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A NONSMOOTH CRITICAL POINT THEORY APPROACH TO SOME NONLINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^n$

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Abstract. We determine nontrivial solutions of some semilinear and quasilinear elliptic problems on $\mathbb{R}^n$; we make use of two different nonsmooth critical point theories which allow to treat two kinds of nonlinear problems. A comparison between the possible applications of the two theories is also made.

1. Introduction. Consider a functional J defined on some Banach space B and having a mountain-pass geometry: the celebrated theorem by Ambrosetti-Rabinowitz [1] states that if $J \in C^1(B)$ and J satisfies the Palais-Smale condition (PS condition in the sequel) then J admits a nontrivial critical point. In this paper we drop these two assumptions: in order to determine nontrivial solutions of some nonlinear elliptic equations in $\mathbb{R}^n$ $(n \geq 3)$, we use the mountain-pass principle for a class of nonsmooth functionals which do not satisfy the PS condition. More precisely, we consider a model elliptic problem first studied by Rabinowitz [13] with the C^1-theory and we extend his results by means of the nonsmooth critical point theories of Clarke [5, 6] and Degiovanni et al. [8, 9]; one of the purposes of this paper is to emphasize some differences between these two theories. This study was inspired by previous work on the existence of standing wave solutions of nonlinear Schrödinger equations: after making a standing wave ansatz, Rabinowitz reduces the problem to that of studying the semilinear elliptic equation

$$-\Delta u + b(x)u = f(x, u) \qquad \text{in } \mathbb{R}^n \tag{1}$$

under suitable conditions on b and assuming that f is smooth, superlinear and subcritical.

To explain our results we introduce some functional spaces. We denote by L^p the space of measurable functions u of p-th power absolutely summable on $\mathbb{R}^n$, that

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is, satisfying $\|u\|_p^p := \int_{\mathbb{R}^n} |u|^p < +\infty$; by H^1 we denote the Sobolev space normed by $\|u\|_{H^1}^2 := \int_{\mathbb{R}^n}(|Du|^2 + |u|^2)$. We will assume that the function b in (1) is greater than some positive constant; then we define the Hilbert space E of all functions $u : \mathbb{R}^n \to \mathbb{R}$ with $\|u\|_E^2 := \int_{\mathbb{R}^n}(|Du|^2 + b(x)u^2) < \infty$. We denote by E^* the dual space of E: as E is continuously embedded in H^1 we also have $H^{-1} \subset E^*$.

We first consider the case where $(-\Delta)$ in (1) is replaced by a quasilinear elliptic operator: we seek positive weak solutions $u \in E$ of the problem

$$-\sum_{i,j=1}^n D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju + b(x)u = f(x,u) \quad \text{in } \mathbb{R}^n. \tag{2}$$

Note that if $a_{ij}(x,s) \equiv \delta_{ij}$, then (2) reduces to (1). Here and in the sequel, by positive solution we mean a nonnegative nontrivial solution. To determine weak solutions of (2) we look for critical points of the functional $J : E \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n}\sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju + \frac{1}{2}\int_{\mathbb{R}^n} b(x)u^2 - \int_{\mathbb{R}^n} F(x,u) \quad \forall u \in E,$$

where $F(x,s) = \int_0^s f(x,t)dt$. Under reasonable assumptions on a_{ij}, b, f, the functional J is continuous but not even locally Lipschitz, see [3], therefore, we cannot work in the classical framework of critical point theory. Nevertheless, the Gâteaux-derivative of J exists in the smooth directions, i.e., for all $u \in E$ and $\varphi \in C_c^\infty$ we can define

$$J'(u)[\varphi] = \int_{\mathbb{R}^n}\Big(\sum_{i,j=1}^n \big[a_{ij}(x,u)D_iuD_j\varphi + \frac{1}{2}\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju\varphi\big] + b(x)u\varphi - f(x,u)\varphi\Big).$$

According to the nonsmooth critical point theory developed in [8, 9] we know that critical points u of J satisfy $J'(u)[\varphi] = 0$ for all $\varphi \in C_c^\infty$ and hence solve (2) in distributional sense; moreover, since

$$-\sum_{i,j=1}^n D_j(a_{ij}(x,u)D_iu) + b(x)u - f(x,u) \in E^*$$

we also have

$$\frac{1}{2}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju \in E^*$$

and (2) is solved in the weak sense ($\forall \varphi \in E$). We refer to [3] for the adaptation of this theory to quasilinear equations of the kind of (2) and to [7, 10] for applications in the case of unbounded domains and for further references. Under suitable assumptions on a_{ij}, b, f and by using the above mentioned tools we will prove that (2) admits a positive weak solution. Next, we take into account the case where f is not continuous: let $f(x,\cdot) \in L^\infty_{\text{loc}}(\mathbb{R})$ and denote

$$\underline{f}(x,s) = \lim_{\varepsilon \searrow 0} \ \text{essinf} \ \{f(x,t);\ |t-s| < \varepsilon\}$$
$$\overline{f}(x,s) = \lim_{\varepsilon \searrow 0} \ \text{esssup} \ \{f(x,t);\ |t-s| < \varepsilon\};$$

our aim is to determine $u \in E$ such that

$$-\Delta u + b(x)u \in [\underline{f}(x,u), \overline{f}(x,u)] \quad \text{in } \mathbb{R}^n. \tag{3}$$

Positive solutions u of (3) satisfy $0 \in \partial I(u)$, where

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n} (|Du|^2 + b(x)u^2) - \int_{\mathbb{R}^n} F(x,u^+) \qquad \forall u \in E$$

and $\partial I(u)$ stands for the Clarke gradient [5, 6] of the locally Lipschitz energy functional I; more precisely,

$$\partial I(u) = \big\{\zeta \in E^*;\ I^0(u;v) \geq \langle \zeta, v\rangle, \quad \forall v \in E\big\},$$

where

$$I^0(u;v) = \limsup_{\substack{w \to u \\ \lambda \searrow 0}} \frac{I(w+\lambda v) - I(w)}{\lambda}. \tag{4}$$

This problem may be reformulated, equivalently, in terms of hemivariational inequalities as follows: find $u \in E$ such that

$$\int_{\mathbb{R}^n} (DuDv + b(x)uv) + \int_{\mathbb{R}^n} (-F)^0(x,u;v) \geq 0 \qquad \forall v \in E, \tag{5}$$

where $(-F)^0(x,u;v)$ denotes the Clarke directional derivative of $(-F)$ at $u(x)$ with respect to $v(x)$ and is defined as in (4). So, when $f(x,\cdot)$ is not continuous, Clarke's theory will enable us to prove that (3) admits a positive solution. The two existence results stated in next section have several points in common; in both cases we first prove that the corresponding functional has a mountain-pass geometry and that a PS sequence can be built at a suitable inf-max level. Then we prove that the PS sequence is bounded and that its weak limit is a solution of the problem considered; the final step is to prove that this solution is not the trivial one; to this end we use the concentration-compactness principle [11] and the behavior of the function b at infinity. However, the construction of a PS sequence and the proof that its weak limit is a solution are definitely different. They highlight the different tools existing in the two theories.

2. Main results. Let us first state our results concerning (2). We require the coefficients a_{ij} $(i,j=1,...,n)$ to satisfy

$$\begin{cases} a_{ij} \equiv a_{ji} \\ a_{ij}(x,\cdot) \in C^1(\mathbb{R}) \quad \text{for a.e. } x \in \mathbb{R}^n \\ a_{ij}(x,s),\ \frac{\partial a_{ij}}{\partial s}(x,s) \in L^\infty(\mathbb{R}^n \times \mathbb{R}); \end{cases} \tag{6}$$

moreover, on the matrices $[a_{ij}(x,s)]$ and $[s\frac{\partial a_{ij}}{\partial s}(x,s)]$ we make the following assumptions:

$$\exists \nu > 0 \ \text{ such that } \sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j \geq \nu|\xi|^2 \text{ for a.e. } x \in \mathbb{R}^n,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^n \tag{7}$$

$$\begin{cases} \exists \mu \in (2,2^*), \quad \gamma \in (0,\mu-2) \quad \text{such that} \\ 0 \leq s\displaystyle\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,s)\xi_i\xi_j \leq \gamma \sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j \text{ for a.e. } x \in \mathbb{R}^n, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^n. \end{cases} \tag{8}$$

We require that $b \in L^\infty_{\rm loc}(\mathbb{R}^n)$ and that

$$\begin{cases} \exists \underline{b} > 0 \quad \text{such that} \quad b(x) \geq \underline{b} \quad \text{for a.e. } x \in \mathbb{R}^n \\ \text{ess} \displaystyle\lim_{|x|\to\infty} b(x) = +\infty. \end{cases} \tag{9}$$

Let μ be as in (8), assume that $f(x,s) \not\equiv 0$ and

$$\begin{cases} f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function} \\ f(x,0) = 0 \quad \text{for a.e. } x \in \mathbb{R}^n \\ 0 \leq \mu F(x,s) \leq sf(x,s)\ \forall s \geq 0 \quad \text{and for a.e. } x \in \mathbb{R}^n; \end{cases} \tag{10}$$

moreover, we require f to be subcritical

$$\begin{cases} \forall \varepsilon > 0 \quad \exists f_\varepsilon \in L^{\frac{2n}{n+2}}(\mathbb{R}^n) \text{ such that} \\ |f(x,s)| \leq f_\varepsilon(x) + \varepsilon|s|^{\frac{n+2}{n-2}}\ \forall s \in \mathbb{R} \text{ and for a.e. } x \in \mathbb{R}^n. \end{cases} \tag{11}$$

Finally, for all $\delta \in (2,2^*)$ define $q(\delta) = \frac{2n}{2n+(2-n)\delta}$: then we assume[1]

$$\begin{cases} \exists C \geq 0, \quad \exists \delta \in (2,2^*)\ , \quad \exists G \in L^{q(\delta)}(\mathbb{R}^n) \quad \text{such that} \\ F(x,s) \leq G(x)|s|^\delta + C|s|^{2^*} \quad \forall s \in \mathbb{R} \text{ and for a.e. } x \in \mathbb{R}^n. \end{cases} \tag{12}$$

In Section 3 we will prove

[1] One could also consider the case $\delta = 2$: in such case one also needs $\|G\|_{n/2}$ to be sufficiently small.

Theorem 1. *Assume* (6)–(12); *then* (2) *admits a positive weak solution* $\bar{u} \in E$.

Let us turn to the problem (3). We assume that $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a (nontrivial) measurable function such that

$$|f(x,s)| \leq C(|s| + |s|^p) \quad \text{for a.e. } (x,s) \in \mathbb{R}^n \times \mathbb{R}, \tag{13}$$

where C is a positive constant and $1 < p < \frac{n+2}{n-2}$. Here we do not assume that $f(x,\cdot)$ is continuous: nevertheless, if we define $F(x,s) = \int_0^s f(x,t)dt$ we observe that F is a Carathéodory function which is locally Lipschitz with respect to the second variable. We also observe that the functional

$$\Psi(u) = \int_{\mathbb{R}^n} F(x,u)$$

is locally Lipschitz on E. Indeed, by (13), Hölder's inequality and the embedding $E \subset L^{p+1}$,

$$|\Psi(u) - \Psi(v)| \leq C(\|u\|_E, \|v\|_E)\|u - v\|_E,$$

where $C(\|u\|_E, \|v\|_E) > 0$ depends only on $\max\{\|u\|_E, \|v\|_E\}$.

We impose to f the following additional assumptions

$$\lim_{\varepsilon \searrow 0} \text{ esssup} \left\{ \left| \frac{f(x,s)}{s} \right| ; (x,s) \in \mathbb{R}^n \times (-\varepsilon, \varepsilon) \right\} = 0 \tag{14}$$

and there exists $\mu > 2$ such that

$$0 \leq \mu F(x,s) \leq s\underline{f}(x,s) \quad \text{for a.e. } (x,s) \in \mathbb{R}^n \times [0, +\infty). \tag{15}$$

In Section 4 we will prove

Theorem 2. *Under hypotheses* (9), (13)–(15), *problem* (3) *has at least a positive solution in* E.

Remark. The couple of assumptions (11), (12) is equivalent to the couple (13), (14) in the sense that Theorems 1 and 2 hold under any one of these couples of assumptions.

It seems not possible to use the above mentioned nonsmooth critical point theories to obtain an existence result for the quasilinear operator of (2) in the presence of a function f which is discontinuous with respect to the second variable; indeed, to prove that critical points of J (in the sense of [8, 9]) solve (2) in distributional sense, one needs, for all given $\varphi \in C_c^\infty$, the continuity of the map $u \mapsto J'(u)[\varphi]$, see [3]. Even if $J \notin C^1(E)$, we have at least $J \in C^1(W^{1,p} \cap E)$ for $p \geq \frac{3n}{n+1}$: this smoothness property in a finer topology is in fact the basic (hidden) tool used in Theorem 1.5 in [3]; however, one cannot prove the boundedness of the PS sequences in the $W^{1,p}$ norm. On the other hand, the theory developed in [5, 6] only applies to Lipschitz continuous functionals and therefore it does not allow to manage quasilinear operators as that in (2).

3. Proof of Theorem 1. Throughout this section we assume (6)–(12). From (6) and (8) we have

$$u \in E \Longrightarrow \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u u \in L^1(\mathbb{R}^n) \tag{16}$$

and therefore $J'(u)[u]$ can be written in integral form.

We first remark that positive solutions of (2) correspond to critical points of the functional J_+ defined by

$$J_+(u) := \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij}(x,u) D_i u D_j u + \frac{1}{2}\int_{\mathbb{R}^n} b(x)u^2 - \int_{\mathbb{R}^n} F(x,u^+) \quad \forall u \in E,$$

where u^+ denotes the positive part of u, i.e. $u^+(x) = \max(u(x), 0)$.

Lemma 1. *Let $u \in E$ satisfy $J'_+(u)[\varphi] = 0$ for all $\varphi \in C_c^\infty$; then u is a weak positive solution of* (2).

For the proof of this result we refer to [7]; without loss of generality we can therefore suppose that

$$f(x,s) = 0 \quad \forall s \le 0, \quad \text{for a.e. } x \in \mathbb{R}^n$$

and, from now on, we make this assumption; for simplicity we denote J instead of J_+.

Let us establish the following boundedness criterion which applies, in particular, to PS sequences[2]:

Lemma 2. *Every sequence $\{u_m\} \subset E$ satisfying*

$$|J(u_m)| \le C_1 \quad \textit{and} \quad |J'(u_m)[u_m]| \le C_2 \|u_m\|_E$$

is bounded in E.

Proof. Consider $\{u_m\} \subset E$ such that $|J(u_m)| \le C_1$, then by (10) we get

$$I_m := \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij}(x,u_m) D_i u_m D_j u_m - \frac{1}{\mu}\int_{\mathbb{R}^n} f(x,u_m)u_m + \frac{1}{2}\int_{\mathbb{R}^n} b(x)u_m^2 \le C_1;$$

[2] We refer to [3, 8, 9] for the definition of PS sequences in our nonsmooth critical point framework.

by (16) we can evaluate $J'(u_m)[u_m]$ and by the assumptions we have

$$\begin{aligned} &\Big|\int_{\mathbb{R}^n}\sum_{i,j=1}^n a_{ij}(x,u_m)D_iu_mD_ju_m+\frac{1}{2}\int_{\mathbb{R}^n}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_mu_m \\ &\qquad -\int_{\mathbb{R}^n}f(x,u_m)u_m+\int_{\mathbb{R}^n}b(x)u_m^2\Big|\le C_2\|u_m\|_E. \end{aligned}$$

Therefore, by (8) and computing $I_m-\frac{1}{\mu}J'(u_m)[u_m]$ we get

$$\frac{\mu-2-\gamma}{2\mu}\int_{\mathbb{R}^n}\sum_{i,j=1}^n a_{ij}(x,u_m)D_iu_mD_ju_m+\frac{\mu-2}{2\mu}\int_{\mathbb{R}^n}b(x)u_m^2\le C_3\|u_m\|_E+C_1;$$

by (7) this yields $C_4>0$ such that $C_4\|u_m\|_E^2\le C_3\|u_m\|_E+C_1$ and the result follows. $\square$

Let us denote by E_{loc} the space of functions u satisfying $\int_\omega(|Du|^2+b(x)u^2)<\infty$ for all bounded open set $\omega\subset\mathbb{R}^n$ and by E^*_{loc} its dual space; we establish that the weak limit of a PS sequence solves (2):

Lemma 3. *Let $\{u_m\}$ be a bounded sequence in E satisfying*

$$\int_{\mathbb{R}^n}\sum_{i,j=1}^n a_{ij}(x,u_m)D_iu_mD_j\varphi+\frac{1}{2}\int_{\mathbb{R}^n}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_m\varphi=\langle\beta_m,\varphi\rangle$$

$\forall\varphi\in C_c^\infty$ with $\{\beta_m\}$ converging in E^*_{loc} to some $\beta\in E^*_{\mathrm{loc}}$. Then, up to a subsequence, $\{u_m\}\subset E$ converges in E_{loc} to some $u\in E$ satisfying

$$\int_{\mathbb{R}^n}\sum_{i,j=1}^n a_{ij}(x,u)D_iuD_j\varphi+\frac{1}{2}\int_{\mathbb{R}^n}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju\varphi=\langle\beta,\varphi\rangle\ \ \forall\varphi\in C_c^\infty.$$

Proof. As b is uniformly positive and locally bounded, for all bounded open set $\omega\subset\mathbb{R}^n$ we have

$$\int_\omega(|Du|^2+b(x)u^2)<\infty\quad\Longleftrightarrow\quad\int_\omega(|Du|^2+u^2)<\infty;$$

therefore, the proof is essentially the same as Lemma 3 in [7]. The basic tool is Theorem 2.1 in [2] which is used following the idea of [3]. $\square$

The previous results allow to prove

Proposition 1. *Assume that* $\{u_m\} \subset E$ *is a PS sequence for* J*; then there exists* $\bar{u} \in E$ *such that* (*up to a subsequence*)

(i) $u_m \rightharpoonup \bar{u}$ *in* E

(ii) $u_m \to \bar{u}$ *in* E_{loc}

(iii) $\bar{u} \geq 0$ *and* $\bar{u}$ *solves* (2) *in weak sense.*

Proof. By Lemma 2, the sequence $\{u_m\}$ is bounded and (i) follows. To obtain (ii) it suffices to apply Lemma 3 with $\beta_m = \alpha_m + f(x, u_m) - b(x)u_m \in E^*$ where $\alpha_m \to 0$ in E^*: indeed, if $u_m \rightharpoonup u$ in E, then $\beta_m \to \beta$ in E^*_{loc} with $\beta = f(x, u) - b(x)u$. Finally, (iii) follows from Lemmas 1 and 3. □

In order to build a PS sequence for the functional J we apply the mountain-pass Lemma [1] in the nonsmooth version [dm], see also Theorem 2.1 in [9]. Let us check that J has such a geometrical structure.

First note that $J(0) = 0$; as the function F is superquadratic at $+\infty$, we may choose a nonnegative function e such that

$$e \in C_c^\infty, \quad e \geq 0 \quad \text{and} \quad J(te) < 0 \quad \forall t > 1.$$

Moreover, it is easy to check that there exist $\rho, \beta > 0$ such that $\rho < \|e\|_E$ and $J(u) \geq \beta$ if $\|u\|_E = \rho$: indeed by (12) we infer

$$\int_{\mathbb{R}^n} F(x, u) \leq \|G\|_{q(\delta)} \|u\|_{2^*}^\delta + C\|u\|_{2^*}^{2^*};$$

hence, by (7) we have $J(u) \geq C_1\|u\|_E^2 - C_2\|u\|_E^\delta - C_3\|u\|_E^{2^*}$ and the existence of ρ, β follows.

So, J has a mountain pass geometry; if we define the class

$$\Gamma := \{\gamma \in C([0, 1];\ E);\ \gamma(0) = 0,\ \gamma(1) = e\} \tag{17}$$

and the minimax value

$$\alpha := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)), \tag{18}$$

the existence of a PS sequence for J at level α follows by the results of [8, 9].

We have so proved

Proposition 2. *Let* Γ *and* α *be as in* (17), (18)*; then* J *admits a PS sequence* $\{u_m\}$ *at level* α.

As we are on an unbounded domain, the problem lacks compactness and we cannot infer that the above PS sequence converges strongly; however, by using Proposition 1, the weak limit $\bar{u}$ of the PS sequence is a nonnegative solution of (2): the main problem is that it could be $\bar{u} \equiv 0$. To prove that this is not the case we make use of the following technical result:

Lemma 4. *There exist $p \in (2, 2^*)$ and $C > 0$ such that $\|u_m^+\|_p \geq C$.*

Proof. Using the relations $J'(u_m)[u_m] = o(1)$ and $J(u_m) = \alpha + o(1)$, by assumptions (8) and (10) we have

$$
\begin{aligned}
2\alpha &= 2J(u_m) - J'(u_m)[u_m] + o(1) = \int_{\mathbb{R}^n} \left[f(x, u_m^+)u_m - 2F(x, u_m^+)\right] \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x, u_m) D_i u_m D_j u_m u_m + o(1) \\
&\leq \int_{\mathbb{R}^n} f(x, u_m^+)u_m + o(1).
\end{aligned}
$$

Then, by (11), for all $\varepsilon > 0$ there exists $f_\varepsilon \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$ such that

$$2\alpha \leq \int_{\mathbb{R}^n} |f_\varepsilon(x) u_m^+(x)| + \varepsilon \|u_m^+\|_{2^*}^{2^*} :$$

as $\|u_m\|_{2^*}$ is bounded, one can choose $\varepsilon > 0$ so that

$$\alpha \leq \int_{\mathbb{R}^n} |f_\varepsilon(x) u_m^+(x)|. \tag{19}$$

Now take $r \in (\frac{2n}{n+2}, 2)$: then for all $\delta > 0$ there exist $f_\delta \in L^r$ and $f^\delta \in L^{\frac{2n}{n+2}}$ such that

$$f_\varepsilon = f_\delta + f^\delta \quad \text{and} \quad \|f^\delta\|_{\frac{2n}{n+2}} \leq \delta.$$

Then, by (19) and Hölder's inequality we infer

$$\alpha \leq \|f_\delta\|_r \|u_m^+\|_p + \delta \|u_m^+\|_{2^*},$$

where $p = \frac{r}{r-1}$; as $\|u_m\|_{2^*}$ is bounded, one can choose $\delta > 0$ so that

$$\frac{\alpha}{2} \leq \|f_\delta\|_r \|u_m^+\|_p$$

and the result follows. $\square$

By the previous Lemma we deduce that $\{u_m^+\}$ does not converge strongly to 0 in L^p. Taking into account that $\|u_m^+\|_2$ and $\|\nabla u_m^+\|_2$ are bounded, by Lemma I.1 p. 231 in [11], we infer that the sequence $\{u_m^+\}$ "does not vanish" in L^2, i.e., there exists a sequence $\{y_m\} \subset \mathbb{R}^n$ and $C > 0$ such that

$$\int_{y_m + B_R} |u_m^+|^2 \geq C \tag{20}$$

for some R. We claim that the sequence $\{y_m\}$ is bounded: if not, up to a subsequence, it follows by (9) that

$$\int_{\mathbb{R}^n} b(x)u_m^2 \to +\infty$$

which contradicts $J(u_m) = \alpha + o(1)$. Therefore, by (20), there exists an open bounded set $\omega \subset \mathbb{R}^n$ such that

$$\int_\omega |u_m|^2 \geq C > 0. \tag{21}$$

So, consider the PS sequence found in Proposition 2; by Proposition 1, it converges in the L^2_{loc} topology to some nonnegative function $\bar{u}$ which solves (2) in weak sense; finally, (21) entails $\bar{u} \not\equiv 0$.

The proof of Theorem 1 is complete.

4. Proof of Theorem 2. In this section we assume (9) and (13)–(15); moreover, we set $f(x,s) \equiv 0$ for $s \leq 0$.

To prove Theorem 2, it is sufficient to show that the functional I has a critical point $u_0 \in \mathcal{C}$, $\mathcal{C}$ being the cone of positive functions of E. Indeed,

$$\partial I(u) = -\Delta u + b(x)u - \partial\Psi(u) \quad \text{in } E^*,$$

and, by Theorem 2.2 of [4] and Theorem 3 of [12], we have

$$\partial\Psi(u) \subset [\underline{f}(x,u(x)), \overline{f}(x,u(x))] \quad \text{for a.e. } x \in \mathbb{R}^n,$$

in the sense that if $w \in \partial\Psi(u)$ then

$$\underline{f}(x,u(x)) \leq w(x) \leq \overline{f}(x,u(x)) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{22}$$

Thus, if u_0 is a critical point of I, then there exists $w \in \partial\Psi(u_0)$ such that

$$-\Delta u_0 + b(x)u_0 = w \quad \text{in } E^*.$$

The existence of u_0 will be justified by a nonsmooth variant of the mountain-pass Lemma (see Theorem 1 of [14]), even if the PS condition is not fulfilled. More precisely, we verify the following geometric hypotheses:

$$I(0) = 0 \text{ and } \exists v \in E \text{ such that } I(v) \leq 0 \tag{23}$$

$$\exists \beta,\ \rho > 0 \text{ such that } I \geq \beta \text{ on } \{u \in E;\ \|u\|_E = \rho\}. \tag{24}$$

Verification of (23). It is obvious that $I(0) = 0$. For the second assertion we need

Lemma 5. *There exist two positive constants C_1 and C_2 such that*

$$f(x,s) \geq C_1 s^{\mu-1} - C_2 \quad \textit{for a.e.} \quad (x,s) \in \mathbb{R}^n \times [0,+\infty). \tag{25}$$

Proof. From the definition we clearly have

$$\underline{f}(x,s) \leq f(x,s) \quad \text{a.e. in } \mathbb{R}^n \times [0,+\infty). \tag{26}$$

Then, by (15),

$$0 \leq \mu \underline{F}(x,s) \leq s\underline{f}(x,s) \quad \text{for a.e. } (x,s) \in \mathbb{R}^n \times [0,+\infty), \tag{27}$$

where

$$\underline{F}(x,s) = \int_0^s \underline{f}(x,t)dt.$$

By (27), there exist $R > 0$ and $K_1 > 0$ such that

$$\underline{F}(x,s) \geq K_1 s^{\mu} \quad \text{for a.e. } (x,s) \in \mathbb{R}^n \times [R,+\infty). \tag{28}$$

The inequality (25) follows now by (26), (27) and (28). □

Verification of (23) *continued.* Choose $v \in C_c^\infty(\mathbb{R}^n) \setminus \{0\}$ so that $v \geq 0$ in $\mathbb{R}^n$; we obviously have

$$\int_{\mathbb{R}^n} (|Dv|^2 + b(x)v^2) < +\infty.$$

Then, by Lemma 5,

$$\begin{aligned} I(tv) &= \frac{t^2}{2} \int_{\mathbb{R}^n} (|Dv|^2 + b(x)v^2) - \Psi(tv) \\ &\leq \frac{t^2}{2} \int_{\mathbb{R}^n} (|Dv|^2 + b(x)v^2) + C_2 t \int_{\mathbb{R}^n} v - C_1' t^{\mu} \int_{\mathbb{R}^n} v^{\mu} < 0, \end{aligned}$$

for $t > 0$ large enough.

Verification of (24). First observe that (13) and (14) imply that, for any $\varepsilon > 0$, there exists a constant A_ε such that

$$|f(x,s)| \leq \varepsilon |s| + A_\varepsilon |s|^p \quad \text{for a.e. } (x,s) \in \mathbb{R}^n \times \mathbb{R}. \tag{29}$$

By (29) and Sobolev's embedding Theorem we have, for any $u \in E$

$$\Psi(u) \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^n} u^2 + \frac{A_\varepsilon}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} \leq \varepsilon C_3 \, \|u\|_E^2 + C_4 \, \|u\|_E^{p+1},$$

where ε is arbitrary and $C_4 = C_4(\varepsilon)$. Thus, by (9)

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n}(|Du|^2 + b(x)u^2) - \Psi(u) \geq C_5\|u\|_E^2 - \varepsilon C_3\|u\|_E^2 - C_4\|u\|_E^{p+1} \geq \beta > 0,$$

for $\|u\|_E = \rho$, with ρ, ε and β sufficiently small positive constants. Denote

$$\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = 0,\ \gamma(1) \neq 0 \text{ and } I(\gamma(1)) \leq 0\}$$

and

$$c = \inf_{\gamma\in\Gamma}\ \max_{t\in[0,1]}\ I(\gamma(t)).$$

Set

$$\lambda_I(u) = \min_{\zeta\in\partial I(u)} \|\zeta\|_{E^*}.$$

Then, by Theorem 1 of [14], there exists a sequence $\{u_m\} \subset E$ such that

$$I(u_m) \to c \qquad \text{and} \qquad \lambda_I(u_m) \to 0; \tag{30}$$

since $I(|u|) \leq I(u)$ for all $u \in E$ we may assume that $\{u_m\} \subset \mathcal{C}$. So, there exists a sequence $\{w_m\} \subset \partial\Psi(u_m) \subset E^*$ such that

$$-\Delta u_m + b(x)u_m - w_m \to 0 \qquad \text{in } E^*. \tag{31}$$

Note that for all $u \in \mathcal{C}$, by (15) we have

$$\Psi(u) \leq \frac{1}{\mu}\int_{\mathbb{R}^n} u(x)\underline{f}(x, u(x)).$$

Therefore, by (22), for every $u \in \mathcal{C}$ and any $w \in \partial\Psi(u)$,

$$\Psi(u) \leq \frac{1}{\mu}\int_{\mathbb{R}^n} u(x)w(x).$$

Hence, if $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E^* and E, we have

$$\begin{aligned}
I(u_m) =&\frac{\mu-2}{2\mu}\int_{\mathbb{R}^n}(|Du_m|^2 + b(x)u_m^2)\\
&+\frac{1}{\mu}\langle -\Delta u_m + bu_m - w_m, u_m\rangle + \frac{1}{\mu}\langle w_m, u_m\rangle - \Psi(u_m)\\
\geq&\frac{\mu-2}{2\mu}\int_{\mathbb{R}^n}(|Du_m|^2 + b(x)u_m^2) + \frac{1}{\mu}\langle -\Delta u_m + bu_m - w_m, u_m\rangle\\
\geq&\frac{\mu-2}{2\mu}\ \|u_m\|_E^2 - o(1)\ \|u_m\|_E.
\end{aligned}$$

This, together with (30), implies that the Palais-Smale sequence $\{u_m\}$ is bounded in E: thus, it converges weakly (up to a subsequence) in E and strongly in L^2_{loc} to some $u_0 \in \mathcal{C}$. Taking into account that $w_m \in \partial\Psi(u_m)$ for all m, that $u_m \rightharpoonup u_0$ in E and that there exists $w_0 \in E^*$ such that $w_m \rightharpoonup w_0$ in E^* (up to a subsequence), we infer that $w_0 \in \partial\Psi(u_0)$: this follows from the fact that the map $u \mapsto F(x,u)$ is compact from E into L^1. Moreover, if we take $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and let $\Omega := \mathrm{supp}\varphi$, then by (31) we get

$$\int_\Omega (Du_0 D\varphi + b(x)u_0\varphi - w_0\varphi) = 0;$$

as $w_0 \in \partial\Psi(u_0)$, by using (4) p.104 in [4] and by definition of $(-F)^0$, this implies

$$\int_\Omega (Du_0 D\varphi + b(x)u_0\varphi) + \int_\Omega (-F)^0(x, u_0; \varphi) \geq 0.$$

By density, this hemivariational inequality holds for all $\varphi \in E$ and (5) follows; this means that u_0 solves problem (3).

It remains to prove that $u_0 \not\equiv 0$. If w_m is as in (31), then by (15) (recall that $u_m \in \mathcal{C}$) and (30) (for large m) we get

$$\begin{aligned} \frac{c}{2} &\leq I(u_m) - \frac{1}{2}\ \langle -\Delta u_m + bu_m - w_m, u_m\rangle \\ &= \frac{1}{2}\ \langle w_m, u_m\rangle - \int_{\mathbb{R}^n} F(x,u_m) \leq \frac{1}{2}\ \int_{\mathbb{R}^n} u_m \overline{f}(x,u_m). \end{aligned} \tag{32}$$

Now, taking into account its definition, one deduces that $\overline{f}$ verifies (29), too. So, by (32), we obtain

$$\frac{c}{2} \leq \frac{1}{2}\int_{\mathbb{R}^n} (\varepsilon|u_m|^2 + A_\varepsilon|u_m|^{p+1}) = \frac{\varepsilon}{2}\ \|u_m\|_2^2 + \frac{A_\varepsilon}{2}\ \|u_m\|_{p+1}^{p+1};$$

hence, $\{u_m\}$ does not converge strongly to 0 in L^{p+1}. From now on, with the same arguments as in the proof of Theorem 1 (see after Lemma 4), we deduce that $u_0 \not\equiv 0$, which ends our proof.

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