Problem 1224, Elemente der Mathematik 60 (2005), No. 4

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Let f be a positive continuous function defined on $(0,\infty)$ such that $\liminf_{x\to\infty} f(x) > 0$.

Prove that there exists no positive twice differentiable function g defined on $[0,\infty)$ and satisfying $g'' + f \circ g = 0$.

SOLUTION. The original version of this problem, as published in the journal, is the following.

Aufgabe 1224: *Ist f die Identität, so besitzt die Differentialgleichung*

$$y'' + f \circ y = 0 \tag{1}$$

bekanntlich keine Lösung, welche im Intervall $[0,\infty)$ nur positive Werte annimmt. Man zeige, dass für jede für $x > 0$ stetige positive Funktion f mit $\liminf_{x\to\infty} f(x) > 0$ *keine Lösung von (1) auf $[0,\infty)$ nur positive Werte annimmt.*

The problem is inspired by the following classical framework. Consider the **linear** differential equation $g'' + g = 0$ on $[0,\infty)$. All solutions of this equation are of the form $g(x) = C_1 \cos x + C_2 \sin x$, where C_1 and C_2 are real constants. In particular, this implies that there are **no** solutions which are positive on the **whole** semi-axis $[0,\infty)$. The purpose of this problem is to find a class of functions f such that $f(x)/x \neq$ Const. for all $x > 0$ and the **nonlinear** differential equation $g'' + f \circ g = 0$ has no positive solution on the positive semi-axis.

The equality $g'' + f \circ g = 0$ can be rewritten as

$$g' = h \qquad \text{on } [0,\infty) \tag{2}$$

combined with

$$h' + f \circ g = 0 \qquad \text{on } [0,\infty)\,. \tag{3}$$

The following situations can occur.

CASE 1: there exists $x_0 \in [0,\infty)$ such that $h(x_0) < 0$. Thus, by (3), $h(x) < h(x_0)$ for all $x > x_0$. Then, by integration in (2), we find

$$g(x) < g(x_0) + h(x_0)(x - x_0), \qquad \forall\, x > x_0.$$

So, since $h(x_0) < 0$ and $g > 0$ in (x_0,∞), the above relation yields a contradiction, for x sufficiently large.

CASE 2: $h(x_0) = 0$, for some $x_0 \geq 0$. Thus, by (3), it follows that h is decreasing in (x_0,∞). In particular, we have $h < 0$ in (x_0,∞). With the same arguments as in Case 1 we find again a contradiction. Consequently, Cases 1 and 2 can never occur.

CASE 3: $h > 0$ in $[0,\infty)$. In this situation, by (2), it follows that

$$g(x) > g(0) > 0, \qquad \text{for all } x > 0. \tag{4}$$

We will assume that

$$\liminf_{x\to\infty} f(x) > 0\,. \tag{5}$$

So, by (4) and (5), there exists some $A > 0$ (sufficiently small, but **positive**) such that $f(g(x)) > A$ for all $x > 0$. Thus, by (3),

$$h(x) < h(0) - Ax, \qquad \text{for all } x > 0,$$

a contradiction, since h is positive. In conclusion, the required sufficient condition is formulated in relation (5).

We point out that our assumption (5) on f is not necessary. For this purpose, we prove in what follows that if $f(x) = x^{-1}$ (so, $\liminf_{x\to\infty} f(x) = 0$) then the nonlinear differential equation $g''+g^{-1} = 0$ does not have positive solutions on $[0,\infty)$. We argue by contradiction and assume that such a solution g exists. Then, as observed above, only Case 3 can occur. Thus, g is increasing in $(0,\infty)$. In particular, there exists $g_\infty := \lim_{x\to\infty} g(x)$. We claim that g_∞ cannot be finite. Indeed, assuming the contrary, there exists positive numbers m and M such that $m \le g(x) \le M$, for all $x > 0$. By the continuity of f, we have $f \circ g > A$ in $(0,\infty)$, for some $A > 0$. Thus, by (3),

$$h(x) < h(0) - Ax, \qquad \text{for all } x > 0,$$

which contradicts $h > 0$, provided that x is sufficiently large. These arguments show that $g_\infty = +\infty$. Next, multiplying by g' in $g'' + g^{-1} = 0$ and integrating on $[1, x]$ we find

$$\ln g(x) = \ln g(1) + \frac{g'^2(1) - g'^2(x)}{2} \le \ln g(1) + \frac{g'^2(1)}{2}, \qquad \text{for all } x > 1.$$

This inequality shows that g is bounded on $[1,\infty)$, which contradicts $g_\infty = +\infty$. This completes our proof. □