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PERIODIC SOLUTIONS FOR TIME-DEPENDENT SUBDIFFERENTIAL EVOLUTION INCLUSIONS

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Abstract. We consider evolution inclusions driven by a time dependent subdifferential plus a multivalued perturbation. We look for periodic solutions. We prove existence results for the convex problem (convex valued perturbation), for the nonconvex problem (nonconvex valued perturbation) and for extremal trajectories (solutions passing from the extreme points of the multivalued perturbation). We also prove a strong relaxation theorem showing that each solution of the convex problem can be approximated in the supremum norm by extremal solutions. Finally we present some examples illustrating these results.

1. **Introduction.** In this paper we study the existence of periodic solutions for evolution inclusions driven by a time dependent subdifferential and a multivalued perturbation. So, let $T = [0, b]$ and let H be a separable Hilbert space. The problem under consideration is the following

$$\left\{ \begin{array}{ll} -u'(t) \in \partial\varphi(t, u(t)) + F(t, u(t)) & \text{for almost all } t \in T, \\ u(0) = u(b). & \end{array} \right\} \quad (1)$$

In this problem $\varphi : T \times H \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, for every $t \in T$, $\varphi(t, \cdot)$ is proper, convex and lower semicontinuous and by $\partial\varphi(t, x)$ we denote the subdifferential in the sense of convex analysis (see Section 2). Also $F : T \times H \to 2^H \setminus \{\emptyset\}$ is a multivalued perturbation.

Periodic problems for subdifferential evolution equations were studied by Akagi and Stefanelli [1], Brezis [5], Hirano [13], Frigon [8], Qin and Xue [17], Xue and Cheng [18], Yamada [19], Yamazaki [20] and for subdifferential evolution inclusions by Bader and Papageorgiou [3]. Our work here is closely related to that of Bader and Papageorgiou [3], which deals with time independent subdifferential evolution inclusions. Also, our conditions on the multivalued perturbation $F(t, x)$ are more

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general, the methods here are different and in addition we also prove the existence of extremal periodic solutions (that is, solutions moving through the extreme points of $F(t, x)$).

2. **Mathematical background-preliminary results.** Let (Ω, Σ) be a measurable space and X be a separable Banach space. We will be using the following notation:

$$P_{f(c)}(X) = \{E \subseteq X : \ E \text{ is nonempty, closed (and convex)}\},$$
$$P_{(w)k(c)}(X) = \{E \subseteq X : \ E \text{ is nonempty, (weakly-)compact (and convex)}\}.$$

Given a multifunction $F : \Omega \to 2^X \backslash \{\emptyset\}$, the "graph of F" is defined to be the set

$$\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\}.$$

We say that $F(\cdot)$ is "graph measurable", if

$$\operatorname{Gr} F \in \Sigma \times B(X),$$

with $B(X)$ being the Borel σ-field of X. Suppose that $\mu(\cdot)$ is a σ-finite measure on Σ. Then according to the Yankov-von Neumann-Aumann selection theorem (see Hu and Papageorgiou [14, pp. 158-159]), if $F : \Omega \to 2^X \backslash \{\emptyset\}$ is a graph measurable multifunction, then there exists a sequence $f_n : \Omega \to X$ $(n \in \mathbb{N})$ of Σ-measurable selections of $F(\cdot)$ (that is, $f_n(\omega) \in F(\omega)$ for μ-almost all $\omega \in \Omega$) such that

$$F(\omega) \subseteq \overline{\{f_n(\omega)\}} \text{ for } \mu\text{-almost all } \omega \in \Omega.$$

The result is actually true if X is only a Souslin space. Recall that a Souslin space is always separable but need not be metrizable (see Hu and Papageorgiou [14, p. 145]).

A multifunction $F : \Omega \to P_f(X)$ is said to be measurable, if for every $x \in X$, the function $\omega \mapsto d(x, F(\omega)) = \inf[||x - u|| : u \in F(\omega)]$ is Σ-measurable. A $P_f(X)$-valued multifunction which is measurable, it is also graph measurable. The converse is true if Σ is μ-complete (see Hu and Papageorgiou [14, p. 150]).

Suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $F : \Omega \to 2^X \backslash \{\emptyset\}$ is a multifunction. For $1 \leq p \leq \infty$ we define

$$S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega) \ \mu\text{-almost everywhere in } \Omega\}.$$

An easy application of the Yankov-von Neumann-Aumann selection theorem implies that if $F(\cdot)$ is graph measurable, then $S_F^p \neq \emptyset$ if and only if $\inf[||u|| : u \in F(\omega)] \in L^p(\Omega)$. The set S_F^p is "decomposable" in the sense that if $(A, f_1, f_2) \in \Sigma \times S_F^p \times S_F^p$, then

$$\chi_A f_1 + \chi_{\Omega \backslash A} f_2 \in S_F^p.$$

Here for $C \in \Sigma, \chi_C$ is the characteristic function of the set C.

On $P_f(X)$ we can define a generalized metric, known as the "Hausdorff metric", by setting

$$h(E, C) = \sup[|d(x, E) - d(x, C)| : x \in X] \text{ for all } E, C \in P_f(X).$$

Recall that if $D \in P_f(X)$, then $d(x, D) = \inf[||x - d|| : d \in D]$ for all $x \in X$. We know that $(P_f(X), h)$ is a complete metric space and $F : X \to P_f(X)$ is h-continuous, if it is continuous from X into the metric space $(P_f(X), h)$.

If Y and Z are Hausdorff topological spaces, then a multifunction $G : Y \to 2^Z\backslash\{\emptyset\}$ is said to be "lower semicontinuous" (lsc for short), if for every $U \subseteq Z$ open, the set

$$G^-(U) = \{y \in Y : G(y) \cap U \neq \emptyset\}$$

is open. If Z is a metric space, then $G(\cdot)$ is lsc if and only if for every $z \in Z$ the function $y \mapsto d_Z(z, G(y))$ is upper semicontinuous (see Hu and Papageorgiou [14, p. 45]).

Also $G : Y \to 2^Z\backslash\{\emptyset\}$ is said to be "upper semicontinuous" (usc for short) if for all $U \subseteq Z$ open, the set

$$G^+(U) = \{y \in Y : G(y) \subseteq U\}$$

is open. An usc multifunction $G(\cdot)$ has closed graph and the two notions are equivalent if $G(\cdot)$ is locally compact (that is, for every $y \in Y$, there exists a neighborhood W of y such that $\overline{G(W)} \subseteq Z$ is compact; see Hu and Papageorgiou [14, p. 43]).

Now, let V be a Banach space. A function $\varphi : V \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is said to be proper, if it is not identically $+\infty$. By $\Gamma_0(V)$ we denote the cone of functions $\varphi : V \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ which are proper, convex and lower semicontinuous. Given $\varphi \in \Gamma_0(V)$, by $\operatorname{dom} \varphi$ we denote the "effective domain" of φ, defined by

$$\operatorname{dom} \varphi = \{v \in V : \varphi(v) < +\infty\}.$$

The subdifferential of φ at $v \in X$, is the set $\partial\varphi(v) \subseteq V^*$ (V^* is the topological dual of V), defined by

$$\partial\varphi(v) = \{v^* \in V^* : \langle v^*, u - v\rangle \leq \varphi(u) - \varphi(v) \text{ for all } u \in \operatorname{dom} \varphi\}.$$

Here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (V^*, V). If V is a Hilbert space identified with its dual (that is, $V = V^*$), then

$$\langle\cdot,\cdot\rangle = (\cdot,\cdot)_V = \text{the inner product of } V.$$

If φ is Gâteaux differentiable at $v \in V$, then $\partial\varphi(v) = \{\varphi'(v)\}$. We say that φ is of "compact type", if for all $\eta \in \mathbb{R}$, the sublevel set

$$\{v \in V : ||v||^2 + \varphi(v) \leq \eta\}$$

is compact.

Our conditions on the function $\varphi(t, x)$ in problem (1) are the following. Now H is a separable Hilbert space and $T = [0, b]$.

$H(\varphi)$: $\varphi : T \times H \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function such that for all $t \in T$, $\varphi(t,\cdot) \in \Gamma_0(H)$, it is strictly convex with $0 \in \partial\varphi(t, 0)$ for all $t \in T$ and
(i) for every $r > 0$, there exist $K_r > 0, g_r \in W^{1,2}(0, b)$ and $h_r \in W^{1,1}(0, b)$ such that if $t \in T$, $x \in \operatorname{dom} \varphi(t,\cdot)$ with $||x|| \leq r$ and $s \in [t, b]$, then there exists $\hat{x} \in \operatorname{dom} \varphi(s,\cdot)$ such that

$$||\hat{x} - x|| \leq |g_r(s) - g_r(t)|(\varphi(t, x) + K_r)^{1/2},$$
$$\varphi(s, \hat{x}) \leq \varphi(t, x) + |h_r(s) - h_r(t)|(\varphi(t, x) + K_r);$$

(ii) $\liminf\limits_{||x||\to\infty} \frac{\varphi(t,x)}{||x||} = \eta(t)$ for all $t \in T$ with $\eta : T \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ a measurable function such that $\int_0^b \eta(t)dt = +\infty$;
(iii) $\operatorname{dom} \varphi(b,\cdot) \subseteq \operatorname{dom} \varphi(0,\cdot)$
(iv) for every $t \in T$, $\varphi(t,\cdot)$ is of compact type.

Remark 1. Hypothesis $H(\varphi)(i)$ implies that for every $r > 0$, there exists $K'_r > 0$ such that

$$0 \leq \varphi(t,x) + K'_r \text{ for all } t \in T, \text{ all } ||x|| \leq r.$$

Hypotheses $H(\varphi)(i),(ii),(iii)$ were first introduced by Yamada [19] and allow to have $\operatorname{dom}\varphi(t,\cdot) \cap \operatorname{dom}\varphi(s,\cdot) = \emptyset$ for $t \neq s$. In this way we incorporate in our framework problems with time-varying obstacles. A slightly more general version of hypothesis $H(\varphi)(i)$ was considered by Yotsutani [21]. Hypothesis $H(\varphi)(iv)$ implies that for every $\lambda > 0$, the resolvent $J^t_\lambda = (I + \lambda\partial\varphi(t,\cdot))^{-1}$ is compact for all $t \in T$ (see Hu and Papageorgiou [14, p. 412]).

Now let $g \in L^2(T,H)$ and consider the following periodic problem

$$\left\{ \begin{array}{ll} -u'(t) \in \partial\varphi(t,u(t)) + g(t) & \text{for almost all } t \in T, \\ u(0) = u(b). & \end{array} \right\} \tag{2}$$

By a strong solution of problem (2), we understand a function $u \in W^{1,2}((0,b),H)$ such that

$$u(t) \in \operatorname{dom}\varphi(t,\cdot) \text{ for all } t \in T, u(0) = u(b)$$
$$-u'(t) \in \partial\varphi(t,u(t)) + g(t) \text{ for almost all } t \in T.$$

Recall that the function $u \in W^{1,2}((0,b),H)$ has a representative which is absolutely continuous from T into H (see Hu and Papageorgiou [15, p. 6]). So, $u(\cdot)$ is strongly differentiable almost everywhere. For the same reason, the evaluations $u(0)$, $u(b)$ make sense.

From Theorem 1.4 of Yamada [19], we know that problem (2) admits a strong solution $u = \xi(g)$. Moreover, since $\varphi(t,\cdot)$ $(t \in T)$ is strictly convex, this solution is unique. Indeed the strict convexity of $\varphi(t,\cdot)$ implies that $\partial\varphi(t,\cdot)$ is strictly monotone. Let $u, v \in W^{1,2}((0,b),H)$ be two strong solutions of (2). Then we can find $h_u, h_v \in L^2(T,H)$ such that

$$h_u(t) \in \partial\varphi(t,u(t)), h_v(t) \in \partial\varphi(t,v(t)) \text{ for almost all } t \in T,$$
$$-u'(t) = h_u(t) + g(t), -v'(t) = h_v(t) + g(t) \text{ for almost all } t \in T.$$

We have

$$\begin{aligned}
& (u'(t) - v'(t), u(t) - v(t))_H + (h_u(t) - h_v(t), u(t) - v(t))_H = 0 \\
& \text{for almost all } t \in T, \\
\Rightarrow\ & \frac{1}{2}\frac{d}{dt}|u(t) - v(t)|^2 + (h_u(t) - h_v(t), u(t) - v(t))_H = 0 \text{ for almost all } t \in T, \\
\Rightarrow\ & \int_0^b (h_u(t) - h_v(t), u(t) - v(t))_H dt = 0 \text{ (since } (u-v)(0) = (u-v)(b)), \\
\Rightarrow\ & (h_u(t) - h_v(t), u(t) - v(t))_H = 0 \\
& \text{for almost all } t \in T \text{ (from the monotonicity of } \partial\varphi(t,\cdot)) \\
\Rightarrow\ & u \equiv v \text{ (from the strict monotonicity of } \partial\varphi(t,\cdot)).
\end{aligned}$$

So, we can define the map $\xi : L^2(T,H) \to C(T,H)$ which to every input function $g \in L^2(T,H)$ assigns the unique strong solution $u = \xi(g) \in W^{1,2}((0,b),H) \subseteq C(T,H)$ of problem (2).

Proposition 1. *If hypotheses $H(\varphi)$ hold, then $\xi : L^2(T,H) \to C(T,H)$ is completely continuous (that is, if $g_n \xrightarrow{w} g$ in $L^2(T,H)$, then $\xi(g_n) \to \xi(g)$ in $C(T,H)$).*

Proof. Let $g_n \xrightarrow{w} g$ in $L^2(T,H)$ and let $u_n = \xi(g_n),\ n \in \mathbb{N}$. We have

$$-u_n'(t) \in \partial\varphi(t,u_n(t)) + g_n(t) \text{ for almost all } t \in T,\ u_n(0) = u_n(b), n \in \mathbb{N}. \quad (3)$$

First we show that $\{u_n\}_{n\geq 1} \subseteq C(T,H)$ is bounded. Arguing by contradiction, suppose that at least for a subsequence we have $||u_n||_\infty \to +\infty$. We take inner product on (3) with $u_n(t)$. Then

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|^2 \leq (g_n(t),u_n(t)) \text{ for almost all } t \in T \text{ (recall } 0 \in \partial\varphi(t,0)), \quad (4)$$
$$\Rightarrow\ \frac{1}{2}||u_n(t)||^2 \leq \frac{1}{2}||u_n(0)||^2 + \int_0^t ||g_n(s)||\cdot||u_n(s)||ds \text{ for all } t \in T,\ \text{all } n \in \mathbb{N}.$$

Invoking Lemme A.5 of Brezis [5, p. 157] (see also Hu and Papageorgiou [15, p. 12]) we obtain

$$||u_n(t)|| \leq ||u_n(0)|| + \int_0^b ||g_n(s)||ds,$$
$$\Rightarrow\ ||u_n(t)|| \leq ||u_n(0)|| + M_1 \text{ for some } M_1 > 0,\ \text{all } t \in T,\ \text{all } n \in \mathbb{N}. \quad (5)$$

Let $t_n \in T$ be such that $||u_n(t_n)|| = ||u_n||_\infty$ and recall that $||u_n||_\infty \to +\infty$. Then from (5), we have

$$||u_n(0)|| \to +\infty,$$
$$\Rightarrow\ ||u_n(b)|| \to +\infty \text{ (since } u_n(0) = u_n(b), n \in \mathbb{N}). \quad (6)$$

We return to (4) and integrate over $[t,b]$. Then

$$\frac{1}{2}||u_n(b)||^2 \leq \frac{1}{2}||u_n(t)||^2 + \int_0^b ||g_n(s)||ds,$$
$$\Rightarrow\ ||u_n(b)|| \leq ||u_n(t)|| + M_1^{1/2} \text{ for all } t \in T,\ \text{all } n \in \mathbb{N},$$
$$\Rightarrow\ \min_T ||u_n(\cdot)|| \to +\infty \text{ (see (6))}. \quad (7)$$

Note that

$$(u_n(t), -u_n'(t) - g_n(t)) \in \mathrm{Gr}\,\partial\varphi(t,\cdot) \text{ for almost all } t \in T,\ \text{all } n \in \mathbb{N} \quad (8)$$
$$||u_n(t)|| \to +\infty \text{ for all } t \in T \text{ (see (7))}. \quad (9)$$

Then

$$\begin{aligned}\eta(t) &= \liminf_{n\to\infty} \frac{\varphi(t,u_n(t))}{||u_n(t)||}\\ &= \liminf_{n\to\infty} \frac{\varphi(t,u_n(t)) - \varphi(t,0)}{||u_n(t)||}\\ &\leq \liminf_{n\to\infty} \frac{(-u_n'(t) - g_n(t), u_n(t))_H}{||u_n(t)||} \text{ for almost all } t \in T \text{ (see (8), (9))}.\end{aligned}$$

Let $\eta_n(t) = \frac{(-u_n'(t)-g_n(t),u_n(t))_H}{||u_n(t)||}$ for all $t \in T$. Hypothesis $H(\varphi)(i)$ implies that

$$\varphi(t,x) + c_1||x|| + c_2 \geq 0 \text{ for some } c_1, c_2 > 0,\ \text{all } t \in T,\ \text{all } x \in H$$

(see Hu and Papageorgiou [15, p. 117]). Then hypothesis $H(\varphi)(ii)$ and Fatou's lemma imply that

$$+\infty = \int_0^b \eta(t)dt \leq \int_0^b \liminf_{n\to\infty} \eta_n(t)dt \leq \liminf_{n\to\infty} \int_0^b \eta_n(t)dt. \quad (10)$$

On the other hand, since

$$(-u_n'(t), u_n(t))_H = \frac{1}{2}\,\frac{d}{dt}||u_n(t)||^2 \text{ for all } t \in T, \text{ all } n \in \mathbb{N}$$
$$\text{and} \qquad u_n(0) = u_n(b) \text{ for all } n \in \mathbb{N},$$

and using (7), we have

$$\int_0^b \hat{\eta}(t)dt \le \int_0^b \frac{(-g_n(t), u_n(t))_H}{||u_n(t)||}dt \le \int_0^b ||g_n(t)||dt \le M_2 \tag{11}$$
$$\text{for some } M_2 > 0, \text{ all } n \in \mathbb{N} \text{ big.}$$

Comparing (10) and (11) we reach a contradiction.

So, we have proved that $\{u_n\}_{n\ge1} \subseteq C(T,H)$ is bounded. Then from Yotsutani [21] (see (7.5), p. 645), we see that we can find $M_3 > 0$ such that

$$||u_n'||_2 \le M_3 \text{ for all } n \in \mathbb{N}. \tag{12}$$

For all $s, t \in T$ with $s < t$, we have

$$||u_n(t) - u_n(s)|| = \left\|\int_s^t u_n'(\tau)ds\right\| \le \int_s^t ||u_n'(\tau)||d\tau \le (t-s)^{1/2}M_3$$
$$\text{for all } n \in \mathbb{N} \text{ (see (12))},$$
$$\Rightarrow \quad \{u_n\}_{n\ge1} \subseteq C(T,H) \text{ is equicontinuous.}$$

From Yotsutani [21] (see (7.9), p. 645), we see that there exists $M_4 > 0$ such that

$$\varphi(t, u_n(t)) \le M_4 \text{ for all } n \in \mathbb{N}, \text{ all } t \in T.$$

Therefore, we see that there exists $\eta^* > 0$ such that

$$u_n(t) \in \{x \in H : ||x||^2 + \varphi(t,x) \le \eta^*\} \text{ for all } t \in T, \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \quad \overline{\{u_n(t)\}_{n\ge1}} \in P_k(H) \text{ for all } t \in T \text{ (see hypothesis } H(\varphi)(iv)).$$

Invoking the Arzela-Ascoli theorem (see, for example, Gasinski and Papageorgiou [10, p. 232]), we have that

$$\{u_n\}_{n\ge1} \subseteq C(T,H) \text{ is relatively compact.}$$

Hence, we may assume that

$$u_n \to u \text{ in } C(T,H). \tag{13}$$

On account of (12) and by passing to a suitable subsequence if necessary, we may assume that

$$u_n' \xrightarrow{w} v \text{ in } L^2(T,H). \tag{14}$$

We know that

$$u_n(t) = u_n(s) + \int_s^t u_n'(\tau)d\tau \text{ for all } s,t \in T, s < t, \text{ all } n \in \mathbb{N}.$$

Then using (13) and (14), we have

$$u(t) = u(s) + \int_s^t v(\tau)d\tau \text{ for all, } s,t \in T, \ s < t,$$
$$\Rightarrow \quad u \in W^{1,2}((0,b),H) \text{ and } u' = v.$$

Consider the integral functional $J_\varphi : L^2(T,H) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$J_\varphi(v) = \begin{cases} \int_0^b \varphi(t, v(t))dt & \text{if } \varphi(\cdot, v(\cdot)) \in L^1(T) \\ +\infty & \text{otherwise.} \end{cases}$$

By Lemma 3.4 of Yotsutani [21], this integral functional is well-defined and in fact $J_\varphi \in \Gamma_0(L^2(T,H))$. We have

$$(u_n, -u_n' - g_n) \in \operatorname{Gr} J_\varphi \text{ for all } n \in \mathbb{N}$$

(see Gasinski and Papageorgiou [9, p. 570]). Recall that

$$u_n \to u \text{ in } C(T,H).$$

Since $\operatorname{Gr} \partial J_\varphi$ is sequentially closed in $L^2(T,H) \times L^2(T,H)_w$ (by $L^2(T,H)_w$ we denote the Hilbert space $L^2(T,H)$ furnished with the weak topology; see Gasinski and Papageorgiou [9, p. 308]), we infer that

$$\begin{aligned} & (u, -u' - g) \in \operatorname{Gr} \partial J_\varphi, \\ \Rightarrow \quad & u = \xi(g). \end{aligned}$$

Then by Urysohn's criterion (see Gasinski and Papageorgiou [10, p. 33]), for the original sequence, we have

$$\begin{aligned} & u_n \to u \text{ in } C(T,H), \\ \Rightarrow \quad & \xi(g_n) \to \xi(g) \text{ in } C(T,H), \\ \Rightarrow \quad & \xi(\cdot) \text{ is completely continuous.} \end{aligned}$$

□

In the study of the "convex problem", we will use the following multivalued generalization of the Leray-Schauder alternative principle, due to Bader [2].

So, let Y, Z be Banach spaces, $G : Y \to P_{wkc}(Z)$ a multifunction which is usc from Y into Z_w (=the Banach space Z furnished with the weak topology) and let $\xi : Z \to Y$ be a completely continuous map. We set $Q = \xi \circ G$. The result of Bader [2] reads as follows:

Proposition 2. *If Y, Z, Q are as above, then one of the following alternatives holds,*

(a) $S = \{y \in Y : y \in \lambda Q(y)$ for some $0 < \lambda < 1\}$ is unbounded or

(b) Q has a fixed point (that is, there exists $y \in Y$ such that $y \in Q(y)$).

Remark 2. We stress that Q need not have convex values, in contrast to the original multivalued Leray-Schauder alternative principle (see Hu and Papageorgiou [15, p. 231]).

Finally let us introduce some notation which will be used throughout this work. By $\mathcal{L}_T$ we denote the Lebesgue σ-field of T, $B(H)$ is the Borel σ-field of H, if $D \subseteq H$ is nonempty, then

$$|D| = \sup[||d|| : d \in D]$$

and for $M > 0$, $p_M : H \to H$ denotes the M-radial retraction map defined by

$$p_M(u) = \begin{cases} u & \text{if } ||u|| \le M \\ M\frac{u}{||u||} & \text{if } ||u|| > M. \end{cases}$$

We know that $p_M(\cdot)$ is nonexpansive, that is,

$$||p_M(u) - p_M(v)|| \le ||u - v|| \text{ for all } u, v \in H.$$

By $L^1_w(T,H)$ we denote the Lebesgue-Bochner space $L^1(T,H)$ equipped with the "weak norm" $||\cdot||_w$ defined by

$$||u||_w = \sup\left[\left\|\int_s^t u(\tau)d\tau\right\| : 0 \le s \le t \le b\right], \; u \in L^1(T,H).$$

In fact we can equivalently define the weak norm by

$$||u||_w = \sup \left[\left\| \int_0^t u(\tau) d\tau \right\| : 0 \le t \le b \right].$$

In the next section we produce solutions for the case when the multivalued perturbation F is convex-valued.

3. **Convex problem.** In this section, the conditions on the multivalued perturbation $F(t,x)$ are the following. In the sequel, H_w denotes the Hilbert space H equipped with the weak topology

$H(F)_1 : F : T \times H \to P_{f_c}(H)$ is a multifunction such that
(i) for every $x \in H,\ t \mapsto F(t,x)$ is graph measurable;
(ii) for almost all $t \in T,\ \mathrm{Gr}\, F(t,\cdot)$ is sequentially closed in $H \times H_w$;
(iii) $|F(t,x)| \le c_1(t) + c_2(t)|x|$ for almost all $t \in T$, all $x \in H$, with $c_1, c_2 \in L^2(T)$;
(iv) for almost all $t \in T$, all $x \in H$ and all $v \in F(t,x)$, we have

$$(v,x)_H \ge c_3 ||x||^2 - c_4(t)$$

with $c_3 > 0$ and $c_4 \in L^1(T)$;
(v) there exists $M > 0$ such that for almost all $t \in T$, all $x \in H$ with $||x|| = M$ and all $v \in F(t,x)$ we have

$$(v,x)_H \ge 0.$$

Remark 3. These conditions on F are weaker than the corresponding ones in Bader and Papageorgiou [3]. In [3] it is assumed that $F(\cdot,\cdot)$ is jointly measurable and for almost all $t \in T$, $F(t,\cdot)$ is h-usc (see hypotheses $H(F)_2$ in Bader and Papageorgiou [3]) a notion more restrictive than hypothesis $H(F)_1(ii)$. Also, we stress that hypotheses $H(F)_1(i),(ii)$ do not imply the joint measurability of $F(\cdot,\cdot)$ (see Hu and Papageorgiou [14, Example 7.2, p. 227]). Hypothesis $H(F)_1(v)$ is known as "Hartman's condition" and was first used by Hartman [11] in the context of second order Dirichlet systems in $\mathbb{R}^N$. Note that if in hypothesis $H(F)_1(iv)$ we assume $c_4 \in L^\infty(T)_+$, then Hartman's condition (that is, hypothesis $H(F)_1(v)$) is satisfied.

We say that $u \in W^{1,2}((0,b),H)$ is a "strong solution" of problem (1) if

$$u(t) \in \mathrm{dom}\, \varphi(t,\cdot) \text{ for all } t \in T,\ u(0) = u(b) \text{ and}$$
$$\text{there exists } f \in S^2_{F(\cdot,u(\cdot))} \text{ such that}$$
$$-u'(t) \in \partial\varphi(t,u(t)) + f(t) \text{ for almost all } t \in T.$$

Let $M > 0$ be as postulated by hypothesis $H(F)_1(v)$ and consider the following modification of the multiplication $F(t,\cdot)$:

$$F_1(t,x) = \begin{cases} F(t,x) & \text{if } ||x|| \le M \\ (x - p_M(x)) + F(t,p_M(x)) & \text{if } ||x|| > M. \end{cases} \tag{15}$$

Lemma 3.1. *If $F(t,x)$ satisfies hypotheses $H(F)_1$, then so does the multifunction $F_1(t,x)$.*

Proof. Evidently for every $x \in H$, $t \mapsto F_1(t,x)$ is graph measurable. Moreover, the continuity of the M-radial retraction $p_M(\cdot)$, implies that $\mathrm{Gr}\, F_1(t,\cdot)$ is sequentially closed in $H \times H_w$. Also we have

$$|F_1(t,x)| \le 2||x|| + c(t) \text{ for almost all } t \in T,\ \text{all } x \in H,$$

with $c(t) = c_1(t) + c_2(t)M, c \in L^2(T)$.

For almost all $t \in T$, all $x \in H$ with $||x|| > M$ and all $v \in F_1(t,x)$, we have

$$v = x - p_M(x) + v_0 \text{ with } v_0 \in F(t, p_M(x)).$$

Then

$$\begin{aligned}(v,x)_H &= ||x||^2 - (p_M(x),x)_H + (v_0,x)_H \\ &\geq ||x||^2 - M||x|| + \left(v_0, \frac{Mx}{||x||}\right)\frac{||x||}{M} \\ &\geq ||x||^2 - M||x|| \text{ (see hypothesis } H(F)_1(v)) \\ &\geq \frac{1}{2}||x||^2 - \frac{1}{2}M^2. \qquad (16)\end{aligned}$$

For almost all $t \in T$, all $x \in H$ with $||x|| \leq M$ and all $v \in F_1(t,x) = F(t,x)$ (see (15)), we have

$$(v,x)_H \geq c_3||x||^2 - c_4(t) \text{ (see hypothesis } H(F)_1(iv)). \qquad (17)$$

From (16) and (17) we infer that for almost all $t \in T$, all $x \in H$ and all $v \in F(t,x)$, we have

$$(v,x)_H \geq c_5||x||^2 - c_6(t),$$

with $c_5 = \min\{\frac{1}{2}, c_3\}$ and $c_6(t) = \max\{\frac{1}{2}M^2, c_4(t)\}, c_6 \in L^1(T)$.

Finally, for almost all $t \in T$ and all $x \in H$ with $||x|| = M$, we have

$$F_1(t,x) = F(t,x) \text{ (see (15))}$$

and so for all $v \in F_1(t,x)$, we have

$$(v,x)_H \geq 0 \text{ (see hypothesis } H(F)_1(v)).$$

Therefore we have checked that the new multifunction $F_1(t,x)$ satisfies hypotheses $H(F)_1$. ☐

Let $N_1 : C(T,H) \to 2^{L^2(T,H)}$ be the multivalued map defined by

$$N_1(u) = S^2_{F_1(\cdot,u(\cdot))} \text{ for all } u \in C(T,H).$$

Proposition 3. *If hypotheses $H(F)_1$ hold, then $N_1(\cdot)$ has valued in $P_{wkc}(L^2(T,H))$ and it is usc form $C(T,H)$ into $L^2(T,H)_w$.*

Proof. First we show that N_1 has nonempty values. This is not immediately clear, since as we already mentioned in Section 2, hypotheses $H(F)_1(i),(ii)$ (which are also satisfied by F_1, see Lemma 3.1), do not imply joint measurability of F_1. Therefore we cannot say that for $u \in C(T,H)$, $t \mapsto F_1(t,u(t))$ is graph measurable and apply the Yankov-von Neumann-Aumann selection theorem to conclude that $S^2_{F_1(\cdot,u(\cdot))} \neq \emptyset$.

Let $\{s_n\}_{n\geq 1}$ be a sequence of simple functions such that

$$s_n(t) \to u(t) \text{ for almost all } t \in T \text{ and } ||s_n(t)|| \leq ||u||_\infty \text{ for all } t \in T, \text{ all } n \in \mathbb{N}. \qquad (18)$$

The graph measurability of $F_1(\cdot,x)$ implies that $t \mapsto F_1(t,s_n(t))$ $(n \in \mathbb{N})$ is graph measurable. So, we can apply the Yankov-von Neumann-Aumann selection theorem (see Hu and Papageorgiou [14, p. 158]) and produce a measurable map $f_n : T \to H$ such that

$$f_n(t) \in F_1(t,s_n(t)) \text{ for almost all } t \in T, \text{ all } n \in \mathbb{N}.$$

Then (18) and hypothesis $H(F)_1(i)$ imply that

$$\{f_n\}_{n\geq 1} \subseteq L^2(T,H) \text{ is bounded.}$$

So, we may assume that

$$f_n \xrightarrow{w} f \text{ in } L^2(T,H).$$

Using Proposition VII.3.9 of Hu and Papageorgiou [14, p. 694], we have

$$\begin{aligned} f(t) &\in \overline{\text{conv}}\, w - \limsup_{n\to\infty}\{f_n(t)\} \\ &\subseteq \overline{\text{conv}}\, w - \limsup_{n\to\infty} F_1(t,s_n(t)) \\ &\subseteq F_1(t,u(t)) \text{ for almost all } t\in T \\ &\quad \text{(see hypothesis } H(F)_1(ii) \text{ and Lemma 3.1).} \end{aligned}$$

Therefore we have

$$f \in S^2_{F_1(\cdot,u(\cdot))} = N_1(u).$$

Clearly the values of $N_1(\cdot)$ are closed, convex and bounded. Hence

$$N_1(u) \in P_{wkc}(L^2(T,H)) \text{ for all } u \in C(T,H).$$

According to Proposition I.2.23 of Hu and Papageorgiou [14, p. 43] in order to show the upper semicontinuity of $N_1(\cdot)$ from $C(T,H)$ into $L^2(T,H)_w$, if suffices to show that $\text{Gr}\, N_1$ is sequentially closed in $C(T,H)\times L^2(T,H)_w$. To this end, let

$$\{(u_n,f_n)\}_{n\geq 1} \subseteq \text{Gr}\, N_1, u_n \to u \text{ in } C(T,H), f_n \xrightarrow{w} f \text{ in } L^2(T,H). \tag{19}$$

As before, using Proposition VII.3.9 of Hu and Papageorgiou [14, p. 694] and the fact that $\text{Gr}\, F_1(t,\cdot)$ is sequentially closed in $H\times H_w$, we obtain

$$\begin{aligned} & f(t) \in \overline{\text{conv}}\, w - \limsup_{n\to\infty} F_1(t,u_n(t)) \subseteq F_1(t,u(t)) \text{ for almost all } t\in T, \\ \Rightarrow\ & (u,t) \in \text{Gr}\, N_1, \\ \Rightarrow\ & N_1 \text{ is usc from } C(T,H) \text{ into } L^2(T,H)_w. \end{aligned}$$

□

Now we are ready for the first existence theorem covering the case of a convex valued perturbation $F(t,x)$.

Theorem 3.2. *If hypotheses $H(\varphi)$ and $H(F)_1$ hold, then problem (1) admits a strong solution $\hat{u} \in W^{1,2}((0,b),H)$.*

Proof. We consider the following periodic problem

$$\left\{ \begin{array}{ll} -u'(t) \in \partial\varphi(t,u(t)) + F_1(t,u(t)) & \text{for almost all } t\in T, \\ u(0)=u(b). & \end{array} \right\} \tag{20}$$

Problem (20) is equivalent to the following abstract fixed point problem

$$u = (\xi \circ N_1)(u). \tag{21}$$

Because of Proposition 1 and Lemma 3.1, to solve (21) we can use Proposition 2. So, we introduce the set

$$S = \{u \in C(T,H) : u \in \lambda(\xi\circ N_1)(u), \lambda\in(0,1)\}.$$

Claim 1. $||u||_{C(T,H)} \leq M$ *for all* $u\in S$.

We argue by contradiction. So, suppose that the Claim is not true. Then we can find $u\in S$ such that $||u||_{C(T,H)} > M$. We have two possibilities:

(a)

$$||u(t)|| > M \text{ for all } t \in T \tag{22a}$$

or

(b) there exist $\eta, \vartheta \in T$ with $\eta < \vartheta$ such that

$$||u(\eta)|| = M,\ ||u(t)|| > M \text{ for all } t \in (0, \vartheta]\,. \tag{22b}$$

Since $u \in S$, we have

$$\left\{\begin{array}{ll} -\frac{1}{\lambda}u'(t) \in \partial\varphi(t, \frac{1}{\lambda}u(t)) + \hat{f}_\lambda(t) & \text{for almost all } t \in T, \\ u(0) = u(b), \hat{f}_\lambda \in S^2_{F_1(\cdot,u(\cdot))} & \text{with } \lambda \in (0,1). \end{array}\right\} \tag{23}$$

Suppose that (a) holds (see (22a)). Then from (15) we have

$$\hat{f}_\lambda(t) = u(t) - p_M(u(t)) + f_\lambda(t) \text{ for almost all } t \in T \text{ with } f_\lambda \in S^2_{F(\cdot,p_M(u(\cdot)))}. \tag{24}$$

Let $w \in S^2_{\partial\varphi(\cdot,\frac{1}{\lambda}u(\cdot))}$ such that

$$\frac{1}{\lambda}u'(t) + w(t) + u(t) - p_M(u(t)) + f_\lambda(t) = 0 \text{ for almost all } t \in T \text{ (see (23) and (24))}.$$

We take the inner product with $u(t)$. Then for almost all $t \in T$

$$\frac{1}{2\lambda}\,\frac{d}{dt}||u(t)||^2 + ||u(t)||^2 - M||u(t)|| + (f_\lambda(t)), p_M(u(t))_H\frac{||u(t)||}{M} = 0$$

(recall that $0 \in \partial\varphi(t, 0)$ for almost all $t \in T$)

$\Rightarrow \quad ||u(b)|| < ||u(0)||,$ a contradiction (see (22a) and hypothesis $H(F)_1(v)$).

Similarly if (b) holds (see (22b)), then working on the interval $[\eta, \vartheta]$ we obtain

$$||u(\vartheta)|| < ||u(\eta)|| = M,$$

a contradiction (see (22b)).

This proves the Claim.

Then using Proposition 2, we infer that there exists $\hat{u} \in C(T, H)$ such that

$$\hat{u} \in (\xi \circ N_1)(\hat{u}).$$

As above we show that

$$||\hat{u}||_\infty \le M.$$

Therefore $\hat{u} \in W^{1,2}((0, b), H)$ is a strong solution of problem (1). □

4. **Nonconvex problem.** In this section, we investigate the case when the multivalued perturbation $F(t, x)$ has nonconvex values.

The hypotheses on the multifunction $F(t, x)$ are the following:

$H(F)_2 : F : T \times H \to P_f(H)$ is a multifunction such that

(i) $(t, x) \mapsto F(t, x)$ is graph measurable;

(ii) for almost all $t \in T,\ x \mapsto F(t, x)$ is lsc;

hypotheses $H(F)_2(iii), (iv), (v)$ are the same as the corresponding hypotheses $H(F)_1$ $(iii), (iv), (v)$.

Theorem 4.1. *If hypotheses $H(\varphi)$ and $H(F)_2$ hold, then problem (1) admits a strong solution $\hat{u} \in W^{1,2}((0, b), H)$.*

Proof. Again we consider $F_1 : T \times H \to P_f(H)$ the modification of the multifunction F (see (15)). Evidently $F_1(t,x)$ satisfies hypotheses $H(F)_2$ (see Lemma 3.1). Then let $N_1 : C(T,H) \to 2^{L^2(T,H)}$ are defined by

$$N_1(u) = S^2_{F_1(\cdot,u(\cdot))} \text{ for almost all } u \in C(T,H).$$

Note that

$$\begin{aligned} & \operatorname{Gr} F_1(\cdot,u(\cdot)) = (\operatorname{Gr} u \times H) \cap \operatorname{Gr} F_1, \\ \Rightarrow \quad & t \mapsto F_1(t,u(t)) \text{ is graph measurable.} \end{aligned}$$

So, as before, via the Yankov-von Neumann-Aumann selection theorem (see Hu and Papageorgiou [14, p. 158]), we see that $N_1(\cdot)$ has nonempty, closed values, that is,

$$N_1(u) \in P_f(L^2(T,H)) \text{ for all } u \in C(T,H).$$

Also, $N_1(\cdot)$ has decomposable values.

Claim 2. *N_1 is lsc from $C(T,H)$ into $L^2(T,H)$.*

It suffices to show that for every $h \in L^2(T,H)$ the function $u \mapsto d(h,N_1(u))$ is upper semicontinuous. To this end we show that for every $\eta \geq 0$ the superlevel set

$$U_\eta = \{u \in C(T,H) : d(h,N_1(u)) \geq \eta\}$$

is closed. So, let $\{u_n\}_{n\geq 1} \subseteq U_\eta$ and assume that $u_n \to u$ in $C(T,H)$. Using Fatou's lemma (it can be used thanks to hypothesis $H(F)_2(iii)$), we have

$$\limsup_{n\to\infty} \int_0^b d(h(t),F_1(t,u_n(t)))dt \leq \int_0^b \limsup_{n\to\infty} d(h(t),F_1(t,u_n(t)))dt. \tag{25}$$

Since for almost all $t \in T$, $F_1(t,\cdot)$ is lsc (see hypothesis $H(F)_2(ii)$), we have

$$\begin{aligned} & \limsup_{n\to\infty} d(h(t),F_1(t,u_n(t))) \leq d(h(t),F_1(t,u(t))) \text{ for almost all } t \in T, \\ \Rightarrow \quad & \limsup_{n\to\infty} \int_0^b d(h(t),F_1(t,u_n(t)))dt \leq \int_0^b d(h(t),F_1(t,u(t)))dt \text{ (see (25)).} \end{aligned}$$

But from Theorem II.3.24 of Hu and Papageorgiou [14, p. 183], we have

$$\int_0^b d(h(t),F_1(t,u_n(t)))dt = d(h,N_1(u_n)) \text{ and}$$

$$\int_0^b d(h(t),F_1(t,u(t)))dt = d(h,N_1(u)).$$

So, finally we infer that

$$\begin{aligned} & u \in U_\eta, \\ \Rightarrow \quad & N_1(\cdot) \text{ is lsc.} \end{aligned}$$

This proves the Claim.

So, we can use the Bressan-Colombo selection theorem [4] and find a continuous map $g : C(T,H) \to L^2(T,H)$ such that

$$g(u) \in N_1(u) \text{ for all } u \in C(T,H).$$

We consider the following periodic problem

$$\left\{ \begin{array}{ll} -u'(t) \in \partial\varphi(t,u(t)) + g(u)(t) & \text{for almost all } t \in T, \\ u(0) = u(b). & \end{array} \right\}. \tag{26}$$

Problem (26) is equivalent to the following abstract fixed point problem

$$u = (\xi \circ g)(u). \tag{27}$$

We consider the set

$$S = \{u \in C(T,H) : u = \lambda(\xi \circ g)(u), \lambda \in (0,1)\}.$$

As in the proof of Theorem 3.2, we show that $S \subseteq C(T,H)$ is bounded. So, the classical Leray-Schauder alternative principle (see Gasinski and Papageorgiou [9, p. 827]) implies that (27) admits a solution $\hat{u}$. Then $\hat{u} \in W^{1,2}((0,b),H)$ is a strong solution of (1). □

5. **Extremal periodic solutions.** Here we study the existence of extremal periodic solutions. By this we mean solutions of the following periodic problem

$$\left\{ \begin{array}{l} -u'(t) \in \partial\varphi(t,u(t)) + \operatorname{ext} F(t,u(t)) \quad \text{for almost all } t \in T, \\ u(0) = u(b). \end{array} \right\} \tag{28}$$

In (28), $\operatorname{ext} F(t,u(t))$ denotes the set of extreme points of the set $F(t,u(t))$. Such trajectories are important in control theory in connection with the "bang-bang principle".

We assume that F has weakly compact and convex values. Note that although $F(t,\cdot)$ may be a regular multifunction, this regularity is lost when we pass to $\operatorname{ext} F(t,\cdot)$. In addition, in general $\operatorname{ext} F(t,x)$ is neither convex nor closed. So, the previous existence theorems (see Theorems 3.2 and 4.1). can not be used to produce a solution of problem (28).

In this case the hypotheses on the multifunction $F(t,x)$ are the following.

$H(F)_3 : F : T \times H \to P_{wkc}(H)$ is a multifunction such that
(i) for every $x \in H,\ t \mapsto F(t,x)$ is measurable;
(ii) for almost all $t \in T,\ x \mapsto F(t,x)$ is h-continuous;
hypotheses $H(F)_3(iii), (iv), (v)$ are the same as the corresponding hypotheses $H(F)_1$ (iii), $(iv), (v)$.

Remark 4. Hypotheses $H(F)_3(i), (ii)$ imply that $(t,x) \mapsto F(t,x)$ is measurable. Indeed, for every $h \in H$, the function $(t,x) \mapsto d(h,F(t,x))$ is Carathéodory (that is, for all $x \in H$, $t \mapsto d(h,F(t,x)))$ is measurable and for almost all $t \in T$, $x \mapsto d(h,F(t,x))$ is continuous. Then Proposition II.1.6 of Hu and Papageorgiou [14, p. 142] implies that $(t,x) \mapsto d(h,F(t,x))$ is measurable, which in turn means that the multifunction $(t,x) \mapsto F(t,x)$ is measurable. Therefore for every $u \in C(T,H)$, $t \mapsto F(t,u(t))$ is measurable.

Theorem 5.1. *If hypotheses $H(\varphi)$ and $H(F)_3$ hold, then problem (28) admits a strong solution $\hat{u} \in W^{1,2}((0,b),H)$.*

Proof. We consider the multifunction $F_1(t,x)$ defined by (15) and as before we set

$$N_1(u) = S^2_{F_1(\cdot,u(\cdot))} \text{ for all } u \in C(T,H).$$

Again $F_1(t,x)$ satisfies hypotheses $H(F)_3$.

For every $u,v \in C(T,H)$ and every $h \in L^2(T,H)$, we have

$$d(h,N_1(u)) = \int_0^b d(h(t),F_1(t,u(t)))dt \tag{29}$$

$$d(h,N_1(v)) = \int_0^b d(h(t),F_1(t,v(t)))dt \tag{30}$$

(see Hu and Papageorgiou [14, Theorem II.3.24, p. 183]). Then

$$\begin{aligned}
&h(N_1(u), N_1(v)) \\
&= \sup\left[|d(h, N_1(u)) - d(h, N_1(v))| : h \in L^2(T, H)\right] \\
&\leq \sup\left[\int_0^b |d(h(t), F_1(t, u(t))) - d(h(t), F_1(t, v(t)))|dt : h \in L^2(T, H)\right] \\
&\text{(see (29), (30))} \\
&= \int_0^b \sup\left[|d(w, F_1(t, u(t)) - d(w, F_1(t, v(t))))| : w \in H\right] dt \\
&\text{(see Hu and Papageorgiou [14, p. 183])} \\
&= \int_0^b h(F_1(t, u(t)), F_1(t, v(t)))dt, \\
\Rightarrow \quad & u \mapsto N_1(u) \text{ is } h\text{-continuous.}
\end{aligned}$$

So the Bressan-Colombo selection theorem [4] provides a continuous map $g : C(T, H) \to L^2(T, H)$ such that

$$g(u) \in N_1(u) \text{ for all } u \in C(T, H).$$

Using this map $g(\cdot)$ we consider the auxiliary periodic problem (26). As in the proof of Theorem 4.1, using the Leray-Schauder alternative principle, we show that problem (26) has strong solutions and every such strong solution $u \in W^{1,2}((0, b), H)$ satisfies

$$||u(t)|| \leq M \text{ for all } t \in T,$$

with $M > 0$ as postulated by hypothesis $H(F)_3(v)$. So, without any loss of generality, we may assume that

$$|F_1(t, x)| \leq 2M + c(t) = c_7(t) \text{ for almost all } t \in T, \text{ all } x \in H, \text{ with } c_7 \in L^2(T)$$

(recall $c(t) = c_1(t) + c_2(t)M$, see the proof of Lemma 3.1). Otherwise, we replace $F_1(t, x)$ by $\hat{F}_1(t, x) = F_1(t, p_M(x)) = F(t, x)$, see (15). Let

$$D = \{h \in L^2(T, H) : ||h(t)|| \leq c_7(t) \text{ for almost all } t \in T\}.$$

Using Proposition 2, we have that

$$\xi(D) \subseteq C(T, H) \text{ is compact,}$$
$$\Rightarrow \quad K = \overline{\text{conv}}\, \xi(D) \in P_{kc}(C(T, H))$$

(see Gasinski and Papageorgiou [10, p. 852]).

Invoking Theorem II.8.31 of Hu and Papageorgiou [14, p. 260], we can find a continuous map $\gamma : K \to L^1_w(T, H)$ such that

$$\gamma(u) \in \text{ext}\, S^2_{F_1(\cdot, u(\cdot))} = S^2_{\text{ext}\, F_1(\cdot, u(\cdot))} \text{ for all } u \in K \tag{31}$$

(see Hu and Papageorgiou [14, Theorem II.4.5, p. 191]). Using the Dugundji extension theorem (see Gasinski and Papageorgiou [9, p. 270]), we can find $\hat{\gamma} : C(T, H) \to L^1_w(T, H)$ a continuous map such that

$$\hat{\gamma}|_K = \gamma \text{ and } \hat{\gamma}(C(T, H)) \subseteq \text{conv}\, \gamma(K) \subseteq D. \tag{32}$$

We claim that $\hat{\gamma}$ is sequentially continuous from $C(T, H)$ with the norm topology into $L^2(T, H)$ with the weak topology. To this end, let $u_n \to u$ in $C(T, H)$. Then

$$\hat{\gamma}(u_n) \xrightarrow{||\cdot||_w} \hat{\gamma}(u).$$

But from (32) we obtain

$$\sup_{n\geq 1} ||\hat{\gamma}(u_n)||_{L^2(T,H)} < +\infty\,.$$

So, we can use Lemma I.2.8 of Hu and Papageorgiou [15, p. 24] and have that

$$\hat{\gamma}(u_n) \xrightarrow{w} \hat{\gamma}(u) \text{ in } L^2(T,H).$$

This proves the claimed sequential continuity of $\hat{\gamma}$.

We consider the following auxiliary periodic problem

$$-u'(t) \in \partial\varphi(t,u(t)) + \hat{\gamma}(u)(t) \text{ for almost all } t \in T,\; u(0) = u(b).$$

As before (see the proof of Theorem 4.1), using the Leray-Schauder alternative principle, we can find $\hat{u} \in W^{1,2}((0,b),H)$ such that

$$\hat{u} = (\xi \circ \hat{\gamma})(\hat{u}).$$

Then $\hat{\gamma}(\hat{u}) \in D$ (see (32)) and so

$$\begin{aligned} & \hat{u} \in \xi(D) \subseteq K \\ \Rightarrow\quad & \hat{\gamma}(\hat{u}) = \gamma(\hat{u}) \in \operatorname{ext} F_1(t,\hat{u}(t)) \text{ for almost all } t \in T \text{ (see (32) and (31))}, \\ \Rightarrow\quad & \hat{u} \in W^{1,2}((0,b),H) \text{ is a strong solution of (28)}. \end{aligned}$$

□

6. **Strong relaxation.** Let $K_0 = \{u(0) : u \in K\}$ with $K \subseteq C(T,H)$ as in the proof of Theorem 5.1. We know that $K_0 \in P_{kc}(H)$. Let $S_e(v)$ be the solution of the initial value problem

$$-u'(t) \in \partial\varphi(t,u(t)) + \operatorname{ext} F(t,u(t)) \text{ for almost all } t \in T, u(0) = v.$$

As we did in section 4, we can show that $S_e(v) \neq 0$. Also, let S_c be the solution set of the convexified periodic problem. We know that $S_c \neq 0$ (see Theorem 3.2). Let $S_e = \bigcup_{v\in K_0} S_e(v)$. In this section we show that

$$S_c \subseteq \bar{S}_e^{C(T,H)}. \tag{33}$$

Such a result is known as "strong relaxation theorem" and it is useful in control theory, since it implies that the states of the system can be approximated by states originating from the same initial condition and generated by bang-bang controls. Thus we can economize in the use of the control functions.

To prove (33) we introduce the following conditions on the multifunctions $F(t,x)$.

$H_4 : F : T \times H \to P_{wkc}(H)$ is a multifunction such that

(i) for every $x \in H,\; t \mapsto F(t,x)$ is measurable;

(ii) $h(F(t,x),F(t,y)) \leq k(t)||x-y||$ for almost all $t \in T$, all $x,y \in H$, some $k \in L^1(T)_+$;

hypotheses $H_4(iii),(iv),(v)$ are the same as the corresponding hypotheses $H(F)_1$ (iii), (iv), (v).

Theorem 6.1. *If hypotheses $H(\varphi)$ and $H(F)_4$ hold, then for every $u \in S_c$, we can find $\{u_n\}_{n\geq 1} \subseteq S_e(u(0))$ such that $u_n \to u$ in $C(T,H)$.*

Proof. Let $u \in S_c$. Then we have

$$-\,u'(t) = w(t) + f(t) \text{ for almost all } t \in T,\; u(0) = u(b), \tag{34}$$

with $w, f \in L^2(T,H), w(t) \in \partial\varphi(t,u(t))$ and $f(t) \in F(t,u(t))$ for almost all $t \in T$. Let $K \subseteq C(T,H)$ be the compact set produced in the proof of Theorem 4.1. Given $v \in K$ and $\epsilon > 0$, we consider the multifunction $G^v_\epsilon : T \to 2^H \backslash \{\emptyset\}$ defined by

$$G^v_\epsilon(t) = \{h \in H : ||f(t) - h|| < \frac{\epsilon}{2\eta_0 b} + d(f(t), F(t,v(t))),\ h \in F(t,v(t))\},$$

with $\eta_0 > 0$ such that $\sup\left[||y||_{C(T,H)} : y \in K\right] \le \eta_0$. Evidently

$$\operatorname{Gr} G^v_\epsilon \in \mathcal{L}_T \times B(H).$$

Hence using the Yankov-von Neumann-Aumann selection theorem (see Hu and Papageorgiou [14, p. 158]), we produce $h \in S^2_{F(\cdot,v(\cdot))}$ such that

$$h(t) \in G^v_\epsilon(t) \text{ for almost all } t \in T.$$

Then let $R_\epsilon : K \to 2^{L^1(T,H)}$ be the multifunction defined by

$$R_\epsilon(v) = \Big\{h \in S^2_{F(\cdot,v(\cdot))} : ||f(t) - h(t)|| < \frac{\epsilon}{2\eta_0 b} + d(f(t), F(t,v(t))) \\ \text{for almost all } t \in T\Big\}.$$

From the first part of the proof we have

$$R_\epsilon(v) \neq \emptyset \text{ for all } v \in K.$$

Clearly the values of $R_\epsilon(\cdot)$ are decomposable sets and using Lemma II.8.3 of Hu and Papageorgiou [14, p. 239], we see that $R_\epsilon(\cdot)$ is lsc. Hence so is $v \mapsto \overline{R_\epsilon(v)}$ and we can use the Bressan-Colombo selection theorem [4] and produce a continuous map $r_\epsilon : K \to L^2(T,H)$ such that

$$r_\epsilon(v) \in \overline{R_\epsilon(v)} \text{ for all } v \in K.$$

Then we have

$$||f(t) - r_\epsilon(v)(t)|| \le \frac{\epsilon}{2\eta_0 b} + k(t)||u(t) - v(t)|| \text{ for almost all } t \in T \tag{35}$$

(see hypothesis $H(F)_4(ii)$).

Also, on account of Theorem II.8.31 of Hu and Papageorgiou [14, p. 260] we can find $s_\epsilon : K \to L^1_w(T,H)$ a continuous map such that

$$s_\epsilon(v) \in \operatorname{ext} S^2_{F(\cdot,v(\cdot))} = S^2_{\operatorname{ext} F(\cdot,v(\cdot))} \text{ and } ||r_\epsilon(v) - s_\epsilon(v)||_w < \epsilon \text{ for all } v \in K \tag{36}$$

(see Hu and Papageorgiou [14, Theorem II.4.6, p. 192]).

Now let $\epsilon_n \downarrow 0$ and set $r_n = r_{\epsilon_n},\ s_n = s_{\epsilon_n},\ z = u(0) = u(b)$. We consider the following boundary value problem (Dirichlet problem)

$$-u'_n(t) \in \partial\varphi(t,u_n(t)) + s_n(u_n)(t) \text{ for almost all } t \in T, u_n(0) = z. \tag{37}$$

We know that (37) has a strong solution $u_n \in W^{1,2}((0,b),H)$ and $\{u_n\}_{n\ge1} \subseteq C(T,H)$ is relatively compact (see Hu and Papageorgiou [15, p. 137]). So, we may assume that

$$u_n \to \tilde{u} \text{ in } C(T,H) \text{ as } n \to \infty. \tag{38}$$

We subtract (37) from (34), take inner product with $u_n(t) - u(t)$ and use the monotinicity of $\partial\varphi(t,\cdot)$. We obtain

$$(u'_n(t) - u'(t), u_n(t) - u(t))_H + (f(t) - s_n(u_n)(t), u_n(t) - u(t))_H \le 0 \tag{39}$$
$$\text{for almost all } t \in T.$$

We know that

$$(u_n'(t) - u'(t), u_n(t) - u(t))_H = \frac{1}{2}\frac{d}{dt}||u_n(t) - u(t)||^2.$$

Using this in (39), integrating over $[0,t]$ and using that $u_n(0) = z \in K_0$, we obtain

$$\begin{aligned}\frac{1}{2}||u_n(t) - u(t)||^2 &\leq \int_0^t (f(s) - s_n(u_n)(s), u(s) - u_n(s))_H ds \\ &\leq \int_0^t ||f(s) - r_n(u_n)(s)||\ ||u(s) - u_n(s)|| ds \\ &\quad + \int_0^t (r_n(u_n)(s) - s_n(u_n)(s), u(s) - u_n(s))_H ds. \quad (40)\end{aligned}$$

Recall that

$$\begin{aligned}& ||r_n(u_n) - s_n(u_n)||_w \leq \epsilon_n \text{ for all } n \in \mathbb{N} \text{ (see (36))}, \\ \Rightarrow\ & r_n(u_n) - s_n(u_n) \xrightarrow{||\cdot||_w} 0, \\ \Rightarrow\ & r_n(u_n) - s_n(u_n) \xrightarrow{w} 0 \text{ in } L^2(T, H) \\ & \text{(see Hu and Papageorgiou [15, Lemma I.2.8, p. 24])}, \\ \Rightarrow\ & \int_0^t (r_n(u_n)(s) - s_n(u_n)(s), u(s) - u_n(s))_H ds \to 0 \text{ (see (38))}. \quad (41)\end{aligned}$$

Also we have

$$\begin{aligned}& \int_0^t ||f(s) - r_n(u_n)(s)||\ ||u(s) - u_n(s)|| ds \\ & \leq \epsilon_n + \int_0^t k(s)||u(s) - u_n(s)||^2 ds \text{ for all } n \in \mathbb{N} \text{ (see (35))}, \\ \Rightarrow\ & \limsup_{n\to\infty} \int_0^t ||f(s) - r_n(u_n)(s)||\ ||u(s) - u_n(s)|| ds \leq \\ & \qquad \int_0^t k(s)||u(s) - \tilde{u}(s)|| ds \text{ (see (38))}. \quad (42)\end{aligned}$$

So, if we return to (40) and pass to the limit as $n \to \infty$, then on account of (41), (42) and (38), we obtain

$$\begin{aligned}& \frac{1}{2}||u(t) - \tilde{u}(t)||^2 \leq \int_0^t k(s)||u(s) - \tilde{u}(s)||^2 ds, \\ \Rightarrow\ & u = \tilde{u} \text{ (by Gronwall's inequality)}.\end{aligned}$$

Then from (38) we have

$$u_n \to u \text{ in } C(T, H) \text{ with } u_n \in S_e(u(0)) \text{ for all } n \in \mathbb{N}.$$

So, $S_c \subseteq \bar{S}_e^{C(T,H)}$ (that is, (33) holds). □

7. **Examples.** In this section, we present some examples illustrating the results of this paper.

(a) Let $\Omega \subseteq \mathbb{R}^N$ ba a bounded domain with a C^2-boundary $\partial\Omega$. We consider the following periodic parabolic problem

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} - \operatorname{div}\left(a(t,z)|Du|^{p-2}Du\right) = f(t,z,u(t,z)) & \text{in } T \times \Omega, \\ a(t,z)\frac{\partial u(t,z)}{\partial n_p} \in \beta(z,u(t,z)) & \text{on } T \times \partial\Omega, \\ u(0,z) = u(b,z) & \text{for almost all } z \in \Omega. \end{array} \right\} \tag{43}$$

In this problem, the reaction term $f(t,z,x)$ is a measurable function which is discontinuous in $x \in \mathbb{R}$ with jump discontinuities. Following Chang [6], we replace problem (43) by a parabolic inclusion, in which $f(t,z,x)$ is replaced by a multifunction, which is obtained by filling in the gaps at the discontinuity points of $f(t,z,\cdot)$. So, we introduce

$$f_l(t,z,x) = \liminf_{x' \to x} f(t,z,x'),$$
$$f_u(t,z,x) = \limsup_{x' \to x} f(t,z,x').$$

Using f_l and f_u, we define the multifunction

$$\hat{f}(t,z,x) = [f_l(t,z,x), f_u(t,z,x)].$$

Then instead of (43), we investigate the following parabolic inclusion:

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} - \operatorname{div}\left(a(t,z)|Du|^{p-2}Du\right) \in \hat{f}(t,z,u(t,z)) & \text{in } T \times \Omega, \\ a(t,z)\frac{\partial u}{\partial n_p} \in \beta(z,u(t,z)) & \text{on } T \times \partial\Omega \\ u(0,z) = u(b,z) & \text{for almost all } z \in \Omega. \end{array} \right\} \tag{44}$$

Recall that $\frac{\partial u}{\partial n_p}$ denotes the generalized normal derivative defined by

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\mathbb{R}^N}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

The hypotheses on the data of problem (44) are the following:

$H(a) : T \times \overline{\Omega} \to \mathbb{R}$ is a measurable function such that
(i) $0 < \hat{c} \leq a(t,z) \leq M$ for all $(t,z) \in T \times \overline{\Omega}$;
(ii) for almost all $z \in \overline{\Omega},\ t \mapsto a(t,z)$ is Lipschitz continuous and $a(0,z) = a(b,z)$

$H(f) : f : T \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that
(i) $|f(t,z,x)| \leq \hat{c}_1(t,z) + \hat{c}_2(t,z)|x|$ for almost all $(t,z) \in T \times \Omega$ with $\hat{c}_1, \hat{c}_2 \in L^2(T \times \Omega)$;
(ii) both f_l and f_u are superpositionally measurable, that is, if $u : T \times \Omega \to \mathbb{R}$ is measurable, then so are $(t,z) \mapsto f_l(t,z,u(t,z))$ and $(t,z) \mapsto f_u(t,z,u(t,z))$;
(iii) $f_l(t,z,x)x \geq \hat{c}_3 x^2 - \hat{c}_4(t,z)$ for almost all $(t,z) \in T \times \Omega$, all $x \in \mathbb{R}$, with $\hat{c}_3 > 0, \hat{c}_4 \in L^\infty(T \times \Omega)$

$H(\beta) : \beta(z,x) = \partial j(z,x)$ with $j : \partial\Omega \times \mathbb{R} \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ a normal convex integrand (see Hu and Papageorgiou [14, p. 264]) such that

$$h(z) \leq j(z,x) \text{ for almost all } z \in \partial\Omega,\ \text{all } x \in \mathbb{R}$$

with $h \in L^1(\partial\Omega)$ (on $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$).

Let $H = L^2(\Omega)$ and consider the functional $\varphi : T \times H \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi(t,u) = \begin{cases} \frac{1}{p}\int_\Omega a(t,z)|Du|^p dz + \int_{\partial\Omega} j(z,u)d\sigma & \text{if } j(\cdot,u(\cdot)) \in L^1(\partial\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Hypotheses $H(a), H(\beta)$ imply that conditions $H(\varphi)$ are satisfied.

Also, let

$$F(t,u) = S^2_{\hat{f}(t,\cdot,u(\cdot))} \text{ for all } (t,u) \in T \times H.$$

Then hypotheses $H(f)$ imply that conditions $H(F)_1$ hold (see Hu and Papageorgiou [14, p. 38]).

We rewrite (44) as the following equivalent abstract subdifferential inclusion

$$\left\{ \begin{array}{ll} -u'(t) \in \partial\varphi(t,u(t)) + F(t,u(t)) & \text{for almost all } t \in T, \\ u(0) = u(b). & \end{array} \right\}.$$

The equivalence follows from the nonlinear Green's identity (see Gasinski and Papageorgiou [9, p. 211]). We apply Theorem 3.2 and produce a solution $u \in W^{1,2}((0,b), L^2(\Omega))$ for problem (44).

(b) Consider the following parabolic system

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} - \operatorname{div}(A(t,z)Du) \in \operatorname{ext} \hat{f}(t,z,u(t,z)) & \text{in } T \times \Omega, \\ \frac{\partial u}{\partial n}(t,z) = 0 \text{ on } T \times \partial\Omega, u(0,z) = u(b,z) \text{ for almost all } z \in \Omega\,. & \end{array} \right\} \tag{45}$$

The hypotheses on the data of this problem are the following.

$H(A)$: $A : T \times \Omega \to GL(N,\mathbb{R})_+$ is measurable and for almost all $z \in \Omega$, $A(\cdot,z)$ is Lipschitz (here $GL(N,\mathbb{R})_+$ denotes the $N \times N$ invertible matrices which are positive).

$H(\hat{f})$: $\hat{f} : T \times \Omega \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that
(i) for all $x \in \mathbb{R}^N$, $(t,z) \mapsto \hat{f}(t,z,x)$ is measurable;
(ii) for almost all $(t,z) \in T \times \Omega$, $x \mapsto \hat{f}(t,z,x)$ is h-continuous;
(iii) $|\hat{f}(t,z,x)| \leq \hat{c}(t,z)(1+|x|)$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $\hat{c} \in L^2(T \times \Omega)$;
(iv) for almost all $(t,z) \in T \times \Omega$, all $x \in \mathbb{R}^N$ and all $v \in F(t,z,x)$, we have

$$(v,x)_{\mathbb{R}^N} \geq \hat{c}_1|x|^2 - \hat{c}_2(t,z)$$

with $\hat{c}_1 > 0$ and $\hat{c}_2 \in L^\infty(T, L^1(\Omega))$.

Let $H = L^2(\Omega,\mathbb{R}^N)$ and define

$$\varphi(t,u) = \begin{cases} \frac{1}{2}\int_\Omega (A(t,z)Du, Du)_{\mathbb{R}^N} dz & \text{if } u \in H^1(\Omega,\mathbb{R}^N) \\ +\infty & \text{otherwise} \end{cases}$$

$$F(t,u) = S^2_{\hat{f}(t,\cdot,u(\cdot))} \text{ for all } (t,u) \in T \times L^2(\Omega,\mathbb{R}^N).$$

The problem (45) is equivalent to the following subdifferential evolution inclusion

$$\left\{ \begin{array}{ll} -u'(t) \in \partial\varphi(t,u(t)) + F(t,u(t)) & \text{for almost all } t \in T, \\ u(0) = u(b). & \end{array} \right\}.$$

On account of hypotheses $H(A)$ and $H(\hat{f})$, conditions $H(\varphi)$ and $H(F)_3$ are satisfied. So, we can apply Theorem 5.1 and produce a solution $u \in W^{1,2}((0,b), L^2(\Omega,\mathbb{R}^N))$ for problem (45). Moreover, if $\hat{f}(t,z,\cdot)$ is h-Lipschitz with constant $k \in L^1(T \times \Omega)$, then the solutions of problem (45) are $C(T, L^2(\Omega,\mathbb{R}^N))$-dense in the solution set of the convexified problem, that is, problem (45) with ext $\hat{f}(t,z,x)$ replaced by $\overline{\text{conv}}\,\text{ext}\,\hat{f}(t,z,x)$ (see Theorem 6.1).

(c) Finally we consider the following variational differential inequality in $\mathbb{R}^N$

$$\left\{\begin{array}{ll} -u'(t) \in N_{K(t)}(u(t)) + \partial\varphi_1(u(t)) + F(t,u(t)) & \text{for almost all } t \in T, \\ u(0) = u(b). & \end{array}\right\} \tag{46}$$

In this problem, $N_{K(t)}(x)$ denotes the normal cone to the set $K(t) \subseteq \mathbb{R}^N$ at the point $x \in \mathbb{R}^N$ (see Hu and Papageorgiou [14, p. 634]). Systems like (46) arise in mathematical economics in the study of optimal planning of resources (see Henry [12] and Cornet [7]). More generally, such inclusions describe systems with constraints. For such systems, in describing the effect of the constraints on the dynamical equation, in many cases it can be assumed that the velocity $u'(t)$ is projected at each time instant to the set of allowed directions toward the constraint set at $u(t)$. This is true, for example, in electrical networks with diode nonlinearities (see Krasnoselskii and Pokrovskii [16]).

Suppose that $K : T \to P_{kc}(\mathbb{R}^N)$ is an h-Lipschitz multifunction and $\varphi_1 : \mathbb{R}^N \to \mathbb{R}$ is strictly convex and coercive. We set

$$\varphi(t,x) = \delta_{K(t)}(x) + \varphi_1(x).$$

Note that

$$\partial\varphi(t,x) = N_{K(t)}(x) + \partial\varphi_1(x) \text{ for all } (t,x) \in T \times \mathbb{R}^N$$

(see Hu and Papageorgiou [14, Theorem III.4.29, p. 329]). Hence $\varphi(t,x)$ satisfies conditions $H(\varphi)$.

Suppose that the constraint set $K(t)$ is defined by

$$K(t) = \{x \in \mathbb{R}^N : \hat{u}_1(t) \le x \le \hat{u}_2(t)\},$$

with $\hat{u}_1, \hat{u}_2 : T \to \mathbb{R}^N$ Lipschitz continuous maps, $\hat{u}_1 \le \hat{u}_2$.

Assume that $F : T \times \mathbb{R}^N \to P_f(\mathbb{R}^N)$ satisfies hypotheses $H(F)_2$. Using Theorem 4.1 we can find $u \in W^{1,2}((0,b),\mathbb{R}^N)$ such that

$$\begin{aligned} &-u'(t) \in \partial\varphi_1(u(t)) + F(t,u(t)) \text{ for almost all } t \in \{\hat{u}_1 < u < \hat{u}_2\} \\ &-u'(t) \in \partial\varphi_1(u(t)) + F(t,u(t)) + \mathbb{R}^N_+ \text{ for almost all } t \in \{\hat{u}_1 = u\} \\ &-u'(t) \in \partial\varphi_1(u(t)) + F(t,u(t)) - \mathbb{R}^N_+ \text{ for almost all } t \in \{\hat{u}_2 = u\}. \end{aligned}$$

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