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The Ekeland variational principle for equilibrium problems revisited and applications

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ABSTRACT

In this paper, we study the Ekeland variational principle for equilibrium problems under the setting of real Banach spaces. We make use of techniques related to infinite dimensional spaces to solve the weakly compact case and introduce a suitable weakened set of coerciveness to deal with the non weakly compact case. Some old results for the Euclidean space $\mathbb{R}^n$ are generalized under weakened conditions of semicontinuity and applications to countable rather than finite systems of equilibrium problems on real Banach spaces are derived. Applications to quasi-hemivariational inequalities are discussed.

1. Introduction

The Ekeland variational principle is a powerful tool which entails the existence of approximate solutions of minimization problems for lower semicontinuous functions on complete metric spaces. It is widely used to solve different problems of differential equations and partial differential equations, fixed point theory, optimization, mathematical programming and many other problems of nonlinear analysis. Roughly speaking, Ekeland's variational principle states that there exist points which are almost points of minima and where the gradient is small. In particular, it is not always possible to minimize a nonnegative continuous function on a complete metric space. Ekeland's variational principle is a very basic tool that is effective in numerous situations, which led to many new results and strengthened a series of known results in various fields of analysis, geometry, the Hamilton–Jacobi theory, extremal problems, the Ljusternik–Schnirelmann theory, etc. The Ekeland variational principle [1] was established in 1974 and is the nonlinear version of the Bishop–Phelps theorem [2,3], with its main feature of how to use the norm completeness and a partial ordering to obtain a point where a linear functional achieves its supremum on a closed bounded convex set.

The so-called equilibrium problem is a problem of finding $x^* \in A$ such that

$$\Phi\left(x^*, y\right) \geq 0 \quad \text{for all } y \in A, \tag{EP}$$

where A is a given set and $\Phi : A \times A \longrightarrow \mathbb{R}$ is a bifunction, called equilibrium bifunction if $\Phi(x, x) = 0$, for every $x \in E$. Such a problem is also known under the name of equilibrium problem in the sense of Blum, Muu and Oettli (see [4,5]) or Ky Fan equilibrium problem (see [6]).

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It is well known that variational inequalities, mathematical programming, Nash equilibrium, optimization and many other problems arising in nonlinear analysis are special cases of equilibrium problems, see for example [7–9] and the references therein.

In this paper, the set A is not assumed to be necessarily convex which leads to an interesting case of equilibrium problems called sometimes nonconvex equilibrium problems, see [10] and the references therein.

The Ekeland variational principle for equilibrium problems has been first considered in [7,11] in the setting of the Euclidean space $\mathbb{R}^n$. It has also been the subject of study in [12] for vector equilibrium problems defined on complete metric spaces and with values in locally convex spaces ordered by closed convex cones. Latter on, several authors have been interested in such problems and several results have been obtained under various kinds of generalized metric spaces including quasi-metric spaces with different types of functions such as τ-functions and Q-functions. Many connections with fixed point theory such as the Caristi–Kirk-type fixed point theorem for multivalued mappings have been derived.

In [7,11], the authors mention that the results of the Ekeland variational principle for equilibrium problems and systems of equilibrium problems obtained on compact and on closed subsets of Euclidean spaces could be extended to reflexive Banach spaces. Motivated by this question, we establish that this is not systematic and depends also on other properties and especially, on the finite or infinite dimension of the space.

In this paper, we obtain under weakened conditions of semicontinuity an existence result for equilibrium problems defined on weakly compact subsets of real (non necessarily reflexive) Banach spaces. Then, we introduce a weakly compact set of coerciveness rather than closed balls in order to solve equilibrium problems defined on weakly closed subsets of real reflexive Banach spaces.

Instead of finite systems of equilibrium problems on the Euclidean space $\mathbb{R}^n$, we consider here countable systems of equilibrium problems defined on real Banach spaces. Then, we solve the problem when it is defined on weakly sequentially compact subsets and make use of the properties of countable product of real Banach spaces to solve the problems when it is defined on weakly closed subsets.

In the last part of this paper, we give a discussion in order to make connection between the Ekeland variational principle and fixed point theory. Such connection may be useful in certain situations and in particular, when dealing with variational inequalities. To our knowledge, there does not seem to be any result in the literature with application of the Ekeland variational principle in the analysis studies of variational inequalities. In this direction, we consider quasi-hemivariational inequalities introduced in [13]. Quasi-hemivariational inequalities are generalization of multivalued variational inequalities and several connections with equilibrium problems are obtained, see also [14–16,10] for recent and old investigations on the subject. Our approach is based on a result of [15] stating that a point is a solution of a quasi-hemivariational inequality if and only if it is a fixed point of a suitable multivalued mapping.

2. Notations and preliminaries

As it is well known, the Ekeland variational principle characterizes the completeness of a metric space and has many applications in nonlinear analysis. Several forms and variants of the Ekeland variational principle exist in the literature where the following result summarizes the relationship between this famous variational principle and the completeness of metric spaces.

Theorem 2.1. *Let (X, d) be a metric space. Then X is complete if and only if for every lower semicontinuous and lower bounded function $f : X \to \mathbb{R} \cup \{+\infty\}$ and for every $\varepsilon > 0$ there exists a point $x^* \in X$ satisfying*

$$\begin{cases} f\left(x^*\right) \leq \min\limits_{x\in X} f\left(x\right) + \varepsilon, \\ f\left(x\right) - f\left(x^*\right) + \varepsilon d\left(x^*, x\right) \geq 0, \quad \forall x \in X. \end{cases}$$

Since the problem of minimization is a special case of equilibrium problems, an extended form of the Ekeland variational principle called *Ekeland variational principle for equilibrium problems* has been first introduced in [11,7] in the setting of the Euclidean space $\mathbb{R}^n$. It is followed by several extensions under different kinds of conditions by several authors, see for instance, [17].

Here is the complete metric version of the Ekeland variational principle for equilibrium problems.

Theorem 2.2. *Let A be a nonempty closed subset of a complete metric space (X, d) and $\Phi : A \times A \to \mathbb{R}$ be a bifunction. Assume that the following conditions hold:*

(1) *$\Phi\left(x, x\right) = 0$, for every $x \in A$;*
(2) *$\Phi\left(z, x\right) \leq \Phi\left(z, y\right) + \Phi\left(y, x\right)$, for every $x, y, z \in A$;*
(3) *Φ is lower bounded and lower semicontinuous in its second variable.*

Then, for every $\varepsilon > 0$ and for every $x_0 \in A$, there exists $x^* \in A$ such that

$$\begin{cases} \Phi\left(x_0, x^*\right) + \varepsilon d\left(x_0, x^*\right) \leq 0, \\ \Phi\left(x^*, x\right) + \varepsilon d\left(x^*, x\right) > 0, \quad \forall x \in A, \ x \neq x^*. \end{cases}$$

In the sequel, we make use of some techniques of semicontinuity of functions on a subset introduced recently in [18,16,10] to obtain new results concerning the existence of solutions of nonconvex equilibrium problems.

Let X be Hausdorff topological space, $x \in X$ and $f : X \longrightarrow \mathbb{R}$ a function. Following [16], the function f is said to be

(1) *sequentially upper semicontinuous at x* if for every sequence $(x_n)_n$ in X converging to x, we have

$$f(x) \geq \limsup_{n \to +\infty} f(x_n)$$

where $\limsup_{n \to +\infty} f(x_n) = \inf_n \sup_{k \geq n} f(x_k)$.

(2) *sequentially lower semicontinuous at x* if $-f$ is sequentially upper semicontinuous at x.

The function f is said to be *sequentially upper* (resp. *sequentially lower*) *semicontinuous* on a subset S of X if it is sequentially upper (resp. sequentially lower) semicontinuous at every point of S. Obviously, when X is a metrizable space, sequential lower and upper semicontinuity coincide respectively with the classical notions of lower and upper semicontinuity.

In the sequel, for $r > 0$, we denote

$$K_r = \{x \in E \mid \|x\| \leq r\}$$

the closed ball of center 0 and radius r of a real Banach space E.

3. Existence results for nonconvex equilibrium problems

Before going further, let us recall that a Hausdorff topological space is called sequentially compact if every sequence has a converging subsequence. A subset A of a Hausdorff topological space is called sequentially compact if it sequentially compact as a topological subspace. In general, sequentially compact spaces are different from compact spaces. However, in the setting of Banach spaces, a subset is weakly sequentially compact if and only if it is weakly compact (Eberlein–Šmulian theorem, see [19]).

The following result is a generalization of [7, Proposition 2] (see also [11]) to weakly compact (not necessarily convex) subset of real Banach spaces. This result guarantees the existence of solutions to equilibrium problems in the weakly compact case.

Proposition 3.1. *Let A be a nonempty weakly compact subset of a real Banach space E and $\Phi : A \times A \to \mathbb{R}$ be a bifunction. Assume that the following conditions hold:*

(1) $\Phi(x, x) = 0$, *for every* $x \in A$;
(2) $\Phi(z, x) \leq \Phi(z, y) + \Phi(y, x)$, *for every* $x, y, z \in A$;
(3) Φ *is lower bounded and lower semicontinuous in its second variable.*
(4) Φ *is weakly sequentially upper semicontinuous in its first variable.*

Then, the equilibrium problem (EP) has a solution.

Proof. By Theorem 2.2, for every $n \in \mathbb{N}^*$, let $x_n \in A$ be a $\frac{1}{n}$-solution of the equilibrium problem (EP). Therefore

$$\Phi(x_n, y) \geq -\frac{1}{n}\|x_n - y\| \quad \forall y \in A.$$

Since A is weakly compact, the sequence $(x_n)_n$ has a weakly converging subsequence $\left(x_{n_k}\right)_k$ to some $x^* \in A$. Since Φ is weakly sequentially upper semicontinuous in its first variable on A, we have

$$\begin{aligned}\Phi\left(x^*, y\right) &\geq \limsup_{k \to +\infty} \Phi\left(x_{n_k}, y\right) \\ &\geq \limsup_{k \to +\infty} \left(-\frac{1}{n_k}\|x_{n_k} - y\|\right) = 0 \quad \forall y \in A.\end{aligned}$$

This means that x^* is a solution to the equilibrium problem (EP). □

The following result is a generalization of [7, Theorem 15] (see also [11]) and guarantees the existence of solutions to equilibrium problems in the non weakly compact case. Instead of Euclidean space $\mathbb{R}^n$, this generalization makes more clear in the setting of infinite dimensional spaces, the conditions imposed on both A and the subset of coerciveness. This generalization is also obtained under weakened conditions of semicontinuity of the bifunction involved.

Theorem 3.2. *Let A be a nonempty weakly closed subset of a real reflexive Banach space E and $\Phi : A \times A \to \mathbb{R}$ be a bifunction, and suppose the following conditions hold:*

(1) $\Phi(x, x) = 0$, *for every* $x \in A$;
(2) $\Phi(z, x) \leq \Phi(z, y) + \Phi(y, x)$, *for every* $x, y, z \in A$;

(3) *there exists a nonempty weakly compact subset K of A such that*

$$\forall x \in A \setminus K,\ \exists y \in A,\ \|y\| < \|x\|, \quad \Phi(x, y) \le 0;$$

(4) *Φ is weakly sequentially lower semicontinuous in its second variable on K;*
(5) *the restriction of Φ on $K \times K$ is lower bounded in its second variable;*
(6) *the restriction of Φ on $K \times K$ is weakly sequentially upper semicontinuous in its first variable.*

Then, the equilibrium problem (EP) has a solution.

Proof. For every $x \in A$, define the subset

$$L(x) = \{y \in A \mid \|y\| \le \|x\|, \Phi(x, y) \le 0\},$$

and put $S(x) = cl_A(L(x))$, where the closure is taken with respect to the weak topology of A. We have the following properties:

(1) *The subset $S(x)$ is nonempty, for every $x \in A$.* This holds easily from the fact that $x \in L(x)$.
(2) *The subset $S(x)$ is weakly compact, for every $x \in A$.* Indeed, for every $x \in A$, the subset $L(x)$ is contained in the weakly compact subset $K_{\|x\|}$ and then, $S(x)$ is weakly compact.
(3) *For every $x, y \in A$, if $y \in S(x)$, then $S(y) \subset S(x)$.* Indeed, since $L(y)$ is bounded, then for every $z \in S(y)$, there exists a sequence $(z_n)_n$ in $L(y)$ weakly converging to z, (see for example, [20, Proposition 3.6.23]). It follows that $\|z_n\| \le \|y\| \le \|x\|$ and $\Phi(x, z_n) \le \Phi(x, y) + \Phi(y, z_n) \le 0$, for every n. It follows that the sequence $(z_n)_n$ lies in $L(x)$ and then, $z \in S(x)$.

On the other hand, the restriction of Φ on $K \times K$ satisfies all the conditions of Proposition 3.1 and therefore, there exists $x^* \in K$ such that

$$\Phi\left(x^*, y\right) \ge 0 \quad \forall y \in K.$$

Suppose that x^* is not a solution of the equilibrium problem (EP) and let $x \in A$ such that $\Phi(x^*, x) < 0$. Since $S(x)$ is nonempty weakly compact subset, then the norm, which is weakly lower semicontinuous, attains its lower bound on $S(x)$. Let $y_x \in S(x)$ be such that

$$\|y_x\| = \min_{y \in S(x)} \|y\|$$

and since $L(x)$ is bounded, let $(y_n)_n$ be a weakly converging sequence in $L(x)$ to y_x. We have two cases:

- Suppose first that $y_x \in K$. Since $\Phi(x^*, x) < 0$, choose $\varepsilon > 0$ such that $\Phi(x^*, x) \le -\varepsilon$. Since $\Phi(x, y_n) \le 0$, for every n, then

  $$\Phi\left(x^*, y_n\right) \le \Phi\left(x^*, x\right) + \Phi(x, y_n) \le -\varepsilon.$$

  The bifunction Φ being weakly sequentially lower semicontinuous in its second variable on K, we obtain

  $$\Phi\left(x^*, y_x\right) \le \liminf_{n \to +\infty} \Phi\left(x^*, y_n\right) \le -\epsilon < 0$$

  which yields a contradiction.
- Suppose now that $y_x \notin K$. Then, there exists $y_1 \in A$, $\|y_1\| < \|y_x\|$ and $\Phi(y_x, y_1) \le 0$. Thus,

  $$y_1 \in S(y_x) \subset S(x) \quad \text{and} \quad \|y_1\| < \|y_x\| = \min_{y \in S(x)} \|y\|$$

  which is impossible.

The proof is complete. □

4. Existence results for nonconvex countable systems of equilibrium problems

Inspired by the study of systems of variational inequalities, countable and non countable systems of equilibrium problems have been introduced and investigated in the literature, see, for instance, [21,22]. Instead of finite systems of equilibrium problems studied in [11,7] by means of the Ekeland variational principle, we consider here countable systems of equilibrium problems which are usually defined in the following manner.

Let I be a countable index set which could be identified sometimes to the set $\{i \mid i \in \mathbb{N}\}$. By a system of equilibrium problems we understand the problem of finding $x^* = \left(x_i^*\right)_{i \in I} \in A$ such that

$$\Phi_i\left(x^*, y_i\right) \ge 0 \quad \text{for all } i \in I \text{ and all } y_i \in A_i, \tag{SEP}$$

where $\Phi_i : A \times A_i \to \mathbb{R}$, $A = \prod_{i \in I} A_i$ with A_i some given set.

In the sequel, we suppose that A_i is a closed subset of a metric space (X_i, d_i), for every $i \in I$. An element of the set $A^i = \prod_{j \in I, j \ne i} A_j$ will be represented by x^i; therefore, $x \in A$ can be written as $x = \left(x^i, x_i\right) \in A^i \times A_i$. The space $X = \prod_{i \in I} X_i$ will be endowed by the product topology. Without loss of generality, we may assume that d_i is bounded by 1, for every $i \in I$. The distance d on X defined by

$$d(x, y) = \sum_{i\in I} \frac{1}{2^i} d_i(x_i, y_i) \quad \text{for every } x = (x_i)_{i\in I}, \ y = (y_i)_{i\in I} \in X$$

is a complete metric compatible with the topology of X. Thus, the space (X, d) is a complete metric space.

The following result is the Ekeland variational principle for countable systems of equilibrium problems defined on complete metric spaces. It generalizes [7, Theorem 14] (see also [11, Theorem 2.2]) stated for finite systems of equilibrium problems under the setting of the Euclidean space $\mathbb{R}^n$.

Theorem 4.1. *Let A_i be a nonempty closed subset of a complete metric space (X_i, d_i), for every $i \in I$, and assume that the following conditions hold:*

(1) $\Phi_i(x, x_i) = 0$, *for every $i \in I$ and every $x = (x^i, x_i) \in A$;*
(2) $\Phi_i(z, x_i) \le \Phi_i(z, y_i) + \Phi_i(y, x_i)$, *for every $i \in I$, every $x_i, y_i \in A_i$ and every $y, z \in A$ such that $y = (y^i, y_i)$;*
(3) Φ_i *is lower bounded and lower semicontinuous in its second variable, for every $i \in I$.*

Then, for every $\varepsilon > 0$ and for every $x^0 = (x_i^0)_{i\in I} \in A$, there exists $x^* = (x_i^*)_{i\in I} \in A$ such that for each $i \in I$, we have

$$\begin{cases} \Phi_i(x^0, x_i^*) + \varepsilon d_i(x_i^0, x_i^*) \le 0, \\ \Phi_i(x^*, x_i) + \varepsilon d_i(x_i^* - x_i) > 0, \quad \forall x_i \in A_i, \ x_i \ne x_i^*. \end{cases}$$

Proof. By replacing Φ by $\frac{1}{\varepsilon}\Phi$, we may assume without loss of generality that $\varepsilon = 1$. Let $i \in I$ be arbitrarily fixed, and for every $x = (x_i)_{i\in I} \in A$, put

$$F_i(x) = \{y_i \in A_i \mid \Phi_i(x, y_i) + d_i(x_i, y_i) \le 0\}.$$

Clearly, these subsets are closed and nonempty since $x_i \in F_i(x)$, for every $i \in I$. In addition, if $y_i \in F_i(x)$, for some $x \in A$, $y_i \in A_i$ and $i \in I$, then $F_i(y) \subset F_i(x)$, for every $y = (y^i, y_i) \in A$. Indeed, suppose these conditions hold and let $z = (z_i)_{i\in I} \in F_i(y)$. Then, we have

$$\Phi_i(x, y_i) + d_i(x_i, y_i) \le 0 \quad \text{and} \quad \Phi_i(y, z_i) + d_i(y_i, z_i) \le 0.$$

It follows by addition that

$$\Phi_i(x, z_i) + d_i(x_i, z_i) \le \Phi_i(x, y_i) + \Phi_i(y, z_i) + d_i(x_i, y_i) + d_i(y_i, z_i) \le 0$$

and then, $z \in F_i(x)$.

For every $x \in A$, define now

$$v_i(x) = \inf_{z_i\in F_i(x)} \Phi_i(x, z_i)$$

which is finite since Φ_i is lower bounded in its second variable, for every $i \in I$.

For every $z_i \in F_i(x)$, we have

$$d(x_i, z_i) \le -\Phi_i(x, z_i) \le -\inf_{z_i\in F_i(x)} \Phi_i(x, z_i) = -v_i(x).$$

It follows that $\delta(F_i(x)) \le -2v_i(x)$, for every $i \in I$ and every $x \in A$, where $\delta(S)$ stands for the diameter of the set S.

Fix now $x^0 = (x_i^0)_{i\in I} \in A$ and choose for each $i \in I$ an element $x_i^1 \in F_i(x^0)$ such that

$$\Phi_i(x^0, x_i^1) \le v_i(x^0) + 2^{-1}.$$

Put $x^1 = (x_i^1)_{i\in I} \in A$ and for each $i \in I$ an element $x_i^2 \in F_i(x^0)$ such that

$$\Phi_i(x^1, x_i^2) \le v_i(x^1) + 2^{-2}.$$

Put $x^2 = (x_i^2)_{i\in I} \in A$. Proceeding by induction, we construct a sequence $(x^n)_n$ in A such that $x_i^{n+1} \in F_i(x^n)$ and

$$\Phi_i(x^n, x_i^{n+1}) \le v_i(x^n) + 2^{-(n+1)}, \quad \text{for every } i \in I \text{ and every } n \in \mathbb{N}.$$

Note that

$$\begin{aligned} v_i(x^{n+1}) &= \inf_{z_i\in F_i(x^{n+1})} \Phi_i(x^{n+1}, z_i) \\ &\ge \inf_{z_i\in F_i(x^n)} \Phi_i(x^{n+1}, z_i) \end{aligned}$$

$$\begin{aligned}&\geq \inf_{z_i\in F_i(x^n)}\left(\Phi_i\left(x^n,z_i\right)-\Phi_i\left(x^n,x_i^{n+1}\right)\right)\\&=\inf_{z_i\in F_i(x^n)}\Phi_i\left(x^n,z_i\right)-\Phi_i\left(x^n,x_i^{n+1}\right)\\&=v_i\left(x^n\right)-\Phi_i\left(x^n,x_i^{n+1}\right)\end{aligned}$$

which yields

$$\begin{aligned}v_i\left(x^{n+1}\right)&\geq v_i\left(x^n\right)-\Phi_i\left(x^n,x_i^{n+1}\right)\\&\geq v_i\left(x^n\right)-\left(v_i\left(x^n\right)+2^{-(n+1)}\right).\end{aligned}$$

It follows that $v_i\left(x^{n+1}\right)\geq -2^{-(n+1)}$ and then,

$$\delta\left(F_i\left(x^n\right)\right)\leq -2v_i\left(x^n\right)\leq 2\times 2^{-n}.$$

The sequence $(F_i(x^n))_n$ being a decreasing sequence of closed subsets of the complete metric space (E_i,d_i) with diameter tending to zero, then for every $i\in I$, there exists $x_i^*\in A_i$ such that

$$\bigcap_{n\in\mathbb{N}}\delta\left(F_i\left(x^n\right)\right)=\left\{x_i^*\right\}.$$

Put $x^*=\left(x_i^*\right)_{i\in I}\in A$. For every $i\in I$, since $x_i^*\in F_i\left(x^0\right)$, then

$$\Phi_i\left(x^0,x_i^*\right)+d_i\left(x_i^0,x_i^*\right)\leq 0.$$

On the other hand, since $x_i^*\in F_i(x^n)$, then $F_i(x^*)\subset F_i(x^n)$, for every n and then $F_i(x^*)=\left\{x_i^*\right\}$, for every $i\in I$.

Now, if $x_i\in A_i$ is such that $x_i\neq x_i^*$, then $x_i\notin F_i(x^*)$. It follows that

$$\Phi_i\left(x^*,x_i\right)+d_i\left(x_i^*,x_i\right)>0$$

which completes the proof. □

When X_i is replaced by a real Banach space E_i, for every $i\in I$, we denote by $\|\cdot\|_i$ the norm of E_i and by d_i its associate distance. As before, we may assume without loss of generality that each d_i is a bounded metric on E_i, for every $i\in I$. The distance d defined on $E=\prod_{i\in I}E_i$ as above makes E a real complete metric topological vector space. Note that the distance d cannot be induced by a norm since I is infinite. In this case weak sequential compactness and weak compactness need not coincide on E.

In the sequel, E will be endowed with the product of the weak topologies of E_i denoted by σ.

The following result is a generalization of [7, Proposition 2] in the case of real Banach spaces. This result guarantees the existence of solutions to countable systems of equilibrium problems in the weakly compact case. It is also a generalization of Proposition 3.1 and its proof is similar.

Proposition 4.2. *Let A_i be a nonempty weakly closed subset of a real Banach space E_i and $\Phi_i:A\times A_i\to\mathbb{R}$, for every $i\in I$. Assume the following conditions hold:*

(1) *$\Phi_i(x,x_i)=0$, for every $i\in I$ and every $x=\left(x^i,x_i\right)\in A$;*
(2) *$\Phi_i(z,x_i)\leq\Phi_i(z,y_i)+\Phi_i(y,x_i)$, for every $i\in I$, every $x_i,y_i\in A_i$ and every $y,z\in A$ such that $y=\left(y^i,y_i\right)$;*
(3) *Φ_i is lower bounded and lower semicontinuous in its second variable, for every $i\in I$;*
(4) *Φ_i is sequentially upper semicontinuous in its first variable with respect to the topology σ, for every $i\in I$;*
(5) *A is sequentially compact subset of E with respect to the topology σ.*

Then, the system of equilibrium problems (SEP) has a solution.

Proof. The proof is similar to that of Proposition 3.1. By Theorem 4.1, for every $n\in\mathbb{N}^*$, let $x^n=\left(x_i^n\right)_{i\in I}\in A$ be a $\frac{1}{n}$-solution of the system of equilibrium problems (SEP). Therefore

$$\Phi_i\left(x^n,y_i\right)\geq -\frac{1}{n}\|x_i^n-y_i\|_i\quad\forall y_i\in A_i.$$

Since A is sequentially compact subset of E with respect to the topology σ, then the sequence $(x^n)_n$ has a converging subsequence $(x^{n_k})_k$ to some $x^*=\left(x_i^*\right)_{i\in I}\in A$ with respect to the topology σ. It follows that the subsequence $\left(x_i^{n_k}\right)_k$ is weakly converging to $x_i^*\in A_i$, for every $i\in I$. Since Φ_i is sequentially upper semicontinuous in its first variable on A with respect to the topology σ, for every $i\in I$, we have

$$\begin{aligned}\Phi_i\left(x^*,y_i\right)&\geq\limsup_{k\to+\infty}\Phi_i\left(x^{n_k},y_i\right)\\&\geq\limsup_{k\to+\infty}\left(-\frac{1}{n_k}\|x_i^{n_k}-y_i\|\right)=0\quad\forall i\in I,\ \forall y_i\in A.\end{aligned}$$

This means that x^* is a solution to the system of equilibrium problems (SEP). □

The following result is a generalization of [7, Theorem 15] and guarantees the existence of solutions to countable systems of equilibrium problems in the non weakly compact case.

Theorem 4.3. *Let A_i be a nonempty weakly closed subset of a real reflexive Banach space E_i and $\Phi_i : A \times A_i \to \mathbb{R}$, for every $i \in I$. Assume that the following conditions hold:*

(1) $\Phi_i(x, x_i) = 0$, *for every $i \in I$ and every $x = (x^i, x_i) \in A$;*
(2) $\Phi_i(z, x_i) \le \Phi_i(z, y_i) + \Phi_i(y, x_i)$, *for every $i \in I$, every $x_i, y_i \in A_i$ and every $y, z \in A$ such that $y = (y^i, y_i)$;*
(3) *there exists a nonempty closed subset K_i of A_i for every $i \in I$, such that*

$$\text{for every } x = (x^i, x_j) \in A \quad \text{with } x_j \notin K_j, \text{ for some } j \in I,$$
$$\text{there exists } y_j \in A_j \quad \text{such that } \|y_j\| < \|x_j\| \text{ and } \Phi_j(x, y_j) \le 0;$$

(4) *Φ_i is weakly sequentially lower semicontinuous in its second variable on K_i, for every $i \in I$;*
(5) *the restriction of Φ_i on $\left(\prod_{i\in I} K_i\right) \times K_i$ is lower bounded in its second variable, for every $i \in I$;*
(6) *the restriction of Φ_i on $\left(\prod_{i\in I} K_i\right) \times K_i$ is sequentially upper semicontinuous in its first variable with respect to the topology σ, for every $i \in I$;*
(7) *The subset $\prod_{i\in I} K_i$ is sequentially compact subset of E with respect to the topology σ.*

Then, the system of equilibrium problems (SEP) has a solution.

Proof. For every $x = (x^i, x_i) \in A$ and every $i \in I$, define the subset

$$L_i(x) = \{y_i \in A_i \mid \|y_i\| \le \|x_i\|, \Phi_i(x, y_i) \le 0\},$$

and put $S_i(x) = cl_{A_i}(L_i(x))$, where the closure is taken with respect to the weak topology of A_i. By the same argument as in Theorem 3.2, we have the following properties:

(1) *The subset $S_i(x)$ is nonempty, for every $x \in A$ and every $i \in I$.*
(2) *The subset $S_i(x)$ is weakly compact, for every $x \in A$ and every $i \in I$.*
(3) *For every $i \in I$ and every $x = (x^i, x_i)$, $y = (y^i, y_i) \in A$, if $y_i \in S_i(x)$, then $S_i(y) \subset S_i(x)$.*

On the other hand, the restrictions of Φ_i on $\left(\prod_{i\in I} K_i\right) \times K_i$, for every $i \in I$ respectively, satisfy all the conditions of Proposition 4.2 and therefore, there exists $x^* = (x_1^*, \dots, x_m^*) \in \prod_{i\in I} K_i$ such that

$$\Phi_i(x^*, y_i) \ge 0 \quad \text{for all } i \in I \text{ and all } y_i \in K_i.$$

Suppose that x^* is not a solution of the equilibrium problem (SEP) and let $x_j \in A_j$ be such that $\Phi_j(x^*, x_j) < 0$, for some $j \in I$. Let $x^j \in A^j$ be arbitrary and put $x = (x^j, x_j) \in A$. Since $S_j(x)$ is nonempty weakly compact subset, then the norm attains its lower bound on $S_j(x)$. Let $y(x)_j \in S_j(x)$ be such that

$$\|y(x)_j\| = \min_{y_j \in S_j(x)} \|y_j\|$$

and since $L_j(x)$ is bounded, let $(y_n)_n$ be a weakly converging sequence in $L_j(x)$ to $y(x)_j$. We have two cases:

- Suppose first that $y(x)_j \in K_j$. Since $\Phi_j(x^*, x_j) < 0$, choose $\varepsilon > 0$ such that $\Phi_j(x^*, x_j) \le -\varepsilon$. Since $\Phi_j(x, y_n) \le 0$, for every n, then

  $$\Phi_j(x^*, y_n) \le \Phi_j(x^*, x_j) + \Phi_j(x, y_n) \le -\varepsilon.$$

  The bifunction Φ_j being weakly sequentially lower semicontinuous in its second variable on K_j, we obtain

  $$\Phi_j(x^*, y(x)_j) \le \liminf_{n\to+\infty} \Phi_j(x^*, y_n) \le -\epsilon < 0$$

  which yields a contradiction.
- Suppose now that $y(x)_j \notin K_j$. Let $y^j \in A^j$ be arbitrary and put $y_x = (y^j, y(x)_j) \in A$. Then, there exists $y_j \in A_j$, $\|y_j\| < \|y(x)_j\|$ and $\Phi_j(y_x, y_j) \le 0$. Thus,

  $$y_j \in S_j(y_x) \subset S_j(x) \quad \text{and} \quad \|y_j\| < \|y(x)_j\| = \min_{y_j \in S_j(x)} \|y_j\|$$

  which is impossible.

The proof is complete. □

5. Existence results for quasi-hemivariational inequalities

Recall that if E is a real Banach space which is continuously embedded in $L^p(\Omega;\mathbb{R}^n)$, for some $1 < p < +\infty$ and $n \geq 1$, where Ω is a bounded domain in $\mathbb{R}^m$, $m \geq 1$, then a *quasi-hemivariational inequality* is a problem of the form: find $u \in E$ and $z \in A(u)$ such that

$$\langle z, v\rangle + h(u) J^0(iu; iv) - \langle Fu, v\rangle \geq 0 \quad \forall v \in E, \tag{QHVI}$$

where i is the canonical injection of E into $L^p(\Omega;\mathbb{R}^n)$, $A : E \rightrightarrows E^*$ is a nonlinear multivalued mapping, $F : E \to E^*$ is a nonlinear operator, $J : L^p(\Omega;\mathbb{R}^n) \to \mathbb{R}$ is a locally Lipschitz functional and $h : E \to \mathbb{R}$ is a given nonnegative functional. We denote by E^* the dual space of E and by $\langle ., .\rangle$ the duality pairing between E^* and E.

Recall that a function $\phi : E \to \mathbb{R}$ is called *locally Lipschitzian* if for every $u \in E$, there exists a neighborhood U of u and a constant $L_u > 0$ such that

$$|\phi(w) - \phi(v)| \leq L_u \|w - v\|_X \quad \forall w \in U,\ \forall v \in U.$$

If $\phi : E \to \mathbb{R}$ is locally Lipschitzian near $u \in E$, then the *Clarke generalized directional derivative* of ϕ at u in the direction of $v \in E$, denoted by $\Phi^0(u, v)$, is defined by

$$\phi^0(u, v) = \limsup_{\substack{w \to u \\ \lambda \downarrow 0}} \frac{\phi(w + \lambda v) - \phi(w)}{\lambda}.$$

The *Clarke generalized gradient* of a locally Lipschitzian functional $\phi : E \to \mathbb{R}$ at a point $u_0 \in E$, denoted $\partial\phi(u_0)$, is the subset of E defined by

$$\partial\phi(u_0) = \left\{\xi \in E^* \mid \Phi^0(u_0, v) \geq \langle \xi, v\rangle,\ \text{for all } v \in E\right\}.$$

Among several important properties of the generalized directional derivative and the generalized gradient of locally Lipschitzian functions, the following properties are usually used (for proofs and related properties, we refer to [23, Proposition 2.1.1]).

Suppose that $\phi : E \to \mathbb{R}$ is locally Lipschitzian near $u \in E$. Then,

(1) the function $v \longmapsto \phi^0(u, v)$ is finite, positively homogeneous and subadditive;
(2) the function $(u, v) \longmapsto \phi^0(u, v)$ is upper semicontinuous.

Sometimes for technical reasons, the quasi-hemivariational inequality (QHVI) is replaced by the following quasi-hemivariational inequality: Find $u \in E$ and $z \in A(u)$ such that

$$\langle z, v - u\rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u\rangle \geq 0 \quad \forall v \in E.$$

In [16,10], some equilibrium problems' techniques have been employed to solve the quasi-hemivariational inequality (QHVI). There are two bifunctions approach considered in this study. First the equilibrium bifunction $\Phi_1 : E \times E \to \mathbb{R}$ is defined by

$$\Phi_1(u, v) = \inf_{v^* \in A(v)} \langle v^*, v - u\rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u\rangle.$$

Under the condition of lower quasi-hemicontinuity of A on E with respect to the weak* topology of E^*, if the equilibrium problem

$$\text{find } u \in E \text{ such that } \Phi_1(u, v) \geq 0 \quad \forall v \in E$$

has a solution, then the quasi-hemivariational inequality problem (QHVI) has a solution, see [16, Theorem 4.3].

The second equilibrium bifunction $\Phi_2 : E \times E \to \mathbb{R}$ is defined by

$$\Phi_2(u, v) = \sup_{u^* \in A(u)} \langle u^*, v - u\rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u\rangle.$$

Any solution of the quasi-hemivariational inequality problem (QHVI) is a solution of the equilibrium problem.

$$\text{Find } u \in E \text{ such that } \Phi_2(u, v) \geq 0 \quad \forall v \in E.$$

The converse holds under additional conditions on the values of the set-valued mapping A, see [10, Theorem 4.1].

Although the semicontinuity of Φ_1 and Φ_2 in their first and second variable is now known and holds from those of the functions and multivalued mappings involved, it seems to be not easy to verify if Φ_1 or Φ_2 satisfies all the conditions in order to apply the Ekeland variational principle for equilibrium problems. Nothing can ensure that the condition related to the triangle inequality must be satisfied. This means that the condition

$$\Phi(z, x) \leq \Phi(z, y) + \Phi(y, x), \quad \text{for every } x, y, z \in E$$

of Theorem 2.2 or related versions, is far to be clear when it is verified.

It is well-known that fixed point theory is a powerful tool for the existence of solutions to several types of variational inequalities. In [24,25], the existence of solutions of multivalued mixed variational inequalities is obtained by means of fixed points of a suitable multivalued mapping constructed by solving strongly convex programs.

In this approach, recall that by [15, Proposition 2.1], a point $u^* \in E$ is a solution of the quasi-hemivariational inequality problem (QHVI) if and only if u^* is a fixed point of the multivalued mapping H, that is $u^* \in H(u^*)$, where $H : E \rightrightarrows E$ is the multivalued mapping defined by

$$H(u) = F^{-1}\left(A(u) + h(u)\, i^* \partial J(iu)\right).$$

On the other hand and as mentioned in many papers, the Ekeland variational principle is a subject which is very close to the problem of fixed point theory of multivalued mapping, see for instance, [26,17] and the references therein.

Following [26,17], a function $q : X \times X \to \mathbb{R}_+$ where (X, d) is a metric space, is called a Q-function if the following conditions hold:

(1) $q(x, z) \leq q(x, y) + q(y, z)$, for every $x, y, z \in X$;
(2) Let $x \in X$ and $(y_n)_n$ be a converging sequence to some $y \in X$. If there exists $M > 0$ such that $q(x, y_n) \leq M$, for every n, then $q(x, z) \leq M$;
(3) For every $\varepsilon >$, there exists $\delta > 0$ such that

$$(q(x, y) \leq \delta \text{ and } q(x, z) \leq \delta) \Longrightarrow d(y, z) \leq \varepsilon.$$

The following result is only an adapted and partial version of [17, Theorem 10.21] which is much more stronger.

Theorem 5.1. *Let X be a complete metric space, $q : X \times X \to \mathbb{R}_+$ is a Q-function, $\varphi : (-\infty, +\infty] \to \mathbb{R}_+$ a nondecreasing function, and $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower semicontinuous and lower bounded function. Let $T : E \rightrightarrows E$ be a multivalued mapping with nonempty values. If T satisfies the following condition: there exists $y \in T(x)$ such that*

$$q(x, y) \leq \varphi(f(x))(f(x) - f(y)),$$

then T has a fixed point.

In our opinion, one of the advantages of this result is that the triangle inequality is now required on an additional function, namely the function q. It should be interesting to deeply investigate the structure of the multivalued mapping H in order to find sufficient conditions on the involved functions and multivalued mappings leading to satisfaction of all the conditions of the above theorem. Depending on the quasi-hemivariational inequality studied, it may be now possible to construct the functions q, φ and f as in Theorem 5.1 which provides us with a fixed point of the multivalued mapping H and solves the quasi-hemivariational inequality problem (QHVI).

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