underline{u}_\lambda + p\big(d(x)\big)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \leqslant \lambda f(x, \underline{u}_\lambda) \quad \text{in } \omega. \tag{24}$$

Now, relations (21) and (24) show that $\underline{u}_\lambda = Mh(c\varphi_1)$ is a sub-solution of $(P)^+$ provided $\lambda > 0$ satisfies (23). This finishes the proof of our lemma. □

Using Lemma 2.1, it follows that $\underline{u}_\lambda \leqslant U_\lambda$ in Ω and by standard elliptic arguments (see [19]) we obtain a classical solution u_λ of $(P)^+$ such that $\underline{u}_\lambda \leqslant u_\lambda \leqslant U_\lambda$ in Ω.

Step 2: *Nonexistence for $\lambda > 0$ small*. We first remark that

$$\lim_{t\to 0^+}\big(f(x,t) - p\big(d(x)\big)g(t)\big) = -\infty \quad \text{uniformly for } x \in \Omega.$$

Hence, there exists $t_0 > 0$ such that

$$f(x,t) - p\big(d(x)\big)g(t) < 0, \quad \text{for all } (x,t) \in \Omega\times(0,t_0). \tag{25}$$

On the other hand, the assumption $(f1)$ yields

$$\frac{f(x,t) - p(d(x))g(t)}{t} \leqslant \frac{f(x,t)}{t} \leqslant \frac{f(x,t_0)}{t_0}, \tag{26}$$

for all $(x,t) \in \Omega\times[t_0,\infty)$. Let $m = \max_{x\in\overline{\Omega}} \frac{f(x,t_0)}{t_0}$. Combining (25) and (26) we find

$$f(x,t) - p\big(d(x)\big)g(t) < mt, \quad \text{for all } (x,t) \in \Omega\times(0,+\infty). \tag{27}$$

Set $\lambda_0 = \min\{1, \lambda_1/2m\}$. We claim that problem $(P)^+$ has no classical solution for $0 < \lambda \leqslant \lambda_0$. Indeed, assume by contradiction that u_0 is a classical solution of $(P)^+$ with $\lambda \in (0,\lambda_0]$. Then, according to (27), u_0 is a sub-solution of

$$\begin{cases} -\Delta u = \frac{\lambda_1}{2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{28}$$

By Lemma 2.1 we have $u_0 \leqslant U_\lambda$ in Ω. Furthermore, from (14) and (16) we get $cu_0 \leqslant \varphi_1$ in Ω for some positive constant $c > 0$. Note that cu_0 is still a sub-solution of (28) while φ_1 is a super-solution of (28). By standard elliptic arguments, problem (28) has a solution $u \in C^2(\overline{\Omega})$. Multiplying by φ_1 in (28) and integrating on Ω, we have:

$$-\int_\Omega \varphi_1\Delta u\,\mathrm{d}x = \frac{\lambda_1}{2}\int_\Omega u\varphi_1\,\mathrm{d}x,$$

that is,

$$\lambda_1\int_\Omega u\varphi_1\,\mathrm{d}x = -\int_\Omega u\Delta\varphi_1\,\mathrm{d}x = \frac{\lambda_1}{2}\int_\Omega u\varphi_1\,\mathrm{d}x.$$

The above equality yields $\int_\Omega u\varphi_1\,\mathrm{d}x = 0$, but this is clearly a contradiction, since both u and φ_1 are positive in Ω. It follows that $(P)^+$ has no classical solutions for $0 < \lambda \leqslant \lambda_0$.

Step 3: *Dependence on* $\lambda > 0$. Set,

$$A = \{\lambda > 0;\ \text{problem } (P)^+ \text{ has at least one classical solution}\}.$$

From the above arguments we deduce that A is nonempty and $\lambda^* := \inf A$ is positive. We show that if $\lambda \in A$, then $(\lambda, \infty) \subseteq A$. To this aim, let $\lambda_1 \in A$ and $\lambda_2 > \lambda_1$. If u_{λ_1} is a solution of $(P)^+$ with $\lambda = \lambda_1$, then u_{λ_1} is a sub-solution of $(P)^+$ with $\lambda = \lambda_2$ while U_{λ_2} defined in (13) for $\lambda = \lambda_2$ is a super-solution. Moreover, we have:

$$\begin{gathered}\Delta U_{\lambda_2} + \lambda_2 f(x, U_{\lambda_2}) \leqslant 0 \leqslant \Delta u_{\lambda_1} + \lambda_2 f(x, u_{\lambda_1}) \quad \text{in } \Omega,\\ U_{\lambda_2}, u_{\lambda_1} > 0 \quad \text{in} \Omega,\\ U_{\lambda_2} = u_{\lambda_1} = 0 \quad \text{on } \partial\Omega,\\ \Delta U_{\lambda_2} \in L^1(\Omega).\end{gathered}$$

Again by Lemma 2.1 we get $u_{\lambda_1} \leqslant U_{\lambda_2}$ in Ω. Therefore, problem $(P)^+$ with $\lambda = \lambda_2$ has at least one classical solution. Since $\lambda \in A$ was arbitrary, we conclude that $(\lambda^*, \infty) \subset A$. This completes the proof of (i).

(ii) *Step 1*: *Existence of a solution for* λ *large*.

According to Lemma 2.2, there exists $\lambda^* > 0$ such that $(P)^+$ has a sub-solution $\underline{u}_\lambda$ for $\lambda > \lambda^*$ and $\mu = -1$. Then $\underline{u}_\lambda$ is also a sub-solution in case $\mu = +1$, provided $\lambda > \lambda^*$. Let us construct now a super-solution. By [28, Lemma 2.4], for all $\lambda > \lambda^*$ there exists $v_\lambda \in C^2(\overline{\Omega})$ a solution of

$$\begin{cases} -\Delta v = \lambda f(x, v) + 1 & \text{in } \Omega,\\ v > 0 & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $0 < a < 1$, we can choose $M = M(\lambda) > 1$ large enough such that $M > M^a |\nabla v_\lambda|^a$ in Ω. Then, using $(f1)$, we obtain:

$$-\Delta(M v_\lambda) = \lambda M f(x, v_\lambda) + M \geqslant \lambda f(x, M v_\lambda) + \left|\nabla(M v_\lambda)\right|^a \quad \text{in } \Omega.$$

Hence $\overline{u}_\lambda := M v_\lambda \in C^2(\overline{\Omega})$ is a super-solution of $(P)^+$ for all $\lambda > \lambda^*$. On the other hand, since $\Delta \overline{u}_\lambda + \lambda f(x, \overline{u}_\lambda) \leqslant 0 \leqslant \Delta \underline{u}_\lambda + \lambda f(x, \underline{u}_\lambda)$ in Ω, by Lemma 2.1 we get $\underline{u}_\lambda \leqslant \overline{u}_\lambda$ and finally, problem $(P)^+$ has at least one solution for all $\lambda > \lambda^*$.

Step 2: *Nonexistence for* $\lambda > 0$ *small*. We first extend Lemma 2.1 in the following way:

Lemma 2.3. *Let* $0 < a < 1$ *and* $\Psi : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ *be a Hölder continuous function such that the mapping* $i(0, \infty) \ni s \mapsto \frac{\Psi(x,s)}{s}$ *is strictly decreasing for each* $x \in \Omega$. *Assume that there exist* $v,\ w \in C^2(\Omega) \cap C(\overline{\Omega})$ *such that*

(a) $\Delta w + \Psi(x, w) + |\nabla w|^a \leqslant 0 \leqslant \Delta v + \Psi(x, v) + |\nabla v|^a$ *in* Ω;
(b) $v, w > 0$ *in* Ω *and* $v < w$ *on* $\partial\Omega$.

Then $v \leqslant w$ *in* Ω.

Proof. Assume by contradiction that the inequality $v \leqslant w$ does not hold throughout Ω and let $\varphi = \frac{v}{w}$. Clearly $\varphi < 1$ on $\partial\Omega$ and

$$-\nabla \cdot \left[w^2 \nabla \varphi\right] = -w\Delta v + v\Delta w \quad \text{in } \Omega.$$

Let $x_0 \in \Omega$ be a point where φ achieves its maximum. In particular $\nabla\varphi(x_0) = 0$, $-\Delta\varphi(x_0) \geqslant 0$ and it follows that

$$0 \leqslant [-w\Delta v + v\Delta w](x_0).$$

Since $w(x_0) < v(x_0)$, it follows from assumption (a), the properties of Ψ and the above inequality that

$$0 < \left[|\nabla v|^a w - |\nabla w|^a v\right](x_0).$$

Since $\nabla\varphi(x_0) = 0$, we finally obtain:

$$0 < \left[\left(\frac{v}{w}\right)^a w - v\right] |\nabla w|^a(x_0) = v^a \left(w^{1-a} - v^{1-a}\right) |\nabla w|^a(x_0),$$

contradicting $w(x_0) < v(x_0)$. This concludes the proof of our lemma. □

Next, we assume by contradiction that there exists a sequence of solutions u_n of (P^+) associated to a parameter $\lambda_n \to 0^+$. A simple calculation shows that $w(x) = A(R^2 - |x|^2)$ is positive and satisfies the inequality $\Delta w + f(x, w) + |\nabla w|^a \leqslant 0$ in Ω, where $A, R > 0$ are large constants. In particular, it follows from Lemma 2.3 that $0 < u_n \leqslant w$ in Ω whenever $\lambda_n \leqslant 1$. Let $x_n \in \Omega$ be a maximum point of u_n. Then $\nabla u_n(x_n) = 0$ and $-\Delta u_n(x_n) \geqslant 0$. Letting $d_n = d(x_n)$, $M_n = u_n(x_n)$, it follows from (P^+) that

$$p(d_n)g(M_n) \leqslant \lambda_n f(x_n, M_n) \leqslant C\lambda_n,$$

which yields a contradiction as $n \to \infty$.

The rest of the proof of (ii) follows in the same manner as in the case $\mu = -1$. This completes the proof of Theorem 2.2. □

3. The problem $(P)^-$

3.1. A nonexistence result

We first prove the following general nonexistence result which is related to our problem $(P)^-$.

Theorem 3.1. *Assume that* $\int_0^1 tp(t)\,\mathrm{d}t = +\infty$. *Then the inequality boundary value problem,*

$$\begin{cases} -\Delta u + C|\nabla u|^2 \geqslant p(d(x))g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{29}$$

has no classical solutions.

Proof. It suffices to prove the theorem only for $C > 0$. We argue by contradiction and assume that there exists $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of (29). Using $(g1)$, we can find $c_1 > 0$ such that $\underline{u} := c_1\varphi_1$ verifies:

$$-\Delta \underline{u} + C|\nabla \underline{u}|^2 \leqslant p\big(d(x)\big)g(\underline{u}) \quad \text{in } \Omega.$$

Since g is decreasing, we easily obtain:

$$u \geqslant \underline{u} \quad \text{in } \Omega. \tag{30}$$

We make in (29) the change of variable $v = 1 - \mathrm{e}^{-Cu}$. Therefore,

$$\begin{cases} -\Delta v = C(1-v)(C|\nabla u|^2 - \Delta u) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{31}$$

From (31) we easily derive,

$$-\Delta v \geqslant C(1-v)p\big(d(x)\big)g\left(-\frac{\ln(1-v)}{C}\right) \quad \text{in } \Omega.$$

In order to avoid the singularities in (31) let us consider the approximated problem:

$$\begin{cases} -\Delta v = C(1-v)p(d(x))g(\varepsilon - \frac{\ln(1-v)}{C}) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{32}$$

with $0 < \varepsilon < 1$. Clearly v is a super-solution of (32). Furthermore, by (30) and the fact that $\lim_{t\to 0^+} \frac{1-\mathrm{e}^{-Ct}}{t} = C > 0$, there exists $c_2 > 0$ such that $v \geqslant c_2\varphi_1$ in Ω. On the other hand, there exists $0 < c < c_2$ such that $c\varphi_1$ is a sub-solution of (32) and obviously $c\varphi_1 \leqslant v$ in Ω. Then, the problem (32) has a solution $v_\varepsilon \in C^2(\overline{\Omega})$ such that

$$c\varphi_1 \leqslant v_\varepsilon \leqslant v \quad \text{in } \Omega. \tag{33}$$

Multiplying by φ_1 in (32) and integrating, we find:

$$\lambda_1 \int_\Omega \varphi_1 v_\varepsilon \,\mathrm{d}x = C \int_\Omega (1-v_\varepsilon)\varphi_1 p\big(d(x)\big) g\left(\varepsilon - \frac{\ln(1-v_\varepsilon)}{C}\right) \mathrm{d}x.$$

Using (33), we obtain:

$$\begin{aligned} M =: \lambda_1 \int_\Omega \varphi_1 v \,\mathrm{d}x &\geqslant C \int_\Omega (1-v)\varphi_1 p\big(d(x)\big) g\left(1 - \frac{\ln(1-v)}{C}\right) \mathrm{d}x \\ &\geqslant C_1 \int_{\Omega_\delta} \varphi_1 p\big(d(x)\big)\,\mathrm{d}x, \end{aligned} \tag{34}$$

where $\Omega_\delta = \{x \in \Omega;\ d(x) < \delta\}$, for some $\delta > 0$ sufficiently small. Since $\varphi_1(x)$ behaves like $d(x)$ in Ω_δ and $\int_0^1 tp(t)\,\mathrm{d}t = +\infty$, by (34) we find a contradiction. Hence, problem (3.1) has no classical solutions and the proof is now complete. □

A direct consequence of Theorem 3.1 is the following nonexistence property:

Corollary 3.1. *Assume that* $\int_0^1 tp(t)\,\mathrm{d}t = +\infty$ *and conditions* $(g1)$, $0 < a \leqslant 2$ *are fulfilled.*
Then the problem $(P)^-$ *has no classical solutions.*

3.2. Existence results for $(P)^-$ *in the sublinear case on* f

Our aim here is to give existence results concerning $(P)^-$ in case where f is sublinear. Nevertheless, we prove that condition (5) suffices to guarantee the existence of a classical solution for μ belonging to a certain range.

In this case the existence of a solution is strongly dependent on the exponent a of the gradient term. To better understand this dependence, we assume $\lambda = 1$ but the same results hold for any $\lambda > 0$ (note only that the bifurcation point μ^* in the following theorem is dependent on λ).

Theorem 3.2. *Assume* $\lambda = 1$, $\int_0^1 tp(t)\,\mathrm{d}t < +\infty$ *and conditions* $(f1)$, $(f2)$, $(g1)$ *and* $0 < a \leqslant 2$ *are fulfilled.*

(i) *If* $0 < a < 1$, *then problem* $(P)^-$ *has at least one solution, for all* $\mu \in \mathbb{R}$;
(ii) *If* $1 < a \leqslant 2$, *then there exists* $\mu^* > 0$ *such that* $(P)^-$ *has at least one classical solution for all* $\mu < \mu^*$ *and no solution exists if* $\mu > \mu^*$.

Proof. (i) CASE $\mu > 0$. By [28, Lemma 2.4] there exists a classical solution ζ of the problem:

$$\begin{cases} -\Delta\zeta = f(x,\zeta) & \text{in } \Omega, \\ \zeta > 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases} \tag{35}$$

Using the regularity of f we have $\zeta \in C^2(\overline{\Omega})$. Then, ζ is a sub-solution of $(P)^-$ provided $\mu > 0$. We focus now on finding a super-solution $\overline{u}_\mu$ of $(P)^-$ such that $\zeta \leqslant \overline{u}_\mu$ in Ω.

Let H be the solution of (3). Since H is concave, there exists $H'(0+) \in (0,\infty]$. Taking $0 < b < 1$ small enough, we can assume that $H' > 0$ in $(0,b]$, so H is increasing on $[0,b]$. Multiplying by H' in (3) and integrating on $[t,b]$, we find:

$$(H')^2(t) - (H')^2(b) = 2\int_t^b p(s) g\big(H(s)\big) H'(s)\,\mathrm{d}s \leqslant 2p(t) \int_{H(t)}^{H(b)} g(\tau)\,\mathrm{d}\tau. \tag{36}$$

Using the monotonicity of g it follows that

$$(H')^2(t) \leqslant 2H(b)p(t)g\big(H(t)\big) + (H')^2(b), \quad \text{for all } 0 < t \leqslant b. \tag{37}$$

Hence, there exist $C_1, C_2 > 0$ such that

$$(H')(t) \leqslant C_1 p(t) g\big(H(t)\big), \quad \text{for all } 0 < t \leqslant b \tag{38}$$

and

$$(H')^2(t) \leqslant C_2 p(t) g\big(H(t)\big), \quad \text{for all } 0 < t \leqslant b. \tag{39}$$

Now we can proceed to construct a super-solution for $(P)^-$. First, we fix $c > 0$ such that

$$c\varphi_1 \leqslant \min\big\{b, d(x)\big\} \quad \text{in } \Omega. \tag{40}$$

By Hopf's maximum principle, there exist $\omega \Subset \Omega$ and $\delta > 0$ such that

$$|\nabla \varphi_1| > \delta \quad \text{in } \Omega \setminus \omega. \tag{41}$$

Moreover, since

$$\lim_{d(x)\to 0^+} \big\{c^2 p(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 - 3f\big(x, H(c\varphi_1)\big)\big\} = +\infty,$$

we can assume that

$$c^2 p(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant 3f\big(x, H(c\varphi_1)\big) \quad \text{in } \Omega \setminus \omega. \tag{42}$$

Let $M > 1$ be such that

$$Mc^2\delta^2 > 3. \tag{43}$$

Since $H'(0+) > 0$ and $0 < a < 1$, we can choose $M > 1$ such that

$$M\frac{(c\delta)^2}{C_1} H'(c\varphi_1) \geqslant 3\mu\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^a \quad \text{in } \Omega \setminus \omega,$$

where C_1 is the constant appearing in (38). By (38), (41) and (43), we derive:

$$Mc^2 p(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant 3\mu\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^a \quad \text{in } \Omega \setminus \omega. \tag{44}$$

Since g is decreasing and $H'(c\varphi_1) > 0$ in $\overline{\omega}$, there exists $M > 0$ such that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geqslant 3p\big(d(x)\big) g\big(H(c\varphi_1)\big) \quad \text{in } \omega. \tag{45}$$

In the same manner, using $(f2)$ and the fact that $\varphi_1 > 0$ in $\overline{\omega}$, we can choose $M > 1$ large enough such that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geqslant 3\mu\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^a \quad \text{in } \omega, \tag{46}$$

and

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geqslant 3f\big(x, MH(c\varphi_1)\big) \quad \text{in } \omega. \tag{47}$$

For M satisfying (43)–(47), we prove that

$$\overline{u}_\mu(x) := MH\big(c\varphi_1(x)\big), \quad \text{for all } x \in \Omega, \tag{48}$$

is a super-solution of $(P)^-$. We have:

$$-\Delta\overline{u}_\mu = Mc^2 p(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 H'(c\varphi_1) \quad \text{in } \Omega. \tag{49}$$

We first show that

$$Mc^2 p(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant p\big(d(x)\big) g(\overline{u}_\mu) + f(x, \overline{u}_\mu) + \mu|\nabla\overline{u}_\mu|^a \quad \text{in } \Omega \setminus \omega. \tag{50}$$

Indeed, by (40), (41) and (43), we get:

$$\begin{aligned}\frac{M}{3}c^2 p(c\varphi_1) g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 &\geqslant p\big(d(x)\big) g\big(H(c\varphi_1)\big) \\ &\geqslant p\big(d(x)\big) g\big(MH(c\varphi_1)\big) = p\big(d(x)\big) g(\overline{u}_\mu) \quad \text{in } \Omega \setminus \omega.\end{aligned} \tag{51}$$

The assumption $(f1)$ and (42) produce:

$$\frac{M}{3}c^2 p(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant Mf\big(x, H(c\varphi_1)\big) \geqslant f\big(x, MH(c\varphi_1)\big) = f(x,\bar{u}_\mu) \quad \text{in } \Omega\setminus\omega. \tag{52}$$

From (44) we also obtain:

$$\frac{M}{3}c^2 p(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant \mu\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^a = \mu|\nabla\bar{u}_\mu|^a \quad \text{in } \Omega\setminus\omega. \tag{53}$$

Now, relation (50) follows by (51), (52) and (53).

Next we prove that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geqslant p\big(d(x)\big)g(\bar{u}_\mu) + f(x,\bar{u}_\mu) + \mu|\nabla\bar{u}_\mu|^a \quad \text{in } \omega. \tag{54}$$

From (45) and (46), we get:

$$\frac{M}{3}c\lambda_1\varphi_1 H'(c\varphi_1) \geqslant p\big(d(x)\big)g\big(H(c\varphi_1)\big) \geqslant p\big(d(x)\big)g\big(MH(c\varphi_1)\big) = p\big(d(x)\big)g(\bar{u}_\mu) \quad \text{in } \omega \tag{55}$$

and

$$\frac{M}{3}c\lambda_1\varphi_1 H'(c\varphi_1) \geqslant \mu\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^a = \mu|\nabla\bar{u}_\mu|^a \quad \text{in } \omega. \tag{56}$$

Finally, from (47) we derive:

$$\frac{M}{3}c\lambda_1\varphi_1 H'(c\varphi_1) \geqslant f\big(x, MH(c\varphi_1)\big) = f(x,\bar{u}_\mu) \quad \text{in } \omega. \tag{57}$$

Clearly, relation (54) follows from (55), (56) and (57).

Combining (49) with (50) and (54) we conclude that $\bar{u}_\mu$ is a super-solution of $(P)^-$. Thus, by Lemma 2.1 we obtain $\zeta \leqslant \bar{u}_\mu$ in Ω and by sub and super-solution method it follows that $(P)^-$ has at least one classical solution for all $\mu > 0$.

CASE $\mu \leqslant 0$. We fix $\nu > 0$ and let $u_\nu \in C^2(\Omega)\cap C(\overline{\Omega})$ be a solution of $(P)^-$ for $\mu = \nu$. Then u_ν is a super-solution of $(P)^-$ for all $\mu \leqslant 0$. Set

$$m := \inf_{(x,t)\in\overline{\Omega}\times(0,\infty)} \big(p\big(d(x)\big)g(t) + f(x,t)\big).$$

Since $\lim_{t\to 0^+} g(t) = +\infty$ and the mapping $(0,\infty)\ni t \mapsto \min_{x\in\overline{\Omega}} f(x,t)$ is positive and nondecreasing, we deduce that m is a positive real number. Consider the problem:

$$\begin{cases} -\Delta v = m + \mu|\nabla v|^a & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{58}$$

Clearly zero is a sub-solution of (58). Since $\mu \leqslant 0$, the solution w of the problem,

$$\begin{cases} -\Delta w = m & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

is a super-solution of (58). Hence (58) has at least one solution $v \in C^2(\Omega)\cap C(\overline{\Omega})$. We claim that $v > 0$ in Ω. Indeed, if not, we deduce that $\min_{x\in\overline{\Omega}} v$ is achieved at some point $x_0 \in \Omega$. Then $\nabla v(x_0) = 0$ and $-\Delta v(x_0) = m + \mu|\nabla v(x_0)|^a = m > 0$, contradiction. Therefore, $v > 0$ in Ω. It is easy to see that v is a sub-solution of $(P)^-$ and $-\Delta v \leqslant m \leqslant -\Delta u_\nu$ in Ω. This yields $v \leqslant u_\nu$ in Ω. Again by the sub and super-solution method we conclude that $(P)^-$ has at least one classical solution $u_\mu \in C^2(\Omega)\cap C(\overline{\Omega})$.

(ii) The proof follows the same steps as above. The only difference is that (44) and (46) are no more valid for any $\mu > 0$. The main difficulty when dealing with estimates like (44) is that $H'(c\varphi_1)$ may blow-up at the boundary. However, combining the assumption $1 < a \leqslant 2$ with (39), we can choose $\mu > 0$ small enough such that (44) and (46) hold. This implies that the problem $(P)^-$ has a classical solution provided $\mu > 0$ is sufficiently small.

Set:

$$A = \big\{\mu > 0;\ \text{problem } (P)^- \text{ has at least one classical solution}\big\}.$$

From the above arguments, A is nonempty. Let $\mu^* = \sup A$. We first claim that if $\mu \in A$, then $(0, \mu) \subseteq A$. To this aim, let $\mu_1 \in A$ and $0 < \mu_2 < \mu_1$. If u_{μ_1} is a solution of $(P)^-$ with $\mu = \mu_1$, then u_{μ_1} is a super-solution of $(P)^-$ with $\mu = \mu_2$, while ζ defined in (35) is a sub-solution. Using Lemma 2.1 once more, we get $\zeta \leqslant u_{\mu_1}$ in Ω so $(P)^-$ has at least one classical solution for $\mu = \mu_2$. This proves the claim. Since $\mu_1 \in A$ was arbitrary, we conclude that $(0, \mu^*) \subset A$.

Next, we prove that $\mu^* < +\infty$. To this aim, we use the following result which is a consequence of Theorem 2.1 in [3].

Lemma 3.1. *Assume that $a > 1$. Then there exists a positive number $\bar{\sigma}$ such that the problem,*

$$\begin{cases} -\Delta v \geqslant |\nabla v|^a + \sigma & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{59}$$

has no solutions for $\sigma > \bar{\sigma}$.

Consider $\mu \in A$ and let u_μ be a classical solution of $(P)^-$. Set $v = \mu^{1/(a-1)} u_\mu$. Using our assumption $1 < a \leqslant 2$, we deduce that v fulfills:

$$\begin{cases} -\Delta v \geqslant |\nabla v|^a + m\mu^{1/(a-1)} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{60}$$

According to Lemma 3.1, we obtain $m\mu^{1/(a-1)} \leqslant \bar{\sigma}$, that is, $\mu \leqslant (\frac{\bar{\sigma}}{m})^{a-1}$. This means that $\mu^* \leqslant (\frac{\bar{\sigma}}{m})^{a-1}$, hence μ^* is finite. The existence of a solution in the case $\mu \leqslant 0$ can be achieved in the same manner as above.

This finishes the proof of Theorem 3.2. □

In what follows we discuss the case $a = 1$. Note that the method used in Theorem 3.2 does not apply here for large values of μ.

Assume that $\Omega = B_R(0)$ for some $R > 0$, where $B_R(0) = \{x \in \mathbb{R}^N;\ |x| < R\}$. In this case and with $\lambda = 1$, problem $(P)^-$ becomes:

$$\begin{cases} -\Delta u = p(R - |x|)g(u) + f(x, u) + \mu|\nabla u|, & |x| < R, \\ u > 0, & |x| < R, \\ u = 0, & |x| = R. \end{cases} \tag{61}$$

Theorem 3.3. *Assume that $\int_0^1 t p(t)\,\mathrm{d}t < +\infty$. Then the problem* (61) *has at least one solution for all $\mu \in \mathbb{R}$.*

Proof. The case $\mu \leqslant 0$ is the same as in the proof of Theorem 3.2(i). In what follows, we assume that $\mu > 0$. Using Theorem 3.2(i) it is easy to see that there exists $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta \underline{u} = p(R - |x|)g(\underline{u}), & |x| < R, \\ \underline{u} > 0, & |x| < R, \\ \underline{u} = 0, & |x| = R. \end{cases}$$

It is obvious that $\underline{u}$ is a sub-solution of (61) for all $\mu > 0$. In order to provide a super-solution of (61) we consider the problem:

$$\begin{cases} -\Delta u = p(R - |x|)g(u) + 1 + \mu|\nabla u|, & |x| < R, \\ u > 0, & |x| < R, \\ u = 0, & |x| = R. \end{cases} \tag{62}$$

We need the following auxiliary result.

Lemma 3.2. *Problem* (62) *has at least one solution.*

Proof. We are looking for radially symmetric solution u of (62), that is, $u = u(r)$, $0 \leqslant r = |x| \leqslant R$. In this case, problem (62) becomes

$$\begin{cases} -u'' - \frac{N-1}{r}u'(r) = p(R-r)g(u(r)) + 1 + \mu|u'(r)|, & 0 \leqslant r < R, \\ u > 0, & 0 \leqslant r < R, \\ u(R) = 0. \end{cases} \tag{63}$$

This implies

$$-\big(r^{N-1}u'(r)\big)' \geqslant 0 \quad \text{for all } 0 \leqslant r < R,$$

which yields $u'(r) \leqslant 0$ for all $0 \leqslant r < R$. Then (63) gives:

$$-\left(u'' + \frac{N-1}{r}u'(r) + \mu u'(r)\right) = p(R-r)g\big(u(r)\big) + 1, \quad 0 \leqslant r < R.$$

We obtain:

$$-\big(\mathrm{e}^{\mu r}r^{N-1}u'(r)\big)' = \mathrm{e}^{\mu r}r^{N-1}\psi\big(r, u(r)\big), \quad 0 \leqslant r < R, \tag{64}$$

where

$$\psi(r,t) = p(R-r)g(t) + 1, \quad (r,t) \in [0,R) \times (0,\infty).$$

From (64) we get:

$$u(r) = u(0) - \int_0^r \mathrm{e}^{-\mu t}t^{-N+1}\int_0^t \mathrm{e}^{\mu s}s^{N-1}\psi\big(s, u(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad 0 \leqslant r < R. \tag{65}$$

On the other hand, in view of Theorem 3.2 and using the fact that g is decreasing, there exists a unique solution $w \in C^2(B_R(0)) \cap C(\overline{B}_R(0))$ of the problem:

$$\begin{cases} -\Delta w = p(R-|x|)g(w) + 1, & |x| < R, \\ w > 0, & |x| < R, \\ w = 0, & |x| = R. \end{cases} \tag{66}$$

Clearly, w is a sub-solution of (62). Due to the uniqueness and to the symmetry of the domain, w is radially symmetric, so, $w = w(r)$, $0 \leqslant r = |x| \leqslant R$. As above we get:

$$w(r) = w(0) - \int_0^r t^{-N+1}\int_0^t s^{N-1}\psi\big(s, w(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad 0 \leqslant r < R. \tag{67}$$

We claim that there exists a solution $v \in C^2[0,R) \cap C[0,R]$ of (65) such that $v > 0$ in $[0,R)$.

Let $A = w(0)$ and define the sequence $(v_k)_{k\geqslant 0}$ by $v_0 = w$ and

$$v_k(r) = A - \int_0^r \mathrm{e}^{-\mu t}t^{-N+1}\int_0^t \mathrm{e}^{\mu s}s^{N-1}\psi\big(s, v_{k-1}(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad 0 \leqslant r < R, \tag{68}$$

for all $k \geqslant 1$. Note that v_k is decreasing in $[0,R)$ for all $k \geqslant 0$. From (67) and (68) we easily check that $v_1 \geqslant v_0$ and by induction we deduce $v_k \geqslant v_{k-1}$ for all $k \geqslant 1$. Hence

$$w = v_0 \leqslant v_1 \leqslant \cdots \leqslant v_k \leqslant \cdots \leqslant A \quad \text{in } B_R(0).$$

Thus, there exists $v(r) := \lim_{k\to\infty} v_k(r)$, for all $0 \leqslant r < R$ and $v > 0$ in $[0,R)$. We now can pass to the limit in (68) in order to get that v is a solution of (65). By classical regularity results we also obtain $v \in C^2[0,R) \cap C[0,R]$. This proves the claim.

We have obtained a super-solution v of (62) such that $v \geqslant w$ in $B_R(0)$. So, the problem (62) has at least one solution and the proof of our lemma is now complete. □

Let u be a solution of the problem (62). For $M > 1$, we have:

$$\begin{aligned} -\Delta(Mu) &= Mp\big(R-|x|\big)g(u) + M + \mu\big|\nabla(Mu)\big| \\ &\geqslant p\big(R-|x|\big)g(Mu) + M + \mu\big|\nabla(Mu)\big|. \end{aligned} \tag{69}$$

Since f is sublinear, we can choose $M > 1$ such that

$$M \geqslant f\big(x, M|u|_\infty\big) \quad \text{in } B_R(0).$$

Then $\overline{u}_\mu := Mu$ satisfies:

$$-\Delta\overline{u}_\mu \geqslant p\big(R-|x|\big)g(\overline{u}_\mu) + f(x,\overline{u}_\mu) + \mu|\nabla\overline{u}_\mu| \quad \text{in } B_R(0).$$

It follows that $\overline{u}_\mu$ is a super-solution of (61). Since g is decreasing we easily deduce $\underline{u} \leqslant \overline{u}_\mu$ in $B_R(0)$ so, problem $(P)^-$ has at least one solution.

The proof of Theorem 3.3 is now complete. □

3.3. Existence results for $(P)^-$ in the linear case on f

In this section we study the problem $(P)^-$ in which we drop out the sublinearity assumptions $(f1)$, $(f2)$ on f but we require in turn that f is linear. More precisely, we assume that $f(x,t) = t$, for all $(x,t) \in \overline{\Omega} \times [0,\infty)$ and consider the problem:

$$\begin{cases} -\Delta u = p(d(x))g(u) + \lambda u + \mu|\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{70}$$

where $\lambda > 0$ and p, g are as in the previous sections. We assume in what follows that $0 < a < 1$.

Note that the existence results established in [28, Lemma 2.4] or [29] do not apply here since the mapping,

$$\Psi(x,t) = p\big(d(x)\big)g(t) + \lambda t, \quad (x,t) \in \Omega \times (0,\infty),$$

is not defined on $\partial\Omega \times (0,\infty)$.

Theorem 3.4. *Assume that $\int_0^1 tp(t)\,\mathrm{d}t < +\infty$ and conditions $(g1)$, $0 < a < 1$ are fulfilled. Then for $\mu \geqslant 0$ the problem* (70) *has solutions if and only if $\lambda < \lambda_1$.*

Proof. Fix $\lambda \in (0,\lambda_1)$ and $\mu \geqslant 0$. By Theorem 3.2(i) there exists $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the problem:

$$\begin{cases} -\Delta u = p(d(x))g(u) + \mu|\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Obviously, $\underline{u}_{\lambda\mu} := u$ is a sub-solution of (70). Since $\lambda < \lambda_1$, there exists $v \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta v = \lambda v + 2 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $0 < a < 1$, we can choose $M > 0$ large enough such that

$$M > \lambda|u|_\infty \quad \text{and} \quad M > \mu(M|\nabla v|)^a \quad \text{in } \Omega.$$

Then $w := Mv$ satisfies:

$$-\Delta w \geqslant \lambda(u+w) + \mu|\nabla w|^a \quad \text{in } \Omega.$$

We claim that $\overline{u}_{\lambda\mu} := u + w$ is a super-solution of (70). Indeed, we have:

$$-\Delta\overline{u}_{\lambda\mu} \geqslant p\big(d(x)\big)g(u) + \lambda\overline{u}_{\lambda\mu} + \mu|\nabla u|^a + \mu|\nabla w|^a \quad \text{in } \Omega. \tag{71}$$

Using the assumption $0 < a < 1$ one can easily deduce:

$$t_1^a + t_2^a \geqslant (t_1 + t_2)^a, \quad \text{for all } t_1, t_2 \geqslant 0.$$

Hence

$$|\nabla u|^a + |\nabla w|^a \geqslant \big(|\nabla u| + |\nabla w|\big)^a \geqslant \big|\nabla(u+w)\big|^a \quad \text{in } \Omega. \tag{72}$$

Combining (71) and (72), we obtain:

$$-\Delta \bar{u}_{\lambda\mu} \geqslant p\big(d(x)\big)g(\bar{u}_{\lambda\mu}) + \lambda \bar{u}_{\lambda\mu} + \mu|\nabla \bar{u}_{\lambda\mu}|^a \quad \text{in } \Omega.$$

Hence, $(\underline{u}_{\lambda\mu}, \bar{u}_{\lambda\mu})$ is an ordered pair of sub and super-solution of (70), so there exists a classical solution $u_{\lambda\mu}$ of (70), provided $\mu \geqslant 0$ and $0 < \lambda < \lambda_1$. Assume by contradiction that there exist $\lambda \geqslant \lambda_1$ and $\mu \geqslant 0$ such that the problem (70) has a classical solution $u_{\lambda\mu}$. If $m = \min_{x\in\overline{\Omega}} p(d(x))g(u_{\lambda\mu}) > 0$ it follows that $u_{\lambda\mu}$ is a super-solution of

$$\begin{cases} -\Delta u = \lambda u + m & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{73}$$

Clearly zero is a sub-solution of (73), so there exists a classical solution u of (73) such that $u \leqslant u_{\lambda\mu}$ in Ω. By maximum principle and elliptic regularity we get $u > 0$ in Ω and $u \in C^2(\overline{\Omega})$. To raise a contradiction, we proceed as in the proof of Theorem 2.2(ii).

Multiplying by φ_1 in (73) and then integrating over Ω, we find:

$$-\int_\Omega \varphi_1 \Delta u \,\mathrm{d}x = \lambda \int_\Omega u\varphi_1 \,\mathrm{d}x + m\int_\Omega \varphi_1 \,\mathrm{d}x.$$

This implies $\lambda_1 \int_\Omega u\varphi_1 \,\mathrm{d}x = \lambda \int_\Omega u\varphi_1 \,\mathrm{d}x + m\int_\Omega \varphi_1 \,\mathrm{d}x$, which is a contradiction, since $\lambda \geqslant \lambda_1$ and $m > 0$. The proof of Theorem 3.4 is now complete. □

3.4. An application

We are concerned in this section with problem (6). Recall that if $\int_0^1 tp(t)\,\mathrm{d}t < +\infty$ and μ belongs to a certain range, then Theorem 3.2 asserts that (6) has at least one classical solution u_μ satisfying $u_\mu \leqslant MH(c\varphi_1)$ in Ω, for some $M, c > 0$. Here H is the solution of

$$\begin{cases} H''(t) = -t^{-\alpha}H^{-\beta}(t), & \text{for all } 0 < t \leqslant b < 1, \\ H, H' > 0 & \text{in } (0, b], \\ H(0) = 0. \end{cases} \tag{74}$$

With the same idea as in the proof of Theorem 3.2, we can show that there exists $m > 0$ small enough such that $v := mH(c\varphi_1)$ satisfies:

$$-\Delta v \leqslant d(x)^{-\alpha} v^{-\beta} \quad \text{in } \Omega. \tag{75}$$

Indeed, we have:

$$-\Delta v = m\big[c^{2-\alpha}|\nabla\varphi_1|^2\varphi_1^{-\alpha}H^{-\beta}(c\varphi_1) + \lambda_1 c\varphi_1 H'(c\varphi_1)\big] \quad \text{in } \Omega.$$

Using (16) and (38), there exist two positive constants $c_1, c_2 > 0$ such that

$$-\Delta v \leqslant m\big[c_1|\nabla\varphi_1|^2 + c_2\varphi_1\big]d(x)^{-\alpha}H^{-\beta}(c\varphi_1) \quad \text{in } \Omega.$$

Clearly (75) holds if we choose $m > 0$ small enough such that $m[c_1|\nabla\varphi_1|^2 + c_2\varphi_1] < 1$ in Ω. Moreover, v is a sub-solution of (6) for all $\mu > 0$ and one can easily see that $v \leqslant u_\mu$ in Ω. Hence

$$mH(c\varphi_1) \leqslant u_\mu \leqslant MH(c\varphi_1) \quad \text{in } \Omega. \tag{76}$$

Now, a careful analysis of (74) together with (76) is used in order to obtain boundary estimates for the solution of (6). Our estimates complete the results in [20, Theorem 2.1] since here the potential $p(d(x))$ blows-up at the boundary.

Theorem 3.5. *The following properties hold true*:

(i) *If* $\alpha \geqslant 2$, *then the problem* (6) *has no classical solutions.*

(ii) *If* $\alpha < 2$, *then there exists* $\mu^* \in (0,\infty]$ (*with* $\mu^* = +\infty$ *if* $0 < a < 1$) *such that problem* (6) *has at least one classical solution* u_μ, *for all* $-\infty < \mu < \mu^*$. *Moreover, for all* $0 < \mu < \mu^*$, *there exist* $0 < \delta < 1$ *and* $C_1, C_2 > 0$ *such that* u_μ *satisfies*:

(ii1) *If* $\alpha + \beta > 1$, *then*

$$C_1 d(x)^{\frac{2-\alpha}{1+\beta}} \leqslant u_\mu(x) \leqslant C_2 d(x)^{\frac{2-\alpha}{1+\beta}}, \quad \text{for all } x \in \Omega; \tag{77}$$

(ii2) *If* $\alpha + \beta = 1$, *then*

$$C_1 d(x)\big(-\ln d(x)\big)^{\frac{1}{2-\alpha}} \leqslant u_\mu(x) \leqslant C_2 d(x)\big(-\ln d(x)\big)^{\frac{1}{2-\alpha}}, \tag{78}$$

for all $x \in \Omega$ *with* $d(x) < \delta$;

(ii3) *If* $\alpha + \beta < 1$, *then*

$$C_1 d(x) \leqslant u_\mu(x) \leqslant C_2 d(x), \quad \text{for all } x \in \Omega. \tag{79}$$

Proof. The existence and nonexistence of a solution to (6) follows directly from Theorems 3.1 and 3.2. We next prove the boundary estimates (77)–(79).

(ii1) Remark that

$$H(t) = \left(\frac{(1+\beta)^2}{(2-\alpha)(\alpha+\beta-1)}\right)^{1/(1+\beta)} t^{\frac{2-\alpha}{1+\beta}}, \quad t > 0,$$

is a solution of (74) provided $\alpha + \beta > 1$. The conclusion in this case follows now from (76).

(ii2) Note that in this case problem (74) becomes:

$$\begin{cases} H''(t) = -t^{-\alpha} H^{\alpha-1}(t), & \text{for all } 0 < t \leqslant b < 1, \\ H(0) = 0, \\ H > 0 \quad \text{in } (0,b]. \end{cases} \tag{80}$$

Since H is concave, it follows that

$$H(t) > t H'(t), \quad \text{for all } 0 < t \leqslant b. \tag{81}$$

Relations (80) and (81) yield:

$$-H''(t) < t^{-1}\big(H'(t)\big)^{\alpha-1}, \quad \text{for all } 0 < t \leqslant b.$$

Hence

$$-H''(t)\big(H'(t)\big)^{1-\alpha} \leqslant \frac{1}{t}, \quad \text{for all } 0 < t \leqslant b. \tag{82}$$

Integrating in (82) over $[t,b]$ we get:

$$(H')^{2-\alpha}(t) - (H')^{2-\alpha}(b) \leqslant (2-\alpha)(\ln b - \ln t), \quad \text{for all } 0 < t \leqslant b.$$

Hence, there exist $c_1 > 0$ and $\delta_1 \in (0,b)$ such that

$$H'(t) \leqslant c_1(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } 0 < t \leqslant \delta_1. \tag{83}$$

Fix $t \in (0,\delta_1]$. Integrating over $[\varepsilon, t]$, $0 < \varepsilon < t$, in (83), we have:

$$H(t) - H(\varepsilon) \leqslant c_1 t(-\ln t)^{\frac{1}{2-\alpha}} + \frac{c_1}{2-\alpha}\int_\varepsilon^t (-\ln s)^{\frac{\alpha-1}{2-\alpha}}\,\mathrm{d}s. \tag{84}$$

Note that

$$\int_0^t (-\ln s)^{\frac{\alpha-1}{2-\alpha}}\,\mathrm{d}s < +\infty \quad \text{and} \quad \lim_{t\to 0^+} \frac{\int_0^t (-\ln s)^{\frac{\alpha-1}{2-\alpha}}\,\mathrm{d}s}{t(-\ln t)^{\frac{1}{2-\alpha}}} = 0. \tag{85}$$

Therefore, taking $\varepsilon \to 0^+$ in (84) we deduce that there exist $c_2 > 0$ and $\delta_2 \in (0, \delta_1)$ such that

$$H(t) \leqslant c_2 t(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } 0 < t \leqslant \delta_2. \tag{86}$$

From (80) and (86), we obtain:

$$-H''(t) \geqslant c_2^{\alpha-1} t^{-1}(-\ln t)^{\frac{\alpha-1}{2-\alpha}}, \quad \text{for all } 0 < t \leqslant \delta_2.$$

Integrating over $[t, \delta_2]$ in the above inequality, we get:

$$H'(t) \geqslant (2-\alpha)c_2^{\alpha-1}\big[(-\ln t)^{\frac{1}{2-\alpha}} - (-\ln \delta_2)^{\frac{1}{2-\alpha}}\big], \quad \text{for all } 0 < t \leqslant \delta_2.$$

Therefore, there exist $c_3 > 0$ and $\delta_3 \in (0, \delta_2)$ such that

$$H'(t) \geqslant c_3(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } 0 < t \leqslant \delta_3.$$

With the same arguments as in (83)–(86) we obtain $c_4 > 0$ and $\delta_4 \in (0, \delta_3)$ such that

$$H(t) \geqslant c_4 t(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } 0 < t \leqslant \delta_4. \tag{87}$$

The conclusion of (ii) in Theorem 3.5 follows now from (86) and (87).

(ii3) Using the fact that $H'(0+) \in (0, \infty]$ and the inequality (81), we get the existence of $c > 0$ such that

$$H(t) > ct, \quad \text{for all } 0 < t \leqslant b.$$

This yields

$$-H''(t) \leqslant c^{-\beta} t^{-(\alpha+\beta)}, \quad \text{for all } 0 < t \leqslant b.$$

Since $\alpha + \beta < 1$, it follows that $H'(0+) < +\infty$, that is, $H \in C^1[0, b]$. Thus, there exists $c_1, c_2 > 0$ such that

$$c_1 t \leqslant H(t) \leqslant c_2 t, \quad \text{for all } 0 < t \leqslant b. \tag{88}$$

The conclusion in Theorem 3.5(iii) follows directly from (88) and (76).

This completes the proof of Theorem 3.5. □

Acknowledgements

The authors are partially supported by Programme EGIDE-Brancusi PAI 08915PG between Université de Picardie Jules Verne and University of Craiova. M. Ghergu and V. Rădulescu are also supported by Grant 2-CEx06-11-18/2006. The authors also acknowledge the support of CNCSIS grants AT-191/2007 (M. Ghergu) and A-589/2007 (V. Rădulescu).

References

[1] R. Agarwal, D. O'Regan, Existence theory for single and multiple solutions to singular positone boundary value problems, J. Differential Equations 175 (2001) 393–414.
[2] M. Aizenman, B. Simon, Brownian motion and Harnack inequality for Schrödinger operators, Comm. Pure Appl. Math. 35 (1982) 209–273.
[3] N.E. Alaa, M. Pierre, Weak solutions of some quasilinear elliptic equations with data measures, SIAM J. Math. Anal. 24 (1993) 23–35.
[4] P. Bénilan, H. Brezis, M. Crandall, A semilinear equation in $L^1(\mathbb{R}^N)$, Ann. Scuola Norm. Sup. Pisa 4 (1975) 523–555.
[5] C.M. Brauner, B. Nicolaenko, On nonlinear eigenvalue problems which extend into free boundaries, in: Bifurcation and Nonlinear Eigenvalue Problems, Proc., Session, Univ. Paris XIII, Villetaneuse, 1978, in: Lecture Notes in Math., vol. 782, Springer, Berlin/New York, 1980, pp. 61–100.
[6] L. Caffarelli, R. Hardt, L. Simon, Minimal surfaces with isolated singularities, Manuscripta Math. 48 (1984) 1–18.
[7] A. Callegari, A. Nachman, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, SIAM J. Appl. Math. 38 (1980) 275–281.
[8] Y. Choquet-Bruhat, J. Leray, Sur le problème de Dirichlet quasilinéaire d'ordre 2, C. R. Acad. Sci. Paris, Ser. A 274 (1972) 81–85.
[9] F.-C. Cîrstea, M. Ghergu, V. Rădulescu, Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane–Emden–Fowler type, J. Math. Pures Appl. 84 (2005) 493–508.
[10] M.M. Coclite, G. Palmieri, On a singular nonlinear Dirichlet problem, Comm. Partial Differential Equations 14 (1989) 1315–1327.
[11] D.S. Cohen, H.B. Keller, Some positive problems suggested by nonlinear heat generators, J. Math. Mech. 16 (1967) 1361–1376.

[12] M.G. Crandall, P.H. Rabinowitz, L. Tartar, On a Dirichlet problem with a singular nonlinearity, Comm. Partial Differential Equations 2 (1977) 193–222.
[13] J.I. Díaz, Nonlinear Partial Differential Equations and Free Boundaries, vol. I. Elliptic Equations, Research Notes in Mathematics, vol. 106, Pitman (Advanced Publishing Program), Boston, MA, 1985.
[14] J.I. Díaz, J.M. Morel, L. Oswald, An elliptic equation with singular nonlinearity, Comm. Partial Differential Equations 12 (1987) 1333–1344.
[15] M. Ghergu, V. Rădulescu, Sublinear singular elliptic problems with two parameters, J. Differential Equations 195 (2003) 520–536.
[16] M. Ghergu, V. Rădulescu, On a class of sublinear singular elliptic problems with convection term, J. Math. Anal. Appl. 311 (2005) 635–646.
[17] M. Ghergu, V. Rădulescu, Multiparameter bifurcation and asymptotics for the singular Lane–Emden–Fowler equation with convection term, Proc. Royal Soc. Edinburgh Sect. A 135 (2005) 61–84.
[18] M. Ghergu, V. Rădulescu, Ground state solutions for the singular Lane–Emden–Fowler equation with sublinear convection term, J. Math. Anal. Appl., in press, doi:10.1016/j.jmaa.2006.09.074.
[19] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer-Verlag, Berlin/Heidelberg/New York, 1983.
[20] C. Gui, F.H. Lin, Regularity of an elliptic problem with a singular nonlinearity, Proc. Royal Soc. Edinburgh Sect. A 123 (1993) 1021–1029.
[21] Y. Haitao, Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem, J. Differential Equations 189 (2003) 487–512.
[22] J. Hernández, F.J. Mancebo, J.M. Vega, On the linearization of some singular nonlinear elliptic problems and applications, Ann. Inst. H. Poincaré, Anal. Non Linéaire 19 (2002) 777–813.
[23] J. Hernández, F.J. Mancebo, J.M. Vega, Nonlinear singular elliptic problems: recent results and open problems, in: Nonlinear Elliptic and Parabolic Problems, in: Progr. Nonlinear Differential Equations Appl., vol. 64, Birkhäuser, Basel, 2005, pp. 227–242.
[24] J. Kazdan, F.W. Warner, Remarks on some quasilinear elliptic equations, Comm. Pure Appl. Math. 28 (1975) 567–597.
[25] A. Meadows, Stable and singular solutions of the equation $\Delta u = 1/u$, Indiana Univ. Math. J. 53 (2004) 1681–1703.
[26] N.B. Rhouma, M. Mosbah, On the existence of positive solutions for semilinear elliptic equations with indefinite nonlinearities, J. Differential Equations 215 (2005) 37–51.
[27] J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math. 111 (1964) 247–302.
[28] J. Shi, M. Yao, On a singular nonlinear semilinear elliptic problem, Proc. Royal Soc. Edinburgh Sect. A 128 (1998) 1389–1401.
[29] J. Shi, M. Yao, Positive solutions for elliptic equations with singular nonlinearity, Electron. J. Differential Equations 4 (2005) 1–11.
[30] S.D. Taliaferro, A nonlinear singular boundary value problem, Nonlinear Anal. 3 (1979) 897–904.
[31] Z. Zhang, The existence and asymptotic behaviour of the unique solution near the boundary to a singular Dirichlet problem with a convection term, Proc. Royal Soc. Edinburgh Sect. A 136 (2006) 209–222.
[32] Z. Zhang, J. Cheng, Existence and optimal estimates of solutions for singular nonlinear Dirichlet problems, Nonlinear Anal. 57 (2004) 473–484.