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Research Article

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Multiple normalized solutions for fractional elliptic problems

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Abstract: In this article, we are first concerned with the existence of multiple normalized solutions to the following fractional p-Laplace problem:

$$\begin{cases} (-\Delta)_p^s v + \mathcal{V}(\xi x)|v|^{p-2}v = \lambda|v|^{p-2}v + f(v) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^p \, dx = a^p, \end{cases}$$

where $a, \xi > 0$, $p \geq 2$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier, $\mathcal{V} : \mathbb{R}^N \to [0, \infty)$ is a continuous function, and f is a continuous function with L^p-subcritical growth. We prove that there exists the multiplicity of solutions by using the Lusternik–Schnirelmann category. Next, we show that the number of normalized solutions is at least the number of global minimum points of $\mathcal{V}$, as ξ is small enough via Ekeland's variational principle.

Keywords: Lusternik–Schnirelmann category, normalized solutions, fractional p-Laplace, nonlinear Schrödinger equation, variational methods

MSC 2020: 35A15, 35J10, 35B09, 35B33

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1 Introduction

This paper will give some results on the existence of multiple normalized solutions to fractional p-Laplace problems as follows:

$$\begin{cases} (-\Delta)_p^s v + \mathcal{V}(\xi x)|v|^{p-2}v = \lambda|v|^{p-2}v + f(v) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^p \, dx = a^p, \end{cases} \tag{1.1}$$

where $a, \xi > 0$, $p \geq 2$ and $\lambda \in \mathbb{R}$ is an unknown parameter that comes from a Lagrange multiplier. The fractional p-Laplace operator $(-\Delta)_p^s$ is defined along a smooth function (up to a normalizing constant) $v : \mathbb{R}^N \to \mathbb{R}$ by

$$(-\Delta)_p^s v(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{N+ps}} \, dy, \quad x \in \mathbb{R}^N,$$

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where $B_\varepsilon(x)$ is the ball with center x and radius ε. If $\mathcal{V}(x) \equiv 0$ and $p = 2$, then problem (1.1) becomes

$$\begin{cases} (-\Delta)^s v = \lambda v + f(v) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^2 \, dx = a^2, \end{cases} \tag{1.2}$$

which has been intensively considered until now. Namely Luo and Zhang [24] studied problem (1.2) with

$$f(v) = \mu |v|^{q-2} v + |v|^{p-2} v, \quad \text{where } N \geq 2,\ \mu \in \mathbb{R},\ 2 < q < p < 2_s^*.$$

Under different assumptions on $q < p$, $a > 0$ and $\mu \in \mathbb{R}$, they showed the existence of normalized solutions to problem (1.2). In the L^2-subcritical case, they used the monotonicity of the least energy to get the ground state solution for $\mu > 0$. For the L^2-critical or L^2-supercritical case, they used some certain decomposition on Nehari manifolds of problem (1.2) and some homotopy-stable family with extended boundary of a closed set. After that, Zhen and Zhang [38] extended the results of Luo and Zhang [24] for the critical case

$$f(v) = \mu |v|^{q-2} v + |v|^{2_s^*-2} v, \quad \text{where } 2 < q < s_s^* = \frac{2N}{N-2s}.$$

We also refer to Zhang and Han [36], who studied problem (1.2) when

$$f(v) = |v|^{p-2} v + |v|^{2_s^*-2} v, \quad \text{where } 2 < p < 2_s^*.$$

We set

$$\mathcal{S}(a) = \Big\{ v \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |v|^2 \, dx = a^2 \Big\}$$

and

$$V(a) = \{ v \in H^s(\mathbb{R}^N) : \mathcal{P}(v) = 0 \},$$

where

$$\mathcal{P}(v) := |(-\Delta)^{s/2} v|_2^2 - \gamma_{p,s} |v|_p^p - |v|_{2_s^*}^{2_s^*}, \quad \gamma_{p,s} = \frac{N(p-2)}{2ps}.$$

To study their problem, they used the decomposition of the corresponding Pohozaev manifold into three disjoint submanifolds together with the control of the seminorm of $v_\xi = \varphi(x)\mathcal{U}_\xi(x)$, where

$$\mathcal{U}_\xi(x) = \frac{\xi^{\frac{N-2s}{2}}}{(\xi^2 + |x|^2)^{\frac{N-2s}{2}}}$$

and $\varphi \in C_0^\infty(B_2(0))$ is a radial cutoff function such that $\varphi(x) \in [0,1]$ and $\varphi \equiv 1$ on $B_1(0)$.

Li and Zou [23] studied problem (1.2) when

$$f(v) = \mu |v|^{p-2} u + |v|^{2_s^*-2} v, \quad \mu > 0,\ 2 < p < 2_s^*.$$

In the L^2-subcritical case, they obtained multiple normalized solutions for equation (1.2) by using truncation techniques, the concentration-compactness principle and genus theory. In the L^2-supercritical case, they obtained normalized solutions by using a fiber map and the concentration-compactness principle. Yu, Tang and Zhang [35] found the solutions of the following fractional Schrödinger equation:

$$\begin{cases} (-\Delta)^s v = \lambda v + |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} |v|^2 \, dx = a^2, \end{cases}$$

where $s \in (0,1)$ and $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is an exterior domain with smooth boundary $\partial\Omega \neq \emptyset$ such that $\mathbb{R}^N \setminus \Omega$ is bounded. Using the minimax method, barycentric functions and Brouwer degree theory, they showed that there exists a positive normalized solution for any $a > 0$ when $2 < p < 2 + \frac{4s}{N}$ and $\mathbb{R}^N \setminus \Omega$ is contained in a small ball. If Ω is the complement of the unit ball in $\mathbb{R}^N$, using genus theory, they gave the existence of many radial normalized solutions for any $a > 0$.

When $p = 2$ and $\xi = 1$, problem (1.1) becomes

$$\begin{cases} (-\Delta)^s v + \mathcal{V}(x)v = \lambda v + f(v) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^2\,dx = a^2, \end{cases} \tag{1.3}$$

Peng and Xia [27] studied problem (1.3) when

$$f(v) = |v|^{p-2}v, \quad p \in \Big(2 + \frac{4s}{N}, 2_s^*\Big).$$

The potential function $\mathcal{V}$ is positive and vanishing at infinity with possible singularities. They showed that there exists a normalized solution of (1.3) by using a new min-max argument and the splitting lemma for the nonlocal case. Alves and Ji [4] considered the existence of solution for (1.1) for

$$f(v) = |v|^{q-2}v, \quad q \in \Big(2, 2 + \frac{4}{N}\Big), \; N \geq 2.$$

They supposed that the potential $\mathcal{V}$ verifies one of the following conditions:

(v1) $\mathcal{V} : \mathbb{R}^N \to [0, +\infty)$ is a bounded continuous function and it is 1-periodic with the variables $x_1, \ldots, x_N$.

(v2) $\mathcal{V}$ is asymptotically periodic. That means that there exists a 1-periodic function $\mathcal{V}_P : \mathbb{R}^N \to \mathbb{R}$ with variables $x_1, \ldots, x_N$ such that $\mathcal{V}(x) \leq \mathcal{V}_P(x)$ for all $x \in \mathbb{R}^N$ and $|\mathcal{V}(x) - \mathcal{V}_P(x)| \to 0$ as $|x| \to \infty$.

(v3) $\mathcal{V} \in L^\infty(\mathbb{R}^N)$ and

$$\mathcal{V}_\infty = \liminf_{|x|\to+\infty} \mathcal{V}(x) > \mathcal{V}_0 = \inf_{x\in\mathbb{R}^N} \mathcal{V}(x) > 0.$$

(v4) $\mathcal{V}(x) = \mu\mathcal{W}(x)$, where $\mu > 0$ and $\mathcal{W}$ is a nonnegative continuous function satisfying the condition as follows: There exists $M_0 > 0$ such that

$$\operatorname{meas}\big(x \in \mathbb{R}^N : \mathcal{W}(x) > M_0\big) < +\infty$$

and the set $\Omega = \operatorname{int}(\mathcal{W}^{-1}(0))$ is not an empty set.

They studied the existence of solution to problem (1.1) as $\xi = 1$ or $\xi > 0$ is small enough. To obtain the results, they proved a compactness result for minimizing sequences $\{v_n\}$ restricted on $S(a)$. Zuo, Liu and Vetro [39] have studied the existence of normalized solution of the following fractional Schrödinger equation:

$$(-\Delta)^s v + \mu v + \lambda\mathcal{V}(x)v - |v|^{p-2}v = 0 \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where

$$2s < N < 4s, \quad p \in \Big(2, \min\Big\{\frac{N}{N-2s}, 2 + \frac{4s}{N}\Big\}\Big)$$

and the potential $\mathcal{V} \in L^\infty(\mathbb{R}^N)$ satisfies the condition that there exists a positive constant $D_0 > 0$ such that the measure of the set $\Omega = \{x \in \mathbb{R}^N : \mathcal{V}(x) < D_0\}$ is finite. Using the compactness result for minimizing sequences, they showed that the existence of normalized solution to (1.4) when λ is large enough. Note that, when $p = 2$, $s \to 1^-$, our problem (1.2) reduces to

$$\begin{cases} -\Delta v + \mathcal{V}(\xi x)v = \lambda v + f(v) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^2\,dx = a^2, \end{cases} \tag{1.5}$$

where $a, \xi > 0$ and $\lambda \in \mathbb{R}$ is an unknown parameter that comes from a Lagrange multiplier. Many authors studied problem (1.5) and got many nice results. Using genus theory and deformation arguments, the existence of infinitely many normalized solutions has been studied by many authors such as Alves, Chao and Miyagaki [5, 6], Bartsch and Soave [9], Ikoma and Tanaka [17], Cingolani and Jeanjean [11], Jeanjean and Lu [20, 21], Jeanjean and Le [18, 19], and Soave [28, 29]. In 2022, Alves [3] studied the existence of multiple normalized solutions for problem (1.5), where $\mathcal{V} : \mathbb{R}^N \to [0, +\infty)$ is a continuous function verifying condition ($\mathscr{V}$) and the nonlinear reaction has L^2-subcritical growth. Alves proved that the number of normalized solutions is bounded from below by the number of global minimum points of $\mathcal{V}$ for small enough ξ. Alves and Thin [7] studied the existence and concentration of normalized solutions to equation (1.5). They first used the Lusternik–Schnirelmann category to get multiple normalized solutions to that problem. Wang, Zeng and Zhou [32] studied the properties of least energy solutions to fractional Laplacian eigenvalue problems on $\mathbb{R}^N$ as follows:

$$\begin{cases} (-\Delta)^s v + \mathcal{V}(x)v = \mu v + am(x)|v|^{\frac{4s}{N}} v, \\ \displaystyle\int_{\mathbb{R}^N} |v|^2\, dx = 1, \quad v \in H^s(\mathbb{R}^N), \end{cases} \tag{1.6}$$

where $N \geq 2$, $s \in (0, 1)$, $\mu \in \mathbb{R}$, $a > 0$, and $\mathcal{V}$, m are in $L^\infty(\mathbb{R}^N)$. They showed that there exists $b_s^* > 0$ such that problem (1.6) has a least energy solution $u_a(x)$ for each $a \in (0, b_s^*)$ and u_a blows up, as a is increasing to b_s^*, at some points $x_0 \in \mathbb{R}^N$ where $\mathcal{V}$ attains the minimum and m achieves the maximum. When $s \to 1^-$, our problem (1.1) becomes the following p-Laplacian problem:

$$\begin{cases} -\Delta_p v + \mathcal{V}(\xi x)|v|^{p-2}v = \lambda|v|^{p-2}v + f(v) \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |v|^p\, dx = a^p, \end{cases} \tag{1.7}$$

where $-\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ is the p-Laplace operator. Recently, there was no work on the non-autonomous problem (1.7). If $\xi = 1$ in problem (1.7), Wang and Sun [30] have studied the existence of the normalized solution to the following problem:

$$\begin{cases} -\Delta_p v + \mathcal{V}(x)|v|^{p-2}v = \lambda|v|^{r-2}v + |v|^{q-2}v \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |v|^r\, dx = c, \end{cases} \tag{1.8}$$

where $r = p$ or $r = 2$, $1 < p < N$, $p < q < p^* = \frac{pN}{N-p}$, and $\mathcal{V} \in C(\mathbb{R}^N)$ satisfies

$$\inf_{x\in\mathbb{R}^N} \mathcal{V}(x) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \mathcal{V}(x) = +\infty.$$

When $r = p$ and c is small enough, they showed that there exists a ground state solution with positive energy. For $p = 2$, they proved that problem (1.8) has at least two solutions with positive energy, one of which is a ground state and the other one is a high-energy solution. Up to now, there are a few works on normalized solutions of the p-Laplace equation. Wang, Li, Zhou and Li [31] first studied the existence of the L^2-norm constraint:

$$-\Delta_p v + |v|^{p-2}v = \mu v + |v|^{s-2}v \quad \text{in } \mathbb{R}^N,$$

where $1 < p < N$, $\mu \in \mathbb{R}$ and $s \in (\frac{N+2}{N}p, p^*)$. Using the constrained variational methods, they showed that the above problem has a normalized solution. Zhang and Zhang [37] studied the existence of normalized solutions to p-Laplacian equations with the form

$$\begin{cases} -\Delta_p v = \lambda|v|^{p-2}v + \mu|v|^{q-2}v + g(v) \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |v|^p\, dx = a^p, \end{cases}$$

where $N \geq 2$, $a > 0$, $1 < p < q \leq \overline{p} := p + \frac{p^2}{N}$, and $g \in C(\mathbb{R}, \mathbb{R})$ is old and L^p-supercritical. When $q < \overline{p}$ and $\mu > 0$, they got a positive radial ground state solution for suitable μ by using the Schwarz rearrangement and Ekeland variational principle. Applying the fountain theorem, they obtained infinitely many radial solutions for any $N \geq 2$ and obtained the existence of infinitely many non-radial sign-changing solutions for $N = 4$ or $N \geq 6$. In those cases, μ belongs to a suitable range and depends on a. We also refer to [10, 15, 22, 34] for the qualitative analysis of normalized solutions in different local or nonlocal settings.

So far, there is no result for the existence of multiple normalized solutions to Schrödinger equations involving the fractional p-Laplace, especially to the nonautonomous problem (1.1). Motivated by this fact, the main goal of the present paper is to give the first normalized result for Schrödinger equations involving the fractional p-Laplace.

In the following, we give some assumptions on the nonlinear function f:

(f1) f is a continuous and odd function, and there are $q \in (p, p + \frac{p^2 s}{N})$ and $\alpha \in (0, +\infty)$ such that

$$\lim_{t \to 0} \frac{|f(t)|}{|t|^{q-1}} = \alpha.$$

(f2) There exist constants $c_1, c_2 > 0$ and $\mathfrak{p} \in (p, p + \frac{p^2 s}{N})$ such that

$$|f(t)| \leq c_1 + c_2 |t|^{\mathfrak{p}-1} \quad \text{for all } t \in \mathbb{R}.$$

(f3) There is $q_1 \in (p, p + \frac{p^2 s}{N})$ so that $f(t)/t^{q_1 - 1}$ is an increasing function of t on $(0, +\infty)$.

From conditions (f1) and (f3), we have $F(t) \geq 0$ for all $t \in \mathbb{R}$.

The function

$$f(t) = |t|^{q-2} t + |t|^{r-2} t \ln(1 + |t|) \quad \text{for all } t \in \mathbb{R},$$

for some $r, q \in (p, p + \frac{4}{N})$ and $r > q$, satisfies the above conditions. Here, (f2) and (f3) hold with $\mathfrak{p} \in (r, p + \frac{p^2 s}{N})$ and $q_1 = q$.

For the potential function $\mathcal{V}$, we suppose that one of the following conditions holds:

(V) We have $\mathcal{V} \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$, $\mathcal{V}(0) = 0$ and

$$0 = \inf_{x \in \mathbb{R}^N} \mathcal{V}(x) < \liminf_{|x| \to +\infty} \mathcal{V}(x) = \mathcal{V}_\infty.$$

($\mathscr{V}$) The function $\mathcal{V}$ belongs to $C(\mathbb{R}^N, \mathbb{R}^+) \cap L^\infty(\mathbb{R}^N)$, and

$$\mathcal{V}_\infty = \liminf_{|x| \to \infty} \mathcal{V}(x) > 0.$$

Furthermore, $\mathcal{V}^{-1}(0) = \{b_1, \ldots, b_l\}$, $b_1 = 0$ and $b_i \neq b_j$ for all $i \neq j$.

A solution v to problem (1.1) with $\int_{\mathbb{R}^N} |v|^p \, dx = a^p$ is a critical point of the energy function

$$J_\xi(v) = \frac{1}{p} \Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx \, dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x) |v|^p \, dx \Big) - \int_{\mathbb{R}^N} F(v) \, dx, \quad v \in W^{s,p}(\mathbb{R}^N),$$

restricted to the sphere

$$S(a) = \{v \in W^{s,p}(\mathbb{R}^N) : |v|_p = a\},$$

where $F(t) = \int_0^t f(\tau) \, d\tau$ and $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$ for $p \in [2, +\infty]$. Here the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined for any $p > 1$ and $s \in (0, 1)$ by

$$W^{s,p}(\mathbb{R}^N) = \Big\{ v \in L^p(\mathbb{R}^N) : [v]_{s,p} := \Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \Big)^{\frac{1}{p}} < +\infty \Big\},$$

which is a Banach space with norm

$$\|v\| = (|v|_p^p + [v]_{s,p}^p)^{\frac{1}{p}}.$$

It is well known that $J_\xi \in C^1(W^{s,p}(\mathbb{R}^N), \mathbb{R})$ and

$$\langle J_\xi'(v), u \rangle = \iint_{\mathbb{R}^N} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+ps}} \, dx \, dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x) |v|^{p-2} v u \, dx - \int_{\mathbb{R}^N} f(v) u \, dx$$

for all $u \in W^{s,p}(\mathbb{R}^N)$. We refer to the monograph [25] for the theory of fractional Sobolev spaces and related applications.

In the first result, we establish the existence of multiple normalized solutions for (1.1) via the Lusternik–Schnirelmann category. We denote the sets $\mathcal{M}$ and $\mathcal{M}_\delta$ as follows:

$$\mathcal{M} = \{x \in \mathbb{R}^N : \mathcal{V}(x) = 0\}$$

and

$$\mathcal{M}_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \mathcal{M}) \le \delta\}.$$

Let Y be a closed subset of a topological space X. Then the Lusternik–Schnirelmann category $\operatorname{cat}_X(Y)$ is the least number of closed and contractible sets in X which cover Y. If $X = Y$, we denote $\operatorname{cat}_X(X)$ by $\operatorname{cat}(X)$. Our first main result is stated as follows.

Theorem 1.1. *Suppose that f satisfies conditions* (f1)–(f3) *and that $\mathcal{V}$ verifies condition* (V). *Then, for each $\delta > 0$, there exist $\xi_0 > 0$ and $\mathcal{V}_* > 0$ such that* (1.1) *has at least $\operatorname{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ couples*

$$(v_j, \lambda_j) \in W^{s,p}(\mathbb{R}^N) \times \mathbb{R}$$

of weak solutions for $0 < \xi < \xi_0$ and $|\mathcal{V}|_\infty < \mathcal{V}_*$ with

$$\int_{\mathbb{R}^N} |v_j|^p \, dx = a^p, \quad \lambda_j < 0, \quad J_\xi(v_j) < 0.$$

Moreover, if v_ξ denotes one of those solutions and ζ_ξ is the global maximum of $|v_\xi|$, then

$$\lim_{\xi \to 0} \mathcal{V}(\xi \zeta_\xi) = 0.$$

Theorem 1.2. *Assume that f verifies conditions* (f1)–(f3) *and $\mathcal{V}$ satisfies condition* ($\mathscr{V}$). *Then there exists $\xi_0 > 0$ such that* (1.1) *admits l couples $(v_j, \lambda_j) \in W^{s,p}(\mathbb{R}^N) \times \mathbb{R}$ of weak solutions for $0 < \xi < \xi_0$ with $\int_{\mathbb{R}^N} |v_j|^p \, dx = a^p$, $\lambda_j < 0$ and $J_\xi(v_j) < 0$, $j = 1, \dots, l$.*

When $s \to 1^-$ in Theorems 1.1 and 1.2, we get automatically results for the p-Laplace problem (1.7); we leave this for the reader. Theorems 1.1 and 1.2 extend the results in [3, 7] to the nonlocal case. Here, our solution space $W^{s,p}(\mathbb{R}^N)$ is a non-Hilbert space, and thus we need to develop some new steps in the proofs.

The content of the paper is written as follows. In Section 2, we study the autonomous problem. In Section 3, we study the non-autonomous case. There, we study the Palais–Smale condition on the sphere $S(a)$ for the energy functional. In Sections 4 and 5, we show that there exists the multiplicity of solutions for problem (1.1).

2 The autonomous case

In this section, we study the existence of solution to the following problem:

$$\begin{cases} (-\Delta)_p^s v + \mu |v|^{p-2} v = \lambda |v|^{p-2} v + f(v) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^p \, dx = a^p, \end{cases} \tag{2.1}$$

where $N \ge 1$, $a > 0$, $\mu \ge 0$, $\lambda \in \mathbb{R}$ is a Lagrange multiplier, and f is a continuous function verifying conditions (f1)–(f3).

A solution v to problem (2.1) is a critical point of the C^1-energy functional

$$\mathcal{J}_\mu(v) = \frac{1}{p}\Bigg(\iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}} \, dx\, dy + \mu \int_{\mathbb{R}^N} |v|^p \, dx \Bigg) - \int_{\mathbb{R}^N} F(v) \, dx, \quad v \in W^{s,p}(\mathbb{R}^N),$$

constrained to the sphere $S(a)$ given by

$$S(a) = \{v \in W^{s,p}(\mathbb{R}^N) : |v|_p = a\}.$$

Our main result in this section is the following theorem.

Theorem 2.1. *Suppose that f satisfies conditions* (f1)–(f3). *Then there exists $\mathcal{V}_* > 0$ so that problem* (2.1) *has a solution (v, λ) as $0 \le \mu < \mathcal{V}_*$, where v is nonnegative and $\lambda < 0$.*

To prove the above result, we need the following lemmas.

Lemma 2.2. *The functional $\mathcal{J}_\mu$ is coercive and bounded from below in $S(a)$.*

Proof. By conditions (f1) and (f2), there is $C_1, C_2 > 0$ such that

$$|F(t)| \le C_1|t|^q + C_2|t|^{\mathfrak{p}} \quad \text{for all } t \in \mathbb{R}.$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, for any $v \in W^{s,p}(\mathbb{R}^N)$, we get the fractional Gagliardo–Nirenberg inequality [26, Lemma 2.1] as follows:

$$|v|_\tau^\tau \le C_{s,N,\tau}[v]_{s,p}^{\tau\mathfrak{a}}|v|_p^{\tau(1-\mathfrak{a})}$$

for some positive constant $C_{s,N,\tau} \ge 1$, where $\tau > 0$, $0 \le \mathfrak{a} \le 1$ and

$$\frac{1}{\tau} = \mathfrak{a}\Big(\frac{1}{p} - \frac{s}{N}\Big) + \frac{1-\mathfrak{a}}{p}.$$

Then

$$\tau = \frac{pN}{N - \mathfrak{a}ps} \in [p, p_s^*] \quad \text{or} \quad \mathfrak{a} = \frac{N}{s}\Big(\frac{1}{p} - \frac{1}{\tau}\Big).$$

When $\tau\mathfrak{a} = p$, then $\mathfrak{a} = \frac{p}{\tau}$ and we get $\tau = p + \frac{p^2 s}{N}$, which is called L^p-critical exponent for the fractional Gagliardo–Nirenberg inequality. Hence,

$$\begin{aligned}
\mathcal{J}_\mu(v) &\ge \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - C_1\int_{\mathbb{R}^N}|v|^q\,dx - C_2\int_{\mathbb{R}^N}|v|^{\mathfrak{p}}\,dx \\
&\ge \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - C_{s,N,q}C_1 a^{q(1-\mathfrak{a})}\Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\Big)^{\frac{q\mathfrak{a}}{p}} \\
&\quad - C_{s,N,\mathfrak{p}}C_2 a^{\mathfrak{p}(1-\mathfrak{a})}\Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\Big)^{\frac{\mathfrak{p}\mathfrak{a}}{p}}.
\end{aligned} \tag{2.2}$$

As $q, \mathfrak{p} \in (p, p + \frac{p^2 s}{N})$, we have $0 < \frac{t\mathfrak{a}}{p} < 1$ for $t \in \{\mathfrak{p}, q\}$, which implies the coercivity and bounded from below of $\mathcal{J}_\mu$ on $S(a)$. □

From Lemma 2.2, there exists the real number $\mathfrak{J}_{\mu,a} = \inf_{v\in S(a)} \mathcal{J}_\mu(v)$. Next, we show that $\mathfrak{J}_{\mu,a} < 0$ is negative for a suitable range of μ.

Lemma 2.3. *There exists $\mathcal{V}_* > 0$ such that $\mathfrak{J}_{\mu,a} < 0$ for $0 \le \mu < \mathcal{V}_*$.*

Proof. From condition (f1), we have

$$\lim_{t\to 0}\frac{qF(t)}{t^q} = \alpha > 0.$$

Then there exists $\kappa > 0$ so that

$$\frac{qF(t)}{t^q} \ge \frac{\alpha}{2} \quad \text{for all } t \in [0, \kappa]. \tag{2.3}$$

Choose a nonnegative function $v_0 \in S(a) \cap L^\infty(\mathbb{R}^N)$ and set

$$\mathbb{H}(v_0, t)(x) = e^{\frac{Nt}{p}} v_0(e^t x) \quad \text{for all } x \in \mathbb{R}^N \text{ and all } t \in \mathbb{R}.$$

It is easy to get

$$\int_{\mathbb{R}^N} |\mathbb{H}(v_0, t)(x)|^p\,dx = a^p$$

and

$$\int_{\mathbb{R}^N} F(\mathbb{H}(v_0, t)(x))\,dx = e^{-Nt}\int_{\mathbb{R}^N} F(e^{\frac{Nt}{p}} v_0(x))\,dx.$$

Moreover, for $t < 0$ and $|t|$ large enough, we also have

$$0 \le e^{\frac{Nt}{p}} v_0(x) \le \kappa \quad \text{for all } x \in \mathbb{R}^N,$$

which, combined with (2.3), yields

$$\int_{\mathbb{R}^N} F(\mathbb{H}(v_0, t)(x))\,dx \ge \frac{\alpha}{2q} e^{\frac{(q-p)Nt}{p}} \int_{\mathbb{R}^N} |v_0(x)|^q\,dx.$$

Note that

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} \frac{|\mathbb{H}(v_0,t)(x) - \mathbb{H}(v_0,t)(y)|^p}{|x-y|^{N+ps}}\,dx\,dy &= e^{Nt} \iint_{R^{2N}} \frac{|v_0(e^t x) - v_0(e^t y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &= e^{pts} \iint_{\mathbb{R}^{2N}} \frac{|v_0(x) - v_0(y)|^p}{|x-y|^{N+ps}}\,dx\,dy,\end{aligned}$$

and so

$$\mathfrak{I}_\mu(\mathbb{H}(v_0, s)) \le \frac{e^{pts}}{p} \iint_{\mathbb{R}^{2N}} \frac{|v_0(x) - v_0(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{\mu a^p}{p} - \frac{\alpha e^{\frac{(q-p)Nt}{p}}}{2q} \int_{\mathbb{R}^N} |v_0(x)|^q\,dx.$$

Since $q \in (p, p + \frac{p^2 s}{N})$, increasing $|t|$ if necessary, we derive that

$$\frac{e^{pts}}{p}[v_0]_{s,p}^p - \frac{\alpha e^{\frac{(q-p)Nt}{p}}}{2q} \int_{\mathbb{R}^N} |v_0(x)|^q\,dx = A_t < 0.$$

Then

$$\mathfrak{I}_\mu(\mathbb{H}(v_0, t)) \le A_t + \frac{\mu a^p}{p}.$$

Now, we fix $\mathcal{V}_* > 0$ such that

$$A_t + \frac{\mathcal{V}_* a^p}{p} < 0.$$

From that inequality, if $\mu < \mathcal{V}_*$, then

$$\mathfrak{I}_\mu(\mathbb{H}(v_0, t)) < 0 \quad \text{for all } \mu \in [0, V_*),$$

and we have $\mathfrak{I}_{\mu,a} < 0$. Thus Lemma 2.3 is proved. □

Lemma 2.4. *We fix $\mu \in [0, \mathcal{V}_*)$ and let $0 < a_1 < a_2$. Then we have*

$$\frac{a_1^p}{a_2^p} \mathfrak{I}_{\mu,a_2} < \mathfrak{I}_{\mu,a_1} < 0.$$

Proof. First, we see that

$$||v(x)| - |v(y)|| \le |v(x) - v(y)| \quad \text{for all } x, y \in \mathbb{R}^N \text{ and } v \in W^{s,p}(\mathbb{R}^N).$$

Hence we get

$$\iint_{\mathbb{R}^{2N}} \frac{||v(x)| - |v(y)||^p}{|x-y|^{N+ps}}\,dx\,dy \le \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy$$

for all $v \in W^{s,p}(\mathbb{R}^N)$. Let $\varepsilon > 1$ be such that $a_2 = \varepsilon a_1$, and let $(v_n) \subset S(a_1)$ be a nonnegative minimizing sequence with respect to the $\mathfrak{I}_{\mu,a_1}$, which exists due to $\mathfrak{I}_\mu(v) \ge \mathfrak{I}_\mu(|v|)$ for all $v \in W^{s,p}(\mathbb{R}^N)$. Namely

$$\mathfrak{I}_\mu(v_n) \to \mathfrak{I}_{\mu,a_1} \quad \text{as } n \to +\infty.$$

We set $u_n = \varepsilon v_n$. Then $u_n \in S(a_2)$. From condition (f3), the function $F(t)/t^{q_1}$ is increasing on $t \in (0, +\infty)$. Then we get

$$F(tl) \ge t^{q_1} F(l) \quad \text{for all } t, l > 0 \text{ and } t \ge 1.$$

Thus, we deduce

$$\mathfrak{I}_{\mu,a_2} \le \mathfrak{I}_\mu(u_n) = \varepsilon^p \mathfrak{I}_\mu(v_n) + \varepsilon^p \int_{\mathbb{R}^N} F(v_n)\,dx - \int_{\mathbb{R}^N} F(\varepsilon v_n)\,dx \le \varepsilon^p \mathfrak{I}_\mu(v_n) + (\varepsilon^p - \varepsilon^{q_1}) \int_{\mathbb{R}^N} F(v_n)\,dx.$$

Claim 1. *We show that there exist a positive constant $C > 0$ and $n_0 \in \mathbb{N}$ such that $\int_{\mathbb{R}^N} F(v_n)\,dx \geq C$ for all $n \geq n_0$.*

Assume that, by contradiction, there exists a subsequence of (v_n), still denoted by (v_n), such that

$$\int_{\mathbb{R}^N} F(v_n)\,dx \to 0 \quad \text{as } n \to +\infty.$$

From the inequality

$$0 > \mathfrak{I}_{\mu,a_1} + o_n(1) = \mathfrak{I}_\mu(v_n) \geq -\int_{\mathbb{R}^N} F(v_n)\,dx, \quad n \in \mathbb{N},$$

we get $\mathfrak{I}_{\mu,a_1} = 0$. This is a contradiction with Lemma 2.3, and Claim 1 is proved. By Claim 1 and $\xi^p - \xi^{q_1} < 0$, for n large enough, we deduce

$$\mathfrak{I}_{\mu,a_2} \leq \xi^p \mathfrak{I}_\mu(v_n) + (\varepsilon^p - \varepsilon^{q_1})C.$$

Taking $n \to +\infty$, it follows that

$$\mathfrak{I}_{\mu,a_2} \leq \varepsilon^p \mathfrak{I}_{\mu,a_1} + (\varepsilon^p - \varepsilon^{q_1})C < \varepsilon^p \mathfrak{I}_{\mu,a_1},$$

that is,

$$\frac{a_1^p}{a_2^p}\mathfrak{I}_{\mu,a_2} < \mathfrak{I}_{\mu,a_1}.$$

We finish the proof of Lemma 2.4. □

Our following result is a compactness theorem on $S(a)$ for minimizing sequences.

Theorem 2.5. *Let $\mu \in [0, \mathcal{V}_*)$ and let $(v_n) \subset S(a)$ be a minimizing sequence for $\mathfrak{I}_\mu$. Then, up to a subsequence, one of the following assertions holds:*
(i) *(v_n) is strongly convergent in $W^{s,p}(\mathbb{R}^N)$.*
(ii) *There exists $(y_n) \subset \mathbb{R}^N$ with $|y_n| \to +\infty$ such that the sequence $u_n(x) = v_n(x + y_n)$ is strongly convergent to a function $\tilde{v} \in S(a)$ with $\mathfrak{I}_\mu(\tilde{v}) = \mathfrak{I}_{\mu,a}$.*

Proof. Since $\mathfrak{I}_\mu$ is coercive on $S(a)$, the sequence (v_n) is bounded, and so $v_n \rightharpoonup v$ in $W^{s,p}(\mathbb{R}^N)$ for some subsequence. If $v \neq 0$ and $|v|_p = b \neq a$, we must have $b \in (0, a)$. By the Brézis–Lieb lemma (see [33]),

$$|v_n|_p^p = |v_n - v|_p^p + |v|_p^p + o_n(1).$$

By [8, Lemma 2.5], we have

$$[v_n]_{s,p}^p = [v_n - v]_{s,p}^p + [v]_{s,p}^p + o_n(1).$$

Since F is a differentiable function with subcritical growth, we get

$$\int_{\mathbb{R}^N} F(v_n)\,dx = \int_{\mathbb{R}^N} F(v_n - v)\,dx + \int_{\mathbb{R}^N} F(v)\,dx + o_n(1).$$

Setting $u_n = v_n - v$, $d_n = |u_n|_p$ and supposing that $|u_n|_p \to d$, we get $a^p = b^p + d^p$ and $d_n \in (0, a)$ for n large enough. Hence,

$$\mathfrak{I}_{\mu,a} + o_n(1) = \mathfrak{I}_\mu(v_n) = \mathfrak{I}_\mu(u_n) + \mathfrak{I}_\mu(v) + o_n(1) \geq \mathfrak{I}_{\mu,d_n} + \mathfrak{I}_{\mu,b} + o_n(1),$$

and Lemma 2.4 gives that

$$\mathfrak{I}_{\mu,a} + o_n(1) \geq \frac{d_n^p}{a^p}\mathfrak{I}_{\mu,a} + \mathfrak{I}_{\mu,b} + o_n(1).$$

Taking $n \to +\infty$, we obtain

$$\mathfrak{I}_{\mu,a} \geq \frac{d^p}{a^p}\mathfrak{I}_{\mu,a} + \mathfrak{I}_{\mu,b}. \tag{2.4}$$

Since $b \in (0, a)$, using Lemma 2.4 and (2.4), we have

$$\mathfrak{I}_{\mu,a} > \frac{d^p}{a^p}\mathfrak{I}_{\mu,a} + \frac{b^p}{a^p}\mathfrak{I}_{\mu,a} = \Big(\frac{d^p}{a^p} + \frac{b^p}{a^p}\Big)\mathfrak{I}_{\mu,a} = \mathfrak{I}_{\mu,a},$$

which is a contradiction. Then $|v|_p = a$, and we have $u \in S(a)$. Because $|v_n|_p = |v|_p = a$, $v_n \rightharpoonup v$ in $L^p(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ is reflexive, we deduce

$$v_n \to v \quad \text{in } L^p(\mathbb{R}^N). \tag{2.5}$$

By conditions (f1), (f2) and by using the dominated convergence theorem, we get

$$\int_{\mathbb{R}^N} F(v_n)\,dx \to \int_{\mathbb{R}^N} F(v)\,dx. \tag{2.6}$$

These limits together with $\mathfrak{I}_{\mu,a} = \lim_{n\to+\infty} \mathcal{I}_\mu(v_n)$ provide

$$\mathfrak{I}_{\mu,a} \geq \mathcal{I}_\mu(v).$$

Since $v \in S(a)$, we can see that $\mathcal{I}_\mu(v) = \mathfrak{I}_{\mu,a}$. Then

$$\lim_{n\to+\infty} \mathcal{I}_\mu(v_n) = \mathcal{I}_\mu(v) = \mathfrak{I}_{\mu,a}.$$

This, combined with (2.5) and (2.6), gives

$$\|v_n\|^p \to \|v\|^p$$

in $W^{s,p}(\mathbb{R}^N)$. It implies that $v_n \to v$ in $W^{s,p}(\mathbb{R}^N)$.

Now, let us assume that $v = 0$, that is, $v_n \rightharpoonup 0$ in $W^{s,p}(\mathbb{R}^N)$. By arguments as in Claim 1, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} F(v_n)\,dx \geq C \quad \text{for } n \in \mathbb{N} \text{ large}. \tag{2.7}$$

We claim that there are $R, \beta > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\int_{B_R(y_n)} |v_n|^p\,dx \geq \beta \quad \text{for all } n \in \mathbb{R}^N. \tag{2.8}$$

Otherwise, by [8, Lemma 2.1], we must have $v_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (p, p_s^*)$. This implies $F(v_n) \to 0$ in $L^1(\mathbb{R}^N)$, which is a contradiction with (2.7). Since $v = 0$, inequality (2.8) together with the fractional Sobolev embedding implies that (y_n) is unbounded. We set $\tilde{v}_n(x) = v_n(x + y_n)$. Clearly, $(\tilde{v}_n) \subset S(a)$ and it is also a minimizing sequence for $\mathfrak{I}_{\mu,a}$. Moreover, there exists $\tilde{v} \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\tilde{v}_n \rightharpoonup \tilde{v} \quad \text{in } W^{s,p}(\mathbb{R}^N), \qquad \text{and} \qquad \tilde{v}_n(x) \to \tilde{v}(x) \quad \text{a.e. in } \mathbb{R}^N.$$

By arguments as in the above proofs, we get that $\tilde{v}_n \to \tilde{v}$ in $W^{s,p}(\mathbb{R}^N)$. □

2.1 Proof of Theorem 2.1

Using Lemma 2.3, we can get a bounded minimizing sequence $(v_n) \subset S(a)$ such that $\mathcal{I}_\mu(v_n) \to \mathfrak{I}_{\mu,a}$. From Theorem 2.5, there is $v \in S(a)$ with $\mathcal{I}_\mu(v) = \mathfrak{I}_{\mu,a}$. Therefore, there exists $\lambda_a \in \mathbb{R}$ via by the Lagrange multiplier such that

$$\mathcal{I}'_\mu(v) = \lambda_a \Psi'(v) \quad \text{in } (W^{s,p}(\mathbb{R}^N))', \tag{2.9}$$

where $\Psi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ is given by

$$\Psi(v) = \frac{1}{p}\int_{\mathbb{R}^N} |v|^p\,dx, \quad v \in W^{s,p}(\mathbb{R}^N).$$

From (2.9), we deduce

$$(-\Delta)_p^s v + \mu|v|^{p-2}v = \lambda_a|v|^{p-2}v + f(v) \quad \text{in } \mathbb{R}^N.$$

We have $\mathcal{I}_\mu(v) = \mathfrak{I}_{\mu,a} < 0$, and thus $\lambda_a < 0$.

We can assume that v is a nonnegative function. Indeed, it comes from the inequality $\mathcal{I}_\mu(|v|) \leq \mathcal{I}_\mu(v)$. Moreover, from $v \in S(a)$, we also have that $|v| \in S(a)$, and we deduce

$$\mathfrak{I}_{\mu,a} = \mathcal{I}_\mu(v) \geq \mathcal{I}_\mu(|v|) \geq \mathfrak{I}_{\mu,a}.$$

Thus, $\mathcal{I}_\mu(|v|) = \mathfrak{I}_{\mu,a}$, and so we can replace v by $|v|$. Moreover, denoting by v^* the Schwarz symmetrization of v (see [1, Section 9.2] for the first inequality), we know that

$$\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \geq \iint_{\mathbb{R}^{2N}} \frac{|v^*(x)-v^*(y)|}{|x-y|^{N+ps}}\,dx\,dy,$$

$$\int_{\mathbb{R}^N} |v|^p\,dx = \int_{\mathbb{R}^N} |v^*|^p\,dx,$$

$$\int_{\mathbb{R}^N} F(v)\,dx = \int_{\mathbb{R}^N} F(v^*)\,dx.$$

Then $v^* \in S(a)$ and $\mathcal{I}_\mu(v^*) = \mathfrak{I}_{\mu,a}$. Hence, we can replace v by v^*. Note that $v \in C^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ by [16, Corollary 5.5]. The proof of Theorem 2.1 is now finished.

From Theorem 2.1, we get the following corollary.

Corollary 2.6. *Fix $a > 0$ and let $0 \leq \mu_1 < \mu_2 \leq \mathcal{V}_*$. Then we have $\mathfrak{I}_{\mu_1,a} < \mathfrak{I}_{\mu_2,a} < 0$.*

Proof. Let $u_{\mu_2,a} \in S(a)$ satisfy $\mathcal{I}_{\mu_2}(u_{\mu_2,a}) = \mathfrak{I}_{\mu_2,a}$. Then

$$\mathfrak{I}_{\mu_1,a} \leq \mathcal{I}_{\mu_1}(u_{\mu_2,a}) < \mathcal{I}_{\mu_2}(u_{\mu_2,a}) = \mathfrak{I}_{\mu_2,a}. \qquad \square$$

3 The non-autonomous problem

In this section, we will consider the energy function $J_\xi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_\xi(v) = \frac{1}{p}\Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} V(\xi x)|v|^p\,dx\Big) - \int_{\mathbb{R}^N} F(v)\,dx,$$

restricted to the sphere $S(a)$. We suppose that $|\mathcal{V}|_\infty < \mathcal{V}_*$, where $\mathcal{V}_*$ was given in Section 2.

We also define $J_0, J_\infty : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_0(v) = \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \int_{\mathbb{R}^N} F(v)\,dx$$

and

$$J_\infty(v) = \frac{1}{p}\Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + V_\infty\int_{\mathbb{R}^N} |u|^p\,dx\Big) - \int_{\mathbb{R}^N} F(v)\,dx,$$

respectively. Finally, we define

$$J_{V(y)}(v) = \frac{1}{p}\Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + V(y)\int_{\mathbb{R}^N} |v|^p\,dx\Big) - \int_{\mathbb{R}^N} F(v)\,dx$$

for some $y \in \mathbb{R}^N$. We define $\Gamma_{\xi,a}$, $\Gamma_{0,a}$ and $\Gamma_{\infty,a}$ by

$$\Gamma_{\xi,a} = \inf_{v\in S(a)} J_\xi(v), \quad \Gamma_{0,a} = \inf_{v\in S(a)} J_0(u), \quad \Gamma_{\infty,a} = \inf_{v\in S(a)} J_\infty(v),$$

respectively, and set

$$\Gamma_{V(y),a} = \inf_{v\in S(a)} J_{V(y)}(v).$$

Since $0 < \mathcal{V}_\infty < +\infty$, Corollary 2.6 gives that

$$\Gamma_{0,a} < \Gamma_{\infty,a} < 0. \tag{3.1}$$

With that property, we can fix

$$0 < \rho_1 = \frac{1}{2}(\Gamma_{\infty,a} - \Gamma_{0,a}).$$

Our first lemma in this section establishes some relationship between $\Gamma_{\xi,a}$, $\Gamma_{\infty,a}$ and $\Gamma_{0,a}$.

Lemma 3.1. *We have*

$$\limsup_{\xi\to 0^+} \Gamma_{\xi,a} \le \Gamma_{0,a}$$

and there is $\xi_0 > 0$ *such that* $\Gamma_{\xi,a} < \Gamma_{\infty,a}$ *for all* $\xi \in (0, \xi_0)$.

Proof. Let $v_0 \in S(a)$ with $J_0(v_0) = \Gamma_{0,a}$. Then

$$\Gamma_{\xi,a} \le J_\xi(v_0) = \frac{1}{p}\bigg(\iint\limits_{\mathbb{R}^{2N}} \frac{|v_0(x) - v_0(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int\limits_{\mathbb{R}^N} \mathcal{V}(\xi x)|v_0|^p\,dx\bigg) - \int\limits_{\mathbb{R}^N} F(v_0)\,dx.$$

Taking $\xi \to 0^+$, we obtain

$$\limsup_{\xi\to 0^+} \Gamma_{\xi,a} \le \lim_{\xi\to 0^+} J_\xi(v_0) = J_0(v_0) = \Gamma_{a,0}. \tag{3.2}$$

Combining (3.1) and (3.2), we have $\Gamma_{\xi,a} < \Gamma_{\infty,a}$ for ξ small enough. □

In the following two results, we suppose that $\xi \in (0, \xi_0)$, where ξ_0 is given in Lemma 3.1.

Lemma 3.2. *Fix* $\xi \in (0, \xi_0)$ *and let* $(v_n) \subset S(a)$ *be such that* $J_\xi(v_n) \to c$ *with* $c < \Gamma_{0,a} + \rho_1 < 0$. *If* $v_n \rightharpoonup v$ *in* $W^{s,p}(\mathbb{R}^N)$, *then* $v \not\equiv 0$.

Proof. Suppose that $v = 0$. Then

$$\Gamma_{0,a} + \rho_1 + o_n(1) > J_\xi(v_n) = J_\infty(v_n) + \frac{1}{p}\int\limits_{\mathbb{R}^N} (\mathcal{V}(\xi x) - \mathcal{V}_\infty)|v_n|^p\,dx.$$

By condition (V), for any given $\zeta > 0$, there exists $R > 0$ such that

$$\mathcal{V}(x) \ge \mathcal{V}_\infty - \zeta \quad \text{for all } |x| \ge R.$$

Hence,

$$\Gamma_{0,a} + \rho_1 + o_n(1) > J_\xi(v_n) \ge J_\infty(v_n) + \frac{1}{p}\int\limits_{B_{R/\xi}(0)} (\mathcal{V}(\xi x) - \mathcal{V}_\infty)|v_n|^p\,dx - \frac{\zeta}{p}\int\limits_{B^c_{R/\xi}(0)} |v_n|^p\,dx.$$

Note that (v_n) is bounded in $W^{s,p}(\mathbb{R}^N)$, and $v_n \to 0$ in $L^l(B_{R/\xi}(0))$ for all $l \in [1, p_s^*)$. This implies

$$\Gamma_{0,a} + \rho_1 + o_n(1) \ge J_\infty(v_n) - \zeta D \ge \Gamma_{\infty,a} - \zeta D$$

for a suitable constant $D > 0$. Since $\zeta > 0$ is arbitrary, we get

$$\Gamma_{0,a} + \rho_1 \ge \Gamma_{\infty,a},$$

which is a contradiction with the definition of ρ_1. Hence, $v \not\equiv 0$. □

Lemma 3.3. *Assume that* $(v_n) \subset S(a)$ *is a* $(\mathrm{PS})_c$ *sequence for* J_ξ *constrained to* $S(a)$ *with* $c < \Gamma_{0,a} + \rho_1 < 0$ *and* $v_n \rightharpoonup v_\xi$ *in* $W^{s,p}(\mathbb{R}^N)$, *namely*

$$J_\xi(v_n) \to c \quad \text{as } n \to +\infty, \qquad \text{and} \qquad \|J_\xi|'_{S(a)}(v_n)\| \to 0 \quad \text{as } n \to +\infty.$$

Assume that $u_n = v_n - v_\xi \not\to 0$ *in* $W^{s,p}(\mathbb{R}^N)$. *Then there is* $\beta_* > 0$, *independent of* $\xi \in (0, \xi_0)$, *such that*

$$\liminf_{n\to+\infty} |v_n - v_\xi|_p^p \ge \beta_*,$$

where ξ_0 *is given in Lemma 3.1.*

Proof. We denote $\Psi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Psi(v) = \frac{1}{p} \int_{\mathbb{R}^N} |v|^p \, dx,$$

and see that $S(a) = \Psi^{-1}(\{a^p/p\})$. Then, from [33, Proposition 5.12], we can get a sequence $(\lambda_n) \subset \mathbb{R}$ such that

$$\|J'_\xi(v_n) - \lambda_n \Psi'(v_n)\|_{(W^{s,p}(\mathbb{R}^N))'} \to 0 \quad \text{as } n \to +\infty. \tag{3.3}$$

Since (v_n) is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$, we have that the same holds also for (λ_n) and, up to a subsequence, we can suppose that $\lambda_n \to \lambda_\xi$ as $n \to +\infty$. Hence,

$$J'_\xi(v_\xi) - \lambda_\xi \Psi'(v_\xi) = 0 \quad \text{in } (W^{s,p}(\mathbb{R}^N))'. \tag{3.4}$$

To prove (3.4), we need show that

$$\begin{aligned}\int_{\mathbb{R}^{2N}} & \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, dx \, dy \\ & \to \int_{\mathbb{R}^{2N}} \frac{|v_\xi(x) - v_\xi(y)|^{p-2}(v_\xi(x) - v_\xi(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, dx \, dy\end{aligned} \tag{3.5}$$

for all $\varphi \in W^{s,p}(\mathbb{R}^N)$. Using Hölder's inequality, we see

$$\begin{aligned}\int_{\mathbb{R}^{2N}} & \left|\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}}\right| dx \, dy \\ & \le \left(\int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^{2N}} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{\frac{1}{p}} \\ & \le \|v_n\|^{p-1} \|\varphi\| < +\infty.\end{aligned} \tag{3.6}$$

Hence,

$$\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \in L^1(\mathbb{R}^{2N}) \quad \text{for all } n,$$

and there exists a constant $K_* > 0$ satisfying

$$\left|\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}}\right| \le K_*$$

for all $(x, y) \in \mathbb{R}^{2N}$ outside a set with measure zero. For any $\zeta > 0$, there exists $\delta = \frac{\zeta}{K_*}$ such that, for all measurable sets $E \subset \mathbb{R}^{2N}$ such that $|E| < \delta$, we have

$$\int_E \left|\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}}\right| dx \, dy \le K_* |E| < \zeta.$$

Hence,

$$\left\{\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}}\right\}$$

is equi-integrable on $\mathbb{R}^{2N}$. Clearly,

$$\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \to \frac{|v_\xi(x) - v_\xi(y)|^{p-2}(v_\xi(x) - v_\xi(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}}$$

almost everywhere on $\mathbb{R}^{2N}$. Since $\varphi \in W^{s,p}(\mathbb{R}^N)$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{N+ps}} \, dx \, dy < \zeta^p,$$

where $\mathcal{B}_R(0)$ is a ball in $\mathbb{R}^{2N}$ with center 0 and radius R. By arguments as in (3.6), since we only take the integral on $\mathbb{R}^{2N} \setminus \mathcal{B}_R(0)$ and since $\{u_n\}$ is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$, there exists a suitable constant $\mathcal{K}_* > 0$ such that

$$\int_{\mathbb{R}^{2N}\setminus\mathcal{B}_R(0)} \frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy$$
$$\leq \Big(\int_{\mathbb{R}^{2N}\setminus\mathcal{B}_R(0)} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{\frac{p-1}{p}} \times \Big(\int_{\mathbb{R}^{2N}\setminus\mathcal{B}_R(0)} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{\frac{1}{p}} < \mathcal{K}_*\zeta.$$

Therefore, all conditions of Vitali's theorem are satisfied and we get (3.5). Similarly, we also have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(v_n)\varphi\,dx = \int_{\mathbb{R}^N} f(v_\xi)\varphi\,dx \tag{3.7}$$

for all $\varphi \in W^{s,p}(\mathbb{R}^N)$. Combining (3.3), (3.5) and (3.7), we get (3.4). By arguments as in [2, Lemma 2.8 iv], we have

$$J'_\xi(v_n) = J'_\xi(v_\xi) + J'_\xi(u_n) + o_n(1)$$

and

$$\Psi'_\xi(v_n) = \Psi'_\xi(v_\xi) + \Psi'_\xi(u_n) + o_n(1).$$

From those equations and (3.4), we obtain

$$J'_\xi(v_n) - \lambda_\xi\Psi'(v_n) = J'_\xi(v_\xi) - \lambda_\xi\Psi'(v_\xi) + J'_\xi(u_n) - \lambda_\xi\Psi'(u_n) + o_n(1) = J'_\xi(u_n) - \lambda_\xi\Psi'(u_n) + o_n(1).$$

We get

$$\|J'_\xi(u_n) - \lambda_\xi\Psi'(u_n)\|_{(W^{s,p}(\mathbb{R}^N))'} \to 0 \quad \text{as } n \to +\infty. \tag{3.8}$$

From condition (f3), we have $q_1F(t) \leq f(t)t$ for all $t \geq 0$. Then we get

$$0 > \rho_1 + \Gamma_{0,a} \geq \liminf_{n\to+\infty} J_\xi(v_n) = \liminf_{n\to+\infty}\Big(J_\xi(v_n) - \frac{1}{p}J'_\xi(v_n)v_n + \frac{\lambda_n}{p}a^p\Big) \geq \frac{\lambda_\xi}{p}a^p,$$

implying that

$$\limsup_{\xi\to 0} \lambda_\xi \leq \frac{p(\rho_1 + \Gamma_{0,a})}{a^p} < 0.$$

This implies that there exists $\lambda_* < 0$, independent of ξ, so that

$$\lambda_\xi \leq \lambda_* < 0 \quad \text{for all } \xi \in (0, \xi_0). \tag{3.9}$$

From (3.8), we obtain

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x)|u_n|^p\,dx - \lambda_\xi \int_{\mathbb{R}^N} |u_n|^p\,dx = \int_{\mathbb{R}^N} f(u_n)u_n dx + o_n(1). \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x)|u_n|^p\,dx - \lambda_* \int_{\mathbb{R}^N} |u_n|^p\,dx \leq \int_{\mathbb{R}^N} f(u_n)u_n dx + o_n(1). \tag{3.11}$$

By conditions (f1) and (f2), for some $\tau > 0$, there exists $D(\tau) > 0$ so that

$$|f(t)| \leq \tau|t|^{p-1} + D(\tau)|t|^{\mathfrak{p}-1} \quad \text{for all } t \in \mathbb{R}. \tag{3.12}$$

From (3.11) and (3.12), we conclude

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + C_0 \int_{\mathbb{R}^N} |u_n|^p\,dx \leq C_2|u_n|_{\mathfrak{p}}^{\mathfrak{p}} + o_n(1),$$

where C_0 is a constant not depending on $\xi \in (0, \xi_0)$. By the Sobolev embedding

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow L^{\mathfrak{p}}(\mathbb{R}^N),$$

we obtain

$$\|u_n\|^p_{W^{s,p}(\mathbb{R}^N)} \leq D_3|u_n|^{\mathfrak{p}}_{\mathfrak{p}} + o_n(1) \leq D_4\|u_n\|^{\mathfrak{p}}_{W^{s,p}(\mathbb{R}^N)} + o_n(1), \tag{3.13}$$

where D_3 and $D_4 > 0$ are suitable constants independent of ξ. Because $u_n \nrightarrow 0$ in $W^{s,p}(\mathbb{R}^N)$, up to a subsequence of (u_n) still denoted by itself, we can suppose that

$$\liminf_{n\to+\infty}\|u_n\|_{W^{s,p}(\mathbb{R}^N)} > 0.$$

This yields

$$\liminf_{n\to+\infty}\|u_n\|_{W^{s,p}(\mathbb{R}^N)} \geq \Big(\frac{1}{D_4}\Big)^{\frac{1}{\mathfrak{p}-p}}. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\liminf_{n\to+\infty}|u_n|^{\mathfrak{p}}_{\mathfrak{p}} \geq D_5, \tag{3.15}$$

where $D_5 > 0$ is a suitable constant not depending on ξ. Using the fractional Gagliardo–Nirenberg inequality, we obtain

$$|u_n|^{\mathfrak{p}}_{\mathfrak{p}} \leq C_{s,N,\mathfrak{p}}[u_n]^{\mathfrak{p}\alpha}_{s,p}|u_n|^{\mathfrak{p}(1-\alpha)}_p.$$

Then we deduce that

$$\liminf_{n\to+\infty}|u_n|^{\mathfrak{p}}_{\mathfrak{p}} \leq C_{s,N,\mathfrak{p}}\big(\liminf_{n\to+\infty}|u_n|_p\big)^{\mathfrak{p}(1-\alpha)}K^{\mathfrak{p}\alpha}, \tag{3.16}$$

where $K > 0$ is a constant which is independent on $\xi \in (0, \xi_0)$ and satisfies $\|u_n\| \leq K$ for all $n \in \mathbb{N}$. By combining (3.15) and (3.16), there exists a constant $\beta_* > 0$, independent on $\xi \in (0, \xi_0)$, such that

$$\liminf_{n\to+\infty}|v_n - v_\xi|^p_p \geq \beta_*.$$

This concludes the proof. ☐

From here onwards, we will fix the number ρ satisfying

$$0 < \rho < \min\Big\{\frac{1}{2}, \frac{\beta_*}{a^p}\Big\}(\Gamma_{\infty,a} - \Gamma_{0,a}) \leq \rho_1.$$

Lemma 3.4. *For each $\xi \in (0, \xi_0)$, the functional J_ξ satisfies the $(PS)_c$ condition constrained on $S(a)$ for $c < \Gamma_{0,a} + \rho$.*

Proof. Let (v_n) be a $(PS)_c$ sequence for J_ξ restricted to $S(a)$ with $v_n \rightharpoonup v_\xi$ in $W^{s,p}(\mathbb{R}^N)$ and $c < \Gamma_{0,a} + \rho$. Let $\Psi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$\Psi(v) = \frac{1}{p}\int_{\mathbb{R}^N} |v|^p\,dx.$$

Then $S(a) = \Psi^{-1}(\{a^p/p\})$. Then, by [33, Proposition 5.12], there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$\|J'_\xi(v_n) - \lambda_n\Psi'(v_n)\|_{(W^{s,p}(\mathbb{R}^N))'} \to 0 \quad \text{as } n \to +\infty.$$

From Lemma 3.3, if $u_n = v_n - u_\xi \nrightarrow 0$ in $W^{s,p}(\mathbb{R}^N)$, then there exists $\beta_* > 0$ independent on ξ so that

$$\liminf_{n\to+\infty}|u_n|^p_p \geq \beta_*.$$

Set $d_n = |u_n|_p$ and assume that $|u_n|_p \to d > 0$ and $|v_\xi|_p = b$. We get $a^p = b^p + d^p$. From Lemma 3.2, we have $b > 0$ and $J_\xi(u_n) \geq \Gamma_{\infty,d_n} + o_n(1)$. We get $d_n \in (0, a)$ for n large enough. Hence, we deduce

$$c + o_n(1) = J_\xi(v_n) = J_\xi(u_n) + J_\xi(v_\xi) + o_n(1) \geq \Gamma_{\infty,d_n} + \Gamma_{0,b} + o_n(1).$$

By arguments as in Lemma 2.4, we have

$$\rho + \Gamma_{0,a} \geq \frac{d_n^p}{a^p}\Gamma_{\infty,a} + \frac{b^p}{a^p}\Gamma_{0,a}.$$

Letting $n \to +\infty$, we get

$$\rho \geq \frac{d^p}{a^p}(\Gamma_{\infty,a} - \Gamma_{0,a}) \geq \frac{\beta_*}{a^p}(\Gamma_{\infty,a} - \Gamma_{0,a}),$$

which is a contradiction since

$$\rho < \frac{\beta_*}{a^p}(\Gamma_{\infty,a} - \Gamma_{0,a}).$$

From $u_n \to 0$ in $W^{s,p}(\mathbb{R}^N)$, that is, $v_n \to v_\xi$ in $W^{s,p}(\mathbb{R}^N)$, which implies that $|v_\xi|_p = a$ and

$$(-\Delta)_p^s v_\xi + \mathcal{V}(\xi x)|v_\xi|^{p-2}v_\xi = \lambda_\xi |v_\xi|^{p-2}v_\xi + f(v_\xi) \quad \text{in } \mathbb{R}^N,$$

where λ_ξ is the limit of some subsequence of (λ_n). □

4 Multiplicity result of (1.1)

Fix $\delta > 0$ and let w be a nonnegative solution of the problem

$$\begin{cases} (-\Delta)_p^s v = \lambda |v|^{p-2}v + f(v) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^p \, dx = a^p, \end{cases} \tag{P$_0$}$$

with $J_0(w) = \Gamma_{0,a}$. We denote by $\eta : [0,\infty) \to [0,\infty)$ a smooth nonincreasing cut-off function which is defined as follows:

$$\eta(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{\delta}{2}, \\ 0 & \text{if } t \geq \delta. \end{cases}$$

For any $y \in \mathcal{M}$, define

$$\Psi_{\xi,y}(x) = \eta(|\xi x - y|) w\Big(\frac{\xi x - y}{\xi}\Big),$$

$$\tilde{\Psi}_{\xi,y}(x) = a \frac{\Psi_{\xi,y}(x)}{|\Psi_{\xi,y}|_p},$$

and define $\Phi_\xi : \mathcal{M} \to S(a)$ by $\Phi_\xi(y) = \tilde{\Psi}_{\xi,y}$. We see that $\Phi_\xi(y)$ has compact support for any $y \in \mathcal{M}$.

Lemma 4.1. *We have*

$$\lim_{\xi \to 0} J_\xi(\Phi_\xi(y)) = \Gamma_{0,a} \quad \textit{uniformly in } y \in \mathcal{M}.$$

Proof. Conversely, we assume that there exist $\xi_0 > 0$, $(y_n) \subset \mathcal{M}$ and $\xi_n \to 0$ such that

$$|J_{\xi_n}(\Phi_{\xi_n}(y_n)) - \Gamma_{0,a}| \geq \xi_0 \quad \text{for all } n \in \mathbb{N}.$$

By the dominated convergence theorem, we get (see [8, Lemmas 2.2 and 2.5])

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\Psi_{\xi_n,y_n}|^p \, dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\eta(\xi_n z) w(z)|^p \, dx = \int_{\mathbb{R}^N} |w|^p \, dx = a^p,$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} F(\Phi_{\xi_n}(y_n)) \, dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} F\Big(a \frac{\eta(\xi_n z) w(z)}{|\Psi_{\xi_n,y_n}|_p}\Big) \, dx = \int_{\mathbb{R}^N} F(w) \, dx,$$

$$\begin{aligned} &\lim_{n\to+\infty} \int_{\mathbb{R}^{2N}} \frac{|\Phi_{\xi_n}(y_n)(x) - \Phi_{\xi_n}(y_n)(y)|^p}{|x-y|^{N+ps}} \, dx \, dy \\ &\quad = \lim_{n\to+\infty} \int_{\mathbb{R}^{2N}} \frac{a^p}{|\Psi_{\xi_n,y_n}|_p^p} \frac{|\eta(\xi_n z) w(z) - \eta(\xi_n z') w(z')|^p}{|z-z'|^{N+ps}} \, dz \, dz' = [w]_{s,p}^p, \end{aligned}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \mathcal{V}(\xi_n x) |\Phi_{\xi_n}(y_n)|^p \, dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} \frac{a^p}{|\Psi_{\xi_n,y}|_p^p} \mathcal{V}(\xi_n z + y_n) |\eta(\xi_n z) w(z)|^p \, dz = 0.$$

Consequently,

$$\lim_{n\to+\infty} J_{\xi_n}(\Phi_{\xi_n}(y_n)) = J_{0,a}(w) = \Gamma_{0,a},$$

which is a contraction. □

For any $\delta > 0$, we choose $\mathfrak{r} = \mathfrak{r}(\delta) > 0$ satisfying $\mathcal{M}_\delta \subset B_{\mathfrak{r}}(0)$. We also define $\chi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\chi(x) = \begin{cases} x & \text{if } |x| \le \mathfrak{r}, \\ \dfrac{\mathfrak{r}x}{|x|} & \text{if } |x| \ge \mathfrak{r}. \end{cases}$$

Next, we define $\beta_\xi : S(a) \to \mathbb{R}^N$ by

$$\beta_\xi(v) = \frac{\int_{\mathbb{R}^N} \chi(\xi x)|v|^p\,dx}{a^p}, \quad v \in S(a).$$

By arguments as in [7, Lemma 4.2], we have the following result.

Lemma 4.2. *We have*

$$\lim_{\xi\to 0} \beta_\xi(\Phi_\xi(y)) = y \quad \textit{uniformly in } y \in \mathcal{M}.$$

Proposition 4.3. *Assume that $\xi_n \to 0$ and $(v_n) \subset S(a)$ with $J_{\xi_n}(v_n) \to \Gamma_{0,a}$. Then there is $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $u_n(x) = v_n(x + \tilde{y}_n)$ has a convergent subsequence in $W^{s,p}(\mathbb{R}^N)$. Furthermore, up to a subsequence, we have $y_n = \xi_n \tilde{y}_n \to y$ for some $y \in \mathcal{M}$,*

Proof. We show that there are $\mathfrak{r}_0, \tau > 0$ and $\tilde{y}_n \in \mathbb{R}^N$ such that

$$\int_{B_{\mathfrak{r}_0}(\tilde{y}_n)} |v_n|^p\,dx \ge \tau \tag{4.1}$$

for all n large enough. Conversely, we get $v_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (p, p_s^*)$ via [8, Lemma 2.1]. Then

$$\int_{\mathbb{R}^N} F(v_n)\,dx \to 0.$$

Thus, we deduce

$$\lim_{n\to+\infty} J_{\xi_n}(v_n) \ge 0,$$

which is a contradiction to the fact that

$$\lim_{n\to+\infty} J_{\xi_n}(v_n) = \Gamma_{0,a} < 0.$$

Thus, if we set $u_n(x) = v_n(x + \tilde{y}_n)$, then there exists $u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$. Since

$$(u_n) \subset S(a) \quad \text{and} \quad J_{\xi_n}(v_n) \ge J_0(v_n) = J_0(u_n) \ge \Gamma_{0,a},$$

we have that $J_0(u_n) \to \Gamma_{0,a}$. From Theorem 2.5, we have $u_n \to u$ in $W^{s,p}(\mathbb{R}^N)$, and $u \in S(a)$.

Claim 2. *(y_n) is bounded.*

Indeed, if there exists a subsequence of $\{y_n\}$, still denoted by $\{y_n\}$, such that $|y_n| \to +\infty$, then we have

$$\Gamma_{0,a} = \lim_{n\to+\infty} J_{\xi_n}(u_n) = \liminf_{n\to+\infty}\bigg(\frac{1}{p}\bigg[\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi_n x + y_n)|u_n|^p\,dx\bigg] - \int_{\mathbb{R}^N} F(u_n)\,dx\bigg),$$

that is,

$$\Gamma_{0,a} \ge \frac{1}{p}\bigg[\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} \mathcal{V}_\infty|u|^p\,dx\bigg] - \int_{\mathbb{R}^N} F(u)\,dx \ge \Gamma_{\infty,a},$$

which contradicts (3.1). From Claim 2, we can assume that $y_n \to y$ in $\mathbb{R}^N$. By arguments as above, we get

$$\Gamma_{0,a} \geq \frac{1}{p}\Big[\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int\limits_{\mathbb{R}^N} \mathcal{V}(y)|u|^p\,dx\Big] - \int\limits_{\mathbb{R}^N} F(u)\,dx \geq \Gamma_{\mathcal{V}(y),a}.$$

By Corollary 2.6, if $y \notin \mathcal{M}$, then $\Gamma_{\mathcal{V}(y),a} > \Gamma_{0,a}$ since $\mathcal{V}(y) > 0$. This is a contradiction. Therefore, $\mathcal{V}(y) = 0$, that is, $y \in \mathcal{M}$. □

We consider a positive function $\mathfrak{h} : [0,+\infty) \to [0,+\infty)$ satisfying $\mathfrak{h}(\xi) \to 0$ as $\xi \to 0$, and let

$$\tilde{S}(a) = \big\{v \in S(a) : J_\xi(v) \leq \Gamma_{0,a} + \mathfrak{h}(\xi)\big\}. \tag{4.2}$$

By Lemma 4.1, the function

$$\mathfrak{h}(\xi) = \sup_{y\in M}|J_\xi(\Phi_\xi(y)) - \Gamma_{0,a}|$$

satisfies $\mathfrak{h}(\xi) \to 0$ as $\xi \to 0$. Therefore, $\Phi_\xi(y) \in \tilde{S}(a)$ for all $y \in \mathcal{M}$. By arguments as in [7, Lemma 4.5], we have the following result.

Lemma 4.4. *Let $\delta > 0$ and*

$$\mathcal{M}_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x,\mathcal{M}) \leq \delta\}.$$

Then

$$\lim_{\xi\to 0} \sup_{v\in\tilde{S}(a)} \inf_{z\in\mathcal{M}_\delta} |\beta_\xi(v) - z| = 0.$$

4.1 Proof of Theorem 1.1

We first show the existence of multiple normalized solutions to (1.1). Fix $\xi \in (0,\xi_0)$. Then, by Lemmas 4.1, 4.2 and 4.4, and arguments as in [12], we see that $\beta_\xi \circ \Phi_\xi$ is homotopic to the inclusion map $\mathrm{id} : \mathcal{M} \to \mathcal{M}_\delta$, and so

$$\operatorname{cat}(\tilde{S}(a)) \geq \operatorname{cat}_{\mathcal{M}_\delta}(\mathcal{M}).$$

By arguments as in Lemma 2.3, we also have that J_ξ is bounded from below on $S(a)$. From Lemma 3.4, the functional J_ξ verifies the $(\mathrm{PS})_c$ condition for $c \in (\Gamma_{0,a}, \Gamma_{0,a} + \mathfrak{h}(\xi))$. Then we can apply the Lusternik–Schnirelmann category theorem for critical points (see [14, 33]) to get that J_ξ has at last $\operatorname{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ critical points on $S(a)$.

Let v_ξ be a solution of (1.1) with

$$J_\xi(v_\xi) \leq \Gamma_{0,a} + \mathfrak{h}(\xi),$$

where $\mathfrak{h}$ is defined in (4.2). By arguments as in Proposition 4.3, for each $\xi_n \to 0$, there exists a sequence $\tilde{y}_n \in \mathbb{R}^N$ such that $y_n = \xi_n\tilde{y}_n \to y$ with $y \in \mathcal{M}$, and $u_n(x) = v_{\xi_n}(x+\tilde{y}_n)$ converges strongly to $u \in W^{s,p}(\mathbb{R}^N)$ with $u \not\equiv 0$. We know that u_n is a solution of

$$(-\Delta)_p^s u_n + \mathcal{V}(\xi_n x + y_n)|u_n|^{p-2}u_n = \lambda_{\xi_n}|u_n|^{p-2}u_n + f(u_n) \quad \text{in } \mathbb{R}^N,$$

with

$$\limsup_{n\to\infty} \lambda_{\xi_n} \leq \frac{p(\rho_1 + \Gamma_{0,a})}{a^p} < 0.$$

Because $u_n \to u$ in $W^{s,p}(\mathbb{R}^N)$, we have

$$\lim_{|x|\to+\infty} u_n(x) = 0 \quad \text{uniformly in } \mathbb{N}.$$

Thus, given $\tau > 0$, there are $\mathfrak{R}_1 > 0$ and $n_0 \in \mathbb{N}$ such that

$$|u_n(x)| \leq \frac{1}{2}\Big(\frac{\tau}{2|B_{\mathfrak{R}_1}(0)|}\Big)^{\frac{1}{p}} \quad \text{for } |x| \geq \mathfrak{R}_1 \text{ and } n \geq n_0.$$

In the following, we prove that there exists $\delta > 0$ such that $|u_n|_\infty \geq \delta$ for all n large enough. Indeed, from (4.1), we can choose $\mathfrak{R}_1 > \mathfrak{r}_0$ such that

$$0 < \frac{\tau}{2} \leq \int_{B_{\mathfrak{R}_1}(0)} |u_n|^p \, dx \leq |B_{\mathfrak{R}_1}(0)|.|u_n|_\infty^p \tag{4.3}$$

for all n large enough. Here, we choose

$$\delta = \Big(\frac{\tau}{2|B_{\mathfrak{R}_1}(0)|}\Big)^{\frac{1}{p}}.$$

We denote by z_n the global maximum of u_n. Then $|z_n| \leq \mathfrak{R}_1$ for all $n \in \mathbb{N}$ large enough. Now, let us consider $\zeta_n \in \mathbb{R}^N$ such that $|v_n(\zeta_n)| = |v_n|_\infty$ for all $n \in \mathbb{N}$. Then $\zeta_n = z_n + \tilde{y}_n$ and

$$\lim_{n\to+\infty} \mathcal{V}(\xi_n\zeta_n) = \lim_{n\to+\infty} V(\xi_n z_n + \xi_n\tilde{y}_n) = \mathcal{V}(y) = 0.$$

5 Proof of Theorem 1.2

We fix $\rho_0 > 0$ and $r_0 > 0$ such that the following assertions hold:

(i) It holds

$$\overline{B_{\rho_0}(b_i)} \cap \overline{B_{\rho_0}(b_j)} = \emptyset \quad \text{for } i \neq j \text{ and } i, j \in \{1, 2, \dots, l\}.$$

(ii) It holds

$$\bigcup_{i=1}^{l} B_{\rho_0}(b_i) \subset B_{r_0}(0).$$

(iii) It holds

$$\mathcal{K}_{\frac{\rho_0}{2}} = \bigcup_{i=1}^{l} \overline{B_{\frac{\rho_0}{2}}(b_i)}.$$

We define the function

$$\mathcal{Q}_\varepsilon : W^{s,p}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$$

by

$$\mathcal{Q}_\varepsilon(v) = \frac{\int_{\mathbb{R}^N} \Theta(\varepsilon x)|v|^p \, dx}{a^p},$$

where $\Theta : \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$\Theta(x) = \begin{cases} x & \text{if } |x| \leq r_0, \\ r_0 \dfrac{x}{|x|} & \text{if } |x| > r_0. \end{cases}$$

With the above notations, we have the following result.

Lemma 5.1. *Suppose that f satisfies conditions* (f1)–(f3) *and that* ($\mathcal{V}$) *holds. Then there exists $\rho_2 \in (0, \rho)$ such that, if $v \in S(a)$ and $J_\xi(v) \leq \Gamma_{0,a} + \rho_2$, then $\mathcal{Q}_\varepsilon(v) \in \mathcal{K}_{\rho_0/2}$ for all $\xi \in (0, \xi_0)$, where ξ_0 is given in Lemma 3.1 and ρ is defined in Lemma 3.4.*

Proof. Assume that there exist $\rho_n \to 0$, $\xi_n \to 0$ and $\{v_n\} \subset S(a)$ such that

$$J_\xi(u_n) \leq \Gamma_{0,a} + \rho_n \quad \text{and} \quad \mathcal{Q}_{\xi_n}(v_n) \notin \mathcal{K}_{\frac{\rho_0}{2}}. \tag{5.1}$$

Then we get

$$\Gamma_{0,a} \leq J_0(v_n) \leq J_{\xi_n}(v_n) \leq \Gamma_{0,a} + \rho_n,$$

which implies that $J_0(v_n) \to \Gamma_{0,a}$ as $n \to \infty$. By Theorem 2.5, up to a subsequence, still denoted by itself, one of the following assertions holds:

(i) (v_n) is strongly convergent in $W^{s,p}(\mathbb{R}^N)$.

(ii) There exists $(y_n) \subset \mathbb{R}^N$ with $|y_n| \to +\infty$ such that the sequence $u_n(x) = v_n(x + y_n)$ converges strongly to a function $u \in S(a)$ with $J_0(u) = \Gamma_{0,a}$.

When case (i) occurs, using Lebesgue's dominated convergence theorem, we get

$$\mathcal{Q}_{\xi_n}(v_n) = \frac{\int_{\mathbb{R}^N} \Theta(\xi_n x)|v_n|^p \, dx}{\int_{\mathbb{R}^N} |v_n|^p \, dx} \to 0 \in \mathcal{K}_{\frac{\rho_0}{2}} \quad \text{as } n \to \infty,$$

which is a contradiction.

If case (ii) occurs, then, up to a subsequence still denoted by $\{\xi_n y_n\}$, we assume that $\xi_n y_n \to y \in \mathbb{R}^N$ or $|\xi_n y_n| \to \infty$ as $n \to \infty$. If $\xi_n y_n \to y \in \mathbb{R}^N$, then, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} J_{\xi_n}(v_n) &= \frac{1}{p}\Big(\iint_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+ps}} \, dx\, dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x)|v_n|^p \, dx \Big) - \int_{\mathbb{R}^N} F(v_n) \, dx \\ &= \frac{1}{p}\Big(\iint_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \, dx\, dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi_n x + \xi_n y_n)|u_n|^p \, dx \Big) - \int_{\mathbb{R}^N} F(u_n) \, dx \\ &\to J_{\mathcal{V}(y)}(u) \end{aligned} \tag{5.2}$$

as $n \to \infty$. Combining (5.1) and (5.2), we deduce that

$$\Gamma_{0,a} \ge J_{\mathcal{V}(y)}(v) \ge \Gamma_{\mathcal{V}(y),a}. \tag{5.3}$$

We will show that $\mathcal{V}(y) = 0$, which means that $y_0 = b_i$ for some $i \in \{1, \dots, l\}$. We assume that $\mathcal{V}(y) > 0$. Then $\Gamma_{0,a} < \Gamma_{\mathcal{V}(y),a}$, which contradicts with (5.3). Hence, we arrive that

$$\begin{aligned} \mathcal{Q}_{\xi_n}(u_n) &= \frac{\int_{\mathbb{R}^N} \Theta(\xi_n x)|v_n|^p \, dx}{\int_{\mathbb{R}^N} |v_n|^p \, dx} \\ &= \frac{\int_{\mathbb{R}^N} \Theta(\xi_n x + \xi_n y_n)|v_n|^p \, dx}{\int_{\mathbb{R}^N} |v_n|^p \, dx} \\ &\to \frac{\int_{\mathbb{R}^N} \Theta(y)|u|^p \, dx}{\int_{\mathbb{R}^N} |u|^p \, dx} \\ &= \Theta(y) \\ &= b_i \in K_{\frac{\rho_0}{2}} \end{aligned}$$

as $n \to \infty$, for some $i \in \{1, \dots, l\}$, which is a contradiction.

If $|\xi_n y_n| \to \infty$, then, by arguments as above, we get $\Gamma_{0,a} \ge \Gamma_{\infty,a}$, which is impossible due to (3.1). Thus, we conclude the proof of Lemma 5.1. □

Next, we define some useful sets as follows:

$$\begin{aligned} \gamma_\xi^i &= \{v \in S(a) : |\mathcal{Q}_\xi(v) - b_i| \le \rho_0\}, \\ \partial\gamma_\varepsilon^i &= \{v \in S(a) : |\mathcal{Q}_\xi(u) - b_i| = \rho_0\}, \\ \zeta_\xi^i &= \inf_{v \in \gamma_\xi^i} J_\xi(v), \\ \hat{\zeta}_\xi^i &= \inf_{v \in \partial\gamma_\xi^i} J_\xi(v). \end{aligned}$$

Lemma 5.2. *Assume that f satisfies conditions* (f1)–(f3) *and that* ($\mathscr{V}$) *holds. Then*

$$\zeta_\xi^i < \Gamma_{0,a} + \rho_2 \quad \text{and} \quad \zeta_\xi^i < \hat{\zeta}_\xi^i \qquad \text{for all } \xi \in (0, \xi_0).$$

Proof. Assume $v \in W^{s,p}(\mathbb{R}^N)$ such that $J_0(v) = \Gamma_{0,a}$. For each $i \in \{1, \dots, l\}$, we define the function $\hat{v}_\xi^i : \mathbb{R}^N \to \mathbb{R}$ by

$$\hat{v}_\xi^i := v\Big(x - \frac{b_i}{\xi}\Big).$$

Then we see that $\hat{v}^i_\xi \in S(a)$ for all $\xi > 0$ and $1 \le i \le l$. By a simple calculation, we obtain

$$\begin{aligned} J_\xi(\hat{v}^i_\xi) &= \frac{1}{p}\Bigg(\iint\limits_{\mathbb{R}^N} \frac{|\hat{v}^i_\xi(x) - \hat{v}^i_\xi(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int\limits_{\mathbb{R}^N} \mathcal{V}(\xi x)|\hat{v}^i_\xi|^p\,dx\Bigg) - \int\limits_{\mathbb{R}^N} F(\hat{v}^i_\xi)\,dx \\ &= \frac{1}{p}\Bigg(\iint\limits_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int\limits_{\mathbb{R}^N} \mathcal{V}(\xi x + b_i)|v|^p\,dx\Bigg) - \int\limits_{\mathbb{R}^N} F(v)\,dx. \end{aligned}$$

Taking $\xi \to 0^+$ in the previous equation, we get

$$\lim_{\xi \to 0^+} J_\xi(\hat{v}^i_\xi) = J_{\mathcal{V}(b_i)}(v) = J_0(v) = \Gamma_{0,a}. \tag{5.4}$$

By the definition of $\mathcal{Q}_\xi$, we have

$$\mathcal{Q}_\xi(\hat{v}^i_\xi) = \frac{\int_{\mathbb{R}^N} \Theta(\varepsilon x)|\hat{v}^i_\xi|^p\,dx}{\int_{\mathbb{R}^N} |\hat{v}^i_\xi|^p\,dx} = \frac{\int_{\mathbb{R}^N} \Theta(\xi x + b_i)|v|^p\,dx}{\int_{\mathbb{R}^N} |v|^p\,dx} \to \Theta(b_i) = b_i$$

as $\xi \to 0^+$, since $b_i \in B_{r_0}(0)$ for all $i = 1, \dots, l$. It follows that $\hat{v}^i_\varepsilon \in \gamma^i_\xi$ for ξ small enough. From (5.4), we deduce that

$$\Gamma_{0,a} + \rho_2 > J_\xi(\hat{v}^i_\xi) \quad \text{for all } \xi \in (0, \xi_0).$$

Here we can decrease ξ_0 if necessary. Then, by the definition of γ^i_ξ, we arrive at

$$\Upsilon_{0,a} + \rho_2 > \zeta^i_\xi \tag{5.5}$$

for all $i \in \{1, \dots, l\}$.

Next, we prove the second statement. If $v \in \partial\gamma^i_\xi$, then

$$v \in S(a), \quad |\mathcal{Q}_\xi(v) - b_i| = \rho_0 > \frac{\rho_0}{2}, \quad \mathcal{Q}_\xi(v) \notin \mathcal{K}_{\frac{\rho_0}{2}}.$$

From Lemma 5.1, we have $J_\xi(v) > \Gamma_{0,a} + \rho_2$ for all $v \in \partial\gamma^i_\xi$ and $\xi \in (0, \xi_0)$. Using (5.5), we obtain that

$$\hat{\zeta}^i_\xi = \inf_{v \in \partial\gamma^i_\xi} J_\xi(v) \ge \Gamma_{0,a} + \rho_2 > \zeta^i_\xi.$$

This concludes the proof. □

Proof of Theorem 1.2. For each $i \in \{1, \dots, l\}$, by Ekeland's variational principle [13], there exists a sequence $\{v^i_n\} \subset S(a)$ satisfying $J_\xi(v^i_n) \to \zeta^i_\xi$ and

$$J_\xi(v) - J_\xi(v^i_n) \ge -\frac{1}{n}\|v - v^i_n\| \quad \text{for all } v \in \gamma^i_\varepsilon,\ v \ne v^i_n.$$

By Lemma 5.2, we have $\zeta^i_\xi < \hat{\zeta}^i_\xi$. Thus,

$$v^i_n \in \gamma^i_\xi \setminus \partial\gamma^i_\xi$$

for all n large enough. For $\delta > 0$ small enough, we consider the map $\alpha : (-\delta, \delta) \to S(a)$ given by

$$\alpha(t) = a\frac{v^i_n + tv}{|v^i_n + tv|_p},$$

belonging to $C^1((-\delta, \delta), S(a))$ and satisfying

$$\alpha(t) \in \gamma^i_\xi \setminus \partial\gamma^i_\xi \quad \text{for all } t \in (-\delta, \delta),\ \alpha(0) = v^i_n,\ \alpha'(0) = v,$$

where

$$v \in T_{v^i_n} S(a) = \Big\{ w \in W^{s,p}(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} |v^i_n|^{p-2} v^i_n w\,dx = 0 \Big\}.$$

We have

$$J_\xi(\alpha(t)) - J_\xi(v_n^i) \geq -\frac{1}{n}\|\alpha(t) - v_n^i\| \quad \text{for all } t \in (-\delta, \delta),$$

which implies that

$$\frac{J_\xi(\alpha(t)) - J_\xi(\alpha(0))}{t} = \frac{J_\xi(\alpha(t)) - J_\xi(v_n^i)}{t} \geq -\frac{1}{n}\left\|\frac{\alpha(t) - v_n^i}{t}\right\| = -\frac{1}{n}\left\|\frac{\alpha(t) - \alpha(0)}{t}\right\|. \tag{5.6}$$

Since $J_\xi \in C^1(W^{s,p}(\mathbb{R}^N), \mathbb{R})$, taking $t \to 0^+$ in (5.6), we get

$$\langle J_\xi'(v_n^i), v\rangle \geq -\frac{1}{n}\|v\|.$$

Replacing v by $-v$, we deduce

$$\sup_{\|v\|\leq 1}\{|\langle J_\xi'(v_n^i), v\rangle|\} \leq \frac{1}{n},$$

which leads to

$$J_\xi(v_n^i) \to \zeta_\xi^i \quad \text{and} \quad J_\xi'|_{S(a)}(v_n^i) \to 0 \qquad \text{as } n \to \infty.$$

Therefore, v_n^i is a $(\mathrm{PS})_{\zeta_\xi^i}$ of J_ξ. By Lemma 5.2, we have

$$\zeta_\xi^i < \Gamma_{0,a} + \rho_2 < \Gamma_{0,a} + \rho < 0.$$

Then we can apply Lemma 3.4 to show that there exists $v^i \in W^{s,p}(\mathbb{R}^N) \cap S(a)$ satisfying

$$v^i \in \gamma_\xi^i, J_\xi(v^i) = \zeta_\xi^i \quad \text{and} \quad J_\xi|'_{S(a)}(v^i) = 0.$$

Furthermore,

$$\mathcal{Q}_\xi(v^i) \in \overline{B_{\rho_0}(b_i)}, \quad \mathcal{Q}_\varepsilon(v^j) \in \overline{B_{\rho_0}(b_j)}, \quad \overline{B_{\rho_0}(b_i)} \cap \overline{B_{\rho_0}(b_j)} = \emptyset \qquad \text{for all } i \neq j \in \{1, \dots, l\}.$$

Then J_ξ has at least l nontrivial critical points on $S(a)$ for all $\xi \in (0, \xi_0)$. Because $J_\xi(v^i) = \zeta_\xi^i < 0$, and using condition (f3), we get

$$\begin{aligned}
\lambda^i a^p &= \frac{1}{p}\Big(\iint_{\mathbb{R}^N} \frac{|v^i(x) - v^i(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x)|v^i|^p\,dx\Big) - \int_{\mathbb{R}^N} f(v^i)v^i\,dx \\
&= \frac{1}{p}\Big(\iint_{\mathbb{R}^N} \frac{|v^i(x) - v^i(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x)|v^i|^p\,dx\Big) - \int_{\mathbb{R}^N} F(v^i)\,dx + \int_{\mathbb{R}^N} F(v^i)\,dx - \int_{\mathbb{R}^N} f(v^i)v^i\,dx \\
&= J_\xi(v^i) + \int_{\mathbb{R}^N} F(v^i)\,dx - \int_{\mathbb{R}^N} f(v^i)v^i\,dx \\
&< 0.
\end{aligned}$$

Then $\lambda^i < 0$ for all $i = 1, \dots, l$. We conclude the proof of Theorem 1.2. □

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