MULTIPLE NORMALIZED SOLUTIONS FOR FRACTIONAL ELLIPTIC PROBLEMS

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Abstract. In this article, we are first concerned with the existence of multiple normalized solutions to the following fractional p-Laplace problem

\[
\begin{cases}
(-\Delta)_p^s v + \mathcal{V}(\xi x)|v|^{p-2}v = \lambda|v|^{p-2}v + f(v), & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |v|^p dx = a^p,
\end{cases}
\]

where \(a, \xi > 0\), \(p \geq 2\), \(\lambda \in \mathbb{R}\) is an unknown parameter that appears as a Lagrange multiplier, \(\mathcal{V} : \mathbb{R}^N \rightarrow [0, \infty)\) is a continuous function, and \(f\) is a continuous function with \(L^p\)-subcritical growth. We prove that there exists the multiplicity of solutions by using Lusternik-Schnirelmann category. Next, we show that the number of normalized solutions is at least the number of global minimum points of \(\mathcal{V}\) as \(\xi\) is small enough via Ekeland’s variational principle.

1. Introduction

This paper is to give some results on the existence of multiple normalized solutions to fractional p-Laplace problems as follows

\[
\begin{cases}
(-\Delta)_p^s v + \mathcal{V}(\xi x)|v|^{p-2}v = \lambda|v|^{p-2}v + f(v), & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |v|^p dx = a^p,
\end{cases}
\]

(1.1)

where \(a, \xi > 0\), \(p \geq 2\), and \(\lambda \in \mathbb{R}\) is an unknown parameter that comes from a Lagrange multiplier. The fractional p-Laplace operator \((-\Delta)_p^s\) is defined along a smooth function (up to a normalizing constant) \(v : \mathbb{R}^N \rightarrow \mathbb{R}\) as

\[
(-\Delta)_p^s v(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{N+ps}} dy \quad (x \in \mathbb{R}^N),
\]

where \(B_\varepsilon(x)\) is the ball with center \(x\) and radius \(\varepsilon\). If \(\mathcal{V}(x) \equiv 0\) and \(p = 2\), then the problem (1.1) becomes

\[
\begin{cases}
(-\Delta)^s v = \lambda v + f(v), & \text{in } \mathbb{R}^N \\
\int_{\mathbb{R}^N} |v|^2 dx = a^2,
\end{cases}
\]

(1.2)

which has been intensively considered until now. Namely, Luo and Zhang [25] studied the problem (1.2) with \(f(v) = \mu|v|^{q-2}v + |v|^{p-2}v\), where \(N \geq 2, \mu \in \mathbb{R}, 2 < q < p < 2^*_s\). Under different assumptions on \(q < p, a > 0\) and \(\mu \in \mathbb{R}\), they showed the existence of normalized solutions to problem (1.2). In the \(L^2\)-subcritical case, they use the monotonicity of the least energy to get the ground state solution for \(\mu > 0\). For \(L^2\)-critical or \(L^2\)-supercritical, they used

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some certain decomposition on Nehari manifold of problem (1.2) and homotopy-stable family with extended boundary of a closed set. After that, Zhen and Zhang [37] extended the results of Luo and Zhang [25] for the critical case $f(v) = \mu |v|^{p-2}v + |v|^{2^*_s - 2}v$, where $2 < q < s^*_s = \frac{2N}{N-2s}$.

We also refer to Zhang and Han [38] who studied problem (1.2) when $f(v) = |v|^{p-2}v + |v|^{2^*_s - 2}v$, where $2 < p < 2^*_s$. We denote

$$S(a) = \{ v \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |v|^2 dx = a^2 \}$$

and

$$V(a) = \{ v \in H^s(\mathbb{R}^N) : \mathcal{P}(v) = 0 \},$$

where

$$\mathcal{P}(v) := |(-\Delta)^{s/2}v|_2^2 - \gamma_{p,s}|v|^p_2 - |v|^{2^*_s}, \quad \gamma_{p,s} = \frac{N(p-2)}{2ps}.$$"
(v1) $V : \mathbb{R}^N \to [0, +\infty)$ is a bounded continuous function and it has periodic 1 with the variables $x_1, \ldots, x_N$.
(v2) $V$ is asymptotically periodic. It means that there exists a function $V_P : \mathbb{R}^N \to \mathbb{R}$ periodic 1 with variables $x_1, \ldots, x_N$ such that $V(x) \leq V_P(x)$ for all $x \in \mathbb{R}^N$ and $|V(x) - V_P(x)| \to 0$ as $|x| \to \infty$.
(v3) $V \in L^\infty(\mathbb{R}^N)$ and

$$V_\infty = \liminf_{|x| \to +\infty} V(x) > V_0 = \inf_{x \in \mathbb{R}^N} V(x) > 0.$$  

(v4) $V(x) = \mu W(x)$, where $\mu > 0$ and $W$ is a nonnegative continuous function satisfying the condition as follows: There exists $M_0 > 0$ such that 

$$\text{meas}(x \in \mathbb{R}^N : W(x) > M_0) < +\infty$$

and the set $\Omega = \text{int}(W^{-1}(0))$ is not an empty set. They studied the existence of solution to problem (1.1) as $\xi = 1$ or $\xi > 0$ is small enough. To obtain the results, they prove a compactness result for minimizing sequences $\{v_n\}$ restricted on $S(a)$. Zuo, Liu and Vetro [39] have been studied the existence of normalized solution of following fractional Schrödinger equation

$$(-\Delta)^s + \mu v + \lambda V(x)v - |v|^{p-2}v = 0 \text{ in } \mathbb{R}^N,$$

where $2s < N < 4s$, $p \in (2, \min\{N/(N - 2s), 2 + 4s/N\})$ and the potential $V \in L^\infty(\mathbb{R}^N)$ satisfies the condition: There exists a positive constant $D_0 > 0$ such that the measure of the set $\Omega = \{x \in \mathbb{R}^N : V(x) < D_0\}$ is finite. Using compactness result for minimizing sequences, they show that the existence of normalized solution to (1.4) when $\lambda$ large enough. Note that when $p = 2, s \to 1^-$, our problem (1.2) reduces

$$\begin{cases}
- \Delta v + V(x)v = \lambda v + f(v), & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |v|^2dx = a^2,
\end{cases}$$

where $a, \xi > 0$ and $\lambda \in \mathbb{R}$ is an unknown parameter that comes from a Lagrange multiplier. Many authors studied the problem (1.5), and get many nice results. Using genus theory and deformation arguments, the existence of infinitely many normalized solutions has been studied by many authors such as Alves, Chao and Miyagaki [6], Bartsch and Soave [9], Ikoma and Tanaka [18], Cingolani and Jeanjean [11], Jeanjean and Lu [19, 20] and Jeanjean and Le [21, 22], Soave [29, 29]. In 2022, Alves [3] studied the existence of multiple normalized solutions for problem (1.5), where $V : \mathbb{R}^N \to [0, +\infty)$ is a continuous function verifying condition $(\mathcal{V})$ and the nonlinear reaction has $L^2$-subcritical growth. Alves proved that the number of normalized solutions is bounded from below by the number of global minimum points of $V$ for small enough $\xi$. Alves and Thin [7] studied the existence, concentration of normalized solutions to equation (1.5). They first use Lusternik-Schnirelmann category to get multiple normalized solutions to that problem. Wang, Zeng and Zhou [31] studied the properties of least energy solutions to fractional Laplacian eigenvalue problem on $\mathbb{R}^N$ as follows:

$$\begin{cases}
(-\Delta)^s v + V(x)v = \mu v + am(x)|v|^\frac{4s}{N}v \\
\int_{\mathbb{R}^N} |v|^2dx = 1, v \in H^s(\mathbb{R}^N)
\end{cases}$$

where $N \geq 2, s \in (0, 1), \mu \in \mathbb{R}$, $a > 0$, $V$ and $m$ are in $L^\infty(\mathbb{R}^N)$. They showed that there exists $b^*_s > 0$ such that problem (1.6) has a least energy solution $u_a(x)$ for each $a \in (0, b^*_s)$ and $u_a$ blow up, as $a$ increasing to $b^*_s$, at some points $x_0 \in \mathbb{R}^N$ which $V$ attains the minimum and $m$
achieves maximum. When $s \to 1^-$, our problem (1.1) becomes to the $p$-Laplacian problem as follows:

\begin{equation}
\begin{cases}
-\Delta_p v + \mathcal{V}(\xi x)|v|^{p-2}v = \lambda|v|^{p-2}v + f(v), & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |v|^pdx = a^p,
\end{cases}
\end{equation}

where $-\Delta_p v = \text{div}(|\nabla v|^{p-2}\nabla v)$ is the $p$-Laplace operator. Recently, there is not any work on non autonomous problem (1.7). If $\xi = 1$ in the problem (1.7), Wang and Sun [32] have been studied the existence of normalized solution to following problem:

\begin{equation}
\begin{cases}
-\Delta_p v + \mathcal{V}(x)|v|^{p-2}v = \lambda|v|^{p-2}v + |v|^{q-2}v, & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |v|^pdx = c
\end{cases}
\end{equation}

where $r = p$ or $r = 2$, $1 < p < N$, $p < q < p^*$, $\mathcal{V} \in C(\mathbb{R}^N)$ satisfies $\inf_{x \in \mathbb{R}^N} \mathcal{V}(x) = 0$ and $\lim_{|x| \to \infty} \mathcal{V}(x) = +\infty$. When $r = p$ and $c$ is small enough, they showed that there exists ground state solution with positive energy. For $p = 2$, they proved that problem (1.8) has at least two solutions with positive energy which one is a ground state and the other one is a high-energy solution. Up to now, there are a few works on normalized of $p$-Laplace equation. Wang, Li, Zhou and Li [33] first studied the existence of the $L^2$-norm constraint:

$$-\Delta_p v + |v|^{p-2}v = \mu v + |v|^{s-2}v \text{ in } \mathbb{R}^N,$$

where $1 < p < N$, $\mu \in \mathbb{R}$, $s \in \left(\frac{N+2}{N}, p^*\right)$. Using the constrained variational methods, they show that the above problem has a normalized solution. Zhang and Zhang [40] studied the existence of normalized solutions to $p$-Laplacian equations with the form:

\begin{equation}
\begin{cases}
-\Delta_p v = \lambda|v|^{p-2}v + \mu|v|^{q-2}v + g(v) \text{ in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |v|^pdx = a^p
\end{cases}
\end{equation}

where $N \geq 2, a > 0, 1 < p < q \leq \bar{p} := p + \frac{2}{N}, g \in C(\mathbb{R}, \mathbb{R})$ is old and has $L^p$-supercritical. When $q < \bar{p}$ and $\mu > 0$, they got a positive radial ground state solution for suitable $\mu$ by using Schwarz rearrangement and Ekeland variational principle. Applying the Fountain Theorem, they obtained infinitely many radial solutions for any $N \geq 2$ and obtain the existence of infinitely many non-radial sign-changing solutions for $N = 4$ or $N \geq 6$. In that cases, $\mu$ belongs suitable range, depends on $a$. We also refer to [10, 16, 23, 35] for the qualitative analysis of normalized solutions in different local or nonlocal settings.

So far, there is not any result for existence of multiple normalized solutions to Schrödinger equation involving fractional $p$-Laplace, special to the nonautonomous problem (1.1). Motivated by this fact, the main goal of the present paper is to give the first normalized result for Schrödinger equation involving fractional $p$-Laplace.

In the following, we give some assumptions on the nonlinear function $f$:

1. $f$ is a continuous and odd function, and there are $q \in (p, p + \frac{2}{N})$ and $\alpha \in (0, +\infty)$ such

$$\lim_{t \to 0} \frac{|f(t)|}{t^{q-1}} = \alpha;$$

2. There exist constants $c_1, c_2, c_1, c_2 > 0$ and $p \in (p, p + \frac{2}{N})$ such that

$$|f(t)| \leq c_1 + c_2 t^{p-1} \forall t \in \mathbb{R} \text{ and } |f'(t)| \leq c_1 + c_2 t^{p-2} \forall t \in \mathbb{R};$$

3. There is $q_1 \in (p, p + \frac{2}{N})$ so that $f(t)/t^{q_1+1}$ is an increasing function of $t$ on $(0, +\infty)$. 
From the conditions \((f_1)\) and \((f_3)\), we have \(F(t) \geq 0\) for all \(t \in \mathbb{R}\).

The function
\[
f(t) = |t|^{r-2}t + |t|^{r-2}t \ln(1 + |t|), \quad \forall t \in \mathbb{R},
\]
for some \(r, q \in (p, p + \frac{4}{N})\) and \(r > q\), satisfies the above conditions. Here, \((f_2)\) and \((f_3)\) hold with \(r \in (p, p + \frac{2}{N})\) and \(q_1 = q\).

For the potential function \(V\), we suppose that one of these following conditions holds: (V). We have \(V \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N), V(0) = 0\) and
\[
0 = \inf_{x \in \mathbb{R}^N} V(x) < \liminf_{|x| \to +\infty} V(x) = V_\infty.
\]
\((\forall)\). The function \(V\) belongs \(C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N), V_\infty = \liminf_{|x| \to +\infty} V(x) > 0\). Furthermore, \(V^{-1}(0) = \{b_1, \ldots, b_l\}, b_1 = 0\) and \(b_i \neq b_j\) for all \(i \neq j\).

A solution \(v\) to the problem \((1.1)\) with \(\int_{\mathbb{R}^N} |v|^p dx = a^p\) is a critical point of the energy function
\[
J_\xi(v) = \frac{1}{p} \left( \int_{\mathbb{R}^2N} \frac{|v(x) - v(y)|^p}{|x - y|^{N + ps}} dx dy + \int_{\mathbb{R}^N} V(\xi x)|v|^p dx \right) - \int_{\mathbb{R}^N} F(v) dx, \quad v \in W^{s,p}(\mathbb{R}^N),
\]
restricted to the sphere
\[
S(a) = \{v \in W^{s,p}(\mathbb{R}^N) : |v|_p = a\},
\]
where \(F(t) = \int_0^t f(\tau)d\tau\) and \(|\cdot|_p\) is norm in \(L^p(\mathbb{R}^N)\) for \(p \in [2, +\infty]\). Here the fractional Sobolev space \(W^{s,p}(\mathbb{R}^N)\) is defined for any \(p > 1, s \in (0, 1)\) as
\[
W^{s,p}(\mathbb{R}^N) = \left\{v \in L^p(\mathbb{R}^N) : [v]_{s,p} := \left( \int_{\mathbb{R}^2N} \frac{|v(x) - v(y)|^p}{|x - y|^{N + ps}} dxdy \right)^{1/p} < +\infty \right\},
\]
which is a Banach space with norm
\[
||v|| = ([v]_p^p + [v]_{s,p}^p)^{1/p}.
\]

It is well known that \(J_\xi \in C^1(W^{s,p}(\mathbb{R}^N), \mathbb{R})\) and
\[
< J_\xi'(v), u > = \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N + ps}} dx dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x)|v|^{p-2}vudx - \int_{\mathbb{R}^N} f(v)udx,
\]
for all \(u \in W^{s,p}(\mathbb{R}^N)\). We refer to the monograph [26] for the theory of fractional Sobolev spaces and related applications.

In the first result, we establish the existence of multiple normalized solutions for \((1.1)\) via Lusternik-Schnirelmann category. We denote the sets \(\mathcal{M}\) and \(\mathcal{M}_\delta\) as follows:
\[
\mathcal{M} = \{x \in \mathbb{R}^N : V(x) = 0\}
\]
and
\[
\mathcal{M}_\delta = \{x \in \mathbb{R}^N : dist(x, \mathcal{M}) \leq \delta\}.
\]

Let \(Y\) be a closed subset of a topological space \(X\), then Lusternik-Schnirelmann category \(cat_X(Y)\) is the least number of closed and contractible sets in \(X\) which cover \(Y\). If \(X = Y\), we denote \(cat_X(X)\) by \(cat(X)\). Our first main result is stated as follows:
Theorem 1.1. Suppose that $f$ satisfies the conditions $(f_1)-(f_3)$ and that $V$ verifies the condition $(V)$. Then for each $\delta > 0$, there exist $\xi_0 > 0$ and $V_\star > 0$ such that (1.1) has at least $\text{cat}_{M_2}(M)$ couples $(v_j, \lambda_j) \in W^{s,p}(\mathbb{R}^N) \times \mathbb{R}$ of weak solutions for $0 < \xi < \xi_0$ and $|\xi|_\infty < V_\star$ with $\int_{\mathbb{R}^N} |v|^pdx = a^p$, $\lambda_j < 0$ and $J_\xi(v_j) < 0$. Moreover, if $v_\xi$ denotes one of these solutions and $\xi_\xi$ is the global maximum of $|v|$,$$
abla (\xi) = 0.$$Theorem 1.2. Assume that that $f$ verifies the conditions $(f_1)-(f_3)$ and $V$ satisfies the condition $(V')$. Then there exists $\xi_0 > 0$ such that (1.1) admits at least $l$ couples $(v_j, \lambda_j) \in W^{s,p}(\mathbb{R}^N) \times \mathbb{R}$ of weak solutions for $0 < \xi < \xi_0$ and $\int_{\mathbb{R}^N} |v|^pdx = a^p$, $\lambda_j < 0$ and $J_\xi(v_j) < 0, j = 1, \ldots, l$.

When $s \to 1^-$ in Theorems 1.1 and 1.2, we get automatically results for $p$-Laplace problem (1.7), we leave this works for the readers. Theorems 1.1 and 1.2 extend the results in [7] and [3] to the nonlocal case. Here, our solution space $W^{s,p}(\mathbb{R}^N)$ is a non-Hilbert space, then we need give to develop some new steps in the proofs.

The content of paper is written as follows. In Section 2, we study the autonomous case. In Section 3, we study the non-autonomous case. In this section, we study the Palais-Smale condition on the sphere $S(a)$ for the energy functional. Two next sections, we show that there exists the multiplicity of solutions for problem (1.1).

2. THE AUTONOMOUS CASE

In this section, we study the existence of solution to the following problem

$$
\begin{cases}
(-\Delta)^s_p v + \mu |v|^{p-2}v = \lambda |v|^{p-2}v + f(v), & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |v|^pdx = a^p,
\end{cases}
$$

(2.1)
where $N \geq 1$, $a > 0$, $\mu \geq 0$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier and $f$ is a continuous function verifying the conditions $(f_1)-(f_3)$.

A solution $v$ to the problem (2.1) is a critical point of the $C^1$ energy functional

$$
I_\mu(v) = \frac{1}{p} \left( \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}}dxdy + \mu \int_{\mathbb{R}^N} |v|^pdx \right) - \int_{\mathbb{R}^N} F(v)dx,
$$

constrained to the sphere $S(a)$ given by

$$
S(a) = \{v \in W^{s,p}(\mathbb{R}^N) : |v|^p = a \}.
$$

Our main result in this section is the following:

Theorem 2.1. Suppose that $f$ satisfies the conditions $(f_1)-(f_3)$. Then, there exists $V_\star > 0$ so that problem (2.1) has a solution $(v, \lambda)$ as $0 \leq \mu < V_\star$, where $v$ is non-negative and $\lambda < 0$.

To prove above result, we need the following lemmas.

Lemma 2.2. The functional $I_\mu$ is coercive and bounded from below in $S(a)$.

Proof. By the conditions $(f_1)$ and $(f_2)$, there is $C_1, C_2 > 0$ such that

$$
|F(t)| \leq C_1 |t|^q + C_2 |t|^p, \quad \forall t \in \mathbb{R}.
$$

Since $C_0^\infty(\mathbb{R}^N)$ is density in $W^{s,p}(\mathbb{R}^N)$, then for any $v \in W^{s,p}(\mathbb{R}^N)$, we get the fractional Gagliardo-Nirenberg inequality [27, Lemma 2.1] as follows

$$
|v|^r \leq C_{s,N,\tau}[v]_{s,p,r}^\tau |v|_p^{(1-a)},
$$
for some positive constant $C_{s,N,\tau} \geq 1$, where $\tau > 0$, $0 \leq a \leq 1$ and
\[
\frac{1}{\tau} = a \left( \frac{1}{p} - \frac{s}{N} \right) + \frac{1-a}{p}.
\]
Then $\tau = \frac{pN}{N - ap^{s}} \in [p, p_{s}^{*}]$ or $a = \frac{N}{s} \left( 1 - \frac{1}{p} \frac{1}{\tau} \right)$. When $\tau a = p$, then $a = \frac{p}{\tau}$ and we get $\tau = p + \frac{p^{2}s}{N}$, which is called $L^{p}$-critical exponent for fractional Gagliardo-Nirenberg inequality. Hence,
\[
\mathcal{I}_{\mu}(v) \geq \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p}}{|x-y|^{N+ps}} \, dxdy - C_{1} \int_{\mathbb{R}^{N}} |v|^{q} dx - C_{2} \int_{\mathbb{R}^{N}} |v|^{p} dx
\]
\[
\geq \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p}}{|x-y|^{N+ps}} \, dxdy - C_{s,N,q} C_{1} a^{q(1-a)} \left( \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p}}{|x-y|^{N+ps}} \right)^{\frac{q}{p}}
\]
\[
- C_{s,N,p} C_{2} a^{p(1-a)} \left( \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p}}{|x-y|^{N+ps}} \right)^{\frac{p}{q}}.
\]
As $q, p \in (p, p + \frac{p^{2}s}{N})$, then $0 < \frac{ta}{p} < 1$ for $t \in \{p, q\}$, which implies the coercivity and bounded from below of $\mathcal{I}_{\mu}$ on $S(a)$. \hfill \Box

From Lemma 2.3, there exists the real number $\mathcal{J}_{\mu,a} = \inf_{v \in S(a)} \mathcal{I}_{\mu}(v)$. Next, we show that $\mathcal{J}_{\mu,a} < 0$ is negative for suitable range of $\mu$.

**Lemma 2.3.** There exists $\mathcal{V}_{s} > 0$ such that $\mathcal{J}_{\mu,a} < 0$ for $0 \leq \mu < \mathcal{V}_{s}$.

**Proof.** From the condition $(f_{1})$, we have $\lim_{t \to 0} \frac{qF(t)}{t^{q}} = \alpha > 0$, then there exists $\kappa > 0$ so that
\[
(2.2) \quad \frac{qF(t)}{t^{q}} \geq \frac{\alpha}{2}, \quad \forall t \in [0, \kappa].
\]
Choose a nonnegative function $v_{0} \in S(a) \cap L^{\infty}(\mathbb{R}^{N})$ and denote
\[
\mathbb{H}(u_{0}, t)(x) = e^{\frac{Nt}{p}} v_{0}(e^{t}x), \quad \text{for all } x \in \mathbb{R}^{N} \text{ and all } t \in \mathbb{R}.
\]
It is easy to get
\[
\int_{\mathbb{R}^{N}} |\mathbb{H}(v_{0}, t)(x)|^{p} dx = a^{p}
\]
and
\[
\int_{\mathbb{R}^{N}} F(\mathbb{H}(v_{0}, t)(x)) dx = e^{-Nt} \int_{\mathbb{R}^{N}} F(e^{\frac{Nt}{p}} v_{0}(x)) dx.
\]
Moreover, for $t < 0$ and $|t|$ large enough, we also have
\[
0 \leq e^{\frac{Nt}{p}} v_{0}(x) \leq \kappa, \quad \forall x \in \mathbb{R}^{N},
\]
which combines with $(2.2)$ to get that
\[
\int_{\mathbb{R}^{N}} F(\mathbb{H}(v_{0}, t)(x)) dx \geq \frac{\alpha}{2q} e^{\frac{(q-p)Nt}{p}} \int_{\mathbb{R}^{N}} |v_{0}(x)|^{q} dx.
\]
Note that
\[
\iint_{\mathbb{R}^{2N}} \frac{|\mathcal{H}(v_0,t)(x) - \mathcal{H}(v_0,t)(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = e^{Nt} \iint_{\mathbb{R}^{2N}} \frac{|v_0(e^tx) - u_0(e^ty)|^p}{|x-y|^{N+ps}}\,dx\,dy
\]
and so,
\[
\mathcal{I}_\mu(\mathcal{H}(v_0,s)) \leq \frac{e^{pt}}{p} \iint_{\mathbb{R}^{2N}} \frac{|v_0(x) - v_0(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{\mu p}{p} - \frac{\alpha e^{(q-p)Nt}}{2q} \int_{\mathbb{R}^N} |v_0(x)|^q\,dx.
\]
Since \( q \in (p, p + \frac{v^2}{N}) \), increasing \(|t|\) if necessary, we derive that
\[
\frac{e^{pt}}{p} [v_0]^p_p - \frac{\alpha e^{(q-p)Nt}}{2q} \int_{\mathbb{R}^N} |v_0(x)|^q\,dx = A_t < 0,
\]
then,
\[
\mathcal{I}_\mu(\mathcal{H}(v_0,t)) \leq A_t + \frac{\mu p}{p}.
\]
Now, we fix \( \nu > 0 \) such that
\[
A_t + \nu da^p < 0.
\]
From that inequality, if \( \mu < \nu \), then
\[
\mathcal{I}_\mu(\mathcal{H}(v_0,t)) < 0, \quad \forall \mu \in [0, \nu),
\]
and we have \( \mathcal{I}_{\mu,a} < 0 \), then Lemma 2.3 is proved.

**Lemma 2.4.** We fix \( \mu \in [0, \nu) \) and let \( 0 < a_1 < a_2 \). Then, we have \( \frac{\mu}{a_2} \mathcal{I}_{\mu,a_2} < \mathcal{I}_{\mu,a_1} < 0 \).

**Proof.** First, we see that \(||v(x)| - |v(y)||| \leq |v(x) - v(y)|\) for all \( x, y \in \mathbb{R}^N \) and \( v \in \mathcal{W}^{s,p}(\mathbb{R}^N) \). Hence we get
\[
\iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \leq \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy
\]
for all \( v \in \mathcal{W}^{s,p}(\mathbb{R}^N) \). Let \( \varepsilon > 1 \) be such that \( a_2 = \varepsilon a_1 \) and \( (v_n) \subset S(a_1) \) be a nonnegative minimizing sequence with respect to the \( \mathcal{I}_{\mu,a_1} \), which exists due to \( \mathcal{I}_\mu(v) \geq \mathcal{I}_\mu(|v|) \) for all \( v \in \mathcal{W}^{s,p}(\mathbb{R}^N) \). Namely,
\[
\mathcal{I}_\mu(v_n) \to \mathcal{I}_{\mu,a_1} \quad \text{as} \quad n \to +\infty.
\]
We denote \( u_n = \varepsilon v_n \), then \( u_n \in S(a_2) \). From the condition \((f_3)\), the function \( \frac{F(t)}{t^q} \) is increasing on \( t \in (0, +\infty) \), then we get
\[
F(\varepsilon t) \geq \varepsilon^q F(t), \quad \forall t, l > 0 \quad \text{and} \quad t \geq 1.
\]
Thus, we deduce
\[
\mathcal{I}_{\mu,a_2} \leq \mathcal{I}_\mu(u_n) = \varepsilon^p \mathcal{I}_\mu(v_n) + \varepsilon^p \int_{\mathbb{R}^N} F(v_n)\,dx - \int_{\mathbb{R}^N} F(\varepsilon v_n)\,dx \leq \varepsilon^p \mathcal{I}_\mu(v_n) + (\varepsilon^p - \varepsilon^n) \int_{\mathbb{R}^N} F(v_n)\,dx.
\]

**Claim 2.5.** We show that there exist a positive constant \( C > 0 \) and \( n_0 \in \mathbb{N} \) such that
\[
\int_{\mathbb{R}^N} F(v_n)\,dx \geq C \quad \text{for all} \quad n \geq n_0.
\]
Assume that by a contradiction, there exists a subsequence of \((v_n)\), still denoted by \((v_n)\) such that
\[
\int_{\mathbb{R}^N} F(v_n) \, dx \to 0, \quad \text{as } n \to +\infty.
\]
From the inequality
\[
0 > \mathcal{J}_{\mu,a_1} + o_n(1) = \mathcal{I}_\mu(v_n) \geq - \int_{\mathbb{R}^N} F(v_n) \, dx, \quad n \in \mathbb{N},
\]
we get \(\mathcal{J}_{\mu,a_1} = 0\), it is a contradiction with Lemma 2.3, and Claim 2.5 is proved. By Claim 2.5 and \(\xi^p - \xi^{q_1} < 0\), then for \(n\) large enough, we deduce
\[
\mathcal{J}_{\mu,a_2} \leq \xi^p \mathcal{I}_\mu(v_n) + (\xi^p - \xi^{q_1})C.
\]
Taking \(n \to +\infty\), it follows that
\[
\mathcal{J}_{\mu,a_2} \leq \varepsilon^p \mathcal{J}_{\mu,a_1} + (\varepsilon^p - \varepsilon^{q_1})C < \varepsilon^p \mathcal{J}_{\mu,a_1},
\]
that is,
\[
\frac{a_1^p}{a_2^p} \mathcal{J}_{\mu,a_2} < \mathcal{J}_{\mu,a_1}.
\]
We finish the proofs of Lemma 2.4. \(\square\)

Our following result is a compactness theorem on \(S(a)\) for minimizing sequence.

**Theorem 2.6.** Let \(\mu \in [0, \nu_a]\) and \((v_n) \subset S(a)\) be a minimizing sequence for \(\mathcal{I}_\mu\). Then, up to a subsequence, we have either
i) \((v_n)\) is strongly convergent in \(W^{s,p}(\mathbb{R}^N)\),
or
ii) There exists \((y_n) \subset \mathbb{R}^N\) with \(|y_n| \to +\infty\) such that the sequence \(u_n(x) = v_n(x + y_n)\) is strongly convergent to a function \(\hat{v} \in S(a)\) with \(\mathcal{I}_\mu(\hat{v}) = \mathcal{J}_{\mu,a}\).

**Proof.** Since \(\mathcal{I}_\mu\) is coercive on \(S(a)\), the sequence \((v_n)\) is bounded, and so, \(v_n \rightharpoonup v\) in \(W^{s,p}(\mathbb{R}^N)\) for some subsequence. If \(v \neq 0\) and \(|v|^p = b \neq a\), we must have \(b \in (0,a)\), and by the Brézis-Lieb Lemma (see [34]),
\[
|v_n|^p = |v_n - v|^p + |v|^p + o_n(1)
\]
and [8, Lemma 2.5], we have
\[
|[v_n]|_{s,p}^p = |v_n - v|_{s,p}^p + |v|_{s,p}^p + o_n(1).
\]
Since \(F\) is a differentiable function with subcritical growth, then we get
\[
\int_{\mathbb{R}^N} F(v_n) \, dx = \int_{\mathbb{R}^N} F(v_n - v) \, dx + \int_{\mathbb{R}^N} F(v) \, dx + o_n(1).
\]
Denote by \(u_n = v_n - v\), \(d_n = |u_n|\) and supposing that \(|u_n| \to d\), we get \(a^p = b^p + d^p\) and \(d_n \in (0,a)\) for \(n\) large enough. Hence,
\[
\mathcal{J}_{\mu,a} + o_n(1) = \mathcal{I}_\mu(u_n) = \mathcal{I}_\mu(u_n) + \mathcal{I}_\mu(v) + o_n(1) \geq \mathcal{J}_{\mu,d_n} + \mathcal{J}_{\mu,b} + o_n(1),
\]
and Lemma 2.4 gives that
\[
\mathcal{J}_{\mu,a} + o_n(1) \geq \frac{a^p}{a^p} \mathcal{J}_{\mu,a} + \mathcal{J}_{\mu,b} + o_n(1).
\]
Take \(n \to +\infty\), we obtain
\[
(2.3) \quad \mathcal{J}_{\mu,a} \geq \frac{a^p}{a^p} \mathcal{J}_{\mu,a} + \mathcal{J}_{\mu,b}.
\]
Since \( b \in (0, a) \), using Lemma 2.4 and (2.3), we have
\[
\mathcal{J}_{\mu,a} > \frac{d^p}{a^p} \mathcal{J}_{\mu,a} + \frac{b^p}{a^p} \mathcal{J}_{\mu,a} = \left(\frac{d^p}{a^p} + \frac{b^p}{a^p}\right) \mathcal{J}_{\mu,a},
\]
which is a contradiction. Then \(|v|^p = a\), and we have \( u \in S(a) \). Because \(|v_n|^p = |v|^p = a\), \( v_n \rightarrow v \) in \( L^p(\mathbb{R}^N) \) and \( L^p(\mathbb{R}^N) \) is reflexive, we deduce
\[
(2.4) \quad v_n \rightarrow v \quad \text{in} \quad L^p(\mathbb{R}^N).
\]
By the conditions \((f_1), (f_2)\) and using Dominated convergence theorem, we get
\[
(2.5) \quad \int_{\mathbb{R}^N} F(v_n) \, dx \rightarrow \int_{\mathbb{R}^N} F(v) \, dx.
\]
These limits together with \( \mathcal{J}_{\mu,a} = \lim_{n \rightarrow +\infty} \mathcal{I}_\mu(v_n) \) provide
\[
\mathcal{J}_{\mu,a} \geq \mathcal{I}_\mu(v).
\]
Since \( v \in S(a) \), we can see that \( \mathcal{I}_\mu(v) = \mathcal{J}_{\mu,a} \), then
\[
\lim_{n \rightarrow +\infty} \mathcal{I}_\mu(v_n) = \mathcal{I}_\mu(v) = \mathcal{J}_{\mu,a},
\]
that combines with (2.4) and (2.5) to give
\[
||v_n||^p \rightarrow ||v||^p
\]
in \( W^{s,p}(\mathbb{R}^N) \). It implies that \( v_n \rightarrow v \) in \( W^{s,p}(\mathbb{R}^N) \).

Now, let us assume that \( v = 0 \), that is, \( v_n \rightarrow 0 \) in \( W^{s,p}(\mathbb{R}^N) \). By an arguments as in Claim 2.5, there exists \( C > 0 \) such that
\[
(2.6) \quad \int_{\mathbb{R}^N} F(v_n) \, dx \geq C, \quad \text{for } n \in \mathbb{N} \text{ large}.
\]
We claim that there are \( R, \beta > 0 \) and \( y_n \in \mathbb{R}^N \) such that
\[
(2.7) \quad \int_{B_R(y_n)} |v_n|^p \, dx \geq \beta, \quad \text{for all } n \in \mathbb{R}^N.
\]
Otherwise, by \([8, \text{Lemma 2.1}]\), we must have \( v_n \rightarrow 0 \) in \( L^t(\mathbb{R}^N) \) for all \( t \in (p, p^*_s) \). It implies \( F(v_n) \rightarrow 0 \) in \( L^t(\mathbb{R}^N) \), which is a contradiction with (2.6). Since \( v = 0 \), the inequality (2.7) together with the fractional Sobolev embedding implies that \( (y_n) \) is unbounded. We denote by \( v_n(x) = v_n(x+y_n) \), clearly \( (v_n) \subset S(a) \) and it is also a minimizing sequence for \( \mathcal{J}_{\mu,a} \). Moreover, there exists \( \tilde{v} \in W^{s,p}(\mathbb{R}^N) \setminus \{0\} \) such that
\[
\tilde{v}_n \rightarrow \tilde{v} \quad \text{in} \quad W^{s,p}(\mathbb{R}^N) \quad \text{and} \quad \tilde{v}_n(x) \rightarrow \tilde{v}(x) \quad \text{a.e. in} \quad \mathbb{R}^N.
\]
By arguments as above proofs, we get that \( \tilde{v}_n \rightarrow \tilde{v} \) in \( W^{s,p}(\mathbb{R}^N) \).

2.1. Proof of Theorem 2.1. Using Lemma 2.3, we can get a bounded minimizing sequence \( (v_n) \subset S(a) \) such that \( \mathcal{I}_\mu(v_n) \rightarrow \mathcal{J}_{\mu,a} \). From Theorem 2.6, there is \( v \in S(a) \) with \( \mathcal{I}_\mu(v) = \mathcal{J}_{\mu,a} \).

Therefore, there exists \( \lambda_a \in \mathbb{R} \) via by the Lagrange multiplier such that
\[
(2.8) \quad \mathcal{I}_\mu(v) = \lambda_a \Psi'(v) \quad \text{in} \quad (W^{s,p}(\mathbb{R}^N))',
\]
where \( \Psi : W^{s,p}(\mathbb{R}^N) \rightarrow \mathbb{R} \) is given by
\[
\Psi(v) = \frac{1}{p} \int_{\mathbb{R}^N} |v|^p \, dx, \quad v \in W^{s,p}(\mathbb{R}^N).
\]
From (2.8), we deduce
\[
(-\Delta)_p^v + \mu |v|^{p-2}v = \lambda_a |v|^{p-2}v + f(v), \quad \text{in} \quad \mathbb{R}^N.
\]
We have \( I_\mu(v) = J_{\mu,a} < 0 \), and then \( \lambda_0 < 0 \).

We can assume that \( v \) is a nonnegative function. Indeed, it comes from the inequality \( I_\mu(|v|) \leq I_\mu(v) \). Moreover, from \( v \in S(a) \), we also have that \( |v| \in S(a) \), and we deduce

\[
J_{\mu,a} = I_\mu(v) \geq I_\mu(|v|) \geq J_{\mu,a}.
\]

Thus, \( I_\mu(|v|) = J_{\mu,a} \), and so, we can replace \( v \) by \( |v| \). Moreover, denotes \( v^* \) by the Schwarz’s symmetrization of \( v \) (see [1, Section 9.2] for first inequality), we know that

\[
\int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \geq \int_{\mathbb{R}^N} \frac{|v^*(x) - v^*(y)|^p}{|x - y|^{N+ps}} \, dx \, dy,
\]

and

\[
\int_{\mathbb{R}^N} F(v) \, dx = \int_{\mathbb{R}^N} F(v^*) \, dx,
\]

then \( v^* \in S(a) \) and \( I_\mu(v^*) = J_{\mu,a} \). Hence, we can replace \( v \) by \( v^* \). Note that \( v \in C^a(\mathbb{R}^N) \) for some \( \alpha \in (0, 1) \) by [17, Corollary 5.5]. The proof of Theorem 2.1 is now finished.

From Theorem 2.1, we get the following corollary:

**Corollary 2.7.** Fix \( a > 0 \) and let \( 0 \leq \mu_1 < \mu_2 \leq \mathcal{V}_s \). Then we have \( J_{\mu_1,a} < J_{\mu_2,a} < 0 \).

**Proof.** Let \( u_{\mu_2,a} \in S(a) \) satisfying \( I_{\mu_2}(u_{\mu_2,a}) = J_{\mu_2,a} \). Then,

\[
J_{\mu_1,a} \leq I_{\mu_1}(u_{\mu_2,a}) < I_{\mu_2}(u_{\mu_2,a}) = J_{\mu_2,a}.
\]

\[ \square \]

### 3. The Non-Autonomous Problem

In this section, we will consider the energy function \( J_\xi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R} \) given by

\[
J_\xi(v) = \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx \, dy + \int_{\mathbb{R}^N} V(\xi x)|v|^p \, dx \right) - \int_{\mathbb{R}^N} F(v) \, dx,
\]

restricted to the sphere \( S(a) \). We suppose that \( |\mathcal{V}|_{\infty} < \mathcal{V}_s \), where \( \mathcal{V}_s \) was given in Section 2.

We also denote \( J_0, J_\infty : W^{s,p}(\mathbb{R}^N) \to \mathbb{R} \) by

\[
J_0(v) = \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx \, dy - \int_{\mathbb{R}^N} F(v) \, dx \right),
\]

and

\[
J_\infty(v) = \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx \, dy + V_\infty \int_{\mathbb{R}^N} |v|^p \, dx \right) - \int_{\mathbb{R}^N} F(v) \, dx,
\]

respectively. Final, we use the symbol

\[
J_{V(y)}(v) = \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx \, dy + V(y) \int_{\mathbb{R}^N} |v|^p \, dx \right) - \int_{\mathbb{R}^N} F(v) \, dx
\]

for some \( y \in \mathbb{R}^N \). We denote \( \Gamma_{\xi,a}, \Gamma_{0,a} \) and \( \Gamma_{\infty,a} \) by

\[
\Gamma_{\xi,a} = \inf_{v \in S(a)} J_\xi(v), \quad \Gamma_{0,a} = \inf_{v \in S(a)} J_0(u), \quad \Gamma_{\infty,a} = \inf_{v \in S(a)} J_\infty(v),
\]

respectively, and \( \Gamma_{V(y),a} = \inf_{v \in S(a)} J_{V(y)}(v) \). Since \( 0 < \mathcal{V}_\infty < +\infty \), the Corollary 2.7 gives that

\[
\Gamma_{0,a} < \Gamma_{\infty,a} < 0.
\]

With that property, we can fix \( \rho_1 = \frac{1}{2}(\Gamma_{\infty,a} - \Gamma_{0,a}) \). Our first lemma in this section establishes some relationship between \( \Gamma_{\xi,a}, \Gamma_{\infty,a} \) and \( \Gamma_{0,a} \).
Lemma 3.1. We have \( \limsup_{\xi \to 0^+} \Gamma_{\xi,a} \leq \Gamma_{0,a} \) and there is \( \xi_0 > 0 \) such that \( \Gamma_{\xi,a} < \Gamma_{\infty,a} \) for all \( \xi \in (0, \xi_0) \).

Proof. Let \( v_0 \in S(a) \) with \( J_0(v_0) = \Gamma_{0,a} \). Then,

\[
\Gamma_{\xi,a} \leq J_\xi(v_0) = \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^p}{|x - y|^{N + ps}} \, dx dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x)|v_0|^p \, dx \right) - \int_{\mathbb{R}^N} F(v_0) \, dx.
\]

Taking \( \xi \to 0^+ \), we obtain

\[
\limsup_{\xi \to 0^+} \Gamma_{\xi,a} \leq \lim_{\xi \to 0^+} J_\xi(v_0) = J_0(v_0) = \Gamma_{a,0}.
\]

On combining (3.1) and (3.2), we have \( \Gamma_{\xi,a} < \Gamma_{\infty,a} \) for \( \xi \) small enough. \( \square \)

In the two following results, we suppose that \( \xi \in (0, \xi_0) \), where \( \xi_0 \) given in the Lemma 3.1.

Lemma 3.2. Fix \( \xi \in (0, \xi_0) \) and let \( (v_n) \subset S(a) \) such that \( J_\xi(v_n) \to c \) with \( c < \Gamma_{0,a} + \rho_1 < 0 \). If \( v_n \rightharpoonup v \) in \( W^{s,p}(\mathbb{R}^N) \), then \( v \not\equiv 0 \).

Proof. Suppose that \( v = 0 \), then

\[
\Gamma_{0,a} + \rho_1 + o_n(1) > J_\xi(v_n) = J_\infty(v_n) + \frac{1}{p} \int_{\mathbb{R}^N} (\mathcal{V}(\xi x) - \mathcal{V}_\infty)|v_n|^p \, dx.
\]

By the condition (V), for any given \( \zeta > 0 \), there exists \( R > 0 \) such that

\[
\mathcal{V}(x) \geq \mathcal{V}_\infty - \zeta, \quad \text{for all } |x| \geq R.
\]

Hence

\[
\Gamma_{0,a} + \rho_1 + o_n(1) > J_\xi(v_n) \geq J_\infty(v_n) + \frac{1}{p} \int_{B_R/\xi(0)} (\mathcal{V}(\xi x) - \mathcal{V}_\infty)|v_n|^p \, dx - \frac{\zeta}{p} \int_{B_R(0)} |v_n|^p \, dx.
\]

Note that \( (v_n) \) is bounded in \( W^{s,p}(\mathbb{R}^N) \) and \( v_n \to 0 \) in \( L^l(B_{R'/\xi}(0)) \) for all \( l \in [1, p^*_s) \), it implies that

\[
\Gamma_{0,a} + \rho_1 + o_n(1) \geq J_\infty(v_n) - \zeta D \geq \Gamma_{\infty,a} - \zeta D,
\]

for a suitable constant \( D > 0 \). Since \( \zeta > 0 \) is arbitrary, we get

\[
\Gamma_{0,a} + \rho_1 \geq \Gamma_{\infty,a},
\]

which is a contradiction with the definition of \( \rho_1 \). Hence, \( v \not\equiv 0 \). \( \square \)

Lemma 3.3. Assume that \( (v_n) \subset S(a) \) be a \((PS)_c\) sequence for \( J_\xi \) constrained to \( S(a) \) with \( c < \Gamma_{0,a} + \rho_1 < 0 \) and \( v_n \rightharpoonup v_\xi \) in \( W^{s,p}(\mathbb{R}^N) \), namely,

\[
J_\xi(v_n) \to c \quad \text{as} \quad n \to +\infty \quad \text{and} \quad \|J_\xi'(S(a))(v_n)\| \to 0 \quad \text{as} \quad n \to +\infty.
\]

Assume that \( u_n = v_n - v_\xi \not\to 0 \) in \( W^{s,p}(\mathbb{R}^N) \), then there is \( \beta_* > 0 \) independent of \( \xi \in (0, \xi_0) \) such that

\[
\liminf_{n \to +\infty} |v_n - v_\xi|^p \geq \beta_*,
\]

where \( \xi_0 \) is given in Lemma 3.1. 

Proof. We denote \( \Psi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R} \) by

\[
\Psi(v) = \frac{1}{p} \int_{\mathbb{R}^N} |v|^p \, dx,
\]

where
we see that $S(a) = \Psi^{-1}(\{a^p/p\})$. Then, from [34, Proposition 5.12], we can get a sequence $(\lambda_n) \subset \mathbb{R}$ such that

$$||J_\xi'(v_n) - \lambda_n \Psi'(v_n)||_{(W^{s,p}(\mathbb{R}_N))'} \to 0 \quad \text{as} \quad n \to +\infty.$$ \hfill (3.3)

Since $(v_n)$ is bounded sequence in $W^{s,p}(\mathbb{R}_N)$, then $(\lambda_n)$ is also the same and up to a subsequence, we can suppose that $\lambda_n \to \lambda_\xi$ as $n \to +\infty$. Hence,

$$J_\xi'(v_\xi) - \lambda_\xi \Psi'(v_\xi) = 0 \quad \text{in} \quad (W^{s,p}(\mathbb{R}_N))'.$$ \hfill (3.4)

To prove (3.4), we need show that

$$\int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + ps}} \, dxdy \to \int_{\mathbb{R}^{2N}} \frac{|v_\xi(x) - v_\xi(y)|^{p-2}(v_\xi(x) - v_\xi(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + ps}} \, dxdy$$ \hfill (3.5)

for all $\varphi \in W^{s,p}(\mathbb{R}_N)$. Using Hölder’s inequality, we see

$$\int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + ps}} \, dxdy \leq \left( \int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^p}{|x - y|^{N + ps}} \, dxdy \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^{2N}} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{N + ps}} \, dxdy \right)^{\frac{1}{p}}$$ \hfill (3.6)

for all $x, y \in \mathbb{R}^2$ outside a set with measure zero. For any $\zeta > 0$, there exists $\delta = \zeta / K_\ast$ such that for all measurable set $E \subset \mathbb{R}^{2N}$ such that $|E| < \delta$, we have

$$\int_E \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + ps}} \, dxdy \leq K_\ast |E| < \zeta.$$ 

Hence $\left\{ \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + ps}} \right\}$ is equi-integrable on $\mathbb{R}^{2N}$. Clearly,

$$\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + ps}} \to \frac{|v_\xi(x) - v_\xi(y)|^{p-2}(v_\xi(x) - v_\xi(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + ps}}$$

almost everywhere on $\mathbb{R}^{2N}$. Since $\varphi \in W^{s,p}(\mathbb{R}_N)$, then there exists $R > 0$ such that

$$\int_{\mathbb{R}^{2N} \setminus B_R(0)} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{N + ps}} \, dxdy < \zeta^p,$$
where $B_R(0)$ is a ball in $\mathbb{R}^{2N}$ with center 0 and radius $R$. By arguments as (3.6) and we only take integral on $\mathbb{R}^{2N} \setminus B_R(0)$, and $\{u_n\}$ is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$, then there exists a suitable constant $K_\ast > 0$ such that
\[
\int_{\mathbb{R}^{2N} \setminus B_R(0)} |v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y)) \frac{dxdy}{|x-y|^{N+ps}} \leq \left( \int_{\mathbb{R}^{2N} \setminus B_R(0)} |v_n(x) - v_n(y)|^p \frac{dxdy}{|x-y|^{N+ps}} \right)^{\frac{p-1}{p}} \times \left( \int_{\mathbb{R}^{2N} \setminus B_R(0)} |\varphi(x) - \varphi(y)|^p \frac{dxdy}{|x-y|^{N+ps}} \right)^{\frac{1}{p}} < K_\ast \zeta.
\]
Therefore all conditions of Vitali’s theorem are satisfied and we get (3.5). Similarly, we also have
\[
\text{for all } \varphi \in W^{s,p}(\mathbb{R}^N). \text{ Combine (3.3), (3.5) and (3.7), we get (3.4). By arguments as in [2, Lemma 2.8-iv], we have}
\]
\[
J'(v_n) = J'(v_\xi) + J'(u_n) + o_n(1)
\]
and
\[
\Psi'(v_n) = \Psi'(v_\xi) + \Psi'(u_n) + o_n(1).
\]
From that equalities and (3.4), we obtain
\[
J'(v_n) - \lambda_\xi \Psi'(v_n) = J'(v_\xi) - \lambda_\xi \Psi'(v_\xi) + J'(u_n) - \lambda_\xi \Psi'(u_n) + o_n(1) = J'(u_n) - \lambda_\xi \Psi'(u_n) + o_n(1),
\]
we get
\[
||J'(u_n) - \lambda_\xi \Psi'(u_n)||_{(W^{s,p}(\mathbb{R}^N))'} \to 0 \quad \text{as } n \to +\infty.
\]
From the condition $(f_3)$, we have $q_1F(t) \leq f(t)t$ for all $t \geq 0$. Then, we get
\[
0 > \rho_1 + \Gamma_{0,a} \geq \liminf_{n \to +\infty} J_\xi(v_n) = \liminf_{n \to +\infty} \left( J_\xi(v_n) - \frac{1}{p} J'_\xi(v_n) v_n + \frac{\lambda_n}{p} a^p \right) \geq \frac{\lambda_\xi}{p} a^p
\]
implying that
\[
\limsup_{\epsilon \to 0} \lambda_\epsilon \leq \frac{p(\rho_1 + \Gamma_{0,a})}{a^p} < 0.
\]
It implies that there exists $\lambda_* < 0$ independent of $\xi$ so that
\[
\lambda_\xi \leq \lambda_* < 0, \quad \forall \xi \in (0, \xi_0).
\]
From (3.8), we obtain
\[
\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} \nabla(\xi x)|u_n|^p dx - \lambda_\xi \int_{\mathbb{R}^N} |u_n|^p dx = \int_{\mathbb{R}^N} f(u_n)u_n dx + o_n(1).
\]
Combine (3.9) and (3.10), we get
\[
\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} \nabla(\xi x)|u_n|^p dx - \lambda_* \int_{\mathbb{R}^N} |u_n|^p dx \leq \int_{\mathbb{R}^N} f(u_n)u_n dx + o_n(1).
\]
By the conditions $(f_1)$ and $(f_2)$, for some $\tau > 0$, there exists $D(\tau) > 0$ so that
\[
|f(t)| \leq \tau |t|^{p-1} + D(\tau)|t|^{p-1} \quad \text{for all } t \in \mathbb{R}.
\]
From (3.11) and (3.12), we conclude
\[
\int \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N + ps}} \, dx \, dy + C_0 \int_{\mathbb{R}^N} |u_n|^p \, dx \leq C_2 |u_n|^p_1 + o_n(1),
\]
where \( C_0 \) is a constant not depending on \( \xi \in (0, \xi_0) \). By the Sobolev embedding \( W^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \), we obtain
\[
\|u_n\|^p_{W^{s,p}(\mathbb{R}^N)} \leq D_3 |u_n|^p + o_n(1) \leq D_4 \|u_n\|^p_{W^{s,p}(\mathbb{R}^N)} + o_n(1),
\]
where \( D_3 \) and \( D_4 \) > 0 are suitable constants independent of \( \xi \). Because \( u_n \not\to 0 \) in \( W^{s,p}(\mathbb{R}^N) \), then up to a subsequence of \( (u_n) \), still denoted by itself, we can suppose that \( \liminf_{n \to +\infty} \|u_n\|^p_{W^{s,p}(\mathbb{R}^N)} > 0 \). It arrives that
\[
\liminf_{n \to +\infty} \|u_n\|^p_{W^{s,p}(\mathbb{R}^N)} \geq \left( \frac{1}{D_4} \right)^{\frac{1}{p-1}}. \quad (3.14)
\]
From (3.13) and (3.14),
\[
\liminf_{n \to +\infty} |u_n|^p \geq D_5, \quad (3.15)
\]
where \( D_5 > 0 \) is a suitable constant not depending on \( \xi \). Using the fractional Gagliardo-Nirenberg inequality, we obtain
\[
|u_n|^p \leq C_{s,N,p}(|u_n|^p_{s,p})^\beta |u_n|^p(1-\alpha),
\]
then we deduce that
\[
\liminf_{n \to +\infty} |u_n|^p \leq C_{s,N,p}(\liminf_{n \to +\infty} |u_n|^p)^{\beta(1-\alpha)} K^{\beta}, \quad (3.16)
\]
\( K > 0 \) is a constant which is independent on \( \xi \in (0, \xi_0) \) and satisfies \( \|u_n\| \leq K \) for all \( n \in \mathbb{N} \). On combining (3.15) and (3.16), there exists a constant \( \beta_* > 0 \) independent on \( \xi \in (0, \xi_0) \) such that
\[
\liminf_{n \to +\infty} |v_n - v_\xi|^p \geq \beta_*. \quad \square
\]
From here, we will fix the number \( \rho \) satisfying \( 0 < \rho < \min\{\frac{1}{2}, \frac{\beta_*}{\beta}\}(\Gamma_{\infty,a} - \Gamma_{0,a}) \leq \rho_1 \).

**Lemma 3.4.** For each \( \xi \in (0, \xi_0) \), the functional \( J_\xi \) satisfies the \( (PS)_c \) condition constrained on \( S(a) \) for \( c < \Gamma_{0,a} + \rho \).

**Proof.** Let \( (v_n) \) be a \( (PS)_c \) sequence for \( J_\xi \) restricted to \( S(a) \) with \( v_n \to v_\xi \) in \( W^{s,p}(\mathbb{R}^N) \) and \( c < \Gamma_{0,a} + \rho \). Let \( \Psi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R} \) be defined by
\[
\Psi(v) = \frac{1}{p} \int_{\mathbb{R}^N} |v|^p \, dx,
\]
then \( S(a) = \Psi^{-1}(\{a^p/p\}) \). Then, by \cite[Proposition 5.12]{34}, there exists \( (\lambda_n) \subset \mathbb{R} \) such that
\[
||J_\xi'(v_n) - \lambda_n \Psi'(v_n)||_{(W^{s,p}(\mathbb{R}^N))'} \to 0 \quad \text{as} \quad n \to +\infty.
\]
From Lemma 3.3, if \( u_n = v_n - u_\xi \not\to 0 \) in \( W^{s,p}(\mathbb{R}^N) \), then there exists \( \beta_* > 0 \) independent on \( \xi \) so that
\[
\liminf_{n \to +\infty} |u_n|^p \geq \beta_*.
\]
Set $d_n = |u_n|^p$ and assume that $|u_n|^p \to d > 0$ and $|v_n|^p = b$. From Lemma 3.2, we have $b > 0$ and $J_\xi(u_n) \geq \Gamma_{\infty,d_n} + o_n(1)$, we get $d_n \in (0, a)$ for $n$ large enough. Hence, we deduce

$$c + o_n(1) = J_\xi(v_n) = J_\xi(u_n) + J_\xi(v_\xi) + o_n(1) \geq \Gamma_{\infty,d_n} + \Gamma_{0,b} + o_n(1).$$

By arguments as Lemma 2.4, we have

$$\rho + \Gamma_{0,a} \geq \frac{d_n^p}{a^p} \Gamma_{\infty,a} + \frac{b^p}{a^p} \Gamma_{0,a}.$$

Letting $n \to +\infty$, we get

$$\rho \geq \frac{d_n^p}{a^p} (\Gamma_{\infty,a} - \Gamma_{0,a}) \geq \frac{\beta_s}{a^p} (\Gamma_{\infty,a} - \Gamma_{0,a}),$$

which is a contradiction since $\rho < \frac{\beta_s}{a^p} (\Gamma_{\infty,a} - \Gamma_{0,a})$. From $u_n \to 0$ in $W^{s,p}(\mathbb{R}^N)$, that is, $v_n \to v_\xi$ in $W^{s,p}(\mathbb{R}^N)$, which implies that $|v_\xi|^p = a$ and

$$( -\Delta )^s v_\xi + \mathcal{V}(\xi x)|v_\xi|^{p-2}v_\xi = \lambda_\xi |v_\xi|^{p-2}v_\xi + f(v_\xi), \quad \text{in} \quad \mathbb{R}^N,$$

where $\lambda_\xi$ is the limit of some subsequence of $(\lambda_n)$.

\[ \square \]

4. Multiplicity result of (1.1)

Fix $\delta > 0$ and $w$ be a nonnegative solution of the problem

$$\begin{cases} ( -\Delta )^s v = \lambda |v|^{p-2}v + f(v), & \text{in} \ \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^p \, dx = a^p, \end{cases}$$

with $J_0(w) = \Gamma_{0,a}$. We denote $\eta : [0, \infty) \to [0, \infty)$ by a smooth nonincreasing cut-off function which is defined as follows

$$\eta(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{\delta}{2} \\ 0 & \text{if } t \geq \delta \end{cases}.$$

For any $y \in \mathcal{M}$, let us define

$$\Psi_{\xi,y}(x) = \eta(|\xi x - y|)w((\xi x - y)/\xi),$$

$$\tilde{\Psi}_{\xi,y}(x) = a \frac{\Psi_{\xi,y}(x)}{|\Psi_{\xi,y}|_p},$$

and $\Phi_\xi : \mathcal{M} \to S(a)$ by $\Phi_\xi(y) = \tilde{\Psi}_{\xi,y}$. We see that $\Phi_\xi(y)$ has compact support for any $y \in \mathcal{M}$.

**Lemma 4.1.** We have

$$\lim_{\xi \to 0} J_\xi(\Phi_\xi(y)) = \Gamma_{0,a}, \text{ uniformly in } \ y \in \mathcal{M}.$$

**Proof.** Conversely, we assume that there exist $\xi_0 > 0$, $(y_n) \subset \mathcal{M}$ and $\xi_n \to 0$ such that

$$|J_{\xi_n}(\Phi_{\xi_n}(y_n)) - \Gamma_{0,a}| \geq \xi_0, \quad \forall n \in \mathbb{N}.$$

By Dominated convergence theorem, we get

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\Psi_{\xi_n,y_n}|^p \, dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} |\eta(\xi_n z)w(z)|^p \, dx = \int_{\mathbb{R}^N} |w|^p \, dx = a^p,$$

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} F(\Phi_{\xi_n}(y_n)) \, dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} F\left( a \frac{\eta(\xi_n z)w(z)}{|\Psi_{\xi_n,y_n}|_p} \right) \, dx = \int_{\mathbb{R}^N} F(w) \, dx,$$
for all due to the fact that \( \lim \) then we have \( \text{Theorem 2.6}, \)

\[ u_n(x) \]

\[ \text{Claim 4.4}. \quad (\ref{4.1}) \]

(see \[8, \text{Lemmas 2.2 and 2.5}\]), and

\[ \lim_{n \to +\infty} \int_{\mathbb{R}^N} V(\xi_n x)|\Phi_{\xi_n}(y_n)|^p \, dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} \frac{a_p}{|\Psi_{\xi_n y_n}|^p} V(\xi_n z + y_n)|\eta(\xi_n z) w(z)|^p \, dz = 0. \]

Consequently,

\[ \lim_{n \to +\infty} J_{\xi_n}(\Phi_{\xi_n}(y_n)) = J_{0,a}(w) = \Gamma_{0,a}, \]

which is a contradiction.

For any \( \delta > 0 \), we choose \( \tau = \tau(\delta) > 0 \) satisfying \( \mathcal{M}_\delta \subset B_\tau(0) \). We also denote \( \chi : \mathbb{R}^N \to \mathbb{R} \) by

\[ \chi(x) = \begin{cases} x & \text{if } |x| \leq \tau \\ \frac{x}{|x|} \chi & \text{if } |x| \geq \tau. \end{cases} \]

Next, we consider \( \beta_\xi : S(a) \to \mathbb{R}^N \) given by

\[ \beta_\xi(v) = \int_{\mathbb{R}^N} \chi(\xi(x)|v|^p \, dx \bigg/ a_p, \quad v \in S(a). \]

By arguments as \[7, \text{Lemma 4.2}\], we have the following result:

**Lemma 4.2.** We have

\[ \lim_{\xi \to 0} \beta_\xi(\Phi_\xi(y)) = y, \text{ uniformly in } y \in \mathcal{M}. \]

**Proposition 4.3.** Assume that \( \xi_n \to 0 \) and \( (v_n) \subset S(a) \) with \( J_{\xi_n}(v_n) \to \Gamma_{0,a} \). Then, there is \( (\tilde{y}_n) \subset \mathbb{R}^N \) such that \( u_n(x) = v_n(x + \tilde{y}_n) \) has a convergent subsequence in \( W^{s,p}(\mathbb{R}^N) \). Furthermore, up to a subsequence, we have \( y_n = \xi_n \tilde{y}_n \to y \) for some \( y \in \mathcal{M} \).

**Proof.** We show that there are \( r_0, \tau > 0 \) and \( \tilde{y}_n \in \mathbb{R}^N \) such that

\[ (4.1) \quad \int_{B_{r_0}(\tilde{y}_n)} |v_n|^p \, dx \geq \tau \]

for all \( n \) large enough. Conversely, we get \( v_n \to 0 \) in \( L^t(\mathbb{R}^N) \) for all \( t \in (p, p_*)^* \) via \[8, \text{Lemma 2.1}\], then \( \int_{\mathbb{R}^N} F(v_n) \, dx \to 0 \). Thus, we deduce \( \lim_{n \to +\infty} J_{\xi_n}(v_n) \geq 0 \), which is a contradiction due to the fact that \( \lim_{n \to +\infty} J_{\xi_n}(v_n) = \Gamma_{0,a} < 0 \). Thus, if we denote \( u_n(x) = v_n(x + \tilde{y}_n) \), then there exists \( u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\} \) such that up to a subsequence, \( u_n \to u \) in \( W^{s,p}(\mathbb{R}^N) \). Since \( (u_n) \subset S(a) \) and \( J_{\xi_n}(v_n) \geq J_0(v_n) = J_0(u_n) \geq \Gamma_{0,a} \), it implies that \( J_0(u_n) \to \Gamma_{0,a} \). From \text{Theorem 2.6}, \( u_n \to u \) in \( W^{s,p}(\mathbb{R}^N) \) and \( u \in S(a) \).

**Claim 4.4.** \( (y_n) \) is bounded.

Indeed, if there exists a subsequence of \( \{y_n\} \), still denoted by \( \{y_n\} \) such that \( |y_n| \to +\infty \), then we have

\[ \Gamma_{0,a} = \lim_{n \to +\infty} J_{\xi_n}(u_n) \]

\[ = \liminf_{n \to +\infty} \left( \frac{1}{p} \int_{\mathbb{R}^2N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}} \, dx \, dy + \int_{\mathbb{R}^N} \frac{V(\xi_n x + y_n)|u_n|^p \, dx}{|x - y|^{N+ps}} - \int_{\mathbb{R}^N} F(u_n) \, dx \right) \]
that is,
\[ \Gamma_{0,a} \geq \frac{1}{p} \left[ \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^p}{|x - y|^{N + ps}} \, dx \, dy + \int_{\mathbb{R}^N} V_\infty |u|^p \, dx \right] - \int_{\mathbb{R}^N} F(u) \, dx \geq \Gamma_{\infty,a}, \]
which contradicts (3.1). From Claim 4.4, we can assume that \( y_n \rightarrow y \) in \( \mathbb{R}^N \). By arguments as above, we get
\[ \Gamma_{0,a} \geq \frac{1}{p} \left[ \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^p}{|x - y|^{N + ps}} \, dx \, dy + \int_{\mathbb{R}^N} V(y) |u|^p \, dx \right] - \int_{\mathbb{R}^N} F(u) \, dx \geq \Gamma_{\psi(y),a}. \]
By Corollary 2.7, if \( y \notin \mathcal{M} \), then \( \Gamma_{\psi(y),a} > \Gamma_{0,a} \) since \( \psi(y) > 0 \). It is a contradiction, then \( \psi(y) = 0 \), that is, \( y \in \mathcal{M} \).

We consider a positive function \( \mathfrak{h} : [0, +\infty) \rightarrow [0, +\infty) \) satisfying \( \mathfrak{h}(\xi) \rightarrow 0 \) as \( \xi \rightarrow 0 \) and let
\[ (4.2) \quad \tilde{S}(a) = \{ v \in S(a) : J_\xi(v) \leq \Gamma_{0,a} + \mathfrak{h}(\xi) \}. \]
By Lemma 4.1, the function \( \mathfrak{h}(\xi) = \sup_{y \in \mathcal{M}} |J_\xi(\Phi_\xi(y)) - \Gamma_{0,a}| \) satisfies \( \mathfrak{h}(\xi) \rightarrow 0 \) as \( \xi \rightarrow 0 \). Therefore, \( \Phi_\xi(y) \in \tilde{S}(a) \) for all \( y \in \mathcal{M} \). By arguments as [7, Lemma 4.5], we have the result as follows:

**Lemma 4.5.** Let \( \delta > 0 \) and \( \mathcal{M}_\delta = \{ x \in \mathbb{R}^N : \text{dist}(x, \mathcal{M}) \leq \delta \} \). Then,
\[ \lim_{\xi \rightarrow 0} \sup_{v \in \tilde{S}(a)} \inf_{z \in \mathcal{M}_\delta} |\beta_\xi(v) - z| = 0. \]

### 4.1. Proof of Theorem 1.1.
We first show the existence of multiple normalized solutions to (1.1). Fix \( \xi \in (0, \xi_0) \). Then, by Lemmas 4.1, 4.2 and 4.5, and arguments as in [12], we see that \( \beta_\xi \circ \Phi_\xi \) is homotopic with the inclusion map \( id : \mathcal{M} \rightarrow \mathcal{M}_\delta \), and so,
\[ \text{cat}(\tilde{S}(a)) \geq \text{cat}_{\mathcal{M}_\delta}(\mathcal{M}). \]

By arguments as in Lemma 2.3, we also have that \( J_\xi \) is bounded from below on \( S(a) \). From Lemma 3.4, the functional \( J_\xi \) verifies the \((PS)_c\) condition for \( c \in (\Gamma_{0,a}, \Gamma_{0,a} + \mathfrak{h}(\xi)) \). Then we can apply Lusternik-Schnirelmann category theorem for critical points (see [15] and [34]) to get that \( J_\xi \) has at last \( \text{cat}_{\mathcal{M}_\delta}(\mathcal{M}) \) critical points on \( S(a) \).

Let \( v_\xi \) a solution of (1.1) with \( J_\xi(v_\xi) \leq \Gamma_{0,a} + \mathfrak{h}(\xi) \), where \( \mathfrak{h} \) is defined in (4.2). By arguments as in Proposition 4.3, for each \( \xi \rightarrow 0 \), there exists a sequence \( \tilde{y}_n \in \mathbb{R}^N \) such that \( y_n = \xi_n \tilde{y}_n \rightarrow y \) with \( y \in \mathcal{M} \) and \( u_n(x) = v_{\xi_n}(x + \tilde{y}_n) \) is converges strongly to \( u \in W^{s,p}(\mathbb{R}^N) \) with \( u \neq 0 \). We know that \( u_n \) is a solution of
\[ (-\Delta)_p u_n + V(\xi_n x + y_n)|u_n|^{p-2} u_n = \lambda_n |u_n|^{p-2} u_n + f(u_n), \quad \text{in} \quad \mathbb{R}^N, \]
with
\[ \limsup_{\xi \rightarrow 0} \lambda_n \leq \frac{p(\rho_1 + \Gamma_{0,a})}{a^p} < 0. \]
Because \( u_n \rightarrow u \) in \( W^{s,p}(\mathbb{R}^N) \), then
\[ \lim_{|x| \rightarrow +\infty} u_n(x) = 0, \quad \text{uniformly in} \quad \mathbb{N}. \]
Thus, given \( \tau > 0 \), there are \( \mathfrak{R}_1 > 0 \) and \( n_0 \in \mathbb{N} \) such that
\[ |u_n(x)| \leq \frac{1}{2} \left( \frac{\tau}{2|B_{\mathfrak{R}_1}(0)|} \right)^{1/p} \quad \text{for} \quad |x| \geq \mathfrak{R}_1 \quad \text{and} \quad n \geq n_0. \]
In the following, we prove that there exists $\delta > 0$ such that $|u_n|_\infty \geq \delta$ for all $n$ large enough. Indeed, from (4.1), we can choose $R_1 > r_0$ such that

$$0 < \frac{\tau}{2} \leq \int_{B_0(0)} |u_n|^p dx \leq |B_{R_1}(0)| |u_n|_\infty^p$$

for all $n$ large enough. Here, we choose $\delta = \left( \frac{\tau}{2/|B_{R_1}(0)|} \right)^{1/p}$. We denote $z_n$ is global maximum of $u_n$, then $|z_n| \leq R_1$ for all $n \in \mathbb{N}$ large enough. Now, let us consider $\zeta_n \in \mathbb{R}^N$ such that $|v_n(\zeta_n)| = |v_n|_\infty$ for all $n \in \mathbb{N}$. Then, $\zeta_n = z_n + \tilde{y}_n$ and

$$\lim_{n \to +\infty} V(\zeta_n \zeta_n) = \lim_{n \to +\infty} V(\zeta_n z_n + \zeta_n \tilde{y}_n) = V(y) = 0.$$ 

5. Proof of Theorem 1.2

We fix $\rho_0 > 0$ and $r_0 > 0$ such that

(i). $B_{\rho_0}(b_i) \cap B_{\rho_0}(b_j) = \emptyset$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, l\}$;

(ii). $\bigcup_{i=1}^l B_{\rho_0}(b_i) \subset B_{r_0}(0)$;

(iii). $K_{2\rho} = \bigcup_{i=1}^l B_{\rho}(b_i).$ We denote the function $Q_\xi : W^{s,p}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ given by

$$Q_\xi(v) = \int_{\mathbb{R}^N} \Theta(\varepsilon x)|v|^p dx,$$

where $\Theta : \mathbb{R}^N \to \mathbb{R}^N$ is defined as

$$\Theta(x) = \begin{cases} x, & \text{if } |x| \leq r_0 \\ r_0 \frac{x}{|x|}, & \text{if } |x| > r_0. \end{cases}$$

With above notations, we have the first following result.

Lemma 5.1. Suppose that $f$ satisfies the conditions $(f_1)$ -- $(f_3)$ and $(\mathcal{F})$ holds. Then there exists $\rho_2 \in (0, \rho)$ such that if $v \in S(a)$ and $J_\xi(v) \leq \Gamma_{0,a} + \rho_2$, then $Q_\xi(v) \in K_{\rho_2}^{n_0}$ for all $\xi \in (0, \xi_0)$, where $\xi_0$ is given in Lemma 3.1.

Proof. Assume that there exists $\rho_n \to 0$, $\xi_n \to 0$ and $\{v_n\} \subset S(a)$ such that

$$J_\xi(u_n) \leq \Gamma_{0,a} + \rho_n$$

Then we get $\Gamma_{0,a} \leq J_0(v_n) \leq J_{\xi_n}(v_n) \leq \Gamma_{0,a} + \rho_n$, which implies that $J_0(v_n) \to \Gamma_{0,a}$ as $n \to \infty$. By Theorem 2.6, up to a subsequence, still denote by itself, we have either

(i) $(v_n)$ is strongly convergent in $W^{s,p}(\mathbb{R}^N)$,

or

(ii) There exists $(y_n) \subset \mathbb{R}^N$ with $|y_n| \to +\infty$ such that the sequence $u_n(x) = v_n(x + y_n)$ converges strongly to a function $u \in S(a)$ with $J_0(u) = \Gamma_{0,a}$.

When case (i) occurs, using Lebesgue’s dominated convergence theorem, we get

$$Q_{\xi_n}(v_n) = \int_{\mathbb{R}^N} \Theta(\xi_n x)|v_n|^p dx \to \int_{\mathbb{R}^N} |v_n|^p dx \to 0 \in K_{\rho_2}^{n_0}$$

as $n \to \infty$, which is a contradiction.
If case $(ii)$ occurs, then up to a subsequence, still denote by $\{\xi_n y_n\}$, we assume that $\xi_n y_n \to y \in \mathbb{R}^N$ or $|\xi_n y_n| \to \infty$ as $n \to \infty$. If $\xi_n y_n \to y \in \mathbb{R}^N$, then by Lebesgue’s dominated convergence theorem, we have

$$J_{\xi_n}(v_n) = \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x - y|^{N+ps}} \, dx dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x)|v_n|^p \, dx \right) - \int_{\mathbb{R}^N} F(v_n) \, dx$$

(5.2)$$= \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}} \, dx dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x + \xi_n y_n)|u_n|^p \, dx \right) - \int_{\mathbb{R}^N} F(u_n) \, dx \to J_{\mathcal{V}(y)}(u)$$
as $n \to \infty$. Combine (5.1) and (5.2), we deduce that

$$\Gamma_{0,a} \geq J_{\mathcal{V}(y)}(v) \geq \Gamma_{\mathcal{V}(y),a}.$$

We will show that $\mathcal{V}(y) = 0$, it means that $y_0 = b_i$ for some $i \in \{1, \ldots, l\}$. We assume that $\mathcal{V}(y) > 0$, then $\Gamma_{0,a} < \Gamma_{\mathcal{V}(y),a}$, which contradicts with (5.3). Hence, we arrive that

$$\Gamma_{0,a} \geq J_{\mathcal{V}(y)}(v) \geq \Gamma_{\mathcal{V}(y),a}.$$

as $n \to \infty$, for some $i \in \{1, \ldots, l\}$, which is a contradiction.

If $|\xi_n y_n| \to \infty$, then by arguments as above, we get $\Gamma_{0,a} \geq \Gamma_{\mathcal{V}(y),a}$, which is impossible with (3.1). We finish the proof of Lemma 5.1. \hfill \Box

Next, we define some useful sets as follows:

$$\gamma_\xi^i = \{v \in S(a) : |Q_{\xi}(v) - b_i| \leq \rho_0\}, \quad \partial \gamma_\xi^i = \{v \in S(a) : |Q_{\xi}(u) - b_i| = \rho_0\}$$

and

$$\zeta_\xi^i = \inf_{v \in \gamma_\xi^i} J_{\xi}(v), \quad \hat{\zeta}_\xi^i = \inf_{v \in \partial \gamma_\xi^i} J_{\xi}(v).$$

**Lemma 5.2.** Assume that $f$ satisfies the conditions $(f_1)$ – $(f_3)$ and $(\mathcal{V})$ holds. Then

$$\zeta_\xi^i < \Gamma_{0,a} + \rho_2 \text{ and } \zeta_\xi^i < \hat{\zeta}_\xi^i \text{ for all } \xi \in (0, \xi_0).$$

**Proof.** Give $v \in W^{s,p}(\mathbb{R}^N)$ such that $J_0(v) = \Gamma_{0,a}$. For each $i \in \{1, \ldots, l\}$, we denote the function $\tilde{\xi}_\xi^i : \mathbb{R}^N \to \mathbb{R}$ by $\tilde{\xi}_\xi^i := v(x - \frac{b_i}{\xi})$. Then we see that $\tilde{\xi}_\xi^i \in S(a)$ for all $\xi > 0$ and $1 \leq i \leq l$. By a simple caculation, we obtain

$$J_{\xi}(\tilde{\xi}_\xi^i) = \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|\tilde{\xi}_\xi^i(x) - \tilde{\xi}_\xi^i(y)|^p}{|x - y|^{N+ps}} \, dx dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x)|\tilde{\xi}_\xi^i|^p \, dx \right) - \int_{\mathbb{R}^N} F(\tilde{\xi}_\xi^i) \, dx$$

$$= \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx dy + \int_{\mathbb{R}^N} \mathcal{V}(\xi x + \xi_n y_n)|v|^p \, dx \right) - \int_{\mathbb{R}^N} F(v) \, dx.$$

Taking $\xi \to 0^+$ in previous equality, we get

$$\lim_{\xi \to 0^+} J_{\xi}(\tilde{\xi}_\xi^i) = J_{\mathcal{V}(b_i)}(v) = J_0(v) = \Gamma_{0,a}.$$
By the definition of $Q_\xi$, we have
\[
Q_\xi(\tilde{\varphi}_\xi) = \frac{\int_{\mathbb{R}^N} \Theta(\varepsilon x)|\tilde{\varphi}_\xi|^p dx}{\int_{\mathbb{R}^N} |\tilde{\varphi}_\xi|^p dx} = \frac{\int_{\mathbb{R}^N} \Theta(\xi x + b_i)|v|^p dx}{\int_{\mathbb{R}^N} |v|^p dx} \to \Theta(b_i) = b_i
\]
as $\xi \to 0^+$ since $b_i \in B_{r_0}(0)$ for all $i = 1, \ldots, l$. It follows that $\tilde{\varphi}_\xi \in \gamma^i_\xi$ for $\xi$ small enough. From (5.4), we deduce that $\Gamma_{0,a} + \rho_2 > J_\xi(\tilde{\varphi}_\xi)$ for all $\xi \in (0, \xi_0)$. Here, we can decrease $\xi_0$ if necessary. Then from definition of $\gamma^i_\xi$, we arrive at
\[
(5.5) \quad \forall_{0,a} + \rho_2 > \zeta^i_\xi
\]
for all $i \in \{1, \ldots, l\}$.

Next, we prove the second statement. If $v \in \partial\gamma^i_\xi$, then $v \in S(a)$ and $|Q_\xi(v) - b_i| = \rho_0 > \frac{\rho_0}{2}$, and $Q_\xi(v) \notin K_{\rho_0}$. From Lemma 5.1, we have $J_\xi(v) > \Gamma_{0,a} + \rho_2$ for all $v \in \partial\gamma^i_\xi$ and $\xi \in (0, \xi_0)$, and using (5.5) to obtain that
\[
\zeta^i_\xi = \inf_{v \in \partial\gamma^i_\xi} J_\xi(v) \geq \Gamma_{0,a} + \rho_2 > \zeta^i_\xi.
\]
This concludes the proof. 

Proof of Theorem 1.2. For each $i \in \{1, \ldots, l\}$, by Ekeland’s variational principle [14], there exists a sequence $\{v^i_n\} \subset S(a)$ satisfying $J_\xi(v^i_n) \to \zeta^i_\xi$ and
\[
J_\xi(v) - J_\xi(v^i_n) \geq -\frac{1}{n}||v - v^i_n||, \text{ for all } v \in \gamma^i_\xi, v \neq v^i_n.
\]
By Lemma 5.2, we have $\zeta^i_\xi < \hat{\zeta}_i$. Thus $v^i_n \in \gamma^i_\xi \setminus \partial\gamma^i_\xi$ for all $n$ large enough. For $\delta > 0$ small enough, we consider the map $\alpha : (-\delta, \delta) \to S(a)$ given by $\alpha(t) = \frac{v^i_n + tv}{|v^i_n + tv|^p}$, belongs to $C^1((-\delta, \delta), S(a))$, and satisfies
\[
\alpha(t) \in \gamma^i_\xi \setminus \partial\gamma^i_\xi \text{ for all } t \in (-\delta, \delta), \alpha(0) = v^i_n, \alpha'(0) = v,
\]
where
\[
v \in T_{v^i_n} S(a) = \{w \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |v^i_n|^p - 2v^i_n w dx = 0\}.
\]
We have
\[
J_\xi(\alpha(t)) - J_\xi(v^i_n) \geq -\frac{1}{n}||\alpha(t) - v^i_n|| \text{ for all } t \in (-\delta, \delta),
\]
which implies that
\[
(5.6) \quad J_\xi(\alpha(t)) - J_\xi(\alpha(0)) = J_\frac{\xi(\alpha(t)) - J_\xi(v^i_n)}{t} \geq -\frac{1}{n}||\alpha(t) - v^i_n||
\]
\[
(5.7) \quad = -\frac{1}{n}||\alpha(t) - \alpha(0)||.
\]
Since $J_\xi \in C^1(W^{s,p}(\mathbb{R}^N), \mathbb{R})$, taking $t \to 0^+$ in (5.6), we get
\[
< J_\xi'(v^i_n), v > \geq -\frac{1}{n}||v||.
\]
Replace $v$ by $-v$, we deduce
\[
\sup_{||v|| \leq 1} \{< J_\xi'(v^i_n), v > \} \leq \frac{1}{n},
\]
which leads to that
\[ J_{\xi}(v_{n}^{i}) \to c^{i}_{\xi} \text{ and } J'_{\xi}|_{S(a)}(v_{n}^{i}) \to 0 \text{ as } n \to \infty. \]
Therefore, \( v_{n}^{i} \) is a \((PS)_{c^{i}_{\xi}}\) of \( J_{\xi} \). By Lemma 5.2, we have
\[ c^{i}_{\xi} < \Gamma_{0,a} + \rho_{2} < \Gamma_{0,a} + \rho < 0. \]
Then we can apply Lemma 3.4 to show that there exists \( v^{i} \in W^{s,p}(\mathbb{R}^{N}) \cap S(a) \) satisfying
\[ v^{i} \in \gamma^{i}_{\xi}, J_{\xi}(v^{i}) = c^{i}_{\xi} \text{ and } J'_{\xi}|_{S(a)}(v^{i}) = 0. \]
Furthermore, \( Q_{\xi}(v^{i}) \in \overline{B_{\rho_{0}}(b_{i})} \), \( Q_{\xi}(v^{j}) \in \overline{B_{\rho_{0}}(b_{j})} \) and \( \overline{B_{\rho_{0}}(b_{i})} \cap \overline{B_{\rho_{0}}(b_{j})} = \emptyset \) for all \( i \neq j \in \{1, \ldots, l\} \), then \( J_{\xi} \) has at least \( l \) nontrivial critical points on \( S(a) \) for all \( \xi \in (0, \xi_{0}) \). Because \( J_{\xi}(v^{j}) = c^{j}_{\xi} < 0 \), and using the condition \((f_{3})\), we get
\[
\lambda^{i} a^{p} = \frac{1}{p} \left( \int_{\mathbb{R}^{N}} |v^{i}(x) - v^{i}(y)|^{p} \, dx \right)^{\frac{p}{p-1}} \int_{\mathbb{R}^{N}} \nabla(x)|v^{i}|^{p} \, dx - \int_{\mathbb{R}^{N}} f(v^{i})v^{i} \, dx
\]
\[
= \frac{1}{p} \left( \int_{\mathbb{R}^{N}} |v^{i}(x) - v^{i}(y)|^{p} \, dx \right)^{\frac{p}{p-1}} \int_{\mathbb{R}^{N}} \nabla(x)|v^{i}|^{p} \, dx - \int_{\mathbb{R}^{N}} F(v^{i}) \, dx
\]
\[+ \int_{\mathbb{R}^{N}} F(v^{i}) \, dx - \int_{\mathbb{R}^{N}} f(v^{i})v^{i} \, dx.
\]
\[= J_{\xi}(v^{i}) + \int_{\mathbb{R}^{N}} F(v^{i}) \, dx - \int_{\mathbb{R}^{N}} f(v^{i})v^{i} \, dx < 0.
\]
Then \( \lambda^{i} < 0 \) for all \( i = 1, \ldots, l \). We finish the proof of Theorem 1.2.

\[\square\]

Declarations

Data availability. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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